Electronic Journal of Differential Equations, Vol. 2005(2005), No. 43, pp. 1–5.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

UNIQUENESS OF CRITICAL POINTS FOR SEMI-LINEAR DIRICHLET PROBLEMS IN CONVEX DOMAINS

JAIME ARANGO

Abstract. We establish sufficient conditions for the existence of a unique critical point for the solution to the semi linear elliptic problem $\Delta u = f(u) + w$ with zero Dirichlet boundary condition.

1. Introduction

Regarding the St. Vennat elastic torsion problem

$$\begin{aligned} \Delta u &= w \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

in a convex domain Ω, Makar-Limanov [10] proved that for a solution u, the function $z = \sqrt{-u}$ is concave provided w is a positive constant. From this, it follows immediately that u has exactly one critical point (point of vanishing gradient), which turns out to be an absolute minimum of u on Ω.

The elastic torsion problem (1.1) is a classical issue in PDE with references dating back to St. Vennat (1856). In addition to its importance in elasticity, it arises in fluid mechanics, where it describes the steady unidirectional flow of a viscous fluid down a pipe of cross section Ω, the pressure of the gradient along the pipe being constant. It appears also in connection with vortex streets (see [5, 6]) and isoperimetrical inequalities [11].

An important question related to problem (1.1) is to determine the minimum of u and its location in Ω. This task can be greatly simplified if it is assured – and here lies the importance of Makar-Limanov's result – that u has a single minimum. In this paper we consider the following generalization of the above problem,

$$\begin{aligned} \Delta u &= f(u) + w \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

and the question of whether u possesses a unique critical point. Addressing this question, Kawohl [4] shows the concavity of $g \circ u$ (for an appropriated choice of g). Yet Kawohl's results are strongly based on conditions on the second derivative of the nonlinearity f. In a more recent paper Ma [9] showed, in a slightly different context, the convexity of the solution u for specific nonlinearities.

2000 *Mathematics Subject Classification.* 74K15, 35J05, 65M06.
Key words and phrases. Critical points; semi linear Dirichlet problems.

Submitted February 3, 2005. Published April 7, 2005.

The aim of this paper is to show that the solution to (1.2) possesses a unique critical point (a fortiori an absolute minimum) provided the following assumptions hold:

(A1) Ω is convex planar domain with a C^2 smooth boundary $\partial\Omega$ having a positive curvature.

(A2) $w > 0$, $f \in C^\infty(\mathbb{R}, \mathbb{R})$, $f(0) = 0$, $f'(x) > 0$ for $x < 0$.

Our approach is strongly based in the geometrical properties of eigenfunctions of some Laplace-related operators, and is very closed to the one used by Cabré and Chanillo in [1].

Before going into details, we remark that the solution of (1.2) could be interpreted as the deflection of a membrane fixed in its border hanging under the force $f(u) + w$, where $w > 0$ is a constant proportional to the density of the membrane. Having this in mind it seems very plausible that u has a single minimum.

2. Preliminaries

Existence, uniqueness and regularity of solutions to semilinear elliptic differential equation of second order are relatively well understood issues in the theory of PDE. We refer the reader to the classical work of Gilbarg and Trudinger [3] for a thorough treatment of these topics. For instance, the existence of a negative solutions u to the problem (1.2) follows from standard techniques of upper and lower solutions, whereas the uniqueness of negative solutions can be obtained via maximum principle. We limit ourselves to analyze negative solutions to (1.2) for they plausibly model the deflections of membranes under its own weight. Therefore until further notice, all solutions to (1.2) we consider are supposed to be negative on Ω. Regarding the regularity, it can be shown that a solution u to the problem (1.2) satisfies $u \in C^\infty(\Omega) \cap C^1(\overline{\Omega})$.

To begin the discussion of critical points let us suppose $x_m \in \Omega$ is an absolute minimum of a solution to (1.2). Then for all $x \in \Omega$ we have

$$0 \le \Delta u(x_m) = f(u(x_m)) + w \le f(u(x)) + w = \Delta u(x), \tag{2.1}$$

the last inequality being a consequence of assumption (A2).

We go on by proving that the solution u has no critical points on $\partial\Omega$. The argument is based on Hopf's boundary point lemma. For the reader's convenience we present here a slightly modification of Hopf's result (see[12] and [3]):

Lemma 2.1 (Hopf, 1952). *Let $G \subset \mathbb{R}^2$ be a bounded region. Further assume that $u \in C^2(G) \cap C(\overline{G})$ satisfies $\Delta u \ge 0$ and $u \le 0$ in G. If G fulfills an interior sphere condition [1] at $x_0 \in \partial G$, and $u \in C^1(G \cup \{x_0\})$, with $u(x) < u(x_0)$, for $x \in G$, then for every direction ν pointing into an interior sphere we have*

$$\frac{\partial u}{\partial \nu}(x_0) < 0.$$

We remark that the region Ω along with every point $x_0 \in \partial\Omega$ and the solution u to (1.2), satisfy the assumptions of Hopf's lemmas. Therefore it may be concluded that there are no critical points of u at the border $\partial\Omega$. The reader may have noticed that the border $\partial\Omega$ is a level set of u. Unfortunately we cannot repeat the same argument for any level set of $u^{-1}(c)$ since there is no guaranty that such level sets fulfill the interior sphere condition.

[1] There is a ball $B \subset G$ with $x_0 \in \partial G$

In a very influential paper, Cheng [2] studies the topology of particular level sets, more precisely he studies the set $v^{-1}(0)$, called nodal lines of v, when the real valued function v is a solution to an elliptic equation on a Riemann manifold. The following summarizes Cheng's results when Ω is a bounded domain of $\mathbb{R}^2$:

Theorem 2.2 (Cheng, 1976)**.** *Let h be C^∞ on Ω. If v satisfies*

$$(\Delta + h(x))\, v = 0 \quad \text{in } \Omega, \tag{2.2}$$

then for all $p \in \Omega$, there exist a spherical harmonic P_N in $\mathbb{R}^2$ of degree $N \geq 1$, and a C^1 diffeomorphism Φ from a neighborhood of p in Ω onto a neighborhood of $\Phi(p) = 0$ in $\mathbb{R}^2$, such that

$$v(x) = P_N(\Phi(x)).$$

For the proof we refer the reader to [2]. Roughly speaking, Theorem 2.2 says that the nodal lines of v are locally diffeomorphic to the nodal lines of spherical harmonics of degree N at the origin, for some $N \geq 1$. Now, as it is well known, the nodal lines of spherical harmonics are a system of equiangular rays at the origin. Accordingly, if v satisfies (2.2), then following are true:

(a) The critical points on the nodal lines are isolated.
(b) The nodal lines meet only at critical points forming there an equiangular system of more than three angles.

3. Main results

Let $\tau \in S^1 \subset \mathbb{R}^2$ and assume u solves problem (1.2). We shall consider the nodal lines of u_τ defined by (cf. Cabré and Chanillo [1])

$$u_\tau(x) = Du(x) \cdot \tau, \quad x \in \overline{\Omega}.$$

Let us write $J(x) \equiv D^2 u(x)$. A regular point $p \in \overline{\Omega}$ of the nodal lines u_τ satisfies $J(p)\,\tau \neq 0$, and the nodal lines can be locally parametrized at p by the ODE

$$\dot{x}(t) = R\, J(x(t))\, \tau, \qquad x(0) = p, \tag{3.1}$$

where R is the $\frac{\pi}{2}$ rotation 2×2-matrix.

Lemma 3.1. *Fix $\tau \in S^1$ and denote by U the nodal lines of u_τ. If $p \in U \cap \partial\Omega$, then p is a regular point of U and U is nowhere tangent to $\partial\Omega$. Moreover, U split Ω in a finite number of connected subregions.*

Proof. Since $\partial\Omega$ has a positive curvature (assumption (A1)), for all $p \in \partial\Omega$ we have $|J(p)\,\alpha \cdot \alpha| > 0$, where α is any unitary tangent vector to $\partial\Omega$ at p. On the one hand, $Du(p)$ is orthogonal to α; on the other hand $Du(p)$ is orthogonal to τ since $p \in U$, and therefore α and τ are collinear and $|J(p)\,\tau \cdot \tau| > 0$. Thus $J(p)\,\tau \neq 0$, and this implies that p is a regular point of U.

Let us locally parametrize U at $p \in U \cap \partial\Omega$ by ODE (3.1). If U is tangent to $\partial\Omega$ at p then $R\,\dot{x}(0) \cdot \tau = 0$. From this it follows $-J(p)\,\tau \cdot \tau = 0$ which is a contradiction. □

We remark that that u_τ satisfies

$$(\Delta + h(x))u_\tau = 0 \quad \text{in } \Omega, \quad \text{where} quad h(x) \equiv -f'(u(x)). \tag{3.2}$$

Since any solution of (1.2) is C^∞ we have, in view of assumption (A2), that h is C^∞ and negative on Ω. Notice that u_τ cannot vanish on the border ∂G of a sub domain $G \subset\subset \Omega$. For in that case, u_τ satisfies elliptic equation (3.2) in G and the boundary condition $u_\tau\big|_{\partial G} = 0$, then a straightforward application of the maximum principle (recall that h is negative) yields $u_\tau \equiv 0$ on G, and a fortiori, $u_\tau \equiv 0$ on the whole domain $\bar\Omega$ which contradicts Hopf's lemma.

Lemma 3.2. *There are no critical points of u_τ along the nodal lines $u_\tau(x) = 0$. Moreover, the nodal lines $u_\tau(x) = 0$ reduces to the trace of a single C^∞ curve which intersects $\partial\Omega$ in exactly two points.*

Proof. Let us denote by U the nodal lines of of u_τ and let G be one of the subregions in which U splits Ω. We may rule out the case $G \subset\subset \Omega$, hence ∂G contains a non-trivial connected subset of $\partial\Omega$.

By assumption (A1) and lemma 3.1 there are exactly two points on $\partial\Omega$ having a unit normal direction orthogonal to a fixed $\tau \in S^1$. Hence, U intersects $\partial\Omega$ in exactly two points.

Now suppose there is a critical point of u_τ along U. By remark (b) there are at least four different regions, say G_j for $j = 1, \dots 4$, in which in which U splits Ω. Since for any $j = 1, \dots 4$ we may rule out the case $G_j \subset\subset \Omega$, we conclude that U intersect $\partial\Omega$ in at least four different points, which is again a contradiction.

If U contains no critical points of u_τ, then by theorem 2.2, U is locally diffeomorphic to the nodal lines of an spherical harmonic of degree 1. Therefore, U is either diffeomorphic to a circle or is the trace of a single C^∞ curve intersecting $\partial\Omega$ in exactly two points. But the first case does not apply since this would imply $G \subset\subset \Omega$, where G is the region bounded by U. $\square$

We can rephrase Lemma 3.2 as follows:

$$u_\tau(q) = 0 \quad \text{implies} \quad J(q)\,\tau \neq 0.$$

Next observe that any critical point of u belongs to any of the nodal lines $u_\tau(x) = 0$ for $\tau \in S^1$. Hence

$$Du(q) = 0 \quad \text{implies} \quad J(q)\,\tau \neq 0 \quad \text{for all } \tau \in S^1.$$

In other words, if q is a critical point of u then the 2×2 matrix $J(q)$ is nonsingular. We have thus proved the following result.

Lemma 3.3. *The negative solution u to 1.2 possesses a finite number of critical points.*

By lemma 3.2 the nodal lines are in fact the trace of a single C^∞ curve, thus it makes senses to to write nodal line instead of nodal lines. For now on we shall adhere to this renaming.

Suppose a nodal line $Du(x)\cdot\tau = 0$ passes through $p \in \partial\Omega$, i. e., $Du(p)\cdot\tau = 0$. It follows then $\tau = \pm R\,\mathbf{n}(p)$, where $\mathbf{n}(p)$ is the unit normal outward directions at $p \in \partial\Omega$. In view of (3.1), the nodal line passing through $p \in \partial\Omega$ can be parametrized by the solution to the initial value ODE problem

$$\dot x(t) = R\,J(x(t))\,R\,\mathbf{n}(p), \quad x(0) = p. \tag{3.3}$$

We are now in a position to show the main result of this paper.

Theorem 3.4. *The negative solution u to 1.2 possesses a single critical point which is an absolute minimum to u in Ω.*

Proof. We begin by recalling that the solutions to (3.3) have a uniformly continuous dependence on $p \in \partial\Omega$. Furthermore, notice that any level set meets all critical points, thus by lemma 3.3 it makes sense to define $f(p)$, for $p \in \partial\Omega$, to be the first critical point met by the solutions to (3.3). By a standard continuity argument f must be constant on the whole border $\partial\Omega$.

By lemma 3.2 any nodal line intersects $\partial\Omega$ in exactly two points, hence, for a given $p \in \partial\Omega$ there exists a unique $\bar{p} \in \partial\Omega$, with $\bar{p} \neq p$, such that $\bar{p}$ and p belong to the same nodal line. It is clear that $f(p) \neq f(\bar{p})$ unless u possesses a single critical point. $\square$

Acknowledgments. The author would like to thank professor G. Keady at the University of West Australia for pointing out the important references on the torsion problem, also to professor O. Perdomo at Universidad del Valle for several helpful comments concerning Cheng's theorem, The author also wants to thank Universidad del Valle for financially supporting this investigation.

References

[1] Cabré, X. and Chanillo S.; *Stable solutions of semilinear elliptic problems in convex domains.* Sel. math., New ser. 4, 1998, pp. 1-10.
[2] Cheng S. Y.; *Eigenfunctions and nodal sets*, Comment. Math. Helvetici, 51, 1976, pp.43-55.
[3] Gilbarg, G. and Trudinger, N.; *Elliptic partial differential equations of second order.* Springer, 1983.
[4] Kawohl, B.; *When are solutions to nonlinear elliptic boundary value problems convex?* Comm. in partial differential equations 10(10), 1985, pp. 1213-1225.
[5] Dragomir, S. and Keady, G.; *A Hadamard-Jensen inequality and an application to the elastic torsion problem.* J. App. Anal. 75, 2000, pp. 285-295.
[6] Keady, G. and McNabb, A.; *The elastic torsion problem: solutions in convex domains.* N.Z. Journal of Mathematics, 22, 1993, pp. 43-64.
[7] Kosmodem'yanskii, Jr.; *Sufficient conditions for the concavity of the solution of the Dirichlet problem for the equation Laplacian u =-1.* Math. Notes of Acad. Sci. of U.S.S.R. 42 (translation 1987), 798-801.
[8] Kosmodem'yanskii, Jr.; *The behavior of solutions of the equation Laplacian u =-1 in convex domains.* Soviet. Math. Doklady. 39 (translation 1989), pp. 112-114.
[9] Ma X.-N.; *Concavity estimates for a class of nonlinear elliptic equations in two dimensions.* Math. Z. 240, 2002, pp. 1-11.
[10] Makar-Limanov; *Solutions for the Dirichlet's problem for the equation* $u = 1$ *in a convex region*, Math. Notes of Acad. Sci. of U.S.S.R. 9 (1971), pp. 52-53.
[11] Payne, L. E.; *Some isoperimetric inequalities in the torsion problem for multiple connected regions.* Stud. Math. Anal. related Topics, Essays in Honor of G. Polya (1962), pp. 270-280.
[12] Protter, M, and Weinberger H.; *Maximum principles in differential equations.* Springer, 1984.

Jaime Arango
Universidad del Valle, Departamento de Matemáticas, A. A. 25 360 Cali, Colombia
E-mail address: jarango@univalle.edu.co