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INSTABILITY OF SOLUTIONS OF CERTAIN NONLINEAR VECTOR DIFFERENTIAL EQUATIONS OF THIRD ORDER

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Abstract. In this paper, study the system of differential equations

$$\dddot{X} + F(\dot{X})\ddot{X} + G(X)\dot{X} + H(X, \dot{X}, \ddot{X}) = 0\,.$$

We find sufficient conditions for the zero solution to be unstable, and to be the only periodic solution.

1. Introduction and Statement of Results

It is well-known that instability and periodic properties for various third-, fourth-, fifth-, sixth-, seventh- and eighth-order nonlinear differential equations have been discussed by many authors. In this connection, we refer the reader to the papers by Bereketoğlu [3, 4, 5, 6], Ezeilo [7, 8], Kipnis [9], Lu [11], Reissig [14], and Skrapek [16, 17], Tejumola [18], Tiryaki [19, 20, 21], C.Tunç and E.Tunç [25], Tunç [22, 23, 24, 26] and the references cited therein. However, according to our investigation of the relevant literature, in the case $n = 1$, the instability properties of nonlinear differential equations of the third order have been discussed by discussed by Bereketoğlu [4], Lu [12], Kipnis [9] and Skrapek [17]. Bereketoğlu [4], Kipnis [9], and Skrapek [17] studied the third order scalar differential equations

$$\dddot{x} + f(\dot{x})\ddot{x} + g(x)\dot{x} + h(x, \dot{x}, \ddot{x}) = 0,$$
$$\dddot{x} + p(t)x = 0,$$
$$\dddot{x} + f_1(\ddot{x}) + f_2(\dot{x}) + f_3(x) + f_4(x, \dot{x}, \ddot{x}) = 0\,.$$

We did not find literature relevant to the instability of solutions of nonlinear vector differential equations of third order. This is perhaps due to the difficulty of constructing proper Lyapunov functions for higher-order nonlinear vector differential equations. The papers mentioned above are the motivation for the present work. Our purpose is to obtain sufficient conditions under which the trivial solution $X = 0$ of vector differential equation

$$\dddot{X} + F(\dot{X})\ddot{X} + G(X)\dot{X} + H(X, \dot{X}, \ddot{X}) = 0 \tag{1.1}$$

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is unstable, and that the nontrivial solutions of equation (1.1) can not be periodic. It is assumed that $X \in \mathbb{R}^n$, that F and G are $n \times n$-symmetric matrix functions; and that $H : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ with $H(0, 0, 0) = 0$. Let F, G, H be continuous, for the uniqueness theorem to be valid.

Equation (1.1) represents a systems of real-valued third-order differential equations of the form

$$\begin{aligned}\dddot{x}_i &+ \sum_{k=1}^{n} f_{ik}(\dot{x}_1, \dot{x}_2, \dots, \dot{x}_n)\ddot{x}_k + \sum_{k=1}^{n} g_{ik}(x_1, x_2, \dots, x_n)\dot{x}_k \\ &+ h_i(x_1, x_2, \dots, x; \dot{x}_1, \dot{x}_2, \dots, \dot{x}_n; \ddot{x}_1, \ddot{x}_2, \dots, \ddot{x}_n) = 0,\end{aligned}$$

$(i = 1, 2, \dots, n)$. Let $J_F(\dot{X})$ and $J_G(X)$ denote the Jacobian matrices corresponding to $F(\dot{X})$ and $G(X)$ respectively, that is,

$$J_F(\dot{X}) = \frac{\partial f_i}{\partial \dot{x}_j}, \quad J_G(X) = \frac{\partial g_i}{\partial x_j} \quad (i, j = 1, 2, \dots, n),$$

where $(x_1, x_2, \dots, x_n), (\dot{x}_1, \dot{x}_2, \dots, \dot{x}_n), (f_1, f_2, \dots, f_n)$ and $(g_1, g_2, \dots, g_n)$ are components of X, $\dot{X}$, F and G, respectively. It will be assumed that the Jacobian matrices $J_F(\dot{X})$, $J_G(X)$ exist and are continuous.

Given any X, Y in $\mathbb{R}^n$. The symbol $\langle X, Y \rangle$ will denote the usual scalar product in $\mathbb{R}^n$, that is, $\langle X, Y \rangle = \sum_{i=1}^{n} x_i y_i$; thus $\langle X, X \rangle = \|X\|^2$. The matrix A is said to be negative-definite, when $\langle AX, X \rangle < 0$ for all non-zero X in $\mathbb{R}^n$. $\lambda_i(A)$ $(i = 1, 2, \dots, n)$ will denote the eigenvalues of the $n \times n$ matrix A.

We will use the following differential system which is equivalent to the equation (1.1):

$$\begin{gathered}\dot{X} = Y, \dot{Y} = Z, \\ \dot{Z} = -F(Y)Z - G(X)Y - H(X, Y, Z)\,.\end{gathered} \tag{1.2}$$

Which is obtained as usual by setting $\dot{X} = Y, \ddot{X} = Z$ in (1.1).

It should be noted that the Lyapunov's second (or direct) method, (see [13]), is used to verify the results established here. This method requires the construction of an appropriate Lyapunov function for the equation under study. Namely, this function and its total time derivative satisfy some fundamental inequalities.

2. The main result

We establish the following statements:

Theorem 2.1. *Let the functions F, G, H be as defined above, , and assume the following conditions are fulfilled:*

(i) $\lambda_i(F(Y)) \le 0$, *for all* $X, Y \in \mathbb{R}^n$
(ii) $\sum_{i=1}^{n} x_i h_i(X, Y, Z) > 0$ *for all* $X, Y, Z \in \mathbb{R}^n$, *where*

$$H(X, Y, Z) = (h_1(X, Y, Z), h_2(X, Y, Z), \dots, h_n(X, Y, Z)).$$

Then the trivial solution $X = 0$ of the system (1.2) *is unstable.*

Theorem 2.2. *Under the assumptions of Theorem 2.1, equation* (1.2) *has no periodic solution other than $X = 0$.*

We need the following algebraic result.

Lemma 2.3. *Let A be a real symmetric $n \times n$-matrix and $a' \geq \lambda_i(A) \geq a > 0$ ($i = 1, 2, \dots, n$), where a', a are constants. Then*

$$a'\langle X, X\rangle \geq \langle AX, X\rangle \geq a\langle X, X\rangle,$$
$$(a')^2\langle X, X\rangle \geq \langle AX, AX\rangle \geq a^2\langle X, X\rangle.$$

For the proof of this lemma, see [24].

Preliminaries. The proof of Theorem 2.1 is based on the instability criterion created by Krasovskiii [10]. According to these criteria it is necessary to show that there exists a continuously differentiable function $V(X, Y, Z)$ which has the following there properties:

(P1) In every neighbourhood of $(0, 0, 0)$ there exists a point (ξ, η, ζ) such that $V(\xi, \eta, \zeta) > 0$,

(P2) The time derivative $\frac{d}{dt}V(X, Y, Z)$ along solution path of (1.2) is positive-semidefinite,

(P3) The only solution $(X(t), Y(t), Z(t))$ of (1.2) which satisfies

$$\dot{V}(X(t), Y(t), Z(t)) = 0 \quad (t \geq 0)$$

is the trivial solution $(0, 0, 0)$.

Since the zero solution of (1.2) is isolated, the existence of a function V with the properties (P1), (P2), (P3) is sufficient for the instability of the trivial solution of (1.2).

Proof of the Theorem 2.1. Consider the continuously differentiable function

$$V(X, Y, Z) = -\int_0^1 \langle F(\sigma Y)Y, X\rangle d\sigma - \int_0^1 \langle \sigma G(\sigma X)X, X\rangle d\sigma - \langle X, Z\rangle + \frac{1}{2}\langle Y, Y\rangle, \tag{2.1}$$

which also plays an essential role in the proof of Theorem 2.2. It is clear that

$$V(0, \varepsilon, 0) = \frac{1}{2}\langle \varepsilon, \varepsilon\rangle = \frac{1}{2}\|\varepsilon\|^2 > 0$$

for all arbitrary $\varepsilon \in \mathbb{R}^n$, $\varepsilon \neq 0$. So, in every neighbourhood of $(0, 0, 0)$ there exists a point (ξ, η, ζ) such that $V(\xi, \eta, \zeta) > 0$ for all ξ, η and ζ in $\mathbb{R}^n$. Hence V has the property (P1).

Now let

$$(X, Y, Z) = (X(t), Y(t), Z(t))$$

be any solution of (1.2). An elementary differentiation from (1.2) and (2.1) yields

$$\begin{aligned} \dot{V}(t) &= \frac{d}{dt}V(X(t), Y(t), Z(t)) \\ &= -\frac{d}{dt}\int_0^1 \langle F(\sigma Y)Y, X\rangle d\sigma - \frac{d}{dt}\int_0^1 \langle \sigma G(\sigma X)X, X\rangle d\sigma \\ &\quad - \langle Y, Z\rangle + \langle X, F(Y)Z\rangle + \langle X, G(X)Y\rangle + \langle X, H(X, Y, Z)\rangle + \langle Y, Z\rangle\,. \end{aligned} \tag{2.2}$$

Note that

$$\begin{aligned}
&\frac{d}{dt}\int_0^1 \sigma\langle G(\sigma X)X, X\rangle d\sigma \\
&= \int_0^1 \langle \sigma G(\sigma X)Y, X\rangle d\sigma + \int_0^1 \sigma^2\langle J_G(\sigma X)XY, X\rangle d\sigma + \int_0^1 \sigma\langle G(\sigma X)X, Y\rangle d\sigma \\
&= \int_0^1 \langle \sigma G(\sigma X)Y, X\rangle d\sigma + \int_0^1 \sigma \frac{\partial}{\partial\sigma}\langle \sigma G(\sigma X)Y, X\rangle d\sigma \\
&= \sigma^2\langle G(\sigma X)Y, X\rangle\big|_0^1 = \langle G(X)Y, X\rangle.
\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}
&\frac{d}{dt}\int_0^1 \langle F(\sigma Y)Y, X\rangle d\sigma \\
&= \frac{d}{dt}\int_0^1 \langle F(\sigma Y)X, Y\rangle d\sigma \\
&= \int_0^1 \langle F(\sigma Y)X, Z\rangle d\sigma + \int_0^1 \sigma\langle J_F(\sigma Y)XZ, Y\rangle d\sigma + \int_0^1 \langle F(\sigma Y)Y, Y\rangle d\sigma \\
&= \int_0^1 \langle F(\sigma Y)X, Z\rangle d\sigma + \int_0^1 \sigma\frac{\partial}{\partial\sigma}\langle F(\sigma Y)X, Z\rangle d\sigma + \int_0^1 \langle F(\sigma Y)Y, Y\rangle d\sigma \\
&= \sigma\langle F(\sigma Y)X, Z\rangle\big|_0^1 + \int_0^1 \langle F(\sigma Y)Y, Y\rangle d\sigma \\
&= \langle F(Y)X, Z\rangle + \int_0^1 \langle F(\sigma Y)Y, Y\rangle d\sigma.
\end{aligned} \tag{2.4}$$

Using estimates (2.3) and (2.4) in (2.2), we obtain

$$\dot V = -\int_0^1 \langle F(\sigma Y)Y, Y\rangle d\sigma + \langle X, H(X,Y,Z)\rangle.$$

From (i) and (ii), $\dot V(t) \ge 0$ for all X, Y, Z. Therefore, $\dot V(t)$ is positive-semidefinite so that the property (P2) holds.

Again by (i) and (ii) it is clear that

$$\dot V(X(t), Y(t), Z(t)) = 0 \quad (t \ge 0)$$

implies $X = 0$ for all $t \ge 0$. So by (1.2), we have

$$Y \equiv \dot X = 0, \quad Z \equiv \dot Y = 0, \quad \dot Z = 0 \quad \text{for all } t \ge 0.$$

Thus $\dot V(x, y, z) = 0$ $(t \ge 0)$ implies

$$(X(t), Y(t), Z(t)) = (0,0,0) \quad \text{for all } t \ge 0,$$

which proves the property (P3).

Hence, the function V has the Krasovskii's properties (P1)-(P3), which completes the proof of Theorem 2.1.

Proof of Theorem 2.2. To prove the theorem it is sufficient to show that every ω-periodic solution $(X(t), Y(t), Z(t))$ of (1.2), that is $(X(t), Y(t), Z(t)) = (X(t+\omega), Y(t+\omega), Z(t+\omega))$, satisfies

$$(X(t), Y(t), Z(t)) = (0,0,0) \quad \text{for all } t. \tag{2.5}$$

To verify (2.5), again we employ the function $V(X,Y,Z)$ which is defined by (2.1). Let $(X_1,Y_1,Z_1) \equiv (X_1(t),Y_1(t),Z_1(t))$ be an arbitrary ω-periodic solution of (1.2) and consider the function $V(t) \equiv V(X_1(t),Y_1(t),Z_1(t))$ corresponding to this solution. Differentiating $V(t)$, we have

$$\dot{V}(t) = -\int_0^1 \langle F(\sigma Y_1)Y_1, Y_1\rangle d\sigma + \langle X_1, H(X_1,Y_1,Z_1)\rangle. \tag{2.6}$$

By (i) and (ii),

$$\dot{V}(t) \geq 0 \tag{2.7}$$

for all t which implies $V(t)$ is monotone in t. Since $V(t)$ is monotone and periodic, it must be constant. Therefore,

$$\dot{V}(t) = 0 \text{ for all } t. \tag{2.8}$$

Considering (2.6) and (2.8) together, and then using (i) and (ii) we obtain

$$X_1 = 0 \quad \text{for all } t. \tag{2.9}$$

Because of $\dot{X}_1 = Y_1$ and $\dot{Y}_1 = Z_1$, (2.9) in turn implies that

$$\dot{X}_1 = Y_1 = Z_1 = \dot{Z}_1 = 0 \quad \text{for all } t. \tag{2.10}$$

Hence, we get

$$(X_1(t),Y_1(t),Z_1(t)) = (0,0,0)\text{for all } t$$

which completes the proof.

Remark. Our results give n-dimensional extensions for the results established in [4].

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