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BLOW UP OF SOLUTIONS FOR KLEIN-GORDON EQUATIONS IN THE REISSNER-NORDSTRÖM METRIC

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Abstract. In this paper, we study the solutions to the Cauchy problem

$$(u_{tt} - \Delta u)_{g_s} + m^2 u = f(u), \quad t \in (0,1], x \in \mathbb{R}^3,$$
$$u(1,x) = u_0 \in \dot{B}^{\gamma}_{p,p}(\mathbb{R}^3), \quad u_t(1,x) = u_1 \in \dot{B}^{\gamma-1}_{p,p}(\mathbb{R}^3),$$

where g_s is the Reissner-Nordströ m metric; $p > 1$, $\gamma \in (0,1)$, $m \neq 0$ are constants, $f \in \mathcal{C}^2(\mathbb{R}^1)$, $f(0) = 0$, $2m^2|u| \leq f^{(l)}(u) \leq 3m^2|u|$, $l = 0, 1$. More precisely we prove that the Cauchy problem has unique nontrivial solution in $\mathcal{C}((0,1]\dot{B}^{\gamma}_{p,p}(\mathbb{R}^+))$,

$$u(t,r) = \begin{cases} v(t)\omega(r) & \text{for } t \in (0,1],\ r \leq r_1 \\ 0 & \text{for } t \in (0,1],\ r \geq r_1, \end{cases}$$

where $r = |x|$, and $\lim_{t\to 0} \|u\|_{\dot{B}^{\gamma}_{p,p}(\mathbb{R}^+)} = \infty$.

1. Introduction

In this paper, we study properties of the solutions to the Cauchy problem

$$(u_{tt} - \Delta u)_{g_s} + m^2 u = f(u), \quad t \in (0,1], x \in \mathbb{R}^3, \tag{1.1}$$
$$u(1,x) = u_0 \in \dot{B}^{\gamma}_{p,p}(\mathbb{R}^3), \quad u_t(1,x) = u_1 \in \dot{B}^{\gamma-1}_{p,p}(\mathbb{R}^3), \tag{1.2}$$

where g_s is the Reissner-Nordström [2],

$$g_s = \frac{r^2 - Kr + Q^2}{r^2} dt^2 - \frac{r^2}{r^2 - Kr + Q^2} dr^2 - r^2 d\phi^2 - r^2 \sin^2 \phi d\theta^2,$$

the constants K and Q are positive, $m \neq 0$, $p \in (1,\infty)$ and $\gamma \in (0,1)$ are fixed, $f \in \mathcal{C}^2(\mathbb{R}^1)$, $f(0) = 0$, $2m^2|u| \leq f^{(l)}(u) \leq 3m^2|u|$, $l = 0, 1$. More precisely we prove that the Cauchy problem (1.1)-(1.2) has a unique nontrivial solution u in

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$\mathcal{C}((0,1]\dot{B}^{\gamma}_{p,p}(\mathbb{R}^+))$ such that $\lim_{t\to 0}\|u\|_{\dot{B}^{\gamma}_{p,p}(\mathbb{R}^+)}=\infty$. The Cauchy problem (1.1)-(1.2) may rewrite in the form

$$\begin{aligned}&\frac{r^2}{r^2-Kr+Q^2}u_{tt}-\frac{1}{r^2}\partial_r((r^2-Kr+Q^2)u_r)\\&-\frac{1}{r^2\sin\phi}\partial_\phi(\sin\phi u_\phi)-\frac{1}{r^2\sin^2\phi}u_{\theta\theta}+m^2u=f(u),\end{aligned}\tag{1.3}$$

$$\begin{aligned}u(1,r,\phi,\theta)&=u_0\in\dot{B}^{\gamma}_{p,p}(\mathbb{R}^+\times[0,2\pi]\times[0,\pi]),\\u_t(1,r,\phi,\theta)&=u_1\in\dot{B}^{\gamma-1}_{p,p}(\mathbb{R}^+\times[0,2\pi]\times[0,\pi]),\end{aligned}\tag{1.4}$$

where $x=r\cos\phi\cos\theta$, $y=r\sin\phi\cos\theta$, $z=r\sin\theta$, $\phi\in[0,2\pi]$, $\theta\in[0,\pi]$.

When g_s is the Riemann metric, $m=0$, $f(u)=|u|^p$; $u_0,u_1\in\mathcal{C}_0^\infty(\mathbb{R}^3)$ in [1, Section 6.3] is proved that there exists $T>0$ and a unique local solution $u\in\mathcal{C}^2([0,T)\times\mathbb{R}^3)$ of (1.1)-(1.2) such that

$$\sup_{t<T,\,x\in\mathbb{R}^3}|u(t,x)|=\infty.$$

When g_s is the Riemann metric, $m=0$, $f(u)=|u|^p$, $1\le p<5$ and initial data are in $\mathcal{C}_0^\infty(\mathbb{R}^3)$, in [1] is proved that the initial value problem (1.1)-(1.2) admits a global smooth solution.

When $\phi\neq 0,\pi,2\pi$, $\theta\neq 0$ are fixed constants we obtain the Cauchy problem

$$\frac{r^2}{r^2-Kr+Q^2}u_{tt}-\frac{1}{r^2}\partial_r((r^2-Kr+Q^2)u_r)+m^2u=f(u),\tag{1.5}$$

$$u(1,r)=u_0\in\dot{B}^{\gamma}_{p,p}(\mathbb{R}^+),u_t(1,r)=u_1\in\dot{B}^{\gamma-1}_{p,p}(\mathbb{R}^+).\tag{1.6}$$

Our main result is as follows.

Theorem 1.1. *Let m be a non-zero constant, $p\in(1,\infty)$, $\gamma\in(0,1)$ and K, Q be positive constants for which*

$$K^2>4Q^2,\quad \frac{1}{1-K+Q^2}>1,\quad 1-K+Q^2>0,$$

with $1-K+Q^2$ is small enough such that

$$\frac{K-\sqrt{K^2-4Q^2}}{2}-2\sqrt{1-K+Q^2}>0.$$

Also let $f\in\mathcal{C}^2(\mathbb{R}^1)$, $f(0)=0$, $2m^2|u|\le f^{(l)}(u)\le 3m^2|u|$, $l=0,1$. Then the Cauchy problem (1.1)-(1.2) has a unique nontrivial solution $u(t,r)=v(t)\omega(r)\in\mathcal{C}((0,1]\dot{B}^{\gamma}_{p,p}(\mathbb{R}^+))$ for which

$$\lim_{t\to 0}\|u\|_{\dot{B}^{\gamma}_{p,p}(\mathbb{R}^+)}=\infty.$$

This paper is organized as follows: In section 2 we prove that the Cauchy problem (1.1)-(1.2) has unique nontrivial solution $\tilde{u}=v(t)\omega(r)\in\mathcal{C}((0,1]\dot{B}^{\gamma}_{p,p}(\mathbb{R}^+))$. In section 3 we prove that

$$\lim_{t\to 0}\|\tilde{u}\|_{\dot{B}^{\gamma}_{p,p}(\mathbb{R}^+)}=\infty,$$

where $\tilde{u}$ is the solution, which is received in section 2.

Let

$$C=\Big(\frac{p\gamma.2^{p\gamma}}{2^{p\gamma}-1}\Big)^{1/p}.$$

Let $A>0$, $Q>0$, $B>0$, $K>0$, $1<\beta<\alpha$ be constants for which

(H1) $\frac{8}{1-K+Q^2}\Big(\frac{8}{1-K+Q^2}\frac{2m^2}{\alpha^2 A}+4m^2\Big)\leq 1$, $\frac{\alpha A}{m}>1$

(H2)
$$\frac{1}{1-\alpha K+\alpha^2Q^2}\Big(\frac{1}{1-\alpha K+\alpha^2Q^2}\frac{m^2}{\alpha^4A^2}-2m^2r_1^2\Big)\Big(r_1-\frac{1}{\beta}\Big)^2\geq 1$$
and $\frac{m^2}{\alpha^4(1-\alpha K+\alpha^2Q^2)A^2}-2m^2r_1^2\geq 0$,

(H3)
$$C\Big(\frac{2^{p(1-\gamma)}}{p(1-\gamma)}\Big)^{1/p}\frac{8}{1-K+Q^2}\Big(\frac{8}{1-K+Q^2}\frac{2m^2}{\alpha^2A^2}+m^2+\frac{6m^2}{AB}\Big)<1$$

(H4) $\frac{1}{\alpha^2}\frac{1}{1-\alpha K+\alpha^2Q^2}\frac{m^2}{\alpha^2A^4}-\frac{1}{\beta^2}\frac{m^2}{A^2}>0$

(H5) $\Big(\frac{2^{p(1-\gamma)}}{p(1-\gamma)}\Big)^{1/p}\frac{16C}{1-K+Q^2}\Big(\frac{8}{1-K+Q^2}\frac{2m^2}{\alpha^2A^2}+4m^2\Big)<1$

(H6) $K^2>4Q^2$, $A\geq\frac{8}{1-K+Q^2}>1$, $\frac{6}{AB}<1$, $1>\frac{2Q^2}{K}>\frac{K-\sqrt{K^2-4Q^2}}{2}$, $1-K+Q^2>0$ is small enough such that
$$1>\frac{K-\sqrt{K^2-4Q^2}}{2}-3\sqrt{1-K+Q^2}>0,$$
$$\frac{2}{K-\sqrt{K^2-4Q^2}-2\sqrt{1-K+Q^2}}\leq\beta<\alpha\leq 3,$$

where
$$r_1=\frac{K-\sqrt{K^2-4Q^2}}{2}-\frac{\sqrt{2}}{4}\sqrt{1-K+Q^2}.$$

Example. Let
$$A=\frac{1}{\epsilon^4},\quad B=\frac{1}{\epsilon},\quad p=\frac{3}{2},\quad \gamma=\frac{1}{3},\quad \alpha=3,$$
$$\frac{1}{\beta}=\frac{K-\sqrt{K^2-4Q^2}}{2}-\frac{3}{2}\sqrt{1-K+Q^2},\quad K=\frac{4}{3}+\frac{1}{6}\epsilon^{20}-\frac{3}{2}\epsilon^2,$$
$$Q^2=\frac{1}{3}+\frac{1}{6}\epsilon^{20}-\frac{1}{2}\epsilon^2,\quad m^2=\epsilon^4,$$
where $0<\epsilon<<1$ is enough small such that (H1)-(H6) hold. Then
$$1-\alpha K+\alpha^2Q^2=1-3K+9Q^2=\epsilon^{20},$$
$$1-K+Q^2=\epsilon^2.$$

Remark 1.2. Let $\epsilon^2=1-K+Q^2$. Note that from (H6) we have $g(r)=r^2-Kr+Q^2>0$ for $r\in[0,r_1]$, $g(r)$ is decrease function for $r\in[0,r_1]$. Also (for $r\in[0,r_1]$) we have
$$\frac{r^2}{r^2-Kr+Q^2}\leq\frac{1}{1-K+Q^2}.$$
In deed, letting $\tilde{r}=\frac{K-\sqrt{K^2-4Q^2}}{2}$, we have $r_1=\tilde{r}-\frac{\sqrt{2}}{4}\epsilon$. Note that function
$$\frac{r^2}{r^2-Kr+Q^2}$$
is increasing for $r\in[0,r_1]$. Therefore,
$$\frac{r^2}{r^2-Kr+Q^2}\leq\frac{r_1^2}{r_1^2-Kr_1+Q^2}\leq\frac{8}{\epsilon^2}=\frac{8}{1-K+Q^2}.$$

Note that the function
$$\frac{1}{r^2-Kr+Q^2}$$
is increasing for $r\in[0,r_1]$. Therefore, for $r\in[0,r_1]$,
$$\frac{1}{r^2-Kr+Q^2}\le\frac{8}{1-K+Q^2}.$$

Here we will use the following definition of the $\dot{B}^{\gamma}_{p,p}(M)$-norm ($\gamma\in(0,1)$, $p>1$) (see [3, p.94, def. 2], [1])
$$\|u\|_{\dot{B}^{\gamma}_{p,p}(M)}=\Big(\int_0^2 h^{-1-p\gamma}\|\Delta_h u\|^p_{L^p(M)}dh\Big)^{1/p},$$
where $\Delta_h u=u(x+h)-u(x)$.

Lemma 1.3. *Let $u(x)\in\mathcal{C}^2([0,r_1])$, $u(x)=0$ for $x\ge r_1$, $0<r_1<1$. Then for $\gamma\in(0,1)$, $p>1$ we have*
$$C\|u\|_{\dot{B}^{\gamma}_{p,p}([0,r_1])}\ge\|u\|_{L^p([0,r_1])}.$$

Proof. We have
$$\begin{aligned}
\|u\|^p_{\dot{B}^{\gamma}_{p,p}([0,r_1])}&=\int_0^2 h^{-1-p\gamma}\|\Delta_h u\|^p_{L^p([0,r_1])}dh\\
&=\int_0^2 h^{-1-p\gamma}\|u(x+h)-u(x)\|^p_{L^p([0,r_1])}dh\\
&\ge\int_1^2 h^{-1-p\gamma}\|u(x+h)-u(x)\|^p_{L^p([0,r_1])}dh\\
&=\int_1^2 h^{-1-p\gamma}\|u(x)\|^p_{L^p([0,r_1])}dh\\
&=\|u(x)\|^p_{L^p([0,r_1])}\int_1^2 h^{-1-p\gamma}dh\\
&=\|u(x)\|^p_{L^p([0,r_1])}\frac{2^{p\gamma}-1}{p\gamma 2^{p\gamma}};
\end{aligned}$$
i.e.,
$$\|u\|^p_{\dot{B}^{\gamma}_{p,p}([0,r_1])}\ge\frac{2^{p\gamma}-1}{p\gamma 2^{p\gamma}}\|u(x)\|^p_{L^p([0,r_1])}.$$
From this estimate, we have $C\|u\|_{\dot{B}^{\gamma}_{p,p}([0,r_1])}\ge\|u(x)\|_{L^p([0,r_1])}$ which completes the proof. □

2. Existence of local solutions to the Cauchy problem (1.1)-(1.2)

Here and below we suppose that the positive constants A, K, Q, B, $1<\beta<\alpha$ satisfy (H1)-(H6). Let $t\in(0,1]$. Let $v(t)$ be function which satisfies the hypotheses:

- (H7) $v(t)\in\mathcal{C}^3[0,\infty)$, $v(t)>0$ for all $t\in[0,1]$
- (H8) $v''(t)>0$ for all $t\in[0,1]$, $v'(1)=v'''(1)=0$, $v(1)\neq 0$

(H9)

$$\min_{t\in[0,1]} v(t) \ge \frac{1}{A}, \quad \max_{t\in[0,1]} v(t) \le \frac{2}{A},$$
$$\min_{t\in[0,1]} \frac{v''(t)}{v(t)} \ge \frac{m^2}{\alpha^2 A^2}, \quad \max_{t\in[0,1]} \frac{v''(t)}{v(t)} \le \frac{2m^2}{\alpha^2 A^2};$$
$$\lim_{t\to 0}[v''(t) - \frac{m^2}{\alpha^2 A^2} v(t)] = +0, \quad v''(t) - \frac{m^2}{\alpha^2 A^2} v(t) \ge 0 \quad \text{for } t \in [0,1].$$

Note that there exist a functions $v(t)$ for which (H7)-(H9) hold. For example consider the function

$$v(t) = \frac{(t-1)^2 + \frac{2\alpha^2 A^2}{m^2} - 1}{A^3 \frac{\alpha^2}{m^2}}. \tag{2.1}$$

Then $v(t) \in \mathcal{C}^3[0,\infty)$; $v(t) > 0$ for all $t \in [0,1]$ because (H1), we have $\frac{\alpha A}{m} > 1$; i.e., (H7) holds. Since

$$v'(t) = \frac{2(t-1)}{A^3 \frac{\alpha^2}{m^2}}, \quad v'(1) = 0,$$
$$v''(t) = \frac{2}{A^3 \frac{\alpha^2}{m^2}} \ge 0 \quad \forall\, t \in [0,1],$$
$$v'''(t) = 0, \quad v'''(1) = 0,$$

it follows (H8). On the other hand

$$\min_{t\in[0,1]} v(t) \ge \frac{1}{A}, \quad \max_{t\in[0,1]} v(t) \le \frac{2}{A}, \quad \frac{v''(t)}{v(t)} = \frac{2}{(t-1)^2 + \frac{2\alpha^2 A^2}{m^2} - 1},$$

which implies

$$\min_{t\in[0,1]} \frac{v''(t)}{v(t)} \ge \frac{m^2}{\alpha^2 A^2}, \quad \max_{t\in[0,1]} \frac{v''(t)}{v(t)} \le \frac{2m^2}{\alpha^2 A^2},$$
$$v''(t) - \frac{m^2}{\alpha^2 A^2} v(t) = \frac{m^4}{\alpha^4 A^5}(2-t)t, \quad \lim_{t\to 0}[v''(t) - \frac{m^2}{\alpha^2 A^2} v(t)] = +0;$$

i.e., (H9) holds.

Here and below we suppose that $v(t)$ is a fixed function satisfying (H7)-(H9). In this section we will prove that the Cauchy problem (1.1)-(1.2) has unique nontrivial solution of the form

$$u(t,r) = \begin{cases} v(t)\omega(r) & \text{for } r \le r_1, \\ 0 & \text{for } r \ge r_1, \end{cases}$$

with t in $(0,1]$ and $u \in \mathcal{C}((0,1]\dot{B}^{\gamma}_{p,p}(\mathbb{R}^+))$.

Let us consider the integral equation

$$u(t,r) = \begin{cases} \int_r^{r_1} \frac{1}{\tau^2 - K\tau + Q^2} \int_\tau^{r_1} \Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} u(t,s) & \\ +s^2 m^2 u(t,s) - f(u(t,s))s^2 \Big) ds\, d\tau, & \text{for } 0 \le r \le r_1, \\ 0 & \text{for} r \ge r_1, \end{cases} \tag{2.2}$$

where $u(t,r) = v(t)\omega(r)$ and $t \in (0,1]$.

Theorem 2.1. *Let $p \in (1,\infty)$, $m \neq 0$ and $\gamma \in (0,1)$ be fixed constants and the positive constants A, B, Q,K, $\alpha > \beta > 1$ satisfy (H1)–(H6) and $f \in \mathcal{C}^2(\mathbb{R}^1)$, $f(0) = 0$, $2m^2|u| \le f^{(l)}(u) \le 3m^2|u|$, $l = 0, 1$. Let also $v(t)$ is function for which (H7)–(H9) hold. Then the equation (2.2) has unique nontrivial solution $u(t,r) = v(t)\omega(r)$ for which $w \in \mathcal{C}^2[0,r_1]$, $u(t,r_1) = u_r(t,r_1) = u_{rr}(t,r_1) = 0$ for $t \in (0,1]$, $u(t,r) \in \mathcal{C}((0,1]\dot{B}^{\gamma}_{p,p}[0,r_1])$, for $r \in [\frac{1}{\alpha},\frac{1}{\beta}]$ and $t \in (0,1]$ $u(t,r) \ge \frac{1}{A^2}$, for $r \in [\frac{1}{\alpha}, r_1]$ and $t \in (0,1]$ $u(t,r) \ge 0$, for $r \in [0,r_1]$ and $t \in (0,1]$ $|u(t,r)| \le \frac{2}{AB}$, $u(t,r) = 0$ for $r \ge r_1$, $t \in (0,1]$.*

Proof. Let $N = \{u(t,r) \in \mathcal{C}([0,r_1]) : t \in (0,1]\}$ with $u(t,r) = u_r(t,r) = u_{rr}(t,r) = 0$ for $t \in (0,1]$, $r \ge r_1$, $u(t,r) \in \mathcal{C}((0,1]\dot{B}^{\gamma}_{p,p}[0,r_1])$. For $r \in [\frac{1}{\alpha},\frac{1}{\beta}]$ and $t \in (0,1]$, we have $u(t,r) \ge \frac{1}{A^2}$. For $r \in [0,r_1]$ and $t \in (0,1]$, we have $|u(t,r)| \le \frac{2}{AB}$. For $r \in [\frac{1}{\alpha}, r_1]$ and $t \in (0,1]$, we have $u(t,r) \ge 0\}$.

We remark that if $u \in N$ is a solution of (2.2), $u \in \mathcal{C}^2([0,r_1])$. We define the operator R as follows

$$\begin{aligned} R(u) = \int_r^{r_1} \frac{1}{\tau^2 - K\tau + Q^2} \int_\tau^{r_1} \Big(&\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} u(t,s) \\ &+ s^2 m^2 u(t,s) - s^2 f(u)\Big) ds\, d\tau, \end{aligned}$$

for $0 \le r \le r_1$ and $t \in (0,1]$.

First we show that $R : N \to N$. For each $u \in N$, we have the following five statements:

(1) Since $u \in \mathcal{C}([0,r_1])$ and $f \in \mathcal{C}^2(\mathbb{R}^1)$, from the definition of the operator R we have $R(u) \in \mathcal{C}^2([0,r_1])$, $R(u)|_{r=r_1} = 0$,

$$\frac{\partial}{\partial r} R(u) = \frac{1}{r^2 - Kr + Q^2} \int_{r_1}^{r} [\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} u + s^2(m^2 u - f(u))] ds,$$

$$\frac{\partial}{\partial r} R(u)\big|_{r=r_1} = 0,$$

$$\begin{aligned} \frac{\partial^2}{\partial r^2} R(u) = &\frac{K - 2r}{(r^2 - Kr + Q^2)^2} \int_{r_1}^{r} [\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} u + s^2(m^2 u - f(u))] ds \\ &+ \frac{r^4}{(r^2 - Kr + Q^2)^2} \frac{v''(t)}{v(t)} u(t,r) + \frac{r^2}{r^2 - Kr + Q^2} (m^2 u(t,r) - f(u)). \end{aligned}$$

Since $u(t,r_1) = 0$, $f(u(t,r_1)) = f(0) = 0$ we obtain

$$\frac{\partial^2}{\partial r^2} R(u)\big|_{r=r_1} = 0.$$

Note that $R(u) = 0$ for $r \ge r_1$, $t \in (0,1]$ because $u(t,r) = 0$ for $r \ge r_1$, $t \in (0,1]$ and $f(u(t,r)) = f(0) = 0$ for $r \ge r_1$, $t \in (0,1]$.

(2) For $r \in [0,r_1]$, $t \in (0,1]$ we have $|u(t,r)| \le \frac{2}{AB}$. Then

$$\begin{aligned} &|R(u)| \\ &= \Big| \int_{r_1}^{r} \frac{1}{\tau^2 - K\tau + Q^2} \int_{r_1}^{\tau} \Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} u + s^2(m^2 u - f(u))\Big) ds d\tau \Big| \\ &\le \int_{r_1}^{r} \frac{1}{\tau^2 - K\tau + Q^2} \int_{r_1}^{\tau} \Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} |u| + s^2(m^2 |u| + |f(u)|)\Big) ds d\tau\,. \end{aligned}$$

Since $|f(u)| \leq 3m^2|u|)$, the above quantity is lees than or equal to

$$\begin{aligned}
&\int_{r_1}^{r} \frac{1}{\tau^2 - K\tau + Q^2} \int_{r_1}^{\tau} \Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} |u| + 4s^2 m^2 |u| \Big) ds d\tau \\
&= \int_{r_1}^{r} \frac{1}{\tau^2 - K\tau + Q^2} \int_{r_1}^{\tau} \Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} + 4s^2 m^2 \Big) |u| ds d\tau \\
&\leq \frac{2}{AB} \int_{r_1}^{r} \frac{1}{\tau^2 - K\tau + Q^2} \int_{r_1}^{\tau} \Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} + 4s^2 m^2 \Big) ds d\tau
\end{aligned}$$

where we use $\frac{r^2}{r^2 - Kr + Q^2} \leq \frac{8}{1-K+Q^2}$, $\frac{1}{r^2 - Kr + Q^2} \leq \frac{8}{1-K+Q^2}$ for $r \in [0, r_1]$. The above estimate is also less than or equal to

$$\begin{aligned}
&\frac{2}{AB} \frac{8}{1-K+Q^2} \Big(\frac{8}{1-K+Q^2} \max_{t \in [0,1]} \frac{v''(t)}{v(t)} + 4m^2 \Big) \\
&= \frac{2}{AB} \frac{8}{1-K+Q^2} \Big(\frac{8}{1-K+Q^2} \frac{2m^2}{\alpha^2 A^2} + 4m^2 \Big) \\
&\leq \frac{2}{AB}.
\end{aligned}$$

In the above inequality we use (H1). Consequently,

$$|R(u)| \leq \frac{2}{AB} \quad \text{for } r \in [0, r_1], t \in (0, 1].$$

(3)For $r \in [\frac{1}{\alpha}, r_1]$ and $t \in (0, 1]$ we have $u(t, r) \geq 0$. Then

$$\begin{aligned}
R(u) = &\int_{r}^{r_1} \frac{1}{\tau^2 - K\tau + Q^2} \int_{\tau}^{r_1} \Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} u(t, s) \\
&+ s^2 m^2 u(t, s) - s^2 f(u) \Big) ds\, d\tau
\end{aligned}$$

(where we use $f(u) \leq 3m^2 u$ for $r \in [\frac{1}{\alpha}, r_1]$, $t \in (0, 1]$. The above quantity is greater than or equal to

$$\begin{aligned}
&\int_{r}^{r_1} \frac{1}{\tau^2 - K\tau + Q^2} \int_{\tau}^{r_1} \Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} u(t, s) + s^2 (m^2 u(t, s) - 3m^2 u) \Big) ds\, d\tau \\
&\geq \int_{r}^{r_1} \frac{1}{\tau^2 - K\tau + Q^2} \int_{\tau}^{r_1} \Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} - 2m^2 s^2 \Big) u(t, s) ds\, d\tau \\
&\geq \int_{r}^{r_1} \frac{1}{\tau^2 - K\tau + Q^2} \int_{\tau}^{r_1} \Big(\frac{1}{\alpha^2(1 - \alpha K + \alpha^2 Q^2)} \min_{t \in [0,1]} \frac{v''(t)}{v(t)} - 2m^2 r_1^2 \Big) u(t, s) ds d\tau \\
&= \int_{r}^{r_1} \frac{1}{\tau^2 - K\tau + Q^2} \int_{\tau}^{r_1} \Big(\frac{m^2}{\alpha^4(1 - \alpha K + \alpha^2 Q^2) A^2} - 2m^2 r_1^2 \Big) u(t, s) ds\, d\tau\,.
\end{aligned}$$

From (H2), we have

$$\frac{m^2}{\alpha^4(1 - \alpha K + \alpha^2 Q^2) A^2} - 2m^2 r_1^2 \geq 0.$$

From this inequality and from $u(t, r) \geq 0$ for $r \in [\frac{1}{\alpha}, r_1]$, $t \in (0, 1]$, $r^2 - Kr + Q^2 > 0$, for $r \in [0, r_1]$, we get

$$R(u) \geq 0 \quad for \quad r \in [\frac{1}{\alpha}, r_1], \quad t \in (0, 1].$$

(4) For $r \in [\frac{1}{\alpha}, \frac{1}{\beta}]$ and $t \in (0,1]$ we have that $u(t,r) \geq \frac{1}{A^2}$. Using $f(u) \leq 3m^2 u$ for $r \in [\frac{1}{\alpha}, \frac{1}{\beta}]$, $t \in (0,1]$, we have

$$\begin{aligned}
R(u) &\geq \int_{r_1}^{r} \frac{1}{\tau^2 - K\tau + Q^2} \int_{r_1}^{\tau} \Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} u - 2s^2 m^2 u \Big) ds d\tau \\
&\geq \int_{r_1}^{r} \frac{1}{\tau^2 - K\tau + Q^2} \int_{r_1}^{\tau} \Big(\frac{s^4}{s^2 - Ks + Q^2} \min_{t\in[0,1]} \frac{v''(t)}{v(t)} u - 2s^2 m^2 u \Big) ds\, d\tau \\
&= \int_{r_1}^{r} \frac{1}{\tau^2 - K\tau + Q^2} \int_{r_1}^{\tau} s^2 \Big(\frac{s^2}{s^2 - Ks + Q^2} \min_{t\in[0,1]} \frac{v''(t)}{v(t)} u - 2m^2 u \Big) ds\, d\tau \\
&\geq \int_{r}^{r_1} \frac{1}{\tau^2 - K\tau + Q^2} \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} s^2 \Big(\frac{s^2}{s^2 - Ks + Q^2} \min_{t\in[0,1]} \frac{v''(t)}{v(t)} u - 2m^2 u \Big) ds\, d\tau \\
&\geq \int_{\frac{1}{\beta}}^{r_1} \frac{1}{\tau^2 - K\tau + Q^2} \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} s^2 \Big(\frac{s^2}{s^2 - Ks + Q^2} \min_{t\in[0,1]} \frac{v''(t)}{v(t)} u - 2m^2 u \Big) ds\, d\tau \\
&\geq \frac{1}{A^2} \Big(\frac{1}{1 - \alpha K + \alpha^2 Q^2} \frac{m^2}{\alpha^4 A^2} - 2m^2 r_1^2 \Big) \Big(r_1 - \frac{1}{\beta} \Big)^2 \frac{1}{1 - \alpha K + \alpha^2 Q^2} \geq \frac{1}{A^2},
\end{aligned}$$

(see (H2)); i.e., for $r \in [\frac{1}{\alpha}, \frac{1}{\beta}]$ and $t \in (0,1]$ we have $R(u) \geq \frac{1}{A^2}$.

(5) We have the estimate

$$\begin{aligned}
\|\Delta_h R(u)\|_{L^p}^p &= \int_0^{r_1} \Big(\Big| \int_r^{r+h} \frac{1}{\tau^2 - K\tau + Q^2} \int_\tau^{r_1} \Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} u(t,s) \\
&\quad + s^2 (m^2 u - f(u)) \Big) ds\, d\tau \Big| \Big)^p dr \\
&\leq \int_0^{r_1} \Big(\int_r^{r+h} \frac{1}{\tau^2 - K\tau + Q^2} \int_\tau^{r_1} \Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} |u| \\
&\quad + 4m^2 |u(t,s)| s^2 \Big) ds, d\tau \Big)^p dr
\end{aligned}$$

where we use that for $s \in [0, r_1]$, $\frac{s^4}{s^2 - Ks + Q^2} \leq \frac{8}{1 - K + Q^2}$, $\frac{1}{s^2 - Ks + Q^2} \leq \frac{8}{1 - K + Q^2}$ and $u(t,r) = 0$ for $r \geq r_1$ and $t \in (0,1]$. By (H9) the above estimate is less than or equal to

$$\begin{aligned}
&\int_0^{r_1} \Big(\int_r^{r+h} \frac{8}{1 - K + Q^2} \int_\tau^{r_1} \Big(\frac{8}{1 - K + Q^2} \max_{t\in[0,1]} \frac{v''(t)}{v(t)} |u| + 4m^2 |u| \Big) ds d\tau \Big)^p dr \\
&\leq \int_0^{r_1} \Big(\int_r^{r+h} \frac{8}{1 - K + Q^2} \int_\tau^{r_1} \Big(\frac{8}{1 - K + Q^2} \frac{2m^2}{\alpha^2 A^2} |u| + 4m^2 |u| \Big) ds d\tau \Big)^p dr \\
&\leq \int_0^{r_1} \Big(\int_r^{r+h} \frac{64}{(1 - K + Q^2)^2} \frac{2m^2}{\alpha^2 A^2} \int_0^{r_1} |u| ds + \frac{8}{1 - K + Q^2} 4m^2 \int_0^{r_1} |u| ds \Big) d\tau \Big)^p dr \\
&\leq h^p \Big(\frac{64}{(1 - K + Q^2)^2} \frac{2m^2}{\alpha^2 A^2} \|u\|_{L^p[0,r_1]} + \frac{8}{1 - K + Q^2} 4m^2 \|u\|_{L^p[0,r_1]} \Big)^p ;
\end{aligned}$$

i.e,,

$$\begin{aligned}
&\|\Delta_h R(u)\|_{L^p[0,r_1]}^p \\
&\leq h^p \Big(\frac{64}{(1 - K + Q^2)^2} \frac{2m^2}{\alpha^2 A^2} \|u\|_{L^p[0,r_1]} + \frac{8}{1 - K + Q^2} 4m^2 \|u\|_{L^p[0,r_1]} \Big)^p .
\end{aligned}$$

Consequently,

$$
\begin{aligned}
&\|R(u)\|^p_{\dot B^\gamma_{p,p}[0,r_1]}\\
&= \int_0^2 h^{-1-p\gamma}\|\Delta_h R(u)\|^p_{L^p[0,r_1]}dh\\
&\le \Big(\frac{64}{(1-K+Q^2)^2}\frac{2m^2}{\alpha^2A^2}\|u\|_{L^p[0,r_1]}+\frac{8.4m^2}{(1-K+Q^2)}\|u\|_{L^p[0,r_1]}\Big)^p\int_0^2 h^{-1+p(1-\gamma)}dh\\
&= \Big(\frac{64}{(1-K+Q^2)^2}\frac{2m^2}{\alpha^2A^2}\|u\|_{L^p[0,r_1]}+\frac{8.4m^2}{(1-K+Q^2)}\|u\|_{L^p[0,r_1]}\Big)^p\frac{2^{p(1-\gamma)}}{p(1-\gamma)}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\|R(u)\|_{\dot B^\gamma_{p,p}[0,r_1]}\\
&\le \Big(\frac{64}{(1-K+Q^2)^2}\frac{2m^2}{\alpha^2A^2}\|u\|_{L^p[0,r_1]}+\frac{8.4m^2}{(1-K+Q^2)}\|u\|_{L^p[0,r_1]}\Big)\Big(\frac{2^{p(1-\gamma)}}{p(1-\gamma)}\Big)^{1/p}.
\end{aligned}
$$

From Lemma 1.3, we have

$$
\begin{aligned}
\|R(u)\|_{\dot B^\gamma_{p,p}[0,r_1]} &\le C\Big(\frac{64}{(1-K+Q^2)^2}\frac{2m^2}{\alpha^2A^2}\|u\|_{\dot B^\gamma_{p,p}[0,r_1]}\\
&\quad+\frac{8.4m^2}{(1-K+Q^2)}\|u\|_{\dot B^\gamma_{p,p}[0,r_1]}\Big)\Big(\frac{2^{p(1-\gamma)}}{p(1-\gamma)}\Big)^{1/p}.
\end{aligned}
$$

From the above inequality, if $u\in\dot B^\gamma_{p,p}[0,r_1]$ we get $R(u)\in\dot B^\gamma_{p,p}[0,r_1]$ for $t\in(0,1]$. From statements (1)–(5) above, $R:N\to N$.

Now, let $u,u_1\in N$. Then

$$
\begin{aligned}
&\|\Delta_h(R(u)-R(u_1))\|^p_{L^p}\\
&=\int_0^{r_1}\Big(\Big|\int_r^{r+h}\frac{1}{\tau^2-K\tau+Q^2}\int_\tau^{r_1}\Big(\frac{s^4}{s^2-Ks+Q^2}\frac{v''(t)}{v(t)}(u(t,s)-u_1)\\
&\quad+s^2(m^2(u-u_1)-(f(u)-f(u_1)))\Big)ds\,d\tau\Big|\Big)^p dr.
\end{aligned}
$$

From the mean value theorem, $|f(u)-f(u_1)|=|u-u_1\|f'(\xi)|$ where $\xi\in(u,u_1)$ or $\xi\in(u_1,u)$. Then

$$
|f(u)-f(u_1)|\le 3m^2|\xi||u-u_1|\le 3m^2|u-u_1\|q|,
$$

where $|q|=\max\{|u|,|u_1|\}$. Since $|u|\le\frac{2}{AB}$ for $r\in[0,r_1]$, $t\in(0,1]$ we have

$$
|f(u)-f(u_1)|\le\frac{6m^2}{AB}|u-u_1|.
$$

Now, we use that $u(t,r)=0$ for $r\ge r_1$ and $t\in(0,1]$, to obtain

$$
\begin{aligned}
&\|\Delta_h(R(u)-R(u_1))\|_{L^p}^p\\
&\le \int_0^{r_1}\Big(\int_r^{r+h}\frac{1}{\tau^2-K\tau+Q^2}\int_\tau^{r_1}\Big(\frac{s^4}{s^2-Ks+Q^2}\frac{v''(t)}{v(t)}|u(t,s)-u_1|\\
&\quad + s^2m^2|u-u_1|+|f(u)-f(u_1)|\Big)ds\,d\tau\Big)^p dr\\
&\le \int_0^{r_1}\Big(\int_r^{r+h}\frac{1}{\tau^2-K\tau+Q^2}\int_\tau^{r_1}\Big(\frac{s^4}{s^2-Ks+Q^2}\max_{t\in[0,1]}\frac{v''(t)}{v(t)}|u(t,s)-u_1|\\
&\quad + s^2m^2|u-u_1|+|f(u)-f(u_1)|\Big)ds\,d\tau\Big)^p dr\\
&\le \int_0^{r_1}\Big(\int_r^{r+h}\frac{1}{\tau^2-K\tau+Q^2}\int_\tau^{r_1}\Big(\frac{8}{1-K+Q^2}\max_{t\in[0,1]}\frac{v''(t)}{v(t)}|u(t,s)-u_1|\\
&\quad + m^2|u-u_1|+\frac{6m^2}{AB}|u-u_1|\Big)ds\,d\tau\Big)^p dr\\
&\le \int_0^{r_1}\Big(\int_r^{r+h}\frac{8}{1-K+Q^2}\int_\tau^{r_1}\Big(\frac{8}{1-K+Q^2}\frac{2m^2}{\alpha^2A^2}+m^2+\frac{6m^2}{AB}\Big)\\
&\quad \times|u-u_1|dsd\tau\Big)^p dr\\
&\le h^p\Big(\frac{8}{1-K+Q^2}\Big)^p\Big(\frac{8}{(1-K+Q^2)}\frac{2m^2}{\alpha^2A^2}+m^2+\frac{6m^2}{AB}\Big)^p\|u-u_1\|_{L^p}^p;
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
&\|\Delta_h(R(u)-R(u_1))\|_{L^p[0,r_1]}^p\\
&\le h^p\Big(\frac{8}{1-K+Q^2}\Big)^p\Big(\frac{8}{(1-K+Q^2)}\frac{2m^2}{\alpha^2A^2}+m^2+\frac{6m^2}{AB}\Big)^p\|u-u_1\|_{L^p}^p.
\end{aligned}
$$

From the last inequality we get

$$
\begin{aligned}
\|R(u)-R(u_1)\|_{\dot B^\gamma_{p,p}[0,r_1]}^p \le \Big(\frac{8}{1-K+Q^2}\Big)^p\Big(\frac{8}{(1-K+Q^2)}\frac{2m^2}{\alpha^2A^2}+m^2+\frac{6m^2}{AB}\Big)^p\\
\times\|u-u_1\|_{L^p}^p\int_0^2 h^{-1+p(1-\gamma)}dh.
\end{aligned}
$$

From the above inequality and Lemma 1.3,

$$
\begin{aligned}
\|R(u)-R(u_1)\|_{\dot B^\gamma_{p,p}[0,r_1]} &\le \Big(\frac{2^{p(1-\gamma)}}{p(1-\gamma)}\Big)^{1/p}\frac{8}{1-K+Q^2}\Big(\frac{8}{(1-K+Q^2)}\frac{2m^2}{\alpha^2A^2}\\
&\quad +m^2+\frac{6m^2}{AB}\Big)\|u-u_1\|_{L^p[0,r_1]}\\
&\le C\Big(\frac{2^{p(1-\gamma)}}{(p(1-\gamma)}\Big)^{1/p}\frac{8}{1-K+Q^2}\Big(\frac{8}{(1-K+Q^2)}\frac{2m^2}{\alpha^2A^2}\\
&\quad +m^2+\frac{6m^2}{AB}\Big)\|u-u_1\|_{\dot B^\gamma_{p,p}[0,r_1]}\\
&< \|u-u_1\|_{\dot B^\gamma_{p,p}[0,r_1]}
\end{aligned}
$$

(see i3)). i.e.,

$$
\|R(u)-R(u_1)\|_{\dot B^\gamma_{p,p}[0,r_1]} < \|u-u_1\|_{\dot B^\gamma_{p,p}[0,r_1]}.
$$

Consequently, the operator $R : N \to N$ is contractive operator. $\square$

Lemma 2.2. *The set N is closed subset of $\mathcal{C}((0,1]\dot{B}^{\gamma}_{p,p}(\mathbb{R}^+))$.*

Proof. Let $t \in (0,1]$ be fixed. Let $\{u_n\}$ be a sequence of elements of the set N for which

$$\lim_{n\to\infty} \|u_n - \tilde{\tilde{u}}\|_{\dot{B}^{\gamma}_{p,p}(\mathbb{R}^+)} = 0,$$

where $\tilde{\tilde{u}} \in \dot{B}^{\gamma}_{p,p}(\mathbb{R}^+)$. We have

$$\lim_{n\to\infty} \|u_n - \tilde{\tilde{u}}\|_{\dot{B}^{\gamma}_{p,p}([0,r_1])} = 0.$$

We define

$$\tilde{u} = \begin{cases} \tilde{\tilde{u}} & \text{for } r \in [0,r_1], \\ 0 & \text{for } r > r_1. \end{cases}$$

We have

$$\lim_{n\to\infty} \|u_n - \tilde{u}\|_{\dot{B}^{\gamma}_{p,p}([0,r_1])} = 0.$$

First we note that for $u \in N$, $R(u)$ is continuous function of u and there exists $R'(u)$ because $f(u) \in \mathcal{C}^2(\mathbb{R}^1)$. In fact,

$$R'(u) = \int_r^{r_1} \frac{1}{\tau^2 - K\tau + Q^2} \int_\tau^{r_1} \Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} + s^2m^2 - s^2 f'(u)\Big) ds\, d\tau.$$

From which,

$$\begin{aligned}
&|R'(u)| \\
&\geq \int_r^{r_1} \frac{1}{\tau^2 - K\tau + Q^2} \int_\tau^{r_1} \Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} + s^2m^2 - s^2 |f'(u)|\Big) ds\, d\tau \\
&\geq \int_r^{r_1} \frac{1}{\tau^2 - K\tau + Q^2} \int_\tau^{r_1} \Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} + s^2m^2 - 3m^2s^2|u|\Big) ds, d\tau \\
&\geq \int_r^{r_1} \frac{1}{\tau^2 - K\tau + Q^2} \int_\tau^{r_1} \Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} + s^2m^2 - \frac{6m^2}{AB}s^2\Big) ds\, d\tau \\
&= \int_r^{r_1} \frac{1}{\tau^2 - K\tau + Q^2} \int_\tau^{r_1} \Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} + s^2m^2\Big(1 - \frac{6}{AB}\Big)\Big) ds\, d\tau\,.
\end{aligned}$$

From (H6), $1 > 6/(AB)$. Therefore, for $s \in [0, r_1]$ we have

$$\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} + s^2m^2\Big(1 - \frac{6}{AB}\Big) \geq 0.$$

Then for $r \in [0, r_1]$ we have

$$\begin{aligned}
&|R'(u)| \\
&\geq \int_{\frac{1}{\alpha}}^{r_1} \frac{1}{\tau^2 - K\tau + Q^2} \int_{\frac{1}{\alpha}}^{r_1} \Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} + s^2m^2\Big(1 - \frac{6}{AB}\Big)s^2\Big) ds\, d\tau \\
&\geq \Big(r_1 - \frac{1}{\alpha}\Big)^2 \frac{m^2}{\alpha^2A^2(1 - \alpha K + \alpha^2Q^2)^2} > 0.
\end{aligned}$$

From this, for $u \in N$, there exists

$$M := \min_{x\in[0,r_1]} |R'(u)(x)| > 0$$

because $R'(u)(x)$ is continuous function of $x \in [0, r_1]$. Let

$$M_1 = \max_{r \in [0,r_1]} \Big| \frac{\partial}{\partial r}(R'(u))(r) \Big|.$$

Now we prove that for each $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that

$$|x - y| < \delta \quad \text{implies} \quad |u_m(x) - u_m(y)| < \epsilon \quad \forall m.$$

We suppose that there exists $\tilde{\epsilon} > 0$ such that for every $\delta > 0$ there exist natural m and $x, y \in [0, r_1]$, $|x - y| < \delta$ for which $|u_m(x) - u_m(y)| \geq \tilde{\epsilon}$. We choose $\tilde{\tilde{\epsilon}} > 0$ such that $\tilde{\tilde{\epsilon}} < M\tilde{\epsilon}$. We note that $R(u_m)(x)$ is uniformly continuous function of $x \in [0, r_1]$(For $u \in N$ the function $R(u)(r)$ is uniformly continuous function of $r \in [0, r_1]$ because $R(u)(r) \in \mathcal{C}^2([0, r_1])$ and as in point (2) we have $\left|\frac{\partial}{\partial r} R(u)(r)\right| \leq \frac{2}{AB}$). Then there exists $\delta_1 = \delta_1(\tilde{\tilde{\epsilon}}) > 0$ such that for every $u \in N$

$$|R(u)(x) - R(u)(y)| < \tilde{\tilde{\epsilon}} \quad \text{for for all } x, y \in [0, r_1]\text{: } |x - y| < \delta_1.$$

Then we may choose $0 < \delta < \min\big\{\delta_1, \frac{(M\tilde{\epsilon} - \tilde{\tilde{\epsilon}})AB}{2M_1}\big\}$ such that there exist natural m and $x_1 \in [0, r_1]$, $x_2 \in [0, r_1]$ for which $|x_1 - x_2| < \delta$ and $|u_m(x_1) - u_m(x_2)| \geq \tilde{\epsilon}$. In particular

$$|R(u_m)(x_1) - R(u_m)(x_2)| < \tilde{\tilde{\epsilon}}. \tag{2.3}$$

Then by the mean value theorem, $R(0) = 0$, $R(u_m)(x_1) = R'(\xi)(x_1)u_m(x_1)$, $R(u_m)(x_2) = R'(\xi)(x_2)u_m(x_2)$,

$$\begin{aligned}
&|R(u_m)(x_1) - R(u_m)(x_2)| \\
&= |R'(\xi)(x_1)u_m(x_1) - R'(\xi)(x_2)u_m(x_2)| \\
&= |R'(\xi)(x_1)u_m(x_1) - R'(\xi)(x_1)u_m(x_2) + R'(\xi)(x_1)u_m(x_2) - R'(\xi)(x_2)u_m(x_2)| \\
&\geq |R'(\xi)(x_1)u_m(x_1) - R'(\xi)(x_1)u_m(x_2)| - |R'(\xi)(x_1) - R'(\xi)(x_2)||u_m(x_2)| \\
&= |R'(\xi)(x_1)||u_m(x_1) - u_m(x_2)| - \Big|\frac{\partial}{\partial r}(R'(\xi))(\eta)\Big||x_1 - x_2||u_m(x_2)| \\
&\geq M\tilde{\epsilon} - M_1\delta\frac{2}{AB} \geq \tilde{\tilde{\epsilon}},
\end{aligned}$$

which is a contradiction to (2.3).

Consequently, for each $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that

$$|x - y| < \delta \quad \text{implies} |u_m(x) - u_m(y)| < \epsilon \quad \forall m. \tag{2.4}$$

On the other hand, from the definition of the set N we have

$$|u_m| \leq \frac{2}{AB} \quad \forall m. \tag{2.5}$$

From (2.4) and (2.5), we conclude that the set $\{u_n\}$ is compact subset of $\mathcal{C}([0, r_1])$. Then there exists subsequence $\{u_{n_k}\}$ and function $u \in \mathcal{C}([0, r_1])$ for which: for every $\epsilon > 0$ there exists $M = M(\epsilon) > 0$ such that for every $n_k > M$ we have $|u_{n_k}(x) - u(x)| < \epsilon$ for every $x \in [0, r_1]$; $u(x) = 0$ for $x > r_1$. From this and from $\lim_{k\to\infty} \|u_{n_k} - \tilde{u}\|_{\dot{B}^\gamma_{p,p}([0,r_1])} = 0$ we have: For every $\epsilon > 0$ $\exists M = M(\epsilon) > 0$ such that for every $n_k > M$ we have

$$\max_{x \in [0,r_1]} |u_{n_k} - u| < \epsilon, \quad \|u_{n_k} - \tilde{u}\|_{\dot{B}^\gamma_{p,p}([0,r_1])} < \epsilon.$$

Then for every $n_k > M$ we have

$$|u-\tilde{u}| \le |u-u_{n_k}| + |u_{n_k}-\tilde{u}| < \epsilon + |\tilde{u}-u_{n_k}|,$$
$$\int_0^{r_1} |u-\tilde{u}|dx < \epsilon r_1 + \int_0^{r_1} |\tilde{u}-u_{n_k}|dx,$$

Using the Hölder's inequality,

$$\|u-\tilde{u}\|_{L^1[0,r_1]} < \epsilon r_1 + r_1^{1/q}\Big(\int_0^{r_1} |\tilde{u}-u_{n_k}|^p dx\Big)^{1/p}, \quad \frac{1}{p}+\frac{1}{q}=1,$$

for $h > 0$, we have

$$h^{-1-p\gamma}\|u-\tilde{u}\|_{L^1[0,r_1]} < h^{-1-p\gamma}\epsilon r_1 + r_1^{1/q} h^{-1-p\gamma}\Big(\int_0^{r_1} |\tilde{u}-u_{n_k}|^p dx\Big)^{1/p},$$

$$\begin{aligned}
&\int_1^2 h^{-1-p\gamma} dh \|u-\tilde{u}\|_{L^1[0,r_1]}\\
&< \int_1^2 h^{-1-p\gamma} dh \epsilon r_1 + r_1^{1/q}\int_1^2 h^{-1-p\gamma}\Big(\int_0^{r_1} |\tilde{u}-u_{n_k}|^p dx\Big)^{1/p} dh,
\end{aligned}$$

Using Hölder's inequality and that for $h > 1$ we have $h^{(-1-p\gamma)p} \le h^{-1-p\gamma}$,

$$\begin{aligned}
&\frac{1-2^{-p\gamma}}{p\gamma}\|u-\tilde{u}\|_{L^1[0,r_1]}\\
&< \frac{1-2^{-p\gamma}}{p\gamma}\epsilon r_1 + r_1^{1/q}\int_1^2 h^{-1-p\gamma}\Big(\int_0^{r_1} |\tilde{u}-u_{n_k}|^p dx\Big)^{1/p} dh\\
&\le \frac{1-2^{-p\gamma}}{p\gamma}\epsilon r_1 + r_1^{1/q}\Big(\int_1^2 h^{(-1-p\gamma)p}\int_0^{r_1} |\tilde{u}-u_{n_k}|^p dx dh\Big)^{1/p}\\
&\le \frac{1-2^{-p\gamma}}{p\gamma}\epsilon r_1 + r_1^{1/q}\Big(\int_1^2 h^{-1-p\gamma}\int_0^{r_1} |\tilde{u}-u_{n_k}|^p dx dh\Big)^{1/p}.
\end{aligned}$$

Using that for $x > r_1$, $u_{n_k}(x) = \tilde{u}(x) = 0$, the above expression equals

$$\begin{aligned}
&\frac{1-2^{-p\gamma}}{p\gamma}\epsilon r_1 + r_1^{1/q}\Big(\int_1^2 h^{-1-p\gamma}\int_0^{r_1} |\Delta_h(\tilde{u}-u_{n_k})|^p dx dh\Big)^{1/p}\\
&\le \frac{1-2^{-p\gamma}}{p\gamma}\epsilon r_1 + r_1^{1/q}\Big(\int_0^2 h^{-1-p\gamma}\int_0^{r_1} |\Delta_h(\tilde{u}-u_{n_k})|^p dx dh\Big)^{1/p}\\
&= \frac{1-2^{-p\gamma}}{p\gamma}\epsilon r_1 + r_1^{1/q}\|\tilde{u}-u_{n_k}\|_{\dot{B}^{\gamma}_{p,p}([0,r_1])}\\
&< \epsilon(\frac{1-2^{-p\gamma}}{p\gamma} r_1 + r_1^{1/q})
\end{aligned}$$

i.e., for every $\epsilon > 0$,

$$\frac{1-2^{-p\gamma}}{p\gamma}\|u-\tilde{u}\|_{L^1[0,r_1]} < \epsilon\Big(\frac{1-2^{-p\gamma}}{p\gamma} r_1 + r_1^{1/q}\Big).$$

Consequently $u = \tilde{u}$ a.e.(almost everywhere) in $[0, r_1]$, $|u|^p = |\tilde{u}|^p$ a.e. in $[0, r_1]$. ¿From here $|u_n - u| = |u_n - \tilde{u}|$ a.e., $|u_n - u|^p = |u_n - \tilde{u}|^p$ a.e. in $[0, r_1]$. Since

$u_n(x) = u(x) = 0$ for $x > r_1$ we have $|\Delta_h(u_n - u)|^p = |\Delta_h(u_n - \tilde{u})|^p$, $|\Delta_h u|^p = |\Delta_h \tilde{u}|^p$ a.e. in $[0, r_1]$, for $h > 0$. Therefore, $u \in \dot{B}^{\gamma}_{p,p}([0, r_1])$ and

$$\int_0^{r_1} |u_n - u|^p dx = \int_0^{r_1} |u_n - \tilde{u}|^p dx.$$

Now, we show that $\lim_{n\to\infty} \|u_n - u\|_{\dot{B}^{\gamma}_{p,p}([0,r_1])} = 0$. Note that

$$\begin{aligned}
&\|u_n - u\|_{\dot{B}^{\gamma}_{p,p}([0,r_1])} \\
&= \Big(\int_0^2 h^{-1-p\gamma} \int_0^{r_1} |\Delta_h(u_n - u)|^p dx\, dh\Big)^{1/p} \\
&= \Big(\int_0^2 h^{-1-p\gamma} \int_0^{r_1} |\Delta_h(u_n - \tilde{u})|^p dx\, dh\Big)^{1/p} \\
&= \|u_n - \tilde{u}\|_{\dot{B}^{\gamma}_{p,p}([0,r_1])} \to_{n\to\infty} 0.
\end{aligned}$$

Consequently, for every sequence $\{u_n\}$ with elements from N, which converges in $\dot{B}^{\gamma}_{p,p}([0, r_1])$ there exists function $u \in \mathcal{C}([0, r_1])$, $u \in \dot{B}^{\gamma}_{p,p}([0, r_1])$, for which $\|u_n - u\|_{\dot{B}^{\gamma}_{p,p}([0,r_1])} \to_{n\to\infty} 0$.

Below we suppose that the sequence $\{u_n\}$ is a sequence of elements of the set N which converges in $\dot{B}^{\gamma}_{p,p}([0, r_1])$. Then there exists $u \in \mathcal{C}([0, r_1])$, $u(x) = 0$ for $x > r_1$, $u \in \dot{B}^{\gamma}_{p,p}([0, r_1])$, $\|u_n - u\|_{\dot{B}^{\gamma}_{p,p}([0,r_1])} \to_{n\to\infty} 0$.

Now we suppose that $u(r_1) \neq 0$. Since $u \in \mathcal{C}([0, r_1])$, $u_n \in \mathcal{C}([0, r_1])$, $u_n(r_1) = 0$ for every natural n, there exist $\epsilon_2 > 0$ and $\Delta_1 \subset [0, r_1]$, $r_1 \in \Delta_1$, such that

$$|u_n| < \frac{\epsilon_2}{2}, \quad |u| > \epsilon_2$$

for every natural n and every $x \in \Delta_1$. Then for every natural n and for every $x \in \Delta_1$,

$$|u_n(x) - u(x)| > \frac{\epsilon_2}{2}.$$

Let $\epsilon_3 > 0$ be such that

$$\epsilon_3 < \frac{\epsilon_2}{2} \frac{1 - 2^{-p\gamma}}{p\gamma} \mu(\Delta_1) r_1^{-\frac{1}{q}}, \quad \frac{1}{p} + \frac{1}{q} = 1, \tag{2.6}$$

where $\mu(\Delta_1)$ is the measure of the set Δ_1. There exists $M > 0$ such that for every $n > M$ we have $\|u_n - u\|_{\dot{B}^{\gamma}_{p,p}([0,r_1])} < \epsilon_3$. Consequently for every $n > M$ and for every $x \in \Delta_1$ we have

$$|u_n(x) - u(x)| > \frac{\epsilon_2}{2}, \quad \|u_n - u\|_{\dot{B}^{\gamma}_{p,p}([0,r_1])} < \epsilon_3.$$

Also using the Hölder's inequality, we have

$$\begin{aligned}
\frac{\epsilon_2}{2}\mu(\Delta_1) &< \int_{\Delta_1} |u_n(x) - u(x)| dx \\
&\le \int_0^{r_1} |u_n(x) - u(x)| dx \\
&\le \Big(\int_0^{r_1} |u_n(x) - u(x)|^p dx\Big)^{1/p} r_1^{1/q}.
\end{aligned}$$

For $h > 0$, we have

$$h^{-1-p\gamma}\frac{\epsilon_2}{2}\mu(\Delta_1) \le h^{-1-p\gamma}\Big(\int_0^{r_1}|u_n(x)-u(x)|^p dx\Big)^{1/p} r_1^{1/q},$$
$$\int_1^2 h^{-1-p\gamma}\frac{\epsilon_2}{2}\mu(\Delta_1)dh \le \int_1^2 h^{-1-p\gamma}\Big(\int_0^{r_1}|u_n(x)-u(x)|^p dx\Big)^{1/p} r_1^{1/q}dh,$$
$$\frac{1-2^{-p\gamma}}{p\gamma}\frac{\epsilon_2}{2}\mu(\Delta_1) \le \Big(\int_1^2 h^{(-1-p\gamma)p}\int_0^{r_1}|u_n(x)-u(x)|^p dxdh\Big)^{1/p} r_1^{1/q}.$$

Since $h > 1$, $h^{(-1-p\gamma)p} \le h^{-1-p\gamma}$ and the above expression is less than or equal to

$$\Big(\int_1^2 h^{-1-p\gamma}\int_0^{r_1}|u_n(x)-u(x)|^p dxdh\Big)^{1/p} r_1^{1/q} \le$$

Now using that $u_n = u = 0$ for $x > r_1$, the above expression is less than or equal to

$$\begin{aligned}
&\Big(\int_1^2 h^{-1-p\gamma}\int_0^{r_1}|\Delta_h(u_n(x)-u(x))|^p dxdh\Big)^{1/p} r_1^{1/q}\\
&\le \Big(\int_0^2 h^{-1-p\gamma}\int_0^{r_1}|\Delta_h(u_n(x)-u(x))|^p dxdh\Big)^{1/p} r_1^{1/q}\\
&= \|u_n-u\|_{\dot{B}^{\gamma}_{p,p}([0,r_1])} r_1^{1/q} < \epsilon_3 r_1^{1/q}.
\end{aligned}$$

Therefore,

$$\frac{1-2^{-p\gamma}}{p\gamma}\frac{\epsilon_2}{2}\mu(\Delta_1) < \epsilon_3 r_1^{1/q},$$

which is a contradiction with (2.6). Consequently, $u(r_1) = 0$. From this, $u(t,r) = 0$ for $r \ge r_1$. Then $u_r(t,r) = u_{rr}(t,r) = 0$ for every $r \ge r_1$.

Now we suppose that the inequality

$$|u(t,r)| \le \frac{2}{AB}$$

is not hold for every $r \in [0,r_1]$. Since $u \in \mathcal{C}([0,r_1])$ we may take $\epsilon_4 > 0$ and $\Delta_2 \subset [0,r_1]$ such that

$$|u| \ge \frac{2}{AB} + \epsilon_4 \quad \text{for} \quad r \in \Delta_2.$$

Then for every natural n and for every $r \in \Delta_2$, we have

$$|u_n - u| \ge |u| - |u_n| \ge \frac{2}{AB} + \epsilon_4 - \frac{2}{AB} = \epsilon_4.$$

Let $\epsilon_5 > 0$ be such that

$$\frac{1-2^{-p\gamma}}{p\gamma}\epsilon_4\mu(\Delta_2) > \epsilon_5 r_1^{1/q}. \tag{2.7}$$

There exist $M > 0$ such that for every $n > M$ we have $\|u_n - u\|_{\dot{B}^{\gamma}_{p,p}([0,r_1])} < \epsilon_5$. Consequently for every $n > M$ and for every $x \in \Delta_2$ we have

$$|u_n(x)-u(x)| \ge \epsilon_4, \quad \|u_n-u\|_{\dot{B}^{\gamma}_{p,p}([0,r_1])} < \epsilon_5.$$

Also using the Hölder's inequality, we have

$$\epsilon_4\mu(\Delta_2) < \int_{\Delta_2}|u_n(x)-u(x)|dx \le \Big(\int_0^{r_1}|u_n(x)-u(x)|^p dx\Big)^{1/p} r_1^{1/q}.$$

For $h>0$ we have

$$h^{-1-p\gamma}\epsilon_4\mu(\Delta_2)\le h^{-1-p\gamma}\Big(\int_0^{r_1}|u_n(x)-u(x)|^p dx\Big)^{1/p} r_1^{1/q},$$

$$\int_1^2 h^{-1-p\gamma}\epsilon_4\mu(\Delta_2)dh\le \int_1^2 h^{-1-p\gamma}\Big(\int_0^{r_1}|u_n(x)-u(x)|^p dx\Big)^{1/p} r_1^{1/q}dh,$$

Using Hölder's inequality and that for $h>1$, $h^{(-1-p\gamma)p}\le h^{-1-p\gamma}$, we have

$$\begin{aligned}
&\frac{1-2^{-p\gamma}}{p\gamma}\epsilon_4\mu(\Delta_2)\\
&\le \Big(\int_1^2 h^{(-1-p\gamma)p}\int_0^{r_1}|u_n(x)-u(x)|^p dxdh\Big)^{1/p} r_1^{1/q}\\
&\le \Big(\int_1^2 h^{-1-p\gamma}\int_0^{r_1}|u_n(x)-u(x)|^p dxdh\Big)^{1/p} r_1^{1/q}.
\end{aligned}$$

Using that $u_n=u=0$ for $x>r_1$, the above expression is less than or equal to

$$\begin{aligned}
&\Big(\int_1^2 h^{-1-p\gamma}\int_0^{r_1}|\Delta_h(u_n(x)-u(x))|^p dxdh\Big)^{1/p} r_1^{1/q}\\
&\le \Big(\int_0^2 h^{-1-p\gamma}\int_0^{r_1}|\Delta_h(u_n(x)-u(x))|^p dxdh\Big)^{1/p} r_1^{1/q}\\
&=\|u_n-u\|_{\dot B^{\gamma}_{p,p}([0,r_1])} r_1^{1/q}<\epsilon_5 r_1^{1/q}.
\end{aligned}$$

Therefore,

$$\frac{1-2^{-p\gamma}}{p\gamma}\epsilon_4\mu(\Delta_2)<\epsilon_5 r_1^{1/q},$$

which is a contradiction with (2.7). Therefore, $|u|\le\frac{2}{AB}$ for every $r\in[0,r_1]$.

Now suppose that the inequality

$$|u(t,r)|\ge\frac{1}{A^2}$$

is not true for every $r\in[\frac{1}{\alpha},\frac{1}{\beta}]$. Since $u\in\mathcal{C}([0,r_1])$ we may take $\epsilon_6>0$ and $\Delta_3\subset[\frac{1}{\alpha},\frac{1}{\beta}]$ such that

$$|u|\le\frac{1}{A^2}-\epsilon_6\quad for\quad r\in\Delta_3.$$

Then for every natural n and for every $r\in\Delta_3$ we have

$$|u_n-u|\ge|u_n|-|u|\ge\frac{1}{A^2}+\epsilon_6-\frac{1}{A^2}=\epsilon_6.$$

Let $\epsilon_7>0$ be such that

$$\frac{1-2^{-p\gamma}}{p\gamma}\epsilon_6\mu(\Delta_3)>\epsilon_7 r_1^{1/q}. \tag{2.8}$$

There exist $M>0$ such that for every $n>M$ we have $\|u_n-u\|_{\dot B^{\gamma}_{p,p}([0,r_1])}<\epsilon_7$. Consequently, for every $n>M$ and for every $x\in\Delta_3$, we have

$$|u_n(x)-u(x)|>\epsilon_6,\quad \|u_n-u\|_{\dot B^{\gamma}_{p,p}([0,r_1])}<\epsilon_7.$$

Also using the Hölder's inequality, we have

$$\epsilon_6\mu(\Delta_3)<\int_{\Delta_3}|u_n(x)-u(x)|dx\le\Big(\int_0^{r_1}|u_n(x)-u(x)|^p dx\Big)^{1/p} r_1^{1/q}.$$

For $h > 0$ we have

$$h^{-1-p\gamma}\epsilon_6\mu(\Delta_3) \le h^{-1-p\gamma}\Big(\int_0^{r_1} |u_n(x)-u(x)|^p dx\Big)^{1/p} r_1^{1/q},$$

$$\int_1^2 h^{-1-p\gamma}\epsilon_6\mu(\Delta_3)dh \le \int_1^2 h^{-1-p\gamma}\Big(\int_0^{r_1} |u_n(x)-u(x)|^p dx\Big)^{1/p} r_1^{1/q}dh,$$

Using the Hölder's inequality and that for $h > 1$, $h^{(-1-p\gamma)p} \le h^{-1-p\gamma}$, we have

$$\begin{aligned}\frac{1-2^{-p\gamma}}{p\gamma}\epsilon_6\mu(\Delta_3) &\le \Big(\int_1^2 h^{(-1-p\gamma)p}\int_0^{r_1} |u_n(x)-u(x)|^p dxdh\Big)^{1/p} r_1^{1/q}\\ &\le \Big(\int_1^2 h^{-1-p\gamma}\int_0^{r_1} |u_n(x)-u(x)|^p dxdh\Big)^{1/p} r_1^{1/q}.\end{aligned}$$

Using that $u_n = u = 0$ for $x > r_1$, the above expression is less than or equal to

$$\begin{aligned}&\Big(\int_1^2 h^{-1-p\gamma}\int_0^{r_1} |\Delta_h(u_n(x)-u(x))|^p dxdh\Big)^{1/p} r_1^{1/q}\\ &\le \Big(\int_0^2 h^{-1-p\gamma}\int_0^{r_1} |\Delta_h(u_n(x)-u(x))|^p dxdh\Big)^{1/p} r_1^{1/q}\\ &= \|u_n-u\|_{\dot B^\gamma_{p,p}([0,r_1])} r_1^{1/q} < \epsilon_7 r_1^{1/q}.\end{aligned}$$

Therefore,

$$\frac{1-2^{-p\gamma}}{p\gamma}\epsilon_6\mu(\Delta_2) < \epsilon_7 r_1^{1/q},$$

which is a contradiction with (2.8). Therefore, $|u| \ge \frac{1}{A^2}$ for every $r \in [\frac{1}{\alpha}, \frac{1}{\beta}]$.

Now suppose that the inequality

$$u(t,r) \ge 0$$

is not true for every $r \in [\frac{1}{\alpha}, r_1]$. Then from $u \in \mathcal{C}([0,r_1])$ and from $u_n \ge 0$ for every natural n and for every $r \in [\frac{1}{\alpha}, r_1]$, we may take $\epsilon_8 > 0$ and $\Delta_4 \subset [\frac{1}{\alpha}, r_1]$ such that for every natural n and for every $r \in \Delta_4$ we have

$$|u_n - u| \ge \epsilon_8.$$

Let $\epsilon_9 > 0$ be such that

$$\frac{1-2^{-p\gamma}}{p\gamma}\epsilon_8\mu(\Delta_3) > \epsilon_9 r_1^{1/q}. \tag{2.9}$$

There exist $M > 0$ such that for every $n > M$ we have $\|u_n - u\|_{\dot B^\gamma_{p,p}([0,r_1])} < \epsilon_9$. Consequently, for every $n > M$ and for every $x \in \Delta_4$ we have

$$|u_n(x) - u(x)| > \epsilon_8, \quad \|u_n - u\|_{\dot B^\gamma_{p,p}([0,r_1])} < \epsilon_9.$$

Also using the Hölder's inequality,

$$\epsilon_8\mu(\Delta_4) < \int_{\Delta_4} |u_n(x)-u(x)|dx \le \Big(\int_0^{r_1} |u_n(x)-u(x)|^p dx\Big)^{1/p} r_1^{1/q}.$$

For $h > 0$,

$$h^{-1-p\gamma}\epsilon_8\mu(\Delta_4) \le h^{-1-p\gamma}\Big(\int_0^{r_1} |u_n(x)-u(x)|^p dx\Big)^{1/p} r_1^{1/q},$$

$$\int_1^2 h^{-1-p\gamma}\epsilon_8\mu(\Delta_4)dh \le \int_1^2 h^{-1-p\gamma}\Big(\int_0^{r_1} |u_n(x)-u(x)|^p dx\Big)^{1/p} r_1^{1/q}dh$$

Using Hölder's inequality and that for $h > 1$, $h^{(-1-p\gamma)p} \le h^{-1-p\gamma}$ we have

$$\begin{aligned} \frac{1-2^{-p\gamma}}{p\gamma}\epsilon_8\mu(\Delta_4) &\le \Big(\int_1^2 h^{(-1-p\gamma)p}\int_0^{r_1}|u_n(x)-u(x)|^p dxdh\Big)^{1/p} r_1^{1/q} \\ &\le \Big(\int_1^2 h^{-1-p\gamma}\int_0^{r_1}|u_n(x)-u(x)|^p dxdh\Big)^{1/p} r_1^{1/q}. \end{aligned}$$

Using that $u_n = u = 0$ for $x > r_1$, the above expression is less than or equal to

$$\begin{aligned} &\Big(\int_1^2 h^{-1-p\gamma}\int_0^{r_1}|\Delta_h(u_n(x)-u(x))|^p dxdh\Big)^{1/p} r_1^{1/q} \\ &\le \Big(\int_0^2 h^{-1-p\gamma}\int_0^{r_1}|\Delta_h(u_n(x)-u(x))|^p dxdh\Big)^{1/p} r_1^{1/q} \\ &= \|u_n - u\|_{\dot{B}^\gamma_{p,p}([0,r_1])} r_1^{1/q} < \epsilon_9 r_1^{1/q}. \end{aligned}$$

Therefore,

$$\frac{1-2^{-p\gamma}}{p\gamma}\epsilon_8\mu(\Delta_4) < \epsilon_9 r_1^{1/q},$$

which is a contradiction with (2.9). Therefore, $|u| \ge 0$ for every $r \in [\frac{1}{\alpha}, r_1]$.

Consequently $u \in N$. Then for every sequence $\{u_n\} \subset N$, which converges in $\dot{B}^\gamma_{p,p}([0,r_1])$ there exists $u \in N$ for which

$$\lim_{n\to\infty} \|u_n - u\|_{\dot{B}^\gamma_{p,p}([0,r_1])} = 0.$$

□

From lemma 2.2 we have that the set N is closed subset of $\mathcal{C}((0,1]\dot{B}^\gamma_{p,p}([0,r_1]))$. Since $\mathcal{C}((0,1]\dot{B}^\gamma_{p,p}([0,r_1]))$ is complete metric space and $R : N \to N$ is contractive operator the equation (2.2) has unique nontrivial solution $\tilde{u} \in N$. From (2.2) we have that $\tilde{u}(r) \in \mathcal{C}^2[0,r_1]$ and $\tilde{u}(t,r_1) = \tilde{u}_r(t,r_1) = \tilde{u}_{rr}(t,r_1) = 0$.

Let $\tilde{u}$ is the solution from the Theorem 2.1, i.e $\tilde{u}$ is the solution to the equation (2.2). Then $\tilde{u}$ is solution to the Cauchy problem (1.1)-(1.2) with initial data

$$u_0 = \begin{cases} \int_r^{r_1} \frac{1}{\tau^2-K\tau+Q^2}\int_\tau^{r_1}\Big(\frac{s^4}{s^2-Ks+Q^2}v''(1)\omega(s) \\ +s^2(m^2v(1)\omega(s) - f(v(1)\omega(s))\Big)ds\,d\tau & \text{for } r \le r_1, \\ 0 & \text{for } r \ge r_1, \end{cases}$$

and

$$u_1 = \begin{cases} \int_r^{r_1} \frac{1}{\tau^2-K\tau+Q^2}\int_\tau^{r_1}\Big(\frac{s^4}{s^2-Ks+Q^2}v'''(1)\omega(s) \\ +s^2(m^2v'(1)\omega(s) - f'(u)v'(1)\omega(s)\Big)dsd\tau = 0 & \text{for } r \le r_1, \\ 0\text{for } r \ge r_1. \end{cases}$$

From the proof of the Theorem 2.1, we have $u_0 \in \dot{B}^\gamma_{p,p}(\mathbb{R}^+)$, $u_1 \in \dot{B}^{\gamma-1}_{p,p}(\mathbb{R}^+)$, $\tilde{u} \in \mathcal{C}((0,1]\dot{B}^\gamma_{p,p}[0,r_1])$.

3. Blow-up of solutions to the Cauchy problem (1.1)-(1.2)

Let $v(t)$ be the same function as in Theorem 2.1.

Theorem 3.1. *Let $m^2 \neq 0$, $\gamma \in (0,1)$, $p > 1$ be fixed and the constants A, B, Q, K, $1 < \beta < \alpha$ satisfy the conditions (H1)-(H6). Let $f \in \mathcal{C}^2(\mathbb{R}^1)$, $f(0) = 0$, $2m^2|u| \leq f^{(l)}(u) \leq 3m^2|u|$, $l = 0,1$. Then the solution $\tilde{u}$ of the Cauchy problem* (1.1)-(1.2) *satisfies*

$$\lim_{t\to 0} \|\tilde{u}\|_{\dot{B}^{\gamma}_{p,p}[0,r_1]} = \infty.$$

Proof. For $t \in (0,1]$, we have

$$\begin{aligned}
&\|\Delta_h R(\tilde{u})\|^p_{L^p}\\
&= \int_0^{r_1} \Big| \int_r^{r+h} \frac{1}{\tau^2 - K\tau + Q^2} \int_\tau^{r_1} [\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} \tilde{u} + s^2(m^2\tilde{u} - f(\tilde{u}))] ds d\tau \Big|^p dr\\
&= \int_0^{\frac{1}{\alpha}} \Big| \int_r^{r+h} \frac{1}{\tau^2 - K\tau + Q^2} \int_\tau^{r_1} [\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} \tilde{u} + s^2(m^2\tilde{u} - f(\tilde{u}))] ds d\tau \Big|^p dr\\
&\quad + \int_{\frac{1}{\alpha}}^{r_1} \Big| \int_r^{r+h} \frac{1}{\tau^2 - K\tau + Q^2} \int_\tau^{r_1} [\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} \tilde{u} + s^2(m^2\tilde{u} - f(\tilde{u}))] ds d\tau \Big|^p dr.
\end{aligned}$$

Let

$$\begin{aligned}
I_1 &= \int_0^{\frac{1}{\alpha}} \Big| \int_r^{r+h} \frac{1}{\tau^2 - K\tau + Q^2} \int_\tau^{r_1} [\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} \tilde{u}\\
&\quad + s^2(m^2\tilde{u} - f(\tilde{u}))] ds d\tau \Big|^p dr,\\
I_2 &= \int_{\frac{1}{\alpha}}^{r_1} \Big| \int_r^{r+h} \frac{1}{\tau^2 - K\tau + Q^2} \int_\tau^{r_1} [\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} \tilde{u}\\
&\quad + s^2(m^2\tilde{u} - f(\tilde{u}))] ds d\tau \Big|^p dr.
\end{aligned}$$

As in proof of Theorem 2.1, for I_1 we have the estimate

$$I_1 \leq C^p \frac{2^{p(1-\gamma)}}{p(1-\gamma)} \frac{8^p}{(1-K+Q^2)^p} \Big(\frac{8}{1-K+Q^2} \frac{2m^2}{\alpha^2 A^2} + 4m^2 \Big)^p \|\tilde{u}\|^p_{\dot{B}^{\gamma}_{p,p}} h^p.$$

Using that $u(t,r) = 0$, $f(u(t,r)) = 0$ for $r \geq r_1$), for I_2, we have

$$\begin{aligned}
I_2 &= \int_{\frac{1}{\alpha}}^{r_1} \Big| \int_r^{r+h} \frac{1}{\tau^2 - K\tau + Q^2} \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} [\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} \tilde{u} + s^2(m^2\tilde{u} - f(\tilde{u}))] ds d\tau\\
&\quad + \int_r^{r+h} \frac{1}{\tau^2 - K\tau + Q^2} \int_{\frac{1}{\beta}}^{r_1} [\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} \tilde{u} + s^2(m^2\tilde{u} - f(\tilde{u}))] ds d\tau \Big|^p dr\\
&\leq \int_{\frac{1}{\alpha}}^{r_1} \Big(\int_r^{r+h} \frac{1}{\tau^2 - K\tau + Q^2} \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} [\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} \tilde{u} + s^2(m^2\tilde{u} - f(\tilde{u}))] ds d\tau\\
&\quad + \Big| \int_r^{r+h} \frac{1}{\tau^2 - K\tau + Q^2} \int_{\frac{1}{\beta}}^{r_1} [\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} \tilde{u}\\
&\quad + s^2(m^2\tilde{u} - f(\tilde{u}))] ds d\tau \Big| \Big)^p dr.
\end{aligned}$$

Let

$$I_{21} = \int_r^{r+h} \frac{1}{\tau^2 - K\tau + Q^2} \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} [\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} \tilde{u} + s^2(m^2\tilde{u} - f(\tilde{u}))] ds d\tau,$$

$$I_{22} = \Big| \int_r^{r+h} \frac{1}{\tau^2 - K\tau + Q^2} \int_{\frac{1}{\beta}}^{r_1} [\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} \tilde{u} + s^2(m^2\tilde{u} - f(\tilde{u}))] ds d\tau \Big|.$$

Then

$$I_2 \le \int_{\frac{1}{\alpha}}^{r_1} \Big(I_{21} + I_{22} \Big)^p dr.$$

For I_{21} we have the following estimate, we use that for $r \in [\frac{1}{\alpha}, \frac{1}{\beta}]$ $u \ge 0$, $f(u) \ge 2m^2 u$, therefore $-f(u) \le -2m^2 u$,

$$\begin{aligned} I_{21} &\le \int_r^{r+h} \frac{1}{\tau^2 - K\tau + Q^2} \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} \tilde{u} - s^2 m^2 \tilde{u} \Big) ds d\tau \\ &= \int_r^{r+h} \frac{1}{\tau^2 - K\tau + Q^2} \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} \frac{A^2\tilde{u}}{A^2} - s^2 m^2 \frac{A^2\tilde{u}}{A^2} \Big) ds d\tau \\ &= \int_r^{r+h} \frac{1}{\tau^2 - K\tau + Q^2} \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} \frac{1}{A^2} - s^2 m^2 \frac{1}{A^2} \Big) A^2 \tilde{u} ds d\tau. \end{aligned}$$

From (H4) we have that for $s \in [\frac{1}{\alpha}, \frac{1}{\beta}]$,

$$\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} \frac{1}{A^2} - \frac{s^2 m^2}{A^2} \ge \frac{1}{\alpha^2} \frac{1}{1 - \alpha K + \alpha^2 Q^2} \frac{m^2}{\alpha^2 A^4} - \frac{1}{\beta^2} \frac{m^2}{A^2} > 0.$$

On the other hand we have $\tilde{u} \ge \frac{1}{A^2}$ for every $t \in (0,1]$ and every $r \in [\frac{1}{\alpha}, \frac{1}{\beta}]$; therefore, $A^2\tilde{u} \ge 1$ and $A^2\tilde{u} \le A^{2p}\tilde{u}^p$. Consequently

$$I_{21} \le \int_r^{r+h} \frac{1}{\tau^2 - K\tau + Q^2} \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} \frac{1}{A^2} - s^2 m^2 \frac{1}{A^2} \Big) A^{2p} \tilde{u}^p ds d\tau.$$

From (H6),

$$\frac{8}{1 - K + Q^2} \le A \le A^2.$$

From this inequality, we have

$$\begin{aligned} I_{21} &\le \int_r^{r+h} \frac{8}{1 - K + Q^2} \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} [\frac{v''(t)}{v(t)} - \frac{m^2}{\alpha^2 A^2}] A^{2p} \tilde{u}^p ds d\tau \\ &\le \frac{8}{1 - K + Q^2} \frac{v''(t) - \frac{m^2}{\alpha^2 A^2} v}{v} A^{2p} \int_0^1 \tilde{u}^p ds h \\ &= \frac{8}{1 - K + Q^2} \frac{v''(t) - \frac{m^2}{\alpha^2 A^2} v}{v} A^{2p} \|\tilde{u}\|_{L^p}^p h \\ &\le \frac{8}{(1 - K + Q^2)^2} \frac{v''(t) - \frac{m^2}{\alpha^2 A^2} v}{v} A^{2p} \|\tilde{u}\|_{L^p}^p h. \end{aligned}$$

Now we use Lemma 1.3 to get

$$I_{21} \le C^p \frac{8^2}{(1 - K + Q^2)^2} \frac{v''(t) - \frac{m^2}{\alpha^2 A^2} v}{v} A^{2p} \|\tilde{u}\|_{\dot{B}^{\gamma}_{p,p}}^p h.$$

As in proof of Theorem 2.1 for I_{22} we have

$$I_{22} \le C\frac{8}{(1-K+Q^2)}\Big(\frac{8}{1-K+Q^2}\frac{2m^2}{\alpha^2A^2}+4m^2\Big)\|\tilde u\|_{\dot B^\gamma_{p,p}}h.$$

Consequently,

$$\begin{aligned} I_2 \le [&C\frac{8}{(1-K+Q^2)}\Big(\frac{8}{1-K+Q^2}\frac{2m^2}{\alpha^2A^2}+4m^2\Big)\|\tilde u\|_{\dot B^\gamma_{p,p}}h \\ &+C^p\frac{8^2}{(1-K+Q^2)^2}\frac{v''(t)-\frac{m^2}{\alpha^2A^2}v}{v}A^{2p}\|\tilde u\|^p_{\dot B^\gamma_{p,p}}h]^p, \end{aligned}$$

$$\begin{aligned} \|\Delta_hR(\tilde u)\|^p_{L^p} \le [&C\frac{8}{(1-K+Q^2)}\Big(\frac{8}{1-K+Q^2}\frac{2m^2}{\alpha^2A^2}+4m^2\Big)\|\tilde u\|_{\dot B^\gamma_{p,p}}h \\ &+C^p\frac{8^2}{(1-K+Q^2)^2}\frac{v''(t)-\frac{m^2}{\alpha^2A^2}v}{v}A^{2p}\|\tilde u\|^p_{\dot B^\gamma_{p,p}}h]^p \\ &+C^p\frac{8^p}{(1-K+Q^2)^p}\Big(\frac{8}{1-K+Q^2}\frac{2m^2}{\alpha^2A^2}+4m^2\Big)^p\|\tilde u\|^p_{\dot B^\gamma_{p,p}}h^p. \end{aligned}$$

Then

$$\begin{aligned} &\|\tilde u\|^p_{\dot B^\gamma_{p,p}} \\ &\le \Big[\Big[C\frac{8}{(1-K+Q^2)}\Big(\frac{8}{1-K+Q^2}\frac{2m^2}{\alpha^2A^2}+4m^2\Big)\|\tilde u\|_{\dot B^\gamma_{p,p}} \\ &\quad +C^p\frac{8^2}{(1-K+Q^2)^2}\frac{v''(t)-\frac{m^2}{\alpha^2A^2}v}{v}A^{2p}\|\tilde u\|^p_{\dot B^\gamma_{p,p}}\Big]^p \\ &\quad +C^p\frac{8^p}{(1-K+Q^2)^p}\Big(\frac{8}{1-K+Q^2}\frac{2m^2}{\alpha^2A^2}+4m^2\Big)^p\|\tilde u\|^p_{\dot B^\gamma_{p,p}}\Big]\int_0^2 h^{-1+p(1-\gamma)}dh \\ &=\frac{2^{p(1-\gamma)}}{p(1-\gamma)}\Big[\Big[C\frac{8}{(1-K+Q^2)}\Big(\frac{8}{1-K+Q^2}\frac{2m^2}{\alpha^2A^2}+4m^2\Big)\|\tilde u\|_{\dot B^\gamma_{p,p}} \\ &\quad +C^p\frac{8^2}{(1-K+Q^2)^2}\frac{v''(t)-\frac{m^2}{\alpha^2A^2}v}{v}A^{2p}\|\tilde u\|^p_{\dot B^\gamma_{p,p}}\Big]^p \\ &\quad +C^p\frac{8^p}{(1-K+Q^2)^p}\Big(\frac{8}{1-K+Q^2}\frac{2m^2}{\alpha^2A^2}+4m^2\Big)^p\|\tilde u\|^p_{\dot B^\gamma_{p,p}}\Big], \end{aligned}$$

and

$$\begin{aligned} \|\tilde u\|_{\dot B^\gamma_{p,p}} \le\ &2C\frac{8.2^{1-\gamma}}{(1-K+Q^2)(p(1-\gamma))^{1/p}}\Big(\frac{8}{1-K+Q^2}\frac{2m^2}{\alpha^2A^2}+4m^2\Big)\|\tilde u\|_{\dot B^\gamma_{p,p}} \\ &+C^p\frac{8^2.2^{1-\gamma}}{(1-K+Q^2)^2(p(1-\gamma))^{1/p}}\frac{v''(t)-\frac{m^2}{\alpha^2A^2}v}{v}A^{2p}\|\tilde u\|^p_{\dot B^\gamma_{p,p}}. \end{aligned}$$

Let

$$\begin{aligned} D&=2C\frac{8.2^{1-\gamma}}{(1-K+Q^2)(p(1-\gamma))^{1/p}}\Big(\frac{8}{1-K+Q^2}\frac{2m^2}{\alpha^2A^2}+4m^2\Big), \\ F&=C^p\frac{8^2.2^{1-\gamma}}{(1-K+Q^2)^2(p(1-\gamma))^{1/p}}\frac{v''(t)-\frac{m^2}{\alpha^2A^2}v}{v}A^{2p}. \end{aligned}$$

From (H5) we have that $D < 1$. Then

$$\|\tilde{u}\|_{\dot{B}^{\gamma}_{p,p}} \leq D\|\tilde{u}\|_{\dot{B}^{\gamma}_{p,p}} + F\|\tilde{u}\|^{p}_{\dot{B}^{\gamma}_{p,p}}.$$

from this inequality,

$$(1-D)\|\tilde{u}\|_{\dot{B}^{\gamma}_{p,p}} \leq \|\tilde{u}\|^{p}_{\dot{B}^{\gamma}_{p,p}}, \quad \frac{1-D}{F} \leq \|\tilde{u}\|^{p-1}_{\dot{B}^{\gamma}_{p,p}}.$$

Since $\lim_{t\to 0} F = +0$, we have

$$\lim_{t\to 0} \|\tilde{u}\|_{\dot{B}^{\gamma}_{p,p}} = \infty.$$

□

References

[1] Shatah, J., M. Struwe; *Geometric wave equation.* Courant lecture notes in mathematics 2(1998).

[2] Taylor, M.; *Partial differential equations III.* Springer-Verlag, 1996.

[3] Runst, T., W. Sickel.; *Sobolev spaces of fractional order, Nemytskij operators and nonlinear partial differential equations*, 1996, New York.

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