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FUNCTIONAL DIFFERENTIAL EQUATIONS WITH NON-LOCAL BOUNDARY CONDITIONS

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Abstract. In this work, we study an abstract boundary-value problem generated by an evolution equation and a non-local boundary condition. We prove the existence and uniqueness of the strong generalized solution and its continuity to respect to the parameters. The proofs are obtained via a priori estimates in non classical functional spaces and on the density of the range of the operator generated by the considered problem.

1. Introduction

The aim of this paper is to study a class of first order equations whose operator coefficients have variable domains and the boundary conditions are non-local. Here boundary-value problems (BVP) are called non-local if certain relations between traces of a solution are set at the boundary of the domain.

Many authors have studied evolution equations with Cauchy conditions; see for example [4, 5, 6, 7, 8]. In most of these papers the operator coefficients are assumed to be infinitesimal generators of analytic semi groups and have constant domains. For similar problems, various important results were proved under different assumptions in [2, 3] for hyperbolic problems and [9] for homogeneous Cauchy boundary conditions. Actually, it is difficult to construct a strict solution of the posed BVP, for this reason we prove the existence and uniqueness of the strong generalized solution, then we establish its continuity to respect to the parameters.

Summary of this article is as follows: In section 1, we give the statement of the problem, the basic assumptions then we define the functional spaces in where the posed problem will be solved, then its abstract formulation. In section 2, we prove the uniqueness and continuous dependence to respect to the data of the strong generalized solution when it exists. Section 3 is devoted to prove the existence Theorem. The continuity of the strong generalized solution to respect to the parameter is proved in section 4. Finally, we give an application of the results obtained in this work for a mixed problem.

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2. Statement of the problem, main assumptions, and functional spaces

In the interval $I =]0, T[$, with $0 < T < \infty$, we consider a BVP generated by the equation

$$\mathcal{L}u(t) = u_t(t) + A(t)u(t) = f(t). \tag{2.1}$$

To the above equation, we attach the non-local boundary condition

$$l_\mu u = u(0) - \mu u(T) = \varphi \in H, \tag{2.2}$$

where the functions u, f and φ belong to the Hilbert space H, in which the norm and the inner product are denoted respectively by $|\cdot|$ and $(\cdot,\cdot)$. The complex parameter μ satisfies

$$|\mu|^2 < e^{-3T}.$$

To study BVP (2.1)-(2.2), we have the following assumptions:

(A1) For each $t \in I$, $A(t)$ is a closed linear operator in H densely defined and satisfies

$$\mathrm{Re}(A(t)u, u) \geq c_1 |u|^2, \quad \forall u \in D(A(t)).$$

The same inequality holds for the adjoint operator $A^*(t)$ of $A(t)$.

(A2) The operator $A^{-1}(t)$ exists for almost all $t \in I$ and its derivative $\dfrac{dA^{-1}}{dt}$ belongs to $L_\infty(I, \mathcal{L}(H))$, where $\mathcal{L}(H)$ is the space of linear bounded operators from H to H equipped with the norm

$$\|A\|_{\mathcal{L}(H)} = \sup_{u \in H\, u \neq 0} \frac{|Au|}{|u|}.$$

Now we describe some functional spaces: Let $D(t)$ be the completion of $D(A(t))$ for almost all $t \in I$, with respect to the norm

$$|u|_t^2 = \mathrm{Re}(A(t)u, u)$$

We denote by L_μ the operator $(\mathcal{L}, l_\mu) = (f, \varphi)$, then the BVP (2.1)-(2.2) can be reformulate as

$$L_\mu u = F.$$

The domains of definition $D(L_\mu)$ of the operators L_μ are

$$D(L_\mu) = \{u \in L_2(I, H), u(t) \in D(A(t)), u_t, A(t)u \in L_2(I, H)\}.$$

By completing $D(L_\mu)$ according to the norm

$$\|u\|_\mu^2 = (1 - e^{3T}|\mu|^2)\Big(\sup_{s \in [0,T]} (|u(s)|^2) + \int_0^T |u|_t^2 dt\Big),$$

we obtain a Banach space which we denote by E_μ.

We denote by E the subspace of $L_2(I, H) \times H$, consisting of elements $F = (f, \varphi)$ such the norm $\|F\|_E^2 = \|f\|^2 + |\varphi|^2$ is finite. Here $\|\cdot\|$ denotes the norm of $L_2(I, H)$).

Regarding (2.1)-(2.2) and condition (A1), we have the the following statement.

Lemma 2.1. *The set $D(L_\mu)$ is dense in $L_2(I, H)$.*

Proof. Let $v \in L_2(I, H)$ and $u \in D(L_\mu)$ such that $\int_0^T (u, v)dt = 0$. An appropriate choice of u, for example $u = A^{-1}(t)v$, and using (A1), we obtain

$$0 = \operatorname{Re} \int_0^T (u, v)dt = \operatorname{Re} \int_0^T (A^{-1}(t)v, v)dt \geq c_1|A^{-1}(t)v|^2,$$

from this we deduce that $v = 0$. □

3. A priori estimate and corollaries

Theorem 3.1. *Assume that (A1) and (A2) hold, then there exists a constant $C > 0$ independent of t and u such that*

$$\|u\|_\mu^2 \leq C\|L_\mu u\|_E^2, \quad \forall u \in D(L_\mu). \tag{3.1}$$

Proof. To obtain the a priori estimate (3.1), we introduce the family of abstract smoothing operators $A_\varepsilon^{-1}(t) = (I + \varepsilon A(t))^{-1}$, $\varepsilon > 0$, with range $D(A(t))$, they have the following properties:

(P1) The operators $A_\varepsilon^{-1}(t)$ are strongly differentiable for almost all t and the derivatives $\frac{dA_\varepsilon^{-1}(t)}{dt} \in L_\infty(I, \mathcal{L}(H))$ and when $\varepsilon \to 0$,

$$|u - A_\varepsilon^{-1}(t)u| = |\varepsilon A(t)A_\varepsilon^{-1}(t)u| \to 0, \quad \forall u \in H$$

(P2) We approximate the unbounded operators $A(t)$ using bounded operators $A(t)A_\varepsilon^{-1}(t)$ which are strongly differentiable for almost all $t \in I$ and

$$\frac{d(A(t)A_\varepsilon^{-1}(t))}{dt} = \frac{-1}{\varepsilon}\frac{dA_\varepsilon^{-1}(t)}{dt}.$$

Note that $(A_\varepsilon^{-1}(t))^*$ has the properties (P1)-(P2).

Now, we multiply equation (2.1) by $B(t) = e^{c(s-t)}(A_\varepsilon^{-1}(t))^* A_\varepsilon^{-1}(t)u$. Then we integrate the double real part of the result equation over the interval $I_s =]0, s[\subset I$, to obtain

$$\begin{aligned} &2\operatorname{Re}\int_0^s e^{c(s-t)}(\mathcal{L}u, (A_\varepsilon^{-1}(t))^* A_\varepsilon^{-1}(t)u)dt \\ &= 2\operatorname{Re}\int_0^s e^{c(s-t)}(u_t, (A_\varepsilon^{-1}(t))^* A_\varepsilon^{-1}(t)u)dt \\ &\quad + 2\operatorname{Re}\int_0^s e^{c(s-t)}(A(t)u, (A_\varepsilon^{-1}(t))^* A_\varepsilon^{-1}(t)u)dt, \end{aligned} \tag{3.2}$$

which is equivalent to

$$\begin{aligned} |A_\varepsilon^{-1}(t)u|_{t=s}^2 &= e^{cs}|A_\varepsilon^{-1}(t)u|_{t=0}^2 + 2\operatorname{Re}\int_0^s e^{c(s-t)}(A_\varepsilon^{-1}(t)\mathcal{L}u, A_\varepsilon^{-1}(t)u)dt \\ &\quad - c\operatorname{Re}\int_0^s e^{c(s-t)}|A_\varepsilon^{-1}(t)u|^2 dt \\ &\quad + \operatorname{Re}\int_0^s e^{c(s-t)}(u, \frac{d}{dt}(A_\varepsilon^{-1*}(t)A_\varepsilon^{-1}(t))u)dt \\ &\quad - 2\operatorname{Re}\int_0^s e^{c(s-t)}(Au, (A_\varepsilon^{-1*}(t))A_\varepsilon^{-1}(t)u)dt\,. \end{aligned} \tag{3.3}$$

Using the Cauchy inequality and passing to the limit as $\varepsilon \to 0$, and using (A2) and the properties of the smoothing operators, we arrive at

$$\begin{aligned} &|u|^2_{t=s} + \int_0^s e^{c(s-t)} |u|^2_t dt \\ &\leq e^{cs} |u|^2_{t=0} + \int_0^s e^{c(s-t)} |\mathcal{L}u|^2 dt + (1-c) \int_0^s e^{c(s-t)} |u|^2 dt, \end{aligned} \tag{3.4}$$

To eliminate the last term on the right hand side of (3.4), we choose $c = 1$, consequently

$$|u|^2_{t=s} + \int_0^s e^{(s-t)} |u|^2_t dt \leq e^s |u|^2_{t=0} + \int_0^s e^{(s-t)} |\mathcal{L}u|^2 dt, \tag{3.5}$$

which implies

$$|u|^2_{t=s} + \int_0^s |u|^2_t dt \leq e^T |u|^2_{t=0} + e^T \int_0^T |\mathcal{L}u|^2 dt. \tag{3.6}$$

Repeating steps already employed, but on the interval $]s, T[$, we obtain

$$e^{s-T} |u|^2_{t=T} + \int_s^T e^{(s-t)} |u|^2_t dt \leq |u|^2_{t=s} + \int_s^T e^{(s-t)} |\mathcal{L}u|^2 dt. \tag{3.7}$$

Since the exponential function is increasing,

$$|u|^2_{t=T} + \int_s^T |u|^2_t dt \leq e^T |u|^2_{t=s} + e^T \int_0^T |\mathcal{L}u|^2 dt. \tag{3.8}$$

Multiplying (3.6) and (3.8) by e^{2T} and then adding up, we have

$$\begin{aligned} &(e^{2T} - e^T) |u|^2_{t=s} + e^{2T} \int_0^s |u|^2_t dt + \int_s^T |u|^2_t dt \\ &\leq e^{3T} |u|^2_{t=0} - |u|^2_{t=T} + (e^{3T} + e^T) \|\mathcal{L}u\|^2. \end{aligned} \tag{3.9}$$

Lemma 3.2 ([1]). *Let g be a function from $\overline{I}$ into H and let h be an element of H such that*

$$h = g(0) - \mu g(T). \tag{3.10}$$

If the parameter μ satisfies $|\mu|^2 < e^{-3T}$, then

$$e^{3T} |g(0)|^2 - |g(T)|^2 \leq \frac{e^{3T}}{1 - e^{3T} |\mu|^2} |h|^2. \tag{3.11}$$

Applying Lemma 3.2 to the first two terms in the left hand side of (3.9) and using elementary estimates, we obtain

$$\begin{aligned} &(e^{2T} - e^T) |u|^2_{t=s} + e^{2T} \int_0^s |u|^2_t dt + \int_s^T |u|^2_t dt \\ &\leq \frac{e^{3T}}{1 - e^{3T} |\mu|^2} |l_\mu u|^2 + (e^{3T} + e^T) \|\mathcal{L}u\|^2, \end{aligned} \tag{3.12}$$

Taking the supremum over the interval $[0, T]$ of (3.12),

$$\alpha(T) \big(\sup_{s \in [0,T]} (|u|^2_{t=s}) + \int_0^T |u|^2_t dt \big) \leq \frac{e^{3T}}{1 - e^{3T} |\mu|^2} |l_\mu u|^2 + (e^{3T} + e^T) \|\mathcal{L}u\|^2,$$

where $\alpha(T) = \min\{(e^{2T} - e^T), 1\}$, multiplying both sides of the above inequality by $(1 - e^{3T}|\mu|^2)$ and taking into account that $0 < 1 - e^{3T}|\mu|^2 < 1$, we have

$$\alpha(T)(1 - e^{3T}|\mu|^2)[\sup_{s\in[0,T]}(|u|^2_{t=s}) + \int_0^T |u|^2_t dt] \le (e^{3T} + e^T)[|l_\mu u|^2 + \|\mathcal{L}u\|^2].$$

Finally, multiplying the both sides of this inequality by $\alpha(T)$, we see that $C = \frac{e^{3T}+e^T}{\alpha(T)}$, then the proof of Theorem 3.1 is complete. □

From the inequality $\|u\|^2_\mu \le C\|L_\mu u\|^2_E$, it follows that there is a bounded inverse L_μ^{-1} on the range $R(L_\mu)$ of L_μ. However, since we have no information concerning $R(L_\mu)$ except that $R(L_\mu) \subset L_2(I, H) \times H$, we must extend L_μ (we construct its closure $\overline{L_\mu}$). In a standard way, we prove the following Lemma.

Lemma 3.3. *Assume that the conditions of Theorem 3.1 hold, then the operator L_μ has a closure which we denote by $\overline{L_\mu}$ with domain of definition $D(\overline{L_\mu}) = \overline{D(L_\mu)}$.*

So a function $u \in E_\mu$ is in $D(\bar{L}_\mu)$ if there exist a sequence $(u_n) \in D(L_\mu)$ and an element $F \in E$ such that $\|u_n - u\|_\mu \to 0$ and $\|L_\mu u_n - F\|_E \to 0$. i.e: $\bar{L}_\mu u = \lim_{n\to\infty} L_\mu u_n$

Definition 3.4. The solution of equation $\bar{L}_\mu u = F$ is called strong generalized solution of the BVP (2.1)-(2.2).

From Lemma 3.3, we extend inequality (3.1) to all element $u \in D(\overline{L_\mu})$

$$\|u\|^2_\mu \le C\|\bar{L}_\mu u\|^2_E, \forall u \in D(\overline{L_\mu})\,. \tag{3.13}$$

Corollary 3.5. *The strong generalized solution of the problem* (2.1)-(2.2)*, when it exists is unique and depends continuously on the data (f, φ).*

Corollary 3.6. *The range $R(\overline{L_\mu})$ of $\overline{L_\mu}$ is closed in E,*

$$R(\overline{L_\mu}) = \overline{R(L_\mu)}; \quad \overline{(L_\mu)}^{-1} = \overline{L_\mu^{-1}}.$$

The proofs of the above Corollaries are the same as in [2].

Note that Corollary 3.6 shows that to prove the existence of the strong generalized solution, it suffices to prove that $R(L_\mu)$ is everywhere dense in the Hilbert space E.

4. Denseness of the set of values and the solvability of the problem

Theorem 4.1. *If (A1) and (A2) are satisfied, then for every $f \in L_2(I, H)$ and $\varphi \in H$, the strong generalized solution of the BVP* (2.1)-(2.2)*, exists, is unique and satisfies*

$$\|u\|^2_\mu \le C\|\overline{L_\mu} u\|^2_E, \forall u \in D(\overline{L_\mu}).$$

Proof. By virtue of Corollary 3.6, to prove the existence of the strong generalized solution, it suffices to prove that $\overline{R(L_\mu)} = E$. Since E is a Hilbert space we can show that if $(F, V) = 0$ where $F = L_\mu u = (\mathcal{L}u, l_\mu u) \in R(L_\mu)$ and $V \in E$ then $V = 0$ i.e:

$$R(L_\mu)^\perp = \{0\}$$

Note that $(F, V) = 0$ is equivalent to

$$(\mathcal{L}u, v)_{L_2(I,H)} + (l_\mu u, \varphi) = 0, \tag{4.1}$$

Step 1. Let $u \in \mathfrak{D}(L_\mu) = \{u \in D(L_\mu) \text{ such that } l_\mu u = 0\}$, then (4.1) becomes

$$(\mathcal{L}u, v)_{L_2(I,H)} = 0. \tag{4.2}$$

We put $u = (A_\varepsilon^{-1}(t))^* A_\varepsilon^{-1}(t)h$, in the above equation, where h is an arbitrary element of $L_2(I, H)$. By integrating by part the double real part of the resultant equation, we obtain

$$\begin{aligned}
&\operatorname{Re}((A_\varepsilon^{-1}(t))^* A_\varepsilon^{-1}(t)h, v)_{t=T} - \operatorname{Re}((A_\varepsilon^{-1}(t))^* A_\varepsilon^{-1}(t)h, v)_{t=0} \\
&+ \operatorname{Re}\int_0^T (\frac{d(A_\varepsilon^{-1}(t))^* A_\varepsilon^{-1}(t)}{dt}h, v)dt + 2\operatorname{Re}\int_0^T (A(t)(A_\varepsilon^{-1}(t))^* A_\varepsilon^{-1}(t)h, v)dt = 0.
\end{aligned}$$

In particular if $h = v$, the above equality becomes

$$\begin{aligned}
&|A_\varepsilon^{-1}(t)v|_{t=T} - |A_\varepsilon^{-1}(t)v|_{t=0} + \operatorname{Re}\int_0^T (\frac{d((A_\varepsilon^{-1}(t))^* A_\varepsilon^{-1}(t))}{dt}v, v)dt \\
&+ 2\operatorname{Re}\int_0^T (A_\varepsilon^{-1}(t)v, A_\varepsilon^{-1}(t)A(t)^* v)dt = 0.
\end{aligned}$$

Applying the properties of the regularizing operators $A_\varepsilon^{-1}(t)$ as $\varepsilon \to 0$, the second term in the right hand side of the above equation approaches zero. Then

$$(1 - |\mu|^2)|v(T)|^2 + 2\int_0^T |v|_t dt = 0,$$

which implies that $v(T) = 0$ and $|v|_t = 0$,so $v = 0$.

Step 2. Let $u \in D(L_\mu)$, from $(F, V) = 0$ and the result of step1, it follows that

$$(\varphi, l_\mu u) = 0.$$

Choosing $u(t) = (T - t)A^{-1}(t)h$, where $h \in L_2(I, H)$, substituting u by its value in the above equation, we obtain $(\varphi, TA^{-1}(0)h) = 0$. Since $D(A(0))$ is dense in H, we conclude that $\varphi = 0$, so $V = 0$. □

5. Continuity of the strong generalized solution with respect to the parameter μ

Theorem 5.1. *Assume that the conditions of Theorem 3.1 are satisfied. If $\mu_n \to \mu$ as $n \to \infty$, then $(\overline{L_{\mu_n}})^{-1} \to (\overline{L_\mu})^{-1}$ in $\mathcal{L}(E_{\mu_n}, E_\mu)$ endowed with the simple convergence topology.*

Proof. Let $\hat{E}$ be the completion of $D(L_\mu)$ with respect to the norm

$$\|u\|_{\hat{E}}^2 = \sup_{s\in[0,T]} (|u(s)|^2) + \int_0^T |u|_t^2 dt.$$

We can see that

$$\|u\|_{\hat{E}}^2 \le \beta\|u\|_\mu^2, \tag{5.1}$$

where the constant β is independent of μ_n. From (5.1) and (3.13), we can write

$$\|u\|_{\hat{E}}^2 \le C'\|\overline{L_{\mu_n}}u\|_E^2, \quad \forall u \in D(\overline{L_{\mu_n}}), \tag{5.2}$$

where $C' = C/\beta$.

Now, using Banach Steinhauss Theorem, to prove Theorem 5.1, it suffices to prove that

(a) $\sup \|(\overline{L_{\mu_n}})^{-1}\|_{\mathcal{L}(E,\hat{E})} < \infty$.

(b) As $n \to \infty$, $(\overline{L_{\mu_n}})^{-1} \to (\overline{L_\mu})^{-1}$ in a subspace $\mathcal{G}$ dense in E.

Part(a) is proved immediately from (5.2). To prove part (b) we choose $\mathcal{G} = R(L_\mu)$ (which is dense in E). Indeed, let $F \in R(L_\mu)$, then $(\overline{L_{\mu_n}})^{-1}F - (\overline{L_\mu})^{-1}F \in D(\overline{L_{\mu_n}})$. We can see that (5.2) is equivalent to

$$\|(\overline{L_{\mu_n}})^{-1}F - (\overline{L_\mu})^{-1}F\|^2_{\breve{E}} \le C'\|F - \overline{L_{\mu_n}}(\overline{L_\mu})^{-1}F\|^2_E, \tag{5.3}$$

setting $g = (L_\mu)^{-1}F \in D(L_{\mu_n}) \subseteq D(\overline{L_{\mu_n}})$ in (25), we obtain

$$\begin{aligned}\|(\overline{L_{\mu_n}})^{-1}F - (\overline{L_\mu})^{-1}F\|^2_{\breve{E}} &\le C'\|L_\mu g - \overline{L_{\mu_n}}g\|^2_E\\ &= C'|\mu_n - \mu|^2 g(T) \underset{n\to\infty}{\rightarrow} 0.\end{aligned}$$

This achieves the proof of Theorem 5.1. □

The results concerning the problem (2.1)-(2.2) can be used to investigate mixed boundary-value problems for partial differential equations.

Example. In $D =]0,\ell[\times]0,T[$ consider the mixed problem

$$\mathcal{L}u(x,t) = \frac{\partial u}{\partial t}(x,t) + (-1)^n t^{-n}\frac{\partial^{2n+1}u}{\partial x^{2n+1}}(x,t) = f(x,t) \tag{5.4}$$

$$\frac{\partial^i u}{\partial x^i}(x,t)\big|_{x=0} = \frac{\partial^j u}{\partial x^j}(x,t)\big|_{x=\ell} = 0, \quad 0 \le i < 2n, 0 \le j \le n \tag{5.5}$$

$$l_\mu u(x,t) = u(x,0) - \mu u(x,T) = \varphi(x). \tag{5.6}$$

The functions $u(x,t)$ and $f(x,t)$ are from D to $H = L_2(]0,\ell[)$. The operator $A(t)$ is generated by the expression

$$\mathcal{A}u(x,t) = (-1)^n t^{-n}\frac{\partial^{2n+1}u}{\partial x^{2n+1}}(x,t)$$

and the boundary condition (5.5) with domain of definition

$$\begin{aligned}D(A(t)) = \Big\{&u(x,t) \in L_2(]0,\ell[), t^{-n}\frac{\partial^{2n+1}u}{\partial x^{2n+1}}(x,t) \in L_2(]0,\ell[),\\ &\frac{\partial^i u}{\partial x^i}(x,t)\big|_{x=0} = \frac{\partial^j u}{\partial x^j}(x,t)\big|_{x=\ell} = 0,\ 0 \le i < 2n; 0 \le j \le n; \forall t \in I\Big\}.\end{aligned}$$

Theorem 5.2. *The mixed problem* (5.4)–(2.2) *has one and only one strong generalized solution.*

Proof. It is sufficient to show that (A1) and (A2) hold. Indeed, the operators $A(t)$ satisfy (A1) with

$$c_1 = T^{-n}\ell^{-2n}(2n)!((2n+1)(4n+1)/2\ell)^{\frac{1}{2}}.$$

The inverse operator $A^{-1}(t)$ exists and

$$A^{-1}(t)v(x,t) = (-1)^{n+1}t^n\int_x^\ell\int_0^{s_{2n}}\dots\int_0^{s_2}\int_0^{s_1} v(r,t)\,dr\,ds_1ds_2\dots ds_{2n}.$$

Furthermore, it satisfies

$$|A^{-1}(t)v|^2 \le T^{2n}\frac{\ell^{4n+1}2^{2n+1}}{((2n)!)^2(2n+1)(4n+1)}|v|^2.$$

The strong derivative $\frac{\partial A^{-1}(t)}{\partial t}$ of the operator $A^{-1}(t)$ are

$$\frac{\partial A^{-1}(t)}{\partial t}v(x,t) = (-1)^{n+1}nt^{n-1}\int_x^\ell\int_0^{s_{2n}}\dots\int_0^{s_2}\int_0^{s_1} v(r,t)\,dr\,ds_1ds_2\dots ds_{2n}\,.$$

By a simple calculation, we obtain

$$|\frac{\partial A^{-1}(t)}{\partial t}v|^2 \le n^2 T^{2n-2} \frac{\ell^{4n+1} 2^{2n+1}}{((2n)!)^2 (2n+1)(4n+1)} |v|^2 < \infty\,.$$

From the above inequality we deduce that the operator $\frac{\partial A^{-1}(t)}{\partial t}$ belongs to the space $L_\infty(I, \mathfrak{L}(H))$. $\square$

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