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PERIODIC SOLUTIONS FOR A CLASS OF SECOND-ORDER HAMILTONIAN SYSTEMS

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Abstract. Multiplicity results for an eigenvalue second-order Hamiltonian system are investigated. Using suitable critical points arguments, the existence of an exactly determined open interval of positive eigenvalues for which the system admits at least three distinct periodic solutions is established. Moreover, when the energy functional related to the Hamiltonian system is not coercive, an existence result of two distinct periodic solutions is given.

1. Introduction

Recently, several authors studied problems of the type

$$\begin{gathered} \ddot{u} = \nabla_u F(t,u) \quad \text{a.e. in } [0,T] \\ u(T) - u(0) = \dot{u}(T) - \dot{u}(0) = 0 \end{gathered} \tag{1.1}$$

establishing, under suitable assumptions, existence or multiplicity of periodic solutions. We refer the reader to the book of Mawhin and Willem [9] for basic results. and to [3, 6, 7, 8, 13, 14] for more recent results. In particular, in [6] Brezis and Nirenberg assumed that:

(a) $F(t,0) = 0$, $\nabla_u F(t,0) = 0$.
(b) $\lim_{|u|\to+\infty} F(t,u) = +\infty$ uniformly in t.
(c) For some constant vector u_0,
$$\int_0^T F(t,u_0)\,dt < \int_0^T F(t,0)\,dt.$$
(d) There exists $r > 0$ and an integer $k \geq 0$ such that
$$-\frac{1}{2}(k+1)^2 w^2 |u|^2 \leq F(t,u) - F(t,0) \leq -\frac{1}{2}k^2 w^2 |u|^2$$
for all $|u| \leq r$, a.e. $t \in [0,T]$, where $w = 2\pi/T$.

Under the previous assumptions, they proved that problem (1.1) admits three periodic solutions (see [6, Theorem 7]). In [13] and [14], relaxing the coercivity of the potential and exploiting assumption (d), three periodic solutions to (1.1) are still ensured (see [14, Theorems 2 and 4] and [13, Theorem 2]). Further, the existence of

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one periodic solution to (1.1) is guaranteed when (d) is not required and a weaker type of coercivity is assumed (see [14, Theorems 1 and 3] and [13, Theorem 1]). Very recently, in [8], if $F(t,u) = \frac{1}{2}A(t)u\cdot u - b(t)G(u)$, the existence of three periodic solutions to (1.1) is ensured without assuming (d), but still requiring a condition that implies the coercivity of the energy functional related to the Hamiltonian system, in addition to the following:

(e) There exist $\sigma > 0$ and $u_0 \in \mathbb{R}^N$ such that

$$|u_0| < \sqrt{\frac{\sigma}{\sum_{i,j=1}^N \|a_{ij}\|_\infty \; T}} \quad \text{and} \quad G(u_0) = \sup_{|u| \le \overline{k}\sqrt{\sigma}} G(u),$$

that is, G achieves its maximum in the interior of the ball of radius $\overline{k}\sqrt{\sigma}$, where $\overline{k}$ is the constant of the Sobolev embedding and a_{ij} are the entries of the matrix A (see [8, assumptions 1 and 3 of Theorem 2.1]).

The aim of this paper is twofold: on the one hand we prove the existence of three periodic solutions to (1.1) (see Theorem 3.1) when neither condition (d) nor condition (e) are required, as Remarks 3.2 and 3.3 show; moreover, in our context, condition (c) together with a limit condition on G at zero imply the key assumption of Theorem 3.1 (see Remark 3.4). On the other hand we establish the existence of two periodic solutions (see Theorem 3.3) when, in addition, condition (b) can be removed, that is the energy functional related to the differential problem need not be coercive (see Remark 3.5). In our approach condition (a) is not required, as Example 3.1 and 3.2 show. To be precise, we study the following problem

$$\begin{gathered} \ddot{u} = A(t)u - \lambda b(t)\nabla G(u) \quad \text{a.e. in } [0,T] \\ u(T) - u(0) = \dot{u}(T) - \dot{u}(0) = 0 \end{gathered} \tag{1.2}$$

and establish the existence of an explicit open interval of positive parameters λ for which (1.2) admits three or two distinct periodic solutions. We also observe that problems of type (1.2) were studied in [3] and [7], but there only an upper bound of the interval of positive parameters λ for which (1.2) admits three distinct periodic solutions was established.

The proofs of the above-mentioned results are all based on critical point theorems. In particular, the results in [6], [13] and [14] are obtained exploiting the critical points theorem of Brezis and Nirenberg ([6, Theorem 4]). In [3] and [7] the main tool is the three critical points theorem of Bonanno [5, Theorem 2.1] (which is a consequence of the three critical points theorem of Ricceri [12, Theorem 3]). While in [8] the scope is achieved putting together the variational principle of Ricceri [11, Theorem 2.5] and the classical mountain pass theorem of Pucci and Serrin [10, Corollary 1]. Here, our results are based on multiple critical points theorems established by Averna and Bonanno [2, Theorem B] and by Bonanno [4, Theorem 2.1] (where the variational principle of Ricceri [11, Theorem 2.1] was applied), that we recall in Section 2 (see Theorems 2.1 and 2.2).

The present paper is organized as follows. Section 2 is devoted to preliminaries and basic results; while in Section 3 we establish the multiplicity results for Problem (1.2).

2. Preliminaries

Let T be a positive real number, N a positive integer and consider a matrix-valued function $A : [0,T] \to \mathbb{R}^{N\times N}$. We assume that A satisfies

(A1) $A : [0,T] \to \mathbb{R}^{N\times N}$ is a map into the space of $N \times N$ symmetric matrices with $A \in L^\infty([0,T])$ and there exists a positive constant μ such that
$$A(t)w \cdot w \geq \mu |w|^2$$
for every $w \in \mathbb{R}^N$ and a.e. in $[0,T]$.

Recall that H^1_T is the Sobolev space of all functions $u \in L^2([0,T], \mathbb{R}^N)$ that admit a weak derivative $\dot u \in L^2([0,T], \mathbb{R}^N)$. We emphasize that, in defining this kind of weak derivative, the *test* functions belong to the space C^∞_T of functions that are infinitely differentiable and $T-$periodic from $\mathbb{R}$ into $\mathbb{R}^N$. Moreover, for each $u \in H^1_T$ one has that $\int_0^T \dot u(t)dt = 0$ and u is absolutely continuous (for more details we refer the reader to [9, pp. 6-7]).

For each $u, v \in H^1_T$, we define
$$\langle u, v\rangle = \int_0^T \dot u(t) \cdot \dot v(t)dt + \int_0^T A(t)u(t) \cdot v(t)dt. \tag{2.1}$$
Since $A(t)$ is symmetric, (2.1) defines an inner product in H^1_T.
Then we define a norm in H^1_T by putting $\|u\| = \langle u, u\rangle^{\frac12}$ for each $u \in H^1_T$.

Observe that
$$A(t)\xi \cdot \xi = \sum_{i,j=1}^N a_{ij}(t)\xi_i\xi_j \leq \sum_{i,j=1}^N |a_{ij}(t)||\xi_i||\xi_j| \leq \sum_{i,j=1}^N \|a_{ij}\|_\infty |\xi|^2. \tag{2.2}$$
Hence, if we put
$$m = \min\{1, \mu\}, \quad M = \max\big\{1, \sum_{i,j=1}^N \|a_{ij}\|_\infty\big\},$$
using (A1) and (2.2), we see that our norm $\|\cdot\|$ is equivalent to the usual norm. Indeed one has
$$\sqrt{m}\|u\|_* \leq \|u\| \leq \sqrt{M}\|u\|_*, \tag{2.3}$$
where, for each $u \in H^1_T$,
$$\|u\|_* = \Big(\int_0^T |\dot u(t)|^2dt + \int_0^T |u(t)|^2dt\Big)^{1/2}.$$
It is well known that $(H^1_T, \|\cdot\|_*)$ is compactly embedded in $C^0([0,T], \mathbb{R}^N)$ (see for instance [1]), hence, from (2.3), we conclude that
$$\bar k = \sup_{u\in H^1_T,\ u\neq 0} \frac{\|u\|_{C^0}}{\|u\|} \tag{2.4}$$

is finite. We are able to give an upper estimate of $\bar{k}$ in the following manner. Fix $u \in H^1_T$ and consider $t_0 \in [0,T]$ such that $|u(t_0)| = \min_{\tau\in[0,T]} |u(\tau)|$. We can write

$$\begin{aligned} |u(t)| &= \Big| \int_{t_0}^{t} \dot{u}(\tau)d\tau + u(t_0)\Big| \\ &\le \int_0^T |\dot{u}(\tau)|d\tau + \frac{1}{T}\int_0^T |u(t_0)|d\tau \\ &\le \int_0^T |\dot{u}(\tau)|d\tau + \frac{1}{T}\int_0^T |u(\tau)|d\tau \\ &\le \sqrt{T}\Big(\int_0^T |\dot{u}(\tau)|^2 d\tau\Big)^{1/2} + \frac{1}{\sqrt{T}}\Big(\int_0^T |u(\tau)|^2 d\tau\Big)^{1/2} \\ &\le \sqrt{2}\max\big\{\sqrt{T}, \frac{1}{\sqrt{T}}\big\}\|u\|_* \end{aligned} \tag{2.5}$$

for each $t \in [0,T]$. Hence, from (2.5) and (2.3), if we put

$$k = \sqrt{\frac{2}{m}}\max\big\{\sqrt{T}, \frac{1}{\sqrt{T}}\big\} \tag{2.6}$$

one has

$$\bar{k} \le k. \tag{2.7}$$

In the sequel we shall make use of the constants

$$L = \frac{1}{k^2 T \sum_{i,j=1}^N \|a_{ij}\|_\infty}, \quad R = \frac{L}{1+L}. \tag{2.8}$$

Now, let $b \in L^1([0,T]) \setminus \{0\}$ which is a.e. nonnegative and $G \in C^1(\mathbb{R}^N)$. Put

$$\Phi(u) = \frac{1}{2}\|u\|^2 \quad \text{and} \quad \Psi(u) = -\int_0^T b(t)G(u(t))dt$$

for each $u \in H^1_T$. There are no difficulties in verifying that Φ is a continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse. In addition, Φ is a continuous and convex functional, so that it is sequentially lower semicontinuous too. Thanks to the Rellich-Kondrachov theorem, Ψ is a well-defined continuously Gâteaux differentiable functional whose Gâteaux derivative is a compact operator. In particular, for $u, v \in H^1_T$, one has

$$\begin{gathered} \Phi'(u)(v) = \int_0^T \dot{u}(t)\cdot\dot{v}(t)dt + \int_0^T A(t)u(t)\cdot v(t)dt, \\ \Psi'(u)(v) = -\int_0^T b(t)\nabla G(u(t))\cdot v(t)dt\,. \end{gathered}$$

Let us recall that a critical point for the functional $\Phi + \lambda\Psi$ is any $u \in H^1_T$ such that

$$\Phi'(u)(v) + \lambda\Psi'(u)(v) = 0 \tag{2.9}$$

for each $v \in H^1_T$. Moreover, a solution for problem (1.2) is any $u \in C^1([0,T], \mathbb{R}^k)$ such that $\dot{u}$ is absolutely continuous and

$$\begin{gathered} \ddot{u} = A(t)u - \lambda b(t)\nabla G(u) \quad \text{a.e. in } [0,T] \\ u(T) - u(0) = \dot{u}(T) - \dot{u}(0) = 0. \end{gathered}$$

We claim that each critical point for the functional $\Phi+\lambda\Psi$ is a solution for problem (1.2). In fact, since C_T^∞ is a subset of H_T^1, we can observe that if u is a critical point for the functional $\Phi+\lambda\Psi$, then $\dot u\in H_T^1$ and, in particular,

$$\ddot u = A(t)u-\lambda b(t)\nabla G(u)\quad \text{a.e. in } [0,T].$$

Hence,

$$\int_0^T \dot u(t)dt = \int_0^T \ddot u(t)dt = 0$$

and $u(T)-u(0)=\dot u(T)-\dot u(0)=0$; that is, u is a solution for problem (1.2).

Let us recall a recent result, due to Averna and Bonanno [2, Theorem B], which is the main tool to reach our goal.

Theorem 2.1 ([2, Theorem B])**.** *Let X be a reflexive Banach Space, $\Phi: X\to\mathbb{R}$ a continuously Gâteaux differentiable, coercive and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi: X\to\mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Put, for each $r>\inf_X\Phi$,*

$$\varphi_1(r)=\inf_{x\in\Phi^{-1}(]-\infty,r[)}\frac{\Psi(x)-\inf_{\overline{\Phi^{-1}(]-\infty,r[)}^w}\Psi}{r-\Phi(x)},$$

$$\varphi_2(r)=\inf_{x\in\Phi^{-1}(]-\infty,r[)}\sup_{y\in\Phi^{-1}([r,+\infty[)}\frac{\Psi(x)-\Psi(y)}{\Phi(y)-\Phi(x)},$$

where $\overline{\Phi^{-1}(]-\infty,r[)}^w$ is the closure of $\Phi^{-1}(]-\infty,r[)$ in the weak topology, and assume that

(i) *There is $r\in\mathbb{R}$ such that $\inf_X\Phi<r$ and $\varphi_1(r)<\varphi_2(r)$.*
Further, assume that:

(ii) $\lim_{\|x\|\to+\infty}(\Phi(x)+\lambda\Psi(x))=+\infty$ *for all $\lambda\in]\frac{1}{\varphi_2(r)},\frac{1}{\varphi_1(r)}[$.*

Then, for each $\lambda\in]\frac{1}{\varphi_2(r)},\frac{1}{\varphi_1(r)}[$ the equation (2.9) has at least three solutions in X.

We also use the following theorem concerning two critical points.

Theorem 2.2 ([4, Theorem 1.1])**.** *Let X be a reflexive real Banach space, and let $\Phi,\Psi: X\to\mathbb{R}$ be two sequentially weakly lower semicontinuous and Gâteaux differentiable functionals. Assume that Φ is (strongly) continuous and satisfies $\lim_{\|x\|\to+\infty}\Phi(x)=+\infty$. Assume also that there exist two constants r_1 and r_2 such that*

(j) $\inf_X\Phi<r_1<r_2$;

(jj) $\varphi_1(r_1)<\varphi_2^*(r_1,r_2)$;

(jjj) $\varphi_1(r_2)<\varphi_2^*(r_1,r_2)$,

where φ_1 is defined as in Theorem 2.1 and

$$\varphi_2^*(r_1,r_2):=\inf_{x\in\Phi^{-1}(]-\infty,r_1[)}\sup_{y\in\Phi^{-1}([r_1,r_2[)}\frac{\Psi(x)-\Psi(y)}{\Phi(y)-\Phi(x)},$$

Then, for each $\lambda\in]\frac{1}{\varphi_2^*(r_1,r_2)},\min\{\frac{1}{\varphi_1(r_1)},\frac{1}{\varphi_1(r_2)}\}[$, the functional $\Phi+\lambda\Psi$ admits at least two critical points which lie in $\Phi^{-1}(]-\infty,r_1[)$ and $\Phi^{-1}([r_1,r_2[)$ respectively.

We recall that Theorem 2.1 and Theorem 2.2 are based on the variational principle stated by Ricceri [11].

3. Main results

For the sake of simplicity, throughout this section we shall assume that $G(0) = 0$. Our main result is the following.

Theorem 3.1. *Let A be a matrix-valued function that satisfies assumption (A1). and let $G \in C^1(\mathbb{R}^N, \mathbb{R})$. Assume that there exist a positive constant γ and a vector $w_0 \in \mathbb{R}^N$ with $\gamma < |w_0|$, such that*

$$\frac{\max_{|w|\le\gamma} G(w)}{\gamma^2} < R\frac{G(w_0)}{|w_0|^2}; \tag{3.1}$$

$$\limsup_{|w|\to\infty} \frac{G(w)}{|w|^2} < \frac{\max_{|w|\le\gamma} G(w)}{\gamma^2}. \tag{3.2}$$

where R is defined in (2.8). Then, for every function $b \in L^1([0,T]) \setminus \{0\}$ that is a.e. nonnegative and for every λ in $\big]\frac{1}{2\|b\|_1 k^2}\frac{1}{R}\frac{|w_0|^2}{G(w_0)}, \frac{1}{2\|b\|_1 k^2}\frac{\gamma^2}{\max_{|w|\le\gamma} G(w)}\big[$, problem (1.2) admits at least three solutions.

Proof. Fix $b \in L^1([0,T]) \setminus \{0\}$ that is a.e. nonnegative. Denote by X the space H^1_T and, for each $u \in X$, put

$$\Phi(u) = \frac{1}{2}\|u\|^2, \quad \Psi(u) = -\int_0^T b(t)G(u(t))dt.$$

As we saw in Section 2, Φ and Ψ are continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functionals. In particular Φ' admits a continuous inverse on X^* and Ψ' is compact.

Since $G(0) = 0$, $\max_{|w|\le\gamma} G(w) \ge 0$. Hence, we distinguish two cases. First, assume $\max_{|w|\le\gamma} G(w) > 0$ and fix λ in $\big]\frac{1}{2\|b\|_1 k^2}\frac{1}{R}\frac{|w_0|^2}{G(w_0)}, \frac{1}{2\|b\|_1 k^2}\frac{\gamma^2}{\max_{|w|\le\gamma} G(w)}\big[$. By assumption (3.2), we can find two positive numbers δ and δ', with

$$\limsup_{|w|\to\infty} \frac{G(w)}{|w|^2} < \delta < \frac{\max_{|w|\le\gamma} G(w)}{\gamma^2}$$

such that $G(w) \le \delta|w|^2 + \delta'$ for each $w \in \mathbb{R}^N$. Fix For each $u \in X$ one has

$$\begin{aligned} \Phi(u) + \lambda\Psi(u) &\ge \frac{1}{2}\|u\|^2 - \lambda\delta\int_0^T b(t)|u(t)|^2 dt - \lambda\delta'\|b\|_1 \\ &\ge \frac{1}{2}\|u\|^2 - \lambda\delta\|b\|_1\|u\|^2_{C^0} - \lambda\delta'\|b\|_1 \\ &\ge \left(\frac{1}{2} - \lambda\delta k^2\|b\|_1\right)\|u\|^2 - \lambda\delta'\|b\|_1 \\ &> \frac{1}{2}\left(1 - \delta\frac{\gamma^2}{\max_{|w|\le\gamma} G(w)}\right)\|u\|^2 - \lambda\delta'\|b\|_1. \end{aligned} \tag{3.3}$$

Hence $\Phi + \lambda\Psi$ is coercive.

Let us consider φ_1 and φ_2 given in Theorem 2.1. We can observe that $\inf_X \Phi = \Phi(0) = 0$ and that, for each $r > 0$, $0 \in \Phi^{-1}(]-\infty, r[)$ and $\overline{\Phi^{-1}(]-\infty, r[)}^w =$

$\Phi^{-1}(]-\infty, r])$. Fix $r > 0$. One has

$$\begin{aligned}
\varphi_1(r) &\leq \frac{-G(0)\|b\|_1 - \inf_{\|v\|^2 \leq 2r}\big(-\int_0^T b(t)G(v(t))dt\big)}{r} \\
&\leq \sup_{\|v\|^2 \leq 2r} \frac{\int_0^T b(t)G(v(t))dt}{r}.
\end{aligned} \tag{3.4}$$

Thanks to (2.6) and (2.7), it is easy to check that

$$\{v \in X : \|v\|^2 \leq 2r\} \subseteq \{v \in C^0 : \|v\|_{C^0}^2 \leq 2k^2 r\}.$$

Hence, from (3.4), bearing in mind that $b \geq 0$ a.e. and that G is continuous, we can write

$$\varphi_1(r) \leq \|b\|_1 \frac{\max_{|w| \leq k\sqrt{2r}} G(w)}{r}. \tag{3.5}$$

Let now $r = \gamma^2/(2k^2)$ and consider the function $v \in X$ defined by putting $v(t) = w_0$ for each $t \in [0, T]$. A simple computation shows that $k\sqrt{\mu T} \geq \sqrt{2}$. Therefore, from $\gamma < |w_0|$ one has $\gamma < k\sqrt{\mu T}|w_0|$ and, in view of condition $(\mathcal{A})$, we obtain

$$\|v\|^2 = \int_0^T A(t)w_0 \cdot w_0 dt \geq T\mu|w_0|^2 > 2r.$$

On the other hand, from (2.2), one has

$$\|v\|^2 \leq T \sum_{i,j=1}^N \|a_{ij}\|_\infty |w_0|^2. \tag{3.6}$$

For each $u \in X$ such that $\|u\|^2 < 2r$ one has

$$\int_0^T b(t)G(u(t))dt \leq \|b\|_1 \max_{|w| \leq k\sqrt{2r}} G(w) = \|b\|_1 \max_{|w| \leq \gamma} G(w) \tag{3.7}$$

and

$$0 < \|v\|^2 - \|u\|^2 \leq \|v\|^2. \tag{3.8}$$

We claim that

$$\frac{\max_{|w| \leq \gamma} G(w)}{\gamma^2} < L \frac{G(w_0) - \max_{|w| \leq \gamma} G(w)}{|w_0|^2}, \tag{3.9}$$

where L is defined in (2.8). In fact, since $G(0) = 0$, $\gamma < |w_0|$ and thanks to assumption (3.1) one has

$$\begin{aligned}
&\frac{\max_{|w| \leq \gamma} G(w)}{\gamma^2} + L \frac{\max_{|w| \leq \gamma} G(w)}{|w_0|^2} \\
&< (1 + L) \frac{\max_{|w| \leq \gamma} G(w)}{\gamma^2} \\
&< R(1 + L) \frac{G(w_0)}{|w_0|^2} = L \frac{G(w_0)}{|w_0|^2}.
\end{aligned} \tag{3.10}$$

Hence (3.9) holds and, consequently,

$$G(w_0) > \max_{|w| \leq \gamma} G(w). \tag{3.11}$$

At this point, putting together (3.7), (3.11), (3.8) and (3.6) we can obtain

$$\begin{aligned}\frac{\int_0^T b(t)G(v(t))dt - \int_0^T b(t)G(u(t))dt}{\|v\|^2 - \|u\|^2} &\geq \|b\|_1 \frac{G(w_0) - \max_{w\leq\gamma} G(w)}{\|v\|^2 - \|u\|^2} \\ &\geq \|b\|_1 \frac{G(w_0) - \max_{|w|\leq\gamma} G(w)}{\|v\|^2} \\ &\geq \|b\|_1 \frac{G(w_0) - \max_{|w|\leq\gamma} G(w)}{T\sum_{i,j=1}^N \|a_{ij}\|_\infty |w_0|^2} \\ &= Lk^2\|b\|_1 \frac{G(w_0) - \max_{|w|\leq\gamma} G(w)}{|w_0|^2}\end{aligned}$$

for each $u \in X$ such that $\|u\|^2 < 2r$. Hence, one has

$$\begin{aligned}\varphi_2(r) &\geq 2 \inf_{\|u\|^2<2r} \frac{\int_0^T b(t)G(v(t))dt - \int_0^T b(t)G(u(t))dt}{\|v\|^2 - \|u\|^2} \\ &\geq 2Lk^2\|b\|_1 \frac{G(w_0) - \max_{|w|\leq\gamma} G(w)}{|w_0|^2}.\end{aligned} \tag{3.12}$$

Making use of (3.5), (3.9) and (3.12), we conclude that

$$\begin{aligned}\varphi_1(r) &\leq \|b\|_1 \frac{\max_{|w|\leq k\sqrt{2r}} G(w)}{r} \\ &= 2k^2\|b\|_1 \frac{\max_{|w|\leq\gamma} G(w)}{\gamma^2} \\ &< 2Lk^2\|b\|_1 \frac{G(w_0) - \max_{|w|\leq\gamma} G(w)}{|w_0|^2} \leq \varphi_2(r).\end{aligned} \tag{3.13}$$

Moreover, in view of (3.9) and (3.13), since $\gamma < |w_0|$ and assumption (3.1) holds, we have

$$\begin{aligned}\frac{1}{\varphi_2(r)} &\leq \frac{1}{2Lk^2\|b\|_1} \frac{|w_0|^2}{G(w_0) - \max_{|w|\leq\gamma} G(w)} \\ &< \frac{1}{2Lk^2\|b\|_1} \frac{|w_0|^2}{G(w_0) - \gamma^2 R\frac{G(w_0)}{|w_0|^2}} \\ &< \frac{1}{2Lk^2\|b\|_1} \frac{|w_0|^2}{G(w_0)} \frac{1}{1-R} \\ &= \frac{1}{2\|b\|_1 k^2} \frac{1}{R} \frac{|w_0|^2}{G(w_0)}\end{aligned} \tag{3.14}$$

and

$$\frac{1}{\varphi_1(r)} \geq \frac{1}{2\|b\|_1 k^2} \frac{\gamma^2}{\max_{|w|\leq\gamma} G(w)}. \tag{3.15}$$

Hence, all assumptions of Theorem 2.1 are satisfied and the proof is complete once observed that, as we saw in Section 2, the critical points of the functional $\Phi + \lambda\Psi$ are solutions for our problem (1.2).

Now, let $\max_{|w|\leq\gamma} G(w) = 0$. By assumption (3.2), we can find a positive number $\bar{\delta}$ such that $G(w) < 0$ for every $w \in \mathbb{R}^N$ with $|w| > \bar{\delta}$. At this point, if $\lambda > 0$, one has

$$\begin{aligned}\Phi(u)+\lambda\Psi(u) &\geq \frac{1}{2}\|u\|^2-\lambda\int_{\{t\in[0,T]:|u(t)|\leq\bar{\delta}\}} b(t)G(u(t))dt\\ &\geq \frac{1}{2}\|u\|^2-\lambda\|b\|_1\max_{|w|\leq\bar{\delta}}G(w)\end{aligned}\tag{3.16}$$

for every $u\in X$. Hence $\Phi+\lambda\Psi$ is coercive.

Due to (3.5), for $r=\gamma^2/(2k^2)$, one has $\varphi_1(r)=0$. As well as, since $G(w_0)>0$, reasoning as in (3.12) we obtain

$$\varphi_2(r)\geq 2Lk^2\|b\|_1\frac{G(w_0)}{|w_0|^2}>2Rk^2\|b\|_1\frac{G(w_0)}{|w_0|^2}>0.$$

At this point we have

$$\frac{1}{\varphi_2(r)}<\frac{1}{2\|b\|_1k^2}\frac{1}{R}\frac{|w_0|^2}{G(w_0)}$$

and we can conclude as in the previous case, where we agree to read $\frac{1}{0}$ as $+\infty$. $\square$

Remark 3.1. We explicitly observe that, from the proof of Theorem 3.1 we obtain that, when $\max_{|w|\leq\gamma}G(w)=0$, the interval of parameters for which problem (1.2) admits at least three solutions is $]\frac{1}{2\|b\|_1k^2}\frac{1}{L}\frac{|w_0|^2}{G(w_0)},+\infty[$. Moreover, in this particular case, the conclusion can be also obtained by standard arguments.

Example 3.1. Let $G:\mathbb{R}^2\to\mathbb{R}$ be defined by

$$G(x,y)=\frac{\left(x^2+y^2\right)^6}{e^{x^2+y^2}}+x$$

for every $(x,y)\in\mathbb{R}^2$. By choosing $\gamma=1$ and $w_0\equiv(\sqrt{6},0)$ all assumptions of Theorem 3.1 are satisfied and so, for every function $b\in L^1([0,1])\setminus\{0\}$ that is a.e. nonnegative and for every $\lambda\in\left]\frac{1}{\|b\|_1}\frac{7}{100},\frac{1}{\|b\|_1}\frac{18}{100}\right[$, the problem

$$\begin{gathered}\ddot{u}=u-\lambda b(t)\nabla G(u)\quad\text{a.e. in }[0,1]\\ u(1)-u(0)=\dot{u}(1)-\dot{u}(0)=0\end{gathered}$$

admits at least three nonzero solutions. In fact, it is enough to observe that

$$\frac{\max_{|w|\leq\gamma}G(w)}{\gamma^2}=\frac{1}{e}+1,$$

$R=1/5$, $G(w_0)=\left(\frac{6}{e}\right)^6+\sqrt{6}$ and

$$\lim_{|w|\to+\infty}\frac{G(w)}{|w|^2}=0.$$

Remark 3.2. Let G be as in Example 3.1, fix $b\in C^0([0,1],\mathbb{R}^+)$ and $\lambda>0$. It is easy to see that, if we put $F(t,w)=\frac{1}{2}|w|^2-\lambda b(t)G(w)$ for every $(t,w)\in[0,1]\times\mathbb{R}^2$, one has that

$$\liminf_{|w|\to 0}\frac{F(t,w)}{|w|^2}=-\infty$$

uniformly with respect to t. Therefore, assumption (d) in the introduction does not hold and, hence by [6, Theorem 7], [14, Theorem 4] and [13, Theorem 2] cannot be applied.

Example 3.2. Let $G:\mathbb{R}\to\mathbb{R}$ be defined by

$$G(w)=\begin{cases} e^{e^w}-e & \text{if } w<2\\ e^{e^2}(e^2w+1-2e^2)-e & \text{if } w\geq 2.\end{cases}$$

By choosing $\gamma=1$ and $w_0=2$ we are able to apply Theorem 3.1 and affirm that for every function $b\in L^1([0,1])\setminus\{0\}$ that is a.e. nonnegative and for every $\lambda\in]\frac{1}{\|b\|_1}\frac{19}{1000},\frac{1}{\|b\|_1}\frac{17}{100}[$, the problem

$$\begin{gathered}\ddot{u}=u-\lambda b(t)\dot{G}(u)\quad \text{a.e. in } [0,1]\\ u(1)-u(0)=\dot{u}(1)-\dot{u}(0)=0\end{gathered}$$

admits at least three nonzero solutions. In fact, a simple computation shows that

$$\frac{\max_{|w|\leq\gamma}G(w)}{\gamma^2}=e^e-e,$$

$R=1/3$ and $G(w_0)=e^{e^2}-e$ so that assumption (3.1) holds. Moreover

$$\lim_{|w|\to+\infty}\frac{G(w)}{|w|^2}=0$$

and (3.2) is also true.

Remark 3.3. By the fact that the function $\overline{\lambda}G$, where $\overline{\lambda}\in]\frac{1}{\|b\|_1}\frac{19}{1000},\frac{1}{\|b\|_1}\frac{17}{100}[$ and G is as in Example 3.2, is increasing, condition (e) in the introduction does not hold. Hence, [8, Theorem 2.1] cannot be applied. Moreover, for fixed $b\in C^0([0,1],\mathbb{R}^+)$, if we consider $F(t,w)=\frac{1}{2}|w|^2-b(t)[\overline{\lambda}G(w)]$ for every $(t,w)\in[0,1]\times\mathbb{R}^2$, it is easy to verify that

$$\liminf_{|w|\to 0}\frac{F(t,w)}{|w|^2}=-\infty$$

uniformly with respect to t and condition (d) in Introduction does not hold.

Remark 3.4. In our context, from (c) in Introduction one has

(b1) $G(w_0)>0$ for some constant vector w_0.

If, in addition, we assume

(b2) $\lim_{w\to 0}\frac{G(w)}{|w|^2}=0$,

then it is easy to verify that (b1) and (b2) imply (3.1) of Theorem 3.1.

As an immediate consequence of Theorem 3.1, we can obtain the following result.

Theorem 3.2. *Let A, G, γ and w_0 be like in Theorem 3.1. Then, for every $b\in L^1([0,T])\setminus\{0\}$ that is a.e. nonnegative and such that $\|b\|_1$ is in the interval $]\frac{1}{2k^2}\frac{1}{R}\frac{|w_0|^2}{G(w_0)},\frac{1}{2k^2}\frac{\gamma^2}{\max_{|w|\leq\gamma}G(w)}[$, the problem*

$$\begin{gathered}\ddot{u}=A(t)u-b(t)\nabla G(u)\quad \text{a.e. in } [0,T]\\ u(T)-u(0)=\dot{u}(T)-\dot{u}(0)=0\end{gathered}$$

admits at least three solutions.

Proof. Fix any $b\in L^1([0,T])\setminus\{0\}$ that is a.e. nonnegative and such that $\|b\|_1$ is in $]\frac{1}{2k^2}\frac{1}{R}\frac{|w_0|^2}{G(w_0)},\frac{1}{2k^2}\frac{\gamma^2}{\max_{|w|\leq\gamma}G(w)}[$. Obviously, one has that

$$1\in\Big]\frac{1}{2\|b\|_1k^2}\frac{1}{R}\frac{|w_0|^2}{G(w_0)},\frac{1}{2\|b\|_1k^2}\frac{\gamma^2}{\max_{|w|\leq\gamma}G(w)}\Big[$$

and we apply Theorem 3.1. $\square$

Here is another multiplicity result in which assumption (3.2) is not required.

Theorem 3.3. *Let A be a matrix-valued function satisfying condition (A1) and $G \in C^1(\mathbb{R}^N)$. Put $l = \min\big\{1, \frac{1}{k(T\sum_{i,j=1}^{k}\|a_{ij}\|_\infty)^{1/2}}\big\}$, where k is defined in (2.6) and assume that there exist two positive constants $\gamma_1,\ \gamma_2$ and a vector $w_0 \in \mathbb{R}^N$ such that $\gamma_1 < |w_0| < l\gamma_2$ and*

$$\max\Big\{\frac{\max_{|w|\le\gamma_1} G(w)}{\gamma_1^2}, \frac{\max_{|w|\le\gamma_2} G(w)}{\gamma_2^2}\Big\} < R\frac{G(w_0)}{|w_0|^2}, \tag{3.17}$$

where R is defined in (2.8). Then, for every $b \in L^1([0,T])\setminus\{0\}$ that is a.e. nonnegative, and every λ in $\big]\frac{1}{2\|b\|_1 k^2}\frac{1}{R}\frac{|w_0|^2}{G(w_0)}, \frac{1}{2\|b\|_1 k^2}\min\big\{\frac{\gamma_1^2}{\max_{|w|\le\gamma_1} G(w)}, \frac{\gamma_2^2}{\max_{|w|\le\gamma_2} G(w)}\big\}\big[$, problem (1.2) admits at least two solutions $u_{1,\lambda}$ and $u_{2,\lambda}$ such that $\|u_{1,\lambda}\|_{C^0} \le \gamma_1$ and $\|u_{2,\lambda}\|_{C^0} \le \gamma_2$.

Proof. Let $b \in L^1([0,T]) \setminus \{0\}$ be a function that is a.e. nonnegative, put $X = H_T^1$ and consider Φ and Ψ as usual. Let us introduce the following two positive numbers $r_1 = \frac{\gamma_1^2}{2k^2}$, $r_2 = \frac{\gamma_2^2}{2k^2}$ and verify that all assumptions of Theorem 2.2 hold. Obviously the functionals Φ and Ψ satisfy the regularity conditions required. Moreover, $\inf_X \Phi < r_1 < r_2$. Consider the function $v \in X$ as follows

$$v(t) = w_0$$

for each $t \in [0,T]$. Arguing as in Theorem 3.1, since $\gamma_1 < |w_0| < l\gamma_2$ and taking into account (2.2) we obtain

$$r_1 < \Phi(v) \le \frac{T}{2}\sum_{i,j=1}^{N}\|a_{ij}\|_\infty |w_0|^2 < r_2,$$
$$G(w_0) > \max_{|w|\le\gamma_1} G(w).$$

Hence, by assumption (3.17) and noting that $\gamma_1 < |w_0|$, one has

$$\begin{aligned}
\varphi_2^*(r_1,r_2) &\ge \inf_{x\in\Phi^{-1}(]-\infty,r_1[)} \frac{\Psi(x)-\Psi(v)}{\Phi(v)-\Phi(x)} \\
&= 2\inf_{\|u\|^2<2r_1} \frac{\int_0^T b(t)G(v(t))dt - \int_0^T b(t)G(u(t))dt}{\|v\|^2-\|u\|^2} \\
&\ge 2Lk^2\|b\|_1 \frac{G(w_0)-\max_{|w|\le\gamma_1} G(w)}{|w_0|^2} \\
&> 2Lk^2\|b\|_1(1-R)\frac{G(w_0)}{|w_0|^2} \\
&= 2Rk^2\|b\|_1\frac{G(w_0)}{|w_0|^2}.
\end{aligned} \tag{3.18}$$

Moreover, as we saw in Theorem 3.1,

$$\varphi(r_1) \le 2k^2\|b\|_1\frac{\max_{|w|\le\gamma_1} G(w)}{\gamma_1^2} \tag{3.19}$$

$$\varphi(r_2) \le 2k^2\|b\|_1\frac{\max_{|w|\le\gamma_2} G(w)}{\gamma_2^2}. \tag{3.20}$$

At this point, combining (3.19), (3.20), assumption (3.17) and (3.18) we obtain

$$\begin{aligned}\max\{\varphi_1(r_1),\varphi_1(r_2)\} &\leq 2k^2\|b\|_1 \max\Big\{\frac{\max_{|w|\leq\gamma_1} G(w)}{\gamma_1^2}, \frac{\max_{|w|\leq\gamma_2} G(w)}{\gamma_2^2}\Big\} \\ &< 2Rk^2\|b\|_1 \frac{G(w_0)}{|w_0|^2} \\ &< \varphi_2^*(r_1,r_2).\end{aligned} \tag{3.21}$$

Therefore all assumptions of Theorem 2.2 are satisfied. Hence, since by (3.21) one has

$$\begin{aligned}\frac{1}{\varphi_2^*(r_1,r_2)} &< \frac{1}{2\|b\|_1k^2}\frac{1}{R}\frac{|w_0|^2}{G(w_0)} \\ &< \frac{1}{2\|b\|_1k^2}\min\Big\{\frac{\gamma_1^2}{\max_{|w|\leq\gamma_1} G(w)}, \frac{\gamma_2^2}{\max_{|w|\leq\gamma_2} G(w)}\Big\} \\ &\leq \min\Big\{\frac{1}{\varphi_1(r_1)}, \frac{1}{\varphi_1(r_2)}\Big\},\end{aligned}$$

for each $\lambda \in \big]\frac{1}{2\|b\|_1k^2}\frac{1}{R}\frac{|w_0|^2}{G(w_0)}, \frac{1}{2\|b\|_1k^2}\min\big\{\frac{\gamma_1^2}{\max_{|w|\leq\gamma_1} G(w)}, \frac{\gamma_2^2}{\max_{|w|\leq\gamma_2} G(w)}\big\}\big[$, problem (1.2) admits at least two solutions $u_{1,\lambda}$ and $u_{2,\lambda}$ such that $\|u_{1,\lambda}\|^2 < 2r_1 \leq \|u_{2,\lambda}\|^2 < 2r_2$. Then thanks to (2.4) and (2.7), we can complete the proof. $\square$

Example 3.3. Let $G : \mathbb{R}^2 \to \mathbb{R}$ be defined as follows

$$G(w) = \begin{cases} \frac{|w|^6}{e^{|w|^2}} & \text{if } |w| \leq \sqrt{3} \\ (\frac{3}{e})^3\cos(|w|^2-3) & \text{if } \sqrt{3} < |w| \leq \sqrt{3+\frac{15}{2}\pi} \\ (\frac{3}{e})^3[e^{|w|^2-3-\frac{15}{2}\pi}-1] & \text{if } |w| > \sqrt{3+\frac{15}{2}\pi}. \end{cases}$$

Theorem 3.3 guarantees that for every $b \in L^1([0,1]) \setminus \{0\}$ that is a.e. nonnegative and for every $\lambda \in]\frac{3}{\|b\|_1}, \frac{4}{\|b\|_1}[$ the problem

$$\begin{gathered}\ddot{u} = u - \lambda b(t)\nabla G(u) \quad \text{a.e. in } [0,1] \\ u(1)-u(0) = \dot{u}(1)-\dot{u}(0) = 0\end{gathered} \tag{3.22}$$

admits at least one nonzero solution u_λ such that $\|u_\lambda\|_{C^0} \leq \sqrt{3+\frac{15}{2}\pi}$. To see this, we can observe that

$$k = \sqrt{2}, \quad \sum_{i,j=1}^{2}\|a_{ij}\|_\infty = 2, \quad l = \frac{1}{2}, \quad R = \frac{1}{5}.$$

Hence, Theorem 3.3 applies with $\gamma_1 = 1/2$, $\gamma_2 = \sqrt{3+\frac{15}{2}\pi}$ and $w_0 \in \mathbb{R}^2$ such that $|w_0| = \sqrt{3}$.

Remark 3.5. We observe that in Example 3.3, for every positive λ, the energy functional related to problem (P_λ^I) is not coercive, that is condition (b) in Introduction fails. Hence, we cannot apply [14, Theorem 1] or [13, Theorem 1].

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