Electronic Journal of Differential Equations, Vol. 2005(2005), No. 138, pp. 1–18.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

NONLINEAR KIRCHHOFF-CARRIER WAVE EQUATION IN A UNIT MEMBRANE WITH MIXED HOMOGENEOUS BOUNDARY CONDITIONS

NGUYEN THANH LONG

Abstract. In this paper we consider the nonlinear wave equation problem

$$
\begin{gathered}
u_{tt} - B\big(\|u\|_0^2, \|u_r\|_0^2\big)(u_{rr} + \frac{1}{r}u_r) = f(r,t,u,u_r), \quad 0 < r < 1,\ 0 < t < T, \\
\big|\lim_{r\to 0^+} \sqrt{r}u_r(r,t)\big| < \infty, \\
u_r(1,t) + hu(1,t) = 0, \\
u(r,0) = \widetilde{u}_0(r), u_t(r,0) = \widetilde{u}_1(r).
\end{gathered}
$$

To this problem, we associate a linear recursive scheme for which the existence of a local and unique weak solution is proved, in weighted Sobolev using standard compactness arguments. In the latter part, we give sufficient conditions for quadratic convergence to the solution of the original problem, for an autonomous right-hand side independent on u_r and a coefficient function B of the form $B = B(\|u\|_0^2) = b_0 + \|u\|_0^2$ with $b_0 > 0$.

1. Introduction

In this paper, we consider the initial and boundary value problem

$$
\begin{gathered}
u_{tt} - B\big(\|u\|_0^2, \|u_r\|_0^2\big)(u_{rr} + \frac{1}{r}u_r) = f(r,t,u,u_r), \quad 0 < x < 1,\ 0 < t < T, \\
\big|\lim_{r\to 0^+} \sqrt{r}u_r(r,t)\big| < \infty, \\
u_r(1,t) + hu(1,t) = 0, \\
u(r,0) = u_0(r), \quad u_t(r,0) = u_1(r),
\end{gathered}
\tag{1.1}
$$

where $B, f, \widetilde{u}_0, \widetilde{u}_1$ are given functions, $\|u\|_0^2 = \int_0^1 r|u(r,t)|^2 dr$, $\|u_r\|_0^2 = \int_0^1 r|u_r(r,t)|^2 dr$ and h is a given positive constant.

2000 *Mathematics Subject Classification.* 35L70, 35Q72.
Key words and phrases. Nonlinear wave equation; Galerkin method; quadratic convergence; weighted Sobolev spaces.

Submitted August 3, 2004. Published December 1, 2005.

Many authors [6, 7, 15, 17, 18] have studied the problem

$$\begin{gathered} v_{tt} - B_1(\|v\|^2, \|\nabla v\|^2) \triangle v = f_1(x,t,v,v_t,\nabla v) \quad \text{in } \Omega_1 \times (0,T), \\ \frac{\partial v}{\partial \nu} + hv = 0 \quad \text{on } \partial\Omega_1 \times (0,T), \\ \text{or } v = 0 \quad \text{on } \partial\Omega_1 \times (0,T), \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x) \quad \text{in } \Omega_1, \end{gathered} \tag{1.2}$$

where Ω_1 is a bounded domain in $\mathbb{R}^N$ with a sufficiently regular boundary $\partial\Omega_1$,

$$\|v\|^2 = \int_{\Omega_1} v^2(x,t)dx, \|\nabla v\|^2 = \int_{\Omega_1} |\nabla v(x,t)|^2 dx = \int_{\Omega_1} \sum_{i=1}^N \Big|\frac{\partial v}{\partial x_i}(x,t)\Big|^2 dx,$$

and ν is the outward unit normal vector on boundary $\partial\Omega_1$. With $N = 1$ and $\Omega_1 = (0,L)$ the first equation in (1.2) has its origin in the nonlinear vibration of an elastic string (c.f. Kirchhoff [7]), for which the associated equation is

$$\rho h v_{tt} - \Big(P_0 + \frac{Eh}{2L}\int_0^L \Big|\frac{\partial v}{\partial y}(y,t)\Big|^2 dy\Big) v_{xx} = 0,$$

where v is the lateral deflection, x is the space coordinate, t is the time, ρ is the mass density, h is the cross-section area, L is the length, E is Young's modulus, and P_0 is the initial axial tension.

Carrier [3] also established the model

$$v_{tt} = \Big(P_0 + P_1 \int_0^L v^2(y,t)dy\Big) v_{xx},$$

where P_0 and P_1 are constants.

In the case Ω_1 is an open unit ball of $\mathbb{R}^N$ and the functions v, f_1, $\widetilde{v}_0$, $\widetilde{v}_1$ depend on x through r with $r^2 = |x|^2 = \sum_{i=1}^N x_i^2$, we put

$$\begin{gathered} v(x,t) = u(|x|,t), \quad f_1(x,t,v,v_t,\nabla v) = \widetilde{f}_1(|x|,t), \\ \widetilde{v}_0(x) = \widetilde{u}_0(|x|), \quad \widetilde{v}_1(x) = \widetilde{u}_1(|x|), \quad \gamma = N-1. \end{gathered}$$

Then

$$-B_1\big(\|v\|^2, \|\nabla v\|^2\big)\triangle v = -B\Big(\int_0^1 u^2(r,t)r^\gamma dr, \int_0^1 |u_r(r,t)|^2 r^\gamma dr\Big)\Big(u_{rr} + \frac{1}{r}u_r\Big),$$

where $B(\xi,\eta) = B_1(\omega_N\xi, \omega_N\eta)$ and ω_N is the area of the unit sphere in $\mathbb{R}^N$. Hence, we can rewrite problem (1.2) as

$$\begin{gathered} u_{tt} - B\Big(\int_0^1 u^2(r,t)r^\gamma dr, \int_0^1 |u_r(r,t)|^2 r^\gamma dr\Big)(u_{rr} + \frac{1}{r}u_r) = \widetilde{f}_1(r,t) \\ \text{in } (0,1)\times(0,T), \\ u_r(1,t) + hu(1,t) = 0 \quad \text{on } (0,T), \\ \text{or} \quad u(1,t) = 0 \quad \text{on } (0,T), \\ u(r,0) = \widetilde{u}_0(r), u_t(r,0) = \widetilde{u}_1(r) \quad \text{in } (0,1). \end{gathered} \tag{1.3}$$

With $N = 2$, the first equation of (1.3) is the bi-dimensional nonlinear wave equation describing nonlinear vibrations of the unit membrane $\Omega_1 = \{(x,y) : x^2 + y^2 < 1\}$. In the vibration process, the area of the unit membrane and the tension at various points change in time. The condition on the boundary $\partial\Omega_1$ describes elastic constraints, where the constant h has a mechanical signification. Boundary condition $(1.1)_2$ is satisfied automatically if u is a classical solution of problem (1.1), (for example, with $u \in C^1(\overline{\Omega} \times (0,T)) \cap C^2(\Omega \times (0,T))$). This condition is also used in connection with Sobolev spaces with weight r (see. [2, 16]).

In the case of equation $(1.3)_1$ not involving the term $\frac{1}{r}u_r$ ($\gamma = 0$), we have

$$u_{tt} - B\Big(\int_0^1 u^2(r,t)dr, \int_0^1 |u_r(r,t)|^2 dr\Big) u_{rr} = f(r,t,u,u_r,u_t). \tag{1.4}$$

When $f = 0$, and $B = B\big(\int_0^1 |u_r(r,t)|^2 dr\big)$ is a function depending only on $\int_0^1 |u_r(r,t)|^2 dr$, the Cauchy or mixed problem for (1.3)) has been studied by many authors; see Ebihara, Medeiros and Miranda [5], Pohozaev [22] and the references therein. A survey of the results about the mathematical aspects of Kirchhoff model can be found in Medeiros, Limaco and Menezes [20], [21]. Medeiros [19] studied problem (1.1) on a bounded open set Ω of $\mathbb{R}^3$ with $f = f(u) = -bu^2$ where $b > 0$ is a given constant. Hosoya and Yamada [6] considered problem $(1.3)_{3,4}$-(1.3) with $f = f(u) = -\delta |u|^\alpha u$ where $\delta > 0$ and $\alpha \geq 0$ are given constants. In [9] the authors studied the existence and uniqueness of the solution of the equation

$$u_{tt} + \lambda \triangle^2 u - B(\|\nabla u\|^2)\triangle u + \varepsilon |u_t|^{\alpha-1} u_t = F(x,t),$$

where $\lambda > 0$, $\varepsilon > 0$ and $0 < \alpha < 1$ are given constants.

In the case of the term $\frac{1}{r}u_r$ appearing in equation $(1.1)_1$ we have to eliminate the coefficient $\frac{1}{r}$ by using Sobolev spaces with appropriate weight (see [11]). On the other hand, problem (1.1) with general nonlinear right-hand side $f(r,t,u,u_r,u_t)$ given as a continuous function of five variables has not been studied completely yet.

In the present paper, we study problem (1.1) with some forms of the right-hand side f. In the first part, we study problem (1.1) with the right-hand side $f(r,t,u,u_r)$ where $f \in C^0([0,1] \times \mathbb{R}_+ \times \mathbb{R}^2)$ satisfies the condition

$$\frac{\partial f}{\partial r}, \frac{\partial f}{\partial u}, \frac{\partial f}{\partial u_r} \quad \text{in} \quad C^0\big([0,1] \times \mathbb{R}_+ \times \mathbb{R}^2\big).$$

It is not necessary that $f \in C^1\big([0,1] \times \mathbb{R}_+ \times \mathbb{R}^2\big)$. First, we shall associate with equation $(1.1)_1$ a linear recurrent sequence which is bounded in a suitable function space. The existence of a local solution is proved by a standard compactness argument. Note that the linearization method in this paper and in papers ([2, 4, 14, 17, 18, 23] cannot be used in papers [5, 9, 12, 13, 15, 16, 19]. In the second part, we consider problem (1.1) corresponding to $f = f(r,u)$ and $B(\eta) = b_0 + \eta$ with given constant $b_0 > 0$. We associate with equation $(1.1)_1$ a recurrent sequence u_m (nonlinear) defined by

$$\begin{aligned}
&\frac{\partial^2 u_m}{\partial t^2} - \Big(b_0 + \int_0^1 \big|\frac{\partial u_m}{\partial r}(r,t)\big|^2 r dr\Big)\big(\frac{\partial^2 u_m}{\partial r^2} + \frac{1}{r}\frac{\partial u_m}{\partial r}\big) \\
&= f(r,u_{m-1}) + (u_m - u_{m-1})\frac{\partial f}{\partial u}(r,u_{m-1}) \quad \text{in } (0,1) \times (0,T),
\end{aligned}$$

with u_m satisfying $(1.1)_{2-3}$. The first term u_0 is chosen as $u_0 = \widetilde{u}_0$. If $f \in C^2\big([0,1] \times \mathbb{R}\big)$, we prove that the sequence u_m converges quadratically. The results obtained here relatively are in part generalizations of those in [2, 4, 14, 17, 18, 23].

2. Preliminary results, notation, function spaces

Put $\Omega = (0,1)$. We omit the definitions of the usual function spaces $L^p(\overline{\Omega})$, $H^m(\Omega)$, $W^{m,p}(\Omega)$. For any function $v \in C^0(\overline{\Omega})$ we define $\|v\|_0$ as

$$\|v\|_0 = \Big(\int_0^1 rv^2(r)dr\Big)^{1/2}$$

and define the space V_0 as the completion of the space $C^0(\overline{\Omega})$ with respect to the norm $\|\cdot\|_0$. Similarly, for any function $v \in C^1(\overline{\Omega})$ we define $\|v\|_1$ as

$$\|v\|_1 = \Big(\int_0^1 r[v^2(r) + |v'(r)|^2]dr\Big)^{1/2}$$

and define the space V_1 as completion of the space $C^1(\overline{\Omega})$ with respect to the norm $\|\cdot\|_1$. Note that the norms $\|\cdot\|_0$ and $\|\cdot\|_1$ can be defined, respectively, from the inner products

$$\langle u,v\rangle = \int_0^1 ru(r)v(r)dr,$$
$$\langle u,v\rangle + \langle u',v'\rangle = \int_0^1 r[u(r)v(r) + u'(r)v'(r)]dr.$$

Identifying V_0 with its dual V_0' we obtain the dense and continuous embedding $V_1 \hookrightarrow V_0 \equiv V_0' \hookrightarrow V_1'$. The inner product notation will be re-utilized to denote the duality pairing between V_1 and V_1'. We then have the following lemmas, the proofs of which can be found in [2]:

Lemma 2.1. *There exist constants $K_1 > 0$ and $K_2 > 0$ such that, for all $v \in C^1(\overline{\Omega})$ and $r \in \overline{\Omega}$,*

(i) $\|v'\|_0^2 + v^2(1) \geq \|v\|_0^2$,
(ii) $|v(1)| \leq K_1\|v\|_1$,
(iii) $\sqrt{r}|v(r)| \leq K_2\|v\|_1$.

Lemma 2.2. *The embedding $V_1 \hookrightarrow V_0$ is compact.*

Remark 2.3. In Lemma 2.1, the constants K_1 and K_2 can be given explicitly as $K_1 = \sqrt{1+\sqrt{2}}$ and $K_2 = \sqrt{1+\sqrt{5}}$. We also note that $lim_{r\to 0_+}\sqrt{r}v(r) = 0$ for all $v \in V_1$ (see [1, Lemma 5.40]). On the other hand, from $H^1(\varepsilon,1) \hookrightarrow C^0([\varepsilon,1]), 0 < \varepsilon < 1$ and $\sqrt{\varepsilon}\|v\|_{H^1(\varepsilon,1)} \leq \|v\|_1$ for all $v \in V_1$, it follows that $v|_{[\varepsilon,1]} \in C^0([\varepsilon,1])$. From both relations we deduce that $\sqrt{r}v \in C^0(\overline{\Omega})$ for all $v \in V_1$.

Now, we define the bilinear form

$$a(u,v) = hu(1)v(1) + \int_0^1 ru'(r)v'(r)dr, \text{ for } u,v \in V_1, \tag{2.1}$$

where h is a positive constant. Then for some uniquely defined bounded linear operator $A : V_1 \to V_1'$ we have $a(u,v) = \langle Au,v\rangle$ for all $u,v \in V_1$. We then have the following lemma.

Lemma 2.4. *The symmetric bilinear form $a(\cdot,\cdot)$ defined by (2.1) is continuous on $V_1 \times V_1$ and coercive on V_1, i.e.,*

(i) $|a(u,v)| \leq C_1\|u\|_1\|v\|_1$
(ii) $a(v,v) \geq C_0\|v\|_1^2$

for all $u, v \in V_1$, *where* $C_0 = \frac{1}{2} min\{1, h\}$ *and* $C_1 = 1 + hK_1^2$.

The proof of Lemma 2.4 is straightforward and we omit it.

Lemma 2.5. *There exists an orthonormal Hilbert basis* $\{\widetilde{w}_j\}$ *of the space* V_0 *consisting of eigenfunctions* $\widetilde{w}_j$ *corresponding to eigenvalues* λ_j *such that*

(i) $0 < \lambda_1 \le \lambda_j \uparrow +\infty$ *as* $j \to +\infty$,
(ii) $a(\widetilde{w}_j, v) = \lambda_j \langle \widetilde{w}_j, v\rangle$ *for all* $v \in V_1$ *and* $j \in \mathbb{N}$.

Note that from (ii) it follows that $\{\widetilde{w}_j/\sqrt{\lambda_j}\}$ *is automatically an orthonormal set in* V_1 *with respect to* $a(\cdot,\cdot)$ *as inner product. The eigensolutions* $\widetilde{w}_j$ *are indeed eigensolutions for the boundary value problem*

$$
\begin{gathered}
A\widetilde{w}_j \equiv \frac{-1}{r}\frac{d}{dr}\big(r\frac{d\widetilde{w}_j}{dr}\big) = \lambda_j \widetilde{w}_j, \quad \text{in } \Omega, \\
\big|\lim_{r\to 0_+} \sqrt{r}\frac{d\widetilde{w}_j}{dr}(r)\big| < +\infty, \\
\frac{d\widetilde{w}_j}{dr}(1) + h\widetilde{w}_j(1) = 0.
\end{gathered}
$$

The proof of the above Lemma can be found in [24, Theorem 6.2.1] with $V = V_1, H = V_0$ and $a(\cdot,\cdot)$ as defined by (2.1).

For functions v in $C^2(\overline{\Omega})$, we define

$$
\|v\|_2 = \Big(\int_0^1 r[v^2(r) + |v'(r)|^2 + |Av(r)|^2]dr\Big)^{1/2},
$$

and define the space V_2 as the completion of $C^2(\overline{\Omega})$ with respect to the norm $\|\cdot\|_2$. Note that V_2 is also a Hilbert space with respect to the scalar product

$$
\langle u, v\rangle + \langle u', v'\rangle + \langle Au, Av\rangle
$$

and that V_2 can be defined also as $V_2 = \{v \in V_1 : Av \in V_0\}$.

We then have the following two lemmas whose proof of which can be found in [2].

Lemma 2.6. *The embedding* $V_2 \hookrightarrow V_1$ *is compact.*

Lemma 2.7. *For all* $v \in V_2$ *we have*

(i) $\|v'\|_{L^\infty(\Omega)} \le \frac{1}{\sqrt{2}}\|Av\|_0$,
(ii) $\|v''\|_0 \le \sqrt{\frac{3}{2}}\|Av\|_0$,
(iii) $\|v\|^2_{L^\infty(\Omega)} \le \big(2\|v\|_0 + \frac{1}{\sqrt{2}}\|Av\|_0\big)\|v\|_0$.

Also the following lemma will be useful in Section 4.

Lemma 2.8. *For all* $u \in V_1$ *and* $v \in V_0$,

$$
\langle u^2, |v|\rangle = \sqrt{2}(1 + K_1^2)\|u\|_1^2\|v\|_0, \tag{2.2}
$$

where the constant K_1 *is given by Lemma 2.1.*

Proof. It suffices to prove that inequality (2.2) holds for $u \in C^1(\overline{\Omega})$ and $v \in C^0(\overline{\Omega})$. We have

$$
u(r) = u(1) - \int_r^1 u'(s)ds.
$$

Hence, it follows from Lemma 2.1 that

$$u^2(r) \le 2u^2(1) + 2\Big(\int_r^1 u'(s)ds\Big)^2 \le 2K_1^2\|u\|_1^2 + 2(1-r)\int_r^1 |u'(s)|^2 ds.$$

This implies

$$\begin{aligned} \langle u^2, |v|\rangle &= \int_0^1 ru^2(r)|v(r)|dr \\ &\le 2K_1^2\|u\|_1^2 \int_0^1 r|v(r)|dr + 2\int_0^1 r(1-r)|v(r)|dr \int_r^1 |u'(s)|^2 ds. \end{aligned} \tag{2.3}$$

Note that the first integral herein can be estimated as

$$\int_0^1 r|v(r)|dr \le \Big(\int_0^1 rdr\Big)^{1/2}\Big(\int_0^1 r|v(r)|^2dr\Big)^{1/2} = \frac{1}{\sqrt{2}}\|v\|_0.$$

Reversing the order of integration in the last integral of (2.3), we estimate that integral as

$$\begin{aligned} &\int_0^1 r(1-r)|v(r)|dr \int_0^1 |u'(s)|^2 ds \\ &= \int_0^1 |u'(s)|^2 ds \int_0^s r(1-r)|v(r)|dr \\ &\le \int_0^1 |u'(s)|^2 ds \Big(\int_0^s r(1-r)^2 dr\Big)^{1/2}\Big(\int_0^s r|v(r)|^2 dr\Big)^{1/2} \\ &\le \frac{1}{\sqrt{2}}\|u'\|_0^2\|v\|_0 \le \frac{1}{\sqrt{2}}\|u\|_1^2\|v\|_0. \end{aligned}$$

From the two estimates above, we obtain (2.2) and the lemma is proved. □

For a Banach space X, we denote by $\|\cdot\|_X$ its norm, by X' its dual space and by $L^p(0,T;X)$,$1 \le p \le \infty$ the Banach space of all real measurable functions $u : (0,T) \to X$ such that

$$\|u\|_{L^p(0,T;X)} = \Big(\int_0^T \|u(t)\|_X^p dt\Big)^{1/p} < \infty \quad \text{for } 1 \le p < \infty,$$
$$\|u\|_{L^\infty(0,T;X)} = \operatorname*{ess\,sup}_{0<t<T} \|u(t)\|_X \quad \text{for } p = \infty.$$

Let

$$u(t), \quad u'(t) = u_t(t) = \dot{u}(t), \quad u''(t) = u_{tt}(t) = \ddot{u}(t), \quad u_r(t) = \nabla u(t), \quad u_{rr}(t)$$

denote

$$u(r,t), \quad \frac{\partial u}{\partial t}(r,t), \quad \frac{\partial^2 u}{\partial t^2}(r,t), \quad \frac{\partial u}{\partial r}(r,t), \quad \frac{\partial^2 u}{\partial r^2}(r,t),$$

respectively.

3. The general case

In this section, we consider initial and boundary value problem (1.1) with general right-hand side $f = f(r,t,u,u_r)$. We make the following assumptions:

(H1) $\widetilde{u}_1 \in V_1$ and $\widetilde{u}_0 \in V_2$,
(H2) $B \in C^1(\mathbb{R}_+^2)$ with $B(\xi,\eta) \ge b_0 > 0$ for all $\xi,\eta \ge 0$,
(H3) $f \in C^0(\overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R}^2)$ and $\partial f/\partial r, \partial f/\partial u, \partial f/\partial u_r \in C^0(\overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R}^2)$.

With B and f satisfying assumptions (H2) and (H3), respectively, we introduce the following constants, for any $M > 0$ and $T > 0$:

$$\begin{gathered}
\widetilde{K}_0 = \widetilde{K}_0(M,B) = \sup\{B(\xi,\eta) : 0 \le \xi,\eta \le M^2\}, \\
\widetilde{K}_1 = \widetilde{K}_1(M,B) = \sup\big\{\big(\big|\frac{\partial B}{\partial \xi}\big| + \big|\frac{\partial B}{\partial \eta}\big|\big)(\xi,\eta) : 0 \le \xi,\eta \le M^2\big\}, \\
\overline{K}_0 = \overline{K}_0(M,T,f) = \sup_{(r,t,u,v)\in A_*} |f(r,t,u,v)|, \\
\overline{K}_1 = \overline{K}_1(M,T,f) = \sup_{(r,t,u,v)\in A_*} \big(\big|\frac{\partial f}{\partial r}\big| + \big|\frac{\partial f}{\partial u}\big| + \big|\frac{\partial f}{\partial v}\big|\big)(r,t,u,v),
\end{gathered} \tag{3.1}$$

where

$$\begin{aligned}
A_* &= A_*(M,T) \\
&= \{(r,t,u,v) : 0 \le r \le 1, 0 \le t \le T, |u| \le M\sqrt{2 + 1/\sqrt{2}}, |v| \le M/\sqrt{2}\}.
\end{aligned}$$

For each $M > 0$ and $T > 0$ we put

$$\begin{aligned}
W(M,T) = \Big\{&v \in L^\infty(0,T;V_2) : \dot{v} \in L^\infty(0,T;V_1) \text{ and } \ddot{v} \in L^2(0,T;V_0), \\
&\text{with } \|v\|_{L^\infty(0,T;V_2)},\ \|\dot{v}\|_{L^\infty(0,T;V_1)}, \|\ddot{v}\|_{L^2(0,T;V_0)} \le M\Big\}, \\
W_1(M,T) &= \big\{v \in W(M,T) : \ddot{v} \in L^\infty(0,T;V_0)\big\}.
\end{aligned}$$

We shall choose as first initial term $u_0 = \widetilde{u}_0$, suppose that

$$u_{m-1} \in W_1(M,T), \tag{3.2}$$

and associate with problem (1.1) the following variational problem: Find u_m in $W_1(M,T)$ ($m \ge 1$) so that

$$\begin{gathered}
\langle \ddot{u}_m(t), v\rangle + b_m(t)a(u_m(t),v) = \langle F_m(t), v\rangle \quad \forall v \in V_1, \\
u_m(0) = \widetilde{u}_0, \quad \dot{u}_m(0) = \widetilde{u}_1,
\end{gathered} \tag{3.3}$$

where

$$\begin{aligned}
b_m(t) &= B\big(\|u_{m-1}(t)\|_0^2, \|\nabla u_{m-1}(t)\|_0^2\big) \\
&= B\Big(\int_0^1 u_{m-1}^2(r,t)rdr, \int_0^1 \big|\frac{\partial u_{m-1}}{\partial r}(r,t)\big|^2 rdr\Big), \\
F_m(r,t) &= f\big(r,t,u_{m-1}(t),\nabla u_{m-1}(t)\big).
\end{aligned} \tag{3.4}$$

Then, we have the following result.

Theorem 3.1. *Let assumptions (H1)–(H3) hold. Then there exist a constant $M > 0$ depending on $\widetilde{u}_0$, $\widetilde{u}_1$, B, h and a constant $T > 0$ depending on $\widetilde{u}_0$, $\widetilde{u}_1$, B, h, f such that, for $u_0 = \widetilde{u}_0$, there exists a linear recurrent sequence $\{u_m\} \subset W_1(M,T)$ defined by* (3.3)-(3.4).

Proof. The proof consists of several steps.
Step 1: The Galerkin approximation (introduced by Lions [10]). Consider as in Lemma 2.5 the basis $w_j = \widetilde{w}_j/\sqrt{\lambda_j}$ for V_1 and put

$$u_m^{(k)}(t) = \sum_{j=1}^{k} c_{mj}^{(k)}(t)w_j, \tag{3.5}$$

where the coefficients $c_{mj}^{(k)}$ satisfy the system of linear differential equations

$$\begin{gathered}\langle \ddot{u}_m^{(k)}(t), w_j\rangle + b_m(t)a\big(u_m^{(k)}(t), w_j\big) = \langle F_m(t), w_j\rangle, \quad 1 \le j \le k, \\ u_m^{(k)}(0) = \widetilde{u}_{0k}, \quad \dot{u}_m^{(k)}(0) = \widetilde{u}_{1k},\end{gathered} \tag{3.6}$$

where

$$\begin{gathered}\widetilde{u}_{0k} \to \widetilde{u}_0 \quad \text{strongly in } V_2, \\ \widetilde{u}_{1k} \to \widetilde{u}_1 \quad \text{strongly in } V_1.\end{gathered} \tag{3.7}$$

Suppose that u_{m-1} satisfies (3.2). Then it is clear that system (3.6) has a unique solution $u_m^{(k)}$ on an interval $0 \le t \le T_m^{(k)} \le T$. The following estimates allows us to the take constant $T_m^{(k)} = T$ for all m and k.

Step 2: *A priori estimates*. Put

$$S_m^{(k)}(t) = X_m^{(k)}(t) + Y_m^{(k)}(t) + \int_0^t \|\ddot{u}_m^{(k)}(s)\|_0^2 ds, \tag{3.8}$$

where

$$\begin{gathered}X_m^{(k)}(t) = \|\dot{u}_m^{(k)}(t)\|_0^2 + b_m(t)a\big(u_m^{(k)}(t), u_m^{(k)}(t)\big), \\ Y_m^{(k)}(t) = a\big(\dot{u}_m^{(k)}(t), \dot{u}_m^{(k)}(t)\big) + b_m(t)\|Au_m^{(k)}(t)\|_0^2,\end{gathered} \tag{3.9}$$

where A is defined by (2.1). Then it follows that

$$\begin{aligned} S_m^{(k)}(t) =& S_m^{(k)}(0) + \int_0^t b_m'(s)\big[a\big(u_m^{(k)}(s), u_m^{(k)}(s)\big) + \|Au_m^{(k)}(s)\|_0^2\big]ds \\ &+ 2\int_0^t \langle F_m(s), \dot{u}_m^{(k)}(s)\rangle ds + 2\int_0^t a\big(F_m(s), \dot{u}_m^{(k)}(s)\big)ds \\ &- \int_0^t b_m(s)\langle Au_m^{(k)}(s), \ddot{u}_m^{(k)}(s)\rangle ds + \int_0^t \langle F_m(s), \ddot{u}_m^{(k)}(s)\rangle ds \\ =& S_m^{(k)}(0) + I_1 + \dots + I_5. \end{aligned}$$

We shall estimate step by step all integrals $I_1, \dots, I_5$.

Integral I_1: Using assumption (H2), we obtain from $(3.1)_2$ and $(3.4)_1$ that

$$\begin{aligned} |b_m'(t)| \le& 2\Big|\frac{\partial B}{\partial \xi}\big(\|u_{m-1}(t)\|_0^2, \|\nabla u_{m-1}(t)\|_0^2\big)\langle u_{m-1}(t), \dot{u}_{m-1}(t)\rangle\Big| \\ &+ 2\Big|\frac{\partial B}{\partial \eta}\big(\|u_{m-1}(t)\|_0^2, \|\nabla u_{m-1}(t)\|_0^2\big)\langle \nabla u_{m-1}(t), \nabla \dot{u}_{m-1}(t)\rangle\Big| \\ \le& 4M^2\widetilde{K}_1. \end{aligned}$$

Combining (3.8)-(3.9) we obtain

$$I_1 \le \frac{4M^2\widetilde{K}_1}{b_0}\int_0^t S_m^{(k)}(s)ds.$$

Integral I_2: Since $u_{m-1} \in W_1(M,T)$, it follows from Lemma 2.7 that

$$|u_{m-1}(r,t)| \le M\sqrt{2 + 1/\sqrt{2}}, |\nabla u_{m-1}(r,t)| \le M/\sqrt{2}, \quad \text{a.e. on } \Omega \times (0,T). \tag{3.10}$$

By the Cauchy-Schwarz inequality, it follows from $(3.1)_3$ that

$$I_2 \le 2\int_0^t \|F_m(s)\|_0\big\|\dot{u}_m^{(k)}(s)\big\|_0 ds \le 2\overline{K}_0\int_0^t \sqrt{X_m^{(k)}(s)}ds.$$

Integral I_3: Using Lemma 2.4, we have

$$I_3 \le 2C_1 \int_0^t \|F_m(s)\|_1 \big\|\dot{u}_m^{(k)}(s)\big\|_1 ds.$$

On the other hand, from $(3.1)_{3-4}$ and (3.10) we obtain

$$\|F_m(s)\|_1^2 \le \frac{1}{2}\overline{K}_0^2 + \frac{1}{2}\overline{K}_1^2\big[1 + (1+\sqrt{3}M\big]^2.$$

Then we deduce, from $(3.9)_2$ that

$$I_3 \le \frac{\sqrt{2}C_1}{\sqrt{C_0}}\big[\overline{K}_0^2 + \big(1+(1+\sqrt{3})M\big)^2\overline{K}_1^2\big]^{1/2} \int_0^t \sqrt{Y_m^{(k)}(s)}ds.$$

Integral I_4: Using the inequality $|ab| \le \frac{3}{4}a^2 + \frac{1}{3}b^2 \quad \forall a,b \in \mathbb{R}$, we get from $(3.1)_1$ and (3.8)-(3.9) that

$$\begin{aligned}
I_4 &\le \widetilde{K}_0 \int_0^t \|Au_m^{(k)}(s)\|_0 \big\|\ddot{u}_m^{(k)}(s)\big\|_0 ds \\
&\le \frac{3}{4}\widetilde{K}_0^2 \int_0^t \|Au_m^{(k)}(s)\|_0^2 ds + \frac{1}{3}\int_0^t \big\|\ddot{u}_m^{(k)}(s)\|_0^2 ds \\
&\le \frac{3\widetilde{K}_0^2}{4b_0}\int_0^t S_m^{(k)}(s)ds + \frac{1}{3}S_m^{(k)}(t).
\end{aligned}$$

Integral I_5: We use again inequality $|ab| \le \frac{3}{4}a^2 + \frac{1}{3}b^2 \quad \forall a,b \in \mathbb{R}$, we get from $(3.1)_3$ and (3.8) that

$$I_5 \le \int_0^t \|F_m(s)\|_0 \big\|\ddot{u}_m^{(k)}(s)\|_0 ds \le \frac{3}{4}T\overline{K}_0^2 + \frac{1}{3}S_m^{(k)}(t).$$

Combining the above estimates for $I_1, \dots, I_5$, we get

$$S_m^{(k)}(t) \le 3S_m^{(k)}(0) + \overline{C}_1(M,T) + \overline{C}_2(M)\int_0^t S_m^{(k)}(s)ds, \tag{3.11}$$

where

$$\begin{gathered}
\overline{C}_1(M,T) = \frac{45}{4}T\overline{K}_0^2 + \frac{9}{2C_0}TC_1^2\big[\overline{K}_0^2 + \big(1+(1+\sqrt{3})M\big)^2\overline{K}_1^2\big], \\
\overline{C}_2(M) = 1 + \frac{3}{4b_0}\big(3\widetilde{K}_0^2 + 16M^2\widetilde{K}_1\big).
\end{gathered} \tag{3.12}$$

Now, we need an estimate on the term $S_m^{(k)}(0)$. We have

$$\begin{aligned}
S_m^{(k)}(0) &= X_m^{(k)}(0) + Y_m^{(k)}(0) \\
&= \|\widetilde{u}_{1k}\|_0^2 + a(\widetilde{u}_{1k}, \widetilde{u}_{1k}) + B\big(\|\nabla\widetilde{u}_0\|_0^2\big)\big(a(\widetilde{u}_{0k}, \widetilde{u}_{0k}) + \|A\widetilde{u}_{0k}\|_0^2\big).
\end{aligned}$$

By means of the convergence (3.7), we can deduce the existence of a constant $M > 0$ independent of k and m such that

$$S_m^{(k)}(0) \le M^2/6. \tag{3.13}$$

Note that, from the assumption (H3), we have $\lim_{T\to 0_+}\sqrt{T}K_i(M,T,f)=0, i=0,1$. Then, from (3.12) we can always choose the constant $T>0$ such that

$$\begin{gathered}\big(M^2/2+\overline{C}_1(M,T)\big)exp\big[T\overline{C}_2(M)\big]\le M^2,\\ \Big(1+\frac{1}{\sqrt{b_0C_0}}\Big)\sqrt{8M^2T\widetilde{K}_1+\sqrt{2}T\overline{K}_1}\exp\big[\frac{1}{\sqrt{2}}T\overline{K}_1+\big(4+\frac{2C_1}{b_0C_0}\big)M^2T\widetilde{K}_1\big]<1.\end{gathered}\tag{3.14}$$

It follows from (3.11) and (3.13)-(3.14) that

$$S_m^{(k)}(t)\le M^2\exp[-T\overline{C}_2(M)]+\overline{C}_2(M)\int_0^t S_m^{(k)}(s)ds$$

for $0\le t\le T_m^{(k)}\le T$. By using Gronwall's lemma we deduce from here that

$$S_m^{(k)}(t)\le M^2\exp[-T\overline{C}_2(M)]\exp[\overline{C}_2(M)t]\le M^2$$

for all $t\in[0,T_m^{(k)}]$. So we can take constant $T_m^{(k)}=T$ for all k and m. Therefore, we have $u_m^{(k)}\in W_1(M,T)$ for all m and k. We can extract from $\{u_m^{(k)}\}$ a subsequence $\{u_m^{(k_i)}\}$ such that

$$\begin{aligned} u_m^{(k_i)} &\to u_m \quad \text{in } L^\infty(0,T;V_2) \text{ weak}^\star,\\ \dot u_m^{(k_i)} &\to \dot u_m \quad \text{in } L^\infty(0,T;V_1) \text{ weak}^\star,\\ \ddot u_m^{(k_i)} &\to \ddot u_m \quad \text{in } L^2(0,T;V_0) \text{ weak},\end{aligned}$$

where $u_m\in W(M,T)$. Passing to the limit in (3.6), we have u_m satisfying (3.3) in $L^2(0,T)$, weak. On the other hand, it follows from (3.2)-(3.3)$_1$ and $u_m\in W(M,T)$ that $\ddot u_m=-b_m(t)Au_m+F_m\in L^\infty(0,T;V_0)$, hence $u_m\in W_1(M,T)$ and the proof of Theorem 3.1 is complete. □

Theorem 3.2. *Let assumptions (H1)-(H3) hold. Then:*

(i) *There exist constants $M>0$ and $T>0$ satisfying* (3.13)-(3.14) *such that problem* (1.1) *has a unique weak solution $u_m\in W_1(M,T)$.*
(ii) *On the other hand, the linear recurrent sequence u_m defined by* (3.2)-(3.4) *converges to the solution u of problem* (1.1) *strongly in the space*

$$W_1(T)=\big\{v\in L^\infty(0,T;V_1):\dot v\in L^\infty(0,T;V_0)\big\}.$$

Furthermore, we have the estimate

$$\|u_m-u\|_{L^\infty(0,T;V_1)}+\|\dot u_m-\dot u\|_{L^\infty(0,T;V_0)}\le Ck_T^m\quad\forall m\ge 1,$$

where

$$\begin{aligned}k_T=&\Big(1+\frac{1}{\sqrt{b_0C_0}}\Big)\sqrt{8M^2T\widetilde{K}_1+\sqrt{2}T\widetilde{K}_1}\\ &\times\exp\Big[\frac{1}{\sqrt{2}}T\overline{K}_1+\big(4+\frac{2C_1}{b_0C_0}\big)M^2T\widetilde{K}_1\Big]<1,\end{aligned}$$

and C is a constant depending only on T,u_0,u_1 and k_T.

Proof. Existence of the solution. First, we note that $W_1(T)$ is a Banach space with respect to the norm (see [10]):

$$\|v\|_{W_1(T)}=\|v\|_{L^\infty(0,T;V_1)}+\|\dot v\|_{L^\infty(0,T;V_0)}.$$

We shall prove that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. For this, set $v_m = u_{m+1} - u_m$. Then v_m satisfies the variational problem

$$\begin{gathered}\langle \ddot{v}_m(t), w\rangle + b_{m+1}(t)a\big(v_m(t), w\big) + \big(b_{m+1}(t) - b_m(t)\big)\langle Au_m(t), w\rangle \\ = \langle F_{m+1}(t) - F_m(t), w\rangle \quad \forall \in w \in V_1, \\ v_m(0) = \dot{v}_m(0) = 0.\end{gathered}$$

Taking $w = \dot{v}_m$ herein, after integrating in t, we get

$$\begin{aligned}X_m(t) = \int_0^t & b'_{m+1}(s)a\big(v_m(s), v_m(s)\big)ds \\ & - 2\int_0^t \big(b_{m+1}(s) - b_m(s)\big)\langle Au_m(s), \dot{v}_m(s)\rangle ds \\ & + 2\int_0^t \langle F_{m+1}(s) - F_m(s), \dot{v}_m(s)\rangle ds,\end{aligned}$$

where

$$X_m(t) = \|\dot{v}_m(t)\|_0^2 + b_{m+1}(t)a\big(v_m(t), v_m(t)\big).$$

On the other hand, from $(3.1)_{2,4}$ and (3.2) we obtain

$$\begin{gathered}|b'_{m+1}(t)| \le 4M^2\widetilde{K}_1, \\ |b_{m+1}(t) - b_m(t)| \le 2\widetilde{K}_1 M\|v_{m-1}(t)\|_0 + 2\widetilde{K}_1 M\|\nabla v_{m-1}(t)\|_0 \\ \le 4\widetilde{K}_1 M\|v_{m-1}(t)\|_1, \\ \|F_{m+1}(t) - F_m(t)\|_0 \le \sqrt{2}\overline{K}_1\|v_{m-1}(t)\|_1.\end{gathered}$$

It follows that

$$\begin{aligned}&\|\dot{v}_m(t)\|_0^2 + b_0C_0\|v_m(t)\|_1^2 \\ &\le 4M^2\widetilde{K}_1C_1\int_0^t \|v_m(s)\|_1^2 ds + 8M\widetilde{K}_1\int_0^t \|v_{m-1}(s)\|_1\|Av_m(s)\|_0\|\dot{v}_m(s)\|_0 ds \\ &\quad + 2\sqrt{2}\overline{K}_1\|v_{m-1}(s)\|_1\|\dot{v}_m(s)\|_0 ds \\ &\le 4M^2\widetilde{K}_1C_1\int_0^t \|v_m(s)\|_1^2 ds \\ &\quad + \big(16M^2\widetilde{K}_1 + 2\sqrt{2}\overline{K}_1\big)\int_0^t \|v_{m-1}(s)\|_1\|\dot{v}_m(s)\|_0 ds \\ &\le \big(8M^2\widetilde{K}_1 + \sqrt{2}\overline{K}_1\big)\|v_{m-1}\|_{W_1(T)}^2 \\ &\quad + 2\Big[\frac{1}{\sqrt{2}}\overline{K}_1 + \Big(4 + \frac{2C_1}{b_0C_0}\Big)M^2\widetilde{K}_1\Big]\int_0^t \big(\|\dot{v}_m(s)\|_0^2 + b_0C_0\|v_m(s)\|_1^2\big)ds.\end{aligned}$$

Using Gronwall's lemma we deduce that

$$\begin{aligned}&\|\dot{v}_m(t)\|_0^2 + b_0C_0\|v_m(t)\|_1^2 \\ &\le \big(8M^2\widetilde{K}_1 + \sqrt{2}\overline{K}_1\big)\|v_{m-1}\|_{W_1(T)}^2 \exp\Big\{2T\Big[\frac{1}{\sqrt{2}}\overline{K}_1 + \Big(4 + \frac{2C_1}{b_0C_0}\Big)M^2\widetilde{K}_1\Big]\Big\},\end{aligned}$$

for $0 \le t \le T$. Hence

$$\|v_m\|_{W_1(T)} \le k_T\|v_{m-1}\|_{W_1(T)} \quad \forall m \ge 1,$$

where

$$k_T = \big(1+\frac{1}{\sqrt{b_0C_0}}\big)\sqrt{8M^2T\widetilde{K}_1+\sqrt{2}T\overline{K}_1}\exp\Big[\frac{1}{\sqrt{2}}T\overline{K}_1+\big(4+\frac{2C_1}{b_0C_0}\big)M^2T\widetilde{K}_1\Big]<1.$$

Hence

$$\|u_{m+p}-u_m\|_{W_1(T)} \le \|u_1-u_0\|_{W_1(T)}\frac{k_T^m}{1-k_T}$$

for all m and p. It follows that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Therefore, there exists $u \in W_1(T)$ such that

$$u_m \to u \quad \text{strongly in } W_1(T). \tag{3.15}$$

We also note that $u \in W_1(M,T)$. Then from the sequence $\{u_m\}$ we can deduce a subsequence $\{u_{m_j}\}$ such that

$$\begin{aligned} u_{m_j} &\to u \quad \text{in } L^\infty(0,T;V_2) \text{ weak}^\star,\\ \dot u_{m_j} &\to \dot u \quad \text{in } L^\infty(0,T;V_1) \text{ weak}^\star,\\ \ddot u_{m_j} &\to \ddot u \quad \text{in } L^2(0,T;V_0) \text{ weak}, \end{aligned}$$

with $u \in W(M,T)$. Noticing $(3.1)_{1-2}$ we have

$$\begin{aligned} &\Big|\int_0^T \langle b_m(t)Au_m(t) - B\big(\|u(t)\|_0^2,\|\nabla u(t)\|_0^2\big)Au(t),w(t)\rangle dt\Big|\\ &\le C_1\widetilde{K}_0\|u_m-u\|_{L^\infty(0,T;V_1)}\|w\|_{L^1(0,T;V_1)}\\ &\quad + 4C_1M\widetilde{K}_1\|u_{m-1}-u\|_{L^\infty(0,T;V_1)}\|u\|_{L^\infty(0,T;V_1)}\|w\|_{L^1(0,T;V_1)} \end{aligned} \tag{3.16}$$

for all $w \in L^1(0,T;V_1)$. It follows from (3.15)-(3.16) that

$$b_m(t)Au_m \to B\big(\|u(t)\|_0^2,\|\nabla u(t)\|_0^2\big)Au \quad \text{in } L^\infty(0,T;V_1') \text{ weak}^\star. \tag{3.17}$$

Similarly

$$\|F_m - f(r,t,u,u_r)\|_{L^\infty(0,T;V_0)} \le \sqrt{2}\overline{K}_1\|u_{m-1}-u\|_{L^\infty(0,T;V_1)}. \tag{3.18}$$

Hence, from (3.15) and (3.18), we obtain

$$F_m \to f(r,t,u,u_r) \quad \text{strongly in } L^\infty(0,T;V_0). \tag{3.19}$$

Then, taking limits in (3.3) with $m = m_j \to +\infty$, there exists $u \in W(M,T)$ satisfying

$$\begin{gathered} \langle \ddot u(t),w\rangle + B\big(\|u(t)\|_0^2,\|\nabla u(t)\|_0^2\big)a\big(u(t),w\big) = \langle f(r,t,u,u_r),w\rangle \quad w\in V_1,\\ u(0)=\widetilde{u}_0, \quad \dot u(0)=\widetilde{u}_1. \end{gathered} \tag{3.20}$$

On the other hand, from (3.17) and (3.19)-(3.20) we have

$$\ddot u = -B\big(\|u\|_0^2,\|\nabla u\|_0^2\big)Au + f(r,t,u,u_r) \in L^\infty(0,T;V_0).$$

Hence, $u \in W_1(M,T)$ and the proof of existence complete.

Uniqueness of the solution. Let u_1, u_2, be weak solutions of problem $(1.1)_{1-3}$ such that u_1 and u_2 are in $W_1(M,T)$. Then $w = u_1 - u_2$ satisfies the variational problem

$$\begin{gathered} \langle \ddot w(t),v\rangle + \widetilde{b}_1(t)a\big(w(t),v\big) + \big(\widetilde{b}_1(t)-\widetilde{b}_2(t)\big)\langle Au_2(t),v\rangle = \langle \widetilde{f}_1(t)-\widetilde{f}_2(t),v\rangle \;\forall v\in V_1,\\ w(0)=\dot w(0)=0, \end{gathered}$$

where

$$\widetilde{b}_i(t) = B\big(\|u_i(t)\|_0^2,\|\nabla u_i(t)\|_0^2\big), \widetilde{f}_i(t) = f(r,t,u_i,\nabla u_i), \quad i=1,2\,.$$

Taking $v = \dot{w}$ and integrating by parts, we obtain

$$\begin{aligned} \|\dot{w}(t)\|_0^2 + \widetilde{b}_1(t)a\big(w(t), w(t)\big) &= \int_0^t \widetilde{b}_1'(s)a\big(w(s), w(s)\big)ds \\ &\quad - 2\int_0^t \big(\widetilde{b}_1(s) - \widetilde{b}_2(s)\big)\langle Au_2(s), \dot{w}(s)\rangle ds \\ &\quad + 2\int_0^t \langle \widetilde{f}_1(s) - \widetilde{f}_2(s), \dot{w}(s)\rangle ds. \end{aligned}$$

Put

$$X(t) = \|\dot{w}(t)\|_0^2 + b_0 C_0 \|w(t)\|_1^2.$$

Then

$$X(t) = \frac{1}{\sqrt{b_0 C_0}}\Big[4\Big(1 + \frac{C_1}{\sqrt{b_0 C_0}}\Big)M^2 \widetilde{K}_1 + \sqrt{2}\overline{K}_1\Big]\int_0^t X(s)ds$$

for all $t \in [0, T]$ follows. Using Gronwall's lemma we deduce $X(t) = 0$, i.e., $u_1 = u_2$ and the proof of Theorem 3.2 is complete. □

Remark 3.3. In the case of $B \equiv 1$ and $f = f(t, u, u_t)$ with $f \in C^1(\mathbb{R}_+ \times \mathbb{R}^2)$ and $f(t, 0, 0) = 0$ for all $t \geq 0$, and with the homogeneous Dirichlet boundary condition instead of $(1.1)_2$, some results have been obtained in [4]. In the case of f being in $C^1(\overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R}^2)$ and $B \equiv 1$ we have previously obtained some results in [2]. We emphasize here that in the above, however, we do not need to assume that f is in $C^1(\overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R}^2)$.

4. A special case

In this section, we consider initial boundary value problem (1.1) with an autonomous right-hand side independent of u_r and an affine coefficient function B. Under these assumptions, we obtain stronger conclusion on the approach results in a quadratic convergence of the approximation (Theorem 4.2).

We make the following assumptions:

(H4) $B(\eta) = b_0 + \eta$ with $b_0 > 0$ a given constant.

(H5) $f \in C^2(\overline{\Omega} \times \mathbb{R})$.

With f satisfying assumption (H5), for any $M > 0$ we put

$$\begin{aligned} \overline{K}_0 &= \overline{K}_0(M, f) = \sup_{(r,u)\in\overline{A}_*} |f(r, u)|, \\ \overline{K}_1 &= \overline{K}_1(M, f) = \sup_{(r,u)\in\overline{A}_*} \Big(\Big|\frac{\partial f}{\partial r}\Big| + \Big|\frac{\partial f}{\partial u}\Big|\Big)(r, u), \\ \overline{K}_2 &= \overline{K}_2(M, f) = \sup_{(r,u)\in\overline{A}_*} \Big(\Big|\frac{\partial^2 f}{\partial r \partial u}\Big| + \Big|\frac{\partial^2 f}{\partial u^2}\Big|\Big)(r, u), \end{aligned}$$

where

$$\overline{A}_* = \overline{A}_*(M) = \big\{(r, u) : 0 \leq r \leq 1, |u| \leq M\sqrt{2 + 1/\sqrt{2}}\big\}.$$

We shall choose as a (constant in time) starting point u_0 the initial data $\widetilde{u}_0$. Assume $u_{m-1} \in W_1(M, T)$ and consider the variational problem (3.3), where

$$\begin{aligned} b_m(t) &= b_0 + \|\nabla u_m(t)\|_0^2, \\ F_m(r, t) &= f_m(r, t, u_m) = f(r, u_{m-1}) + (u_m - u_{m-1})\frac{\partial f}{\partial u}(r, u_{m-1}), \end{aligned} \tag{4.1}$$

with

$$f_m(r,t,u) = f(r,u_{m-1}) + (u - u_{m-1})\frac{\partial f}{\partial u}(r,u_{m-1}).$$

Then we have the following theorem.

Theorem 4.1. *Let (H1), (H4), and (H5) hold. Then there exist constants $M > 0$ and $T > 0$ and the recurrent sequence $\{u_m\} \in W_1(M,T)$ defined by (3.3) and (4.1).*

Proof. The idea is the same as in the proof of Theorem 3.1. As there we define $u_m^{(k)}$ by (3.5)-(3.7), where the functions b_m and F_m appearing in (3.5) are replaced by

$$b_m^{(k)}(t) = b_0 + \|\nabla u_m^{(k)}(t)\|_0^2,$$

$$F_m^{(k)}(r,t) = f_m\big(r,t,u_m^{(k)}\big) = f(r,u_{m-1}) + \big(u_m^{(k)} - u_{m-1}\big)\frac{\partial f}{\partial u}(r,u_{m-1}),$$

respectively. With $S_m^{(k)}, X_m^{(k)}, Y_m^{(k)}$ defined by (3.8)-(3.9), where the function b_m appearing in $X_m^{(k)}$ and $Y_m^{(k)}$ are replaced by $b_m^{(k)}$, it follows that

$$\begin{aligned}
S_m^{(k)}(t) =& S_m^{(k)}(0) + \int_0^t b_m^{(k)\prime}(s)\Big[a\big(u_m^{(k)}(s), u_m^{(k)}(s)\big) + \|Au_m^{(k)}(s)\|_0^2\Big]ds \\
&+ 2\int_0^t \big\langle F_m^{(k)}(s), \dot u_m^{(k)}(s)\big\rangle ds + 2\int_0^t a\big(F_m^{(k)}(s), \dot u_m^{(k)}(s)\big) ds \\
&- \int_0^t b_m^{(k)}(s)\big\langle Au_m^{(k)}(s), \ddot u_m^{(k)}(s)\big\rangle ds + \int_0^t \big\langle F_m^{(k)}(s), \ddot u_m^{(k)}(s)\big\rangle ds.
\end{aligned}$$

We can estimate $S_m^{(k)}$ in a manner similar to (3.11) as

$$S_m^{(k)}(t) \le 3S_m^{(k)}(0) + \widetilde{D}_0(M,T) + D_1(M)\int_0^t S_m^{(k)}(s)ds + D_2\int_0^t \big(S_m^{(k)}(s)\big)^2 ds,$$

where

$$\begin{aligned}
\widetilde{D}_0 =& \widetilde{D}_0(M,T) = \frac{21}{2}(\overline{K}_0 + M\overline{K}_1)^2 \\
&+ 6C_1T\Big(\overline{K}_0 + (1 + M + \sqrt{1+3M^2})\overline{K}_1 + \overline{K}_0 + M\overline{K}_2\sqrt{3+3M^2/2}\Big)^2, \\
D_1 =& D_1(M) \\
=& \frac{9}{4} + 3(1+C_1)/C_0 + 21\overline{K}_1^2/2b_0C_0 + \frac{6C_1}{b_0C_0}\big(4\overline{K}_1^2 + (3+3M^2/2)\overline{K}_2^2\big), \\
D_2 =& \frac{3}{b_0^2}\big(\sqrt{b_0} + \frac{3}{4C_0}\big).
\end{aligned}$$

From convergence (3.7) we can deduce the existence of a constant $M > 0$ independent of k and m such that $S_m^{(k)}(0) \le M^2/6$. Next, we can always choose a constant $T > 0$, so that

$$\widetilde{D}_0(M,T) \le M^2/2 \quad \text{and} \quad \Big(1 + \frac{D_1}{M^2D_2}\Big)exp(TD_1) \le 1 + \frac{4D_1}{3M^2D_2}. \tag{4.2}$$

Then

$$S_m^{(k)}(t) \le \frac{3}{4}M^2 + D_1(M)\int_0^t S_m^{(k)}(s)ds + D_2\int_0^t \big(S_m^{(k)}(s)\big)^2 ds. \tag{4.3}$$

On the other hand, the function

$$S(t) = \frac{D_1 \exp(D_1 t)}{\frac{4D_1}{3M^2} - D_2[\exp(D_1 t) - 1]}, \quad 0 \le t \le T, \tag{4.4}$$

is the maximal solution of the Volterra integral equation with non-decreasing kernel [8]

$$S(t) = \frac{3}{4}M^2 + D_1(M)\int_0^t S(s)ds + D_2\int_0^t S^2(s)ds, \quad 0 \le t \le T.$$

From (4.2)-(4.4)

$$S_m^{(k)}(t) \le S(t) \le M^2, \quad 0 \le t \le T \tag{4.5}$$

follows for all k and m. Hence $u_m^{(k)} \in W_1(M,T)$ for all k and m. Then, in a manner similar to the proof of Theorem 3.1, we can prove that the limit $u_m \in W_1(M,T)$ of the sequence $\{u_m^{(k)}\}$ when $k \to +\infty$ is the unique solution of variational problem (3.3) and (4.1). The proof of Theorem 4.1 is complete. □

The following result gives a quadratic convergence of the sequence $\{u_m\}$ to a weak solution of problem (1.1) corresponding to $f = f(r,u)$ and $B(\eta) = b_0 + \eta$.

Theorem 4.2. *Let assumptions (H1) and (H4)-(H5), hold. Then*

(i) *There exist constants $M > 0$ and $T > 0$ such that problem (1.1) corresponding to $f = f(r,u)$ and $B(\eta) = b_0 + \eta$ has a unique weak solution $u \in W_1(M,T)$.*

(ii) *On the other hand, the recurrent sequence $\{u_m\}$ defined by (3.3) and (4.1) converges quadratically to the solution u strongly in the space $W_1(T)$ in the sense*

$$\|u_m - u\|_{W_1(T)} \le C\|u_{m-1} - u\|^2_{W_1(T)},$$

where C is a suitable constant. Furthermore, we have also the estimation

$$\|u_m - u\|_{W_1(T)} \le \frac{\beta^{2^m}}{\mu_T(1-\beta)} \quad \text{for all} \quad m,$$

where

$$\mu_T = \Big(1 + \frac{1}{\sqrt{b_0 C_0}}\Big)(1 + b_0 C_0)\sqrt{TK_T^{(2)} exp(TK_T^{(1)})}$$

and $\beta = 4M\mu_T < 1$.

Note that the last condition is always satisfied by taking a suitable $T > 0$.

Proof. First, we shall prove that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. For this, set $v_m = u_{m+1} - u_m$. Then v_m satisfies the variational problem

$$\begin{gathered} \langle \ddot v_m(t), w\rangle + b_{m+1}(t)a\big(v_m(t), w\big) + \big(b_{m+1}(t) - b_m(t)\big)\langle Au_m(t), w\rangle \\ = \langle F_{m+1}(t) - F_m(t), w\rangle, \quad \forall w \in V_1, \\ v_m(0) = \dot v_m(0) = 0, \end{gathered} \tag{4.6}$$

with

$$\begin{aligned}
F_{m+1} - F_m &= f(r, u_m) - f(r, u_{m-1}) + (u_{m+1} - u_m)\frac{\partial f}{\partial u}(r, u_m) \\
&\quad - (u_m - u_{m-1})\frac{\partial f}{\partial u}(r, u_{m-1}) \\
&= v_m \frac{\partial f}{\partial u}(r, u_m) + \frac{1}{2} v_{m-1}^2 \frac{\partial^2 f}{\partial u^2}(r, \lambda_m), \\
\lambda_m &= u_{m-1} + \theta v_{m-1}, \quad (0 < \theta < 1), \\
b_{m+1}(t) - b_m(t) &= \|\nabla u_{m+1}(t)\|_0^2 - \|\nabla u_m(t)\|_0^2.
\end{aligned}$$

Taking $w = \dot{v}_m$ in (4.6) and integrating in t, we get

$$\begin{aligned}
&\|\dot{v}_m(t)\|_0^2 + b_{m+1}(t) a\big(v_m(t), v_m(t)\big) \\
&= 2\int_0^t \langle \nabla u_{m+1}(s), \nabla \dot{u}_{m+1}(s)\rangle a\big(v_m(s), v_m(s)\big) ds \\
&\quad - 2\int_0^t \big(\|\nabla u_{m+1}(s)\|_0^2 - \|\nabla u_m(s)\|_0^2\big)\langle A u_m(s), \dot{v}_m(s)\rangle ds \\
&\quad + 2\int_0^t \Big\langle v_m \frac{\partial f}{\partial u}(r, u_m), \dot{v}_m(s)\Big\rangle ds + \int_0^t \Big\langle v_{m-1}^2 \frac{\partial^2 f}{\partial u^2}(r, \lambda_m), \dot{v}_m(s)\Big\rangle ds \\
&= J_1 + \dots + J_4.
\end{aligned}$$

We can estimate herein the integrals $J_1, \dots, J_4$ step by step as

$$\begin{gathered}
J_1 \le 2M^2 \int_0^t a\big(v_m(s), v_m(s)\big) ds \le 2C_1 M^2 \int_0^t \|v_m(s)\|_1^2 ds, \\
J_2 \le 4M^2 \int_0^t \|v_m(s)\|_1 \|\dot{v}_m(s)\|_0 ds, \quad \text{(by (4.5))}, \\
J_3 \le 4\overline{K}_1 \int_0^t \|v_m(s)\|_0 \|\dot{v}_m(s)\|_0 ds \le 4\overline{K}_1 \int_0^t \|v_m(s)\|_1 \|\dot{v}_m(s)\|_0 ds, \\
J_4 \le 4\overline{K}_2 \int_0^t \langle v_{m-1}^2(s), |\dot{v}_m(s)|\rangle ds \le \overline{K}_2 \sqrt{2}(1 + K_1^2) \int_0^t \|v_{m-1}(s)\|_1^2 \|\dot{v}_m(s)\|_0 ds,
\end{gathered}$$

where the last inequality follows from (4.5) and Lemma 2.8. Combining the above estimates, we obtain

$$\begin{aligned}
&\|\dot{v}_m(t)\|_0^2 + b_0 C_0 \|v_m(t)\|_1^2 \\
&\le 2C_1 M^2 \int_0^t \|v_m(s)\|_1^2 ds + 2(2M^2 + \overline{K}_1) \int_0^t \|v_m(s)\|_1 \|\dot{v}_m(s)\|_0 ds \\
&\quad + \overline{K}_2 \sqrt{2}(1 + K_1^2) \int_0^t \|v_{m-1}(s)\|_1^2 \|\dot{v}_m(s)\|_0 ds.
\end{aligned}$$

Letting $Z_m(t) = \|\dot{v}_m(s)\|_0^2 + b_0 C_0 \|v_m(s)\|_1^2$, the above inequality can be written as

$$Z_m(t) \le K_T^{(1)} \int_0^t Z_m(s) ds + K_T^{(2)} \int_0^t Z_{m-1}^2(s) ds, \tag{4.7}$$

where

$$K_T^{(1)} = \frac{2C_1 M^2}{b_0 C_0} + \frac{2M^2 + \overline{K}_1}{\sqrt{b_0 C_0}} + \frac{(1 + K_1^2)\overline{K}_2}{\sqrt{2} b_0 C_0}, \quad K_T^{(2)} = \frac{(1 + K_1^2)\overline{K}_2}{\sqrt{2} b_0 C_0}.$$

Hence, we deduce from (4.7) that

$$\|v_m\|_{W_1(T)} \le \mu_T \|v_{m-1}\|^2_{W_1(T)}, \tag{4.8}$$

where μ_T is the constant

$$\mu_T = \big(1 + \frac{1}{\sqrt{b_0 C_0}}\big)\big(1 + b_0 C_0\big)\sqrt{T K_T^{(2)} \exp(T K_T^{(1)})}.$$

From (4.8), we obtain

$$\|u_m - u_{m+p}\|_{W_1(T)} \le \frac{\beta^{2^m}}{\mu_T(1-\beta)} \tag{4.9}$$

for all m and p where $\beta = 4M_{\mu_T} < 1$. It follows that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Then there exists $u \in W_1(T)$ such that $u_m \to u$ strongly in $W_1(T)$. Thus, and by applying a similar argument as used in the proof of Theorem 3.2, $u \in W_1(M,T)$ is the unique weak solution of problem (1.1) corresponding to $f = f(r,u)$ and $B(\eta) = b_0 + \eta$. Passing to the limit as $p \to +\infty$ for m fixed, we obtain estimate (4.5) from (4.9). This completes the proof of Theorem 4.2. $\square$

Acknowledgments. The author wants to thank Professor Dung Le for this help, and the anonymous referees for their valuable suggestions.

References

[1] Adams, R. A.; *Sobolev Spaces*, Academic Press, NewYork, 1975.

[2] Binh, D. T. T.; Dinh, A. P. N; Long, N. T.; *Linear recursive schemes associated with the nonlinear wave equation involving Bessel's operator*, Math. Comp. Modelling, **34** (2002) No. 5-6, 541-556.

[3] Carrier, G. F; *On the nonlinear vibrations problem of elastic string*, Quart. J. Appl. Math. **3** (1945) 157-165.

[4] Dinh, A. P. N; Long N. T.; *Linear approximation and asymptotic expansion associated to the nonlinear wave equation in one dimension*, Demonstratio Math. **19** (1986), No. 1, 45-63.

[5] Ebihara, Y; Medeiros, L. A.; Milla, M. Minranda; *Local solutions for a nonlinear degenerate hyperbolic equation*, Nonlinear Anal. **10** (1986) 27-40.

[6] Hosoya, M.; Yamada, Y.; *On some nonlinear wave equation I: Local existence and regularity of solutions*, J. Fac. Sci. Univ. Tokyo. Sect. IA, Math. **38** (1991) 225-238.

[7] Kirchhoff, G. R.; *Vorlesungen über Mathematiche Physik: Mechanik, Teuber, Leipzig*, 1876, Section 29.7.

[8] Lakshmikantham, V; Leela, S.; *Differential and Integral Inequalities*, Vol.1. Academic Press, NewYork, 1969.

[9] Lan, H. B.; Thanh, L. T.; Long, N. T.; Bang, N.T,; Cuong T.L., Minh, T.N.; *On the nonlinear vibrations equation with a coefficient containing an integral*, Zh. Vychisl. Mat. i Mat. Fiz. **33** (1993), No. 9, 1324–1332; translation in Comput. Math. Math. Phys. **33** (1993), No. 9, 1171–1178.

[10] Lions, J. L.; *Quelques méthodes de résolution des problè mes aux limites non-linéaires*, Dunod; Gauthier- Villars, Paris, 1969.

[11] Long, N. T.; Dinh, A. P. N.; *Periodic solutions of a nonlinear parabolic equation associated with the penetration of a magnetic field into a substance*, Computers Math. Appl. **30** (1995), No. 1, 63-78.

[12] Long, N.T.; Dinh, A. P. N.; *On the quasilinear wave equation: $u_{tt} - \triangle u + f(u, u_t) = 0$ associated with a mixed nonhomogeneous condition*, Nonlinear Anal. **19** (1992), No. 7, 613-623.

[13] Long, N. T.; Dinh, A. P. N.; *A semilinear wave equation associated with a linear differential equation with Cauchy data*, Nonlinear Anal. **24** (1995), No. 8, 1261-1279.

[14] Long, N. T.; Diem, T. N.; *On the nonlinear wave equation $u_{tt} - u_{xx} = f(x, t, u, u_x, u_t)$ associated with the mixed homogeneous conditions*, Nonlinear Anal. **29** (1997), No. 11, 1217-1230.

[15] Long, N. T.; Thuyet, T. M.; *On the existence, uniqueness of solution of the nonlinear vibrations equation*, Demonstratio Math. **32** (1999), No. 4, 749-758.

[16] Long, N. T.; Dinh, A. P. N.; Binh, D. T. T.; *Mixed problem for some semilinear wave equation involving Bessel's operator*, Demonstratio Math. **32** (1999), No. 1, 77-94.

[17] Long, N. T., Dinh, A. P. N.; Diem, T. N.; *Linear recursive schemes and asymptotic expansion associated with the Kirchhoff-Carrier operator*, J. Math. Anal. Appl. **267** (2002), No. 1, 116-134.

[18] Long, N. T.; *On the nonlinear wave equation $u_{tt} - B(t, \|u_x\|^2)u_{xx} = f(x, t, u, u_x, u_t)$ associated with the mixed homogeneous conditions*, J. Math. Anal. Appl. **274** (2002), No. 1, 102-123.

[19] Medeiros L. A.; *On some nonlinear perturbation of Kirchhoff-Carrier operator*, Comp. Appl. Math. **13** (1994) 225-233.

[20] Medeiros, L.A.; Limaco, J.; Menezes, S. B.; *Vibrations of elastic strings: Mathematical aspects*, Part one, J. Comput. Anal. Appl. **4** (2002), No. 2, 91-127.

[21] Medeiros, L. A.; Limaco, J.; Menezes, S. B.; *Vibrations of elastic strings: Mathematical aspects*, Part two, J. Comput. Anal. Appl. **4** (2002), No. 3, 211-263.

[22] Pohozaev, S. I.; *On a class of quasilinear hyperbolic equation*, Math. USSR. Sb. **25** (1975) 145-158.

[23] Ortiz, E. L., Dinh, A. P. N.; *Linear recursive schemes associated with some nonlinear partial differential equations in one dimension and the Tau method*, SIAM J. Math. Anal. **18** (1987) 452-464.

[24] Raviart, P. A.; Thomas, J. M., *Introduction à l'analyse numérique des equations aux dérivées partielles*, Masson, Paris, 1983.

Nguyen Thanh Long

Department of Mathematics and Computer Science, University of Natural Science, Vietnam National University HoChiMinh City, 227 Nguyen Van Cu Str., Dist. 5, Hochiminh City, Vietnam

E-mail address: `longnt@hcmc.netnam.vn` `longnt2@gmail.com`