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ZEROS OF THE JOST FUNCTION FOR A CLASS OF EXPONENTIALLY DECAYING POTENTIALS

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Abstract. We investigate the properties of a series representing the Jost solution for the differential equation $-y'' + q(x)y = \lambda y$, $x \geq 0$, $q \in \mathrm{L}(\mathbb{R}^+)$. Sufficient conditions are determined on the real or complex-valued potential q for the series to converge and bounds are obtained for the sets of eigenvalues, resonances and spectral singularities associated with a corresponding class of Sturm-Liouville operators. In this paper, we restrict our investigations to the class of potentials q satisfying $|q(x)| \leq ce^{-ax}$, $x \geq 0$, for some $a > 0$ and $c > 0$.

1. Introduction

We consider the differential equation

$$-y'' + q(x)y = \lambda y \quad \text{for } x \geq 0, \tag{1.1}$$

where $q \in \mathrm{L}(\mathbb{R}^+)$ is real or complex-valued, with the boundary condition

$$y(0)\cos(\alpha) + y'(0)\sin(\alpha) = 0 \quad \text{for some } \alpha \in [0, \pi). \tag{1.2}$$

In this paper, we consider the consequences of changes on the potential q rather than on the boundary condition (1.2) and we therefore restrict ourself to the classical case $\alpha \in [0, \pi)$. For an analysis of Sturm-Liouville operators with real valued, exponentially decaying potentials and nonselfadjoint boundary conditions, see for example [6].

Let $z = \sqrt{\lambda}$, $\mathrm{Im}(z) > 0$. Since $q \in \mathrm{L}(\mathbb{R}^+)$, there exists a unique $\mathrm{L}^2(\mathbb{R}^+)$-solution $\chi(x, z)$ of (1.1) satisfying

$$\chi(x, z) = e^{izx}(1 + o(1)) \quad \text{as } x \to +\infty,$$

which is known as the Jost solution [3].

Let $\phi(x, z^2)$ be the solution of (1.1) satisfying $\phi(0, z^2) = 0$, $\phi'(0, z^2) = 1$. Then $\phi(x, z^2)$ satisfies (1.2) with $\alpha = 0$ and we have

$$\mathrm{W}_0\left(\chi(x, z), \phi(x, z^2)\right) = \chi(0, z), \quad \mathrm{Im}(z) > 0,$$

where W_0 denotes the Wronskian evaluated at $x = 0$. Note that $\phi(x, z^2)$ and $\chi(x, z)$ are linearly dependent if and only if $\chi(0, z) = 0$ for some z such that $\mathrm{Im}(z) > 0$. The non-zero eigenvalues of the operator L_0 associated with (1.1) and the Dirichlet

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boundary condition are therefore of the form $\lambda = z^2$, where z is a zero of the Jost function $\chi(z) = \chi(0,z)$ satisfying $\mathrm{Im}(z) > 0$. If q is real-valued these zeros are situated on the segment line $z = it$, $0 < t < +\infty$, giving rise to negative eigenvalues.

Moreover, if q is exponentially decaying, i.e. if q satisfies

$$q(x) = O(e^{-ax}) \quad \text{as } x \to +\infty \tag{1.3}$$

for some $a > 0$ then, whether q is real or complex-valued, the Jost function $\chi(z)$ can be analytically extended to the half plane $\{z \in \mathbb{C} : \mathrm{Im}(z) > -a/2\}$ [9, 10, appendix II] and the part of the expansion in generalised eigenfunctions related to the continuous spectrum contains a spectral-type function of the form

$$\frac{1}{\pi}\Big(\frac{z}{\chi(z)\chi(-z)}\Big), \quad z > 0. \tag{1.4}$$

The expansion in eigenfunctions and generalised eigenfunctions in the case of exponentially decaying, complex-valued potentials was established by Naimark [9]. If q is real-valued the spectral-type function (1.4) is actually the spectral density associated with L_0 since, in this case, $\chi(-z) = \overline{\chi(z)}$ for $\mathrm{Im}(z) = 0$. The latter was proved by Kodaira [8] for a real-valued potential q.

If we set

$$\chi_{\pi/2}(x,z) = \frac{d}{dx}\chi(x,z) \quad \text{and} \quad \chi_{\pi/2}(z) = \chi_{\pi/2}(0,z),$$

then the non-zero eigenvalues of the operator L_α associated with (1.1) and (1.2) are of the form $\lambda = z^2$, where z is a zero of $\chi_\alpha(z)$ satisfying $\mathrm{Im}(z) > 0$, with

$$\chi_\alpha(x,z) = \chi(x,z)\cos(\alpha) + \chi_{\pi/2}(x,z)\sin(\alpha) \quad \text{and} \quad \chi_\alpha(z) = \chi_\alpha(0,z). \tag{1.5}$$

To see this note that $\chi(x,z)$ and $\phi_\alpha(x,z^2)$ are linearly dependent if and only if $\chi_\alpha(z) = 0$, where $\phi_\alpha(x,z^2)$ is a solution of (1.1) satisfying (1.2), more precisely $\phi_\alpha(0,z^2) = -\sin(\alpha)$, $\phi'(0,z^2) = \cos(\alpha)$.

If q satisfies (1.3), then $\chi_\alpha(z)$ can be analytically extended to the half-plane $\{\mathrm{Im}(z) > -a/2\}$ [9, 10, appendix II]. It is then likely that the zeros of $\chi_\alpha(z)$ situated just below the real axis will affect the behaviour of (1.4) [2, 4, 5]. Such a zero is called a resonance and, if q is real valued and if the zero is situated on the semi-axis $-it$, $0 < t < +\infty$, it is said to be an antibound state.

For $\mathrm{Im}(z) = 0$, $z \neq 0$, we also have [10, appendix II]

$$\mathrm{W}_0(\chi_\alpha(x,z), \chi_\alpha(x,-z)) = -2iz,$$

so that $\chi_\alpha(z)$ and $\chi_\alpha(-z)$ cannot vanish at the same time for $\mathrm{Im}(z) = 0$, $z \neq 0$. If q is real-valued, then $\chi_\alpha(-z) = \overline{\chi_\alpha(z)}$ and the equality above implies that $\chi_\alpha(z)$ cannot vanish for $\mathrm{Im}(z) = 0$, $z \neq 0$. On the other hand, if q is complex-valued, then $\chi_\alpha(z)$ can vanish for some z with $\mathrm{Im}(z) = 0$. If z is such a zero of $\chi_\alpha(z)$, then $\lambda = z^2$ is called a spectral singularity.

The form of the expansion in generalised eigenfunctions obtained by Naimark [9, 10, appendix II] depends on whether such spectral singularities do exist. If there is no spectral singularity, then the expansion takes a form similar to that obtained by Kodaira [8].

It is to be noted that, for $q \in \mathrm{L}(\mathbb{R}^+)$, there are no $\mathrm{L}^2(\mathbb{R}^+)$-solutions of (1.1) for $\lambda > 0$ so that the spectral singularities cannot be associated with $\mathrm{L}^2(\mathbb{R}^+)$-solutions of (1.1). Moreover, if q also satisfies (1.3), then the number of spectral singularities is finite [9, 10, appendix II].

The literature available on the study of eigenvalues, resonances and spectral singularities is already abundant but we propose here an alternative method that allows us to view them as a single mathematical object, namely as arising from the zeros of the Jost function. Our method is relatively simple and allows us, in particular, to investigate resonance-free regions for exponentially decaying potentials. More detailed results are obtained on the set of resonances for compactly supported and super-exponentially decaying potentials in [4, 5] and in [2] for a class of exponentially decaying potentials. The relationship between the Jost function and the classical Titchmarsh-Weyl function is briefly outlined in section 5.

2. The Series

It was shown by Eastham [1, 2] that, for a real-valued integrable potential q, the Jost solution $\chi(x,z)$ can be represented in the form (2.1). However, it is not difficult to show that the results below also hold when q is complex-valued and integrable. We have

$$\chi(x,z) = e^{ixz}\Big(1+\sum_{n\ge 1} r_n(x,z)\Big), \tag{2.1}$$

with

$$r_0(x,z)=1, \quad r_n(x,z)=\frac{i}{2z}\int_x^{+\infty} q(t)r_{n-1}(t,z)\left(1-e^{2iz(t-x)}\right)dt, \quad n\ge 1. \tag{2.2}$$

Also,

$$\frac{d}{dx}\chi(x,z) = e^{ixz}\Big(iz+\sum_{n\ge 1} s_n(x,z)\Big), \tag{2.3}$$

with

$$s_n(x,z) = -\frac{1}{2}\int_x^{+\infty} q(t)r_{n-1}(t,z)(1+e^{2iz(t-x)})dt \quad n\ge 1. \tag{2.4}$$

From (2.2) we have

$$\begin{gathered} r_0(x,z)=1,\\ r_1(x,z)=\frac{i}{2z}\int_x^{+\infty} q(t)\left(1-e^{2iz(t-x)}\right)dt \end{gathered}$$

so that, for $\mathrm{Im}(z)>0$,

$$|r_1(x,z)|\le \frac{1}{|z|}\int_0^{+\infty}|q(t)|dt.$$

It is readily seen by induction on n that

$$|r_n(x,z)|\le \big(\frac{\|q\|_1}{|z|}\big)^n, \quad n\ge 0,\ x\ge 0,\ \mathrm{Im}(z)>0,$$

where $\|\cdot\|_1$ is the $L(\mathbb{R}^+)$-norm, from which it follows that

$$\big|1+\sum_{n\ge 1} r_n(x,z)\big| \le \sum_{n\ge 0}\big(\frac{\|q\|_1}{|z|}\big)^n.$$

The series in (2.1) therefore converges absolutely and uniformly for $x \geq 0$, $\operatorname{Im}(z) > 0$ and $|z| > \|q\|_1$. Note that we supposed only that $q \in \mathrm{L}(\mathbb{R}^+)$. This result is similar to the one obtained by Rybkin [11, theorem 3.1].

We now investigate the convergence of (2.1) for a class of exponentially decaying potentials.

3. Main Results

We suppose throughout this section that

$$|q(x)| \leq ce^{-ax}, \quad x \geq 0, \tag{3.1}$$

holds for some $c > 0$ and $a > 0$.

We first consider the case $\alpha = 0$ and then examine the case $\alpha \in (0, \pi)$. In the latter case the details get rather cumbersome but, since we are aware of only few results concerning this case, we mention it anyway.

Let $\delta > 0$ and let

$$\Lambda_{a,\delta} = \{z \in \mathbb{C} : \operatorname{Im}(z) > -a/3,\ |z| > \delta\}.$$

Lemma 3.1. *Suppose that* (3.1) *holds and fix $\delta > 2c/a$. Then*

$$|r_n(x,z)| \leq \frac{1}{n!}\Big(\frac{2c}{|z|a}\Big)^n e^{-nax}, \quad x \geq 0,\ \operatorname{Im}(z) > -a/3,\ n \geq 1$$

and the series (2.1) *converges absolutely and uniformly for $x \geq 0$, $z \in \Lambda_{a,\delta}$.*

Proof. We first prove by induction that

$$|r_n(x,z)| \leq \frac{1}{n!}\left(\frac{c}{|z|a}\right)^n \left(\frac{a + \operatorname{Im}(z)}{a + 2\operatorname{Im}(z)}\right) \dots \left(\frac{na + \operatorname{Im}(z)}{na + 2\operatorname{Im}(z)}\right) e^{-nax}, \quad n \geq 1.$$

According to (2.2) we have $r_0(x,z) = 1$ and, from (2.2) and (3.1),

$$r_1(x,z) \leq \frac{c}{2|z|}\int_x^\infty \left(e^{-at} + e^{-t(a+2\operatorname{Im}(z))+2x\operatorname{Im}(z)}\right) dt,$$

which yields

$$|r_1(x,z)| \leq \frac{c}{a|z|}\left(\frac{a + \operatorname{Im}(z)}{a + 2\operatorname{Im}(z)}\right) e^{-ax}.$$

The result is therefore true for $n = 1$. Suppose that it were true for $1 \leq k \leq n-1$, $n \geq 2$. According to (2.2) we have

$$|r_n(x,z)| \leq \frac{1}{2|z|}\int_x^\infty |q(t) r_{n-1}(t,z)| \left(1 + e^{-2(t-x)\operatorname{Im}(z)}\right) dt,$$

so that, from (3.1) and the induction hypothesis,

$$\begin{aligned} |r_n(x,z)| \leq & \frac{c}{2|z|(n-1)!}\left(\frac{c}{|z|a}\right)^{n-1}\left(\frac{a + \operatorname{Im}(z)}{a + 2\operatorname{Im}(z)}\right) \times \dots \\ & \times \left(\frac{(n-1)a + \operatorname{Im}(z)}{(n-1)a + 2\operatorname{Im}(z)}\right)\int_x^{+\infty} e^{-nat}(1 + e^{-2(t-x)\operatorname{Im}(z)}) dt, \end{aligned}$$

which yields

$$\begin{aligned} &|r_n(x,z)| \\ &\leq \frac{1}{n!}\left(\frac{c}{|z|a}\right)^n \left(\frac{a + \operatorname{Im}(z)}{a + 2\operatorname{Im}(z)}\right) \dots \left(\frac{(n-1)a + \operatorname{Im}(z)}{(n-1)a + 2\operatorname{Im}(z)}\right)\left(\frac{na + \operatorname{Im}(z)}{na + 2\operatorname{Im}(z)}\right) e^{-nax}, \end{aligned}$$

as required. The lemma is proved when we notice that

$$0 < \frac{na + \mathrm{Im}(z)}{na + 2\,\mathrm{Im}(z)} < 2, \quad n \geq 1, \quad \text{and} \quad \frac{2c}{|z|a} < \frac{2c}{\delta a} < 1$$

if $\mathrm{Im}(z) > -a/3$ and $|z| > \delta > 2c/a$. □

We are now in position to identify a region in the z-plane where $\chi(z)$ cannot vanish.

Theorem 3.2. *Suppose* (3.1) *holds and fix* $\delta > 2c/a$. *Then, for* $z \in \Lambda_{a,\delta}$,

$$|\chi(z)| \geq 2 - \exp\left(\frac{2c}{\delta a}\right)$$

In particular, if

$$\delta > \frac{2c}{a\ln(2)},$$

then $\chi(z)$ *cannot vanish inside the set* $\Lambda_{a,\delta}$ *and the operator* L_0 *has*

(i) *no eigenvalue* $\lambda = z^2$ *such that* $z \in \Lambda_{a,\delta} \cap \{z : \mathrm{Im}(z) > 0\}$,
(ii) *no spectral singularity* $\lambda = z^2$ *such that* $z \in (-\infty, \delta) \cup (\delta, +\infty)$,
(iii) *no resonance inside* $\Lambda_{a,\delta} \cap \{z : \mathrm{Im}(z) < 0\}$.

Proof. According to lemma 3.1 we have, for $z \in \Lambda_{a,\delta}$,

$$|r_n(x,z)| \leq \frac{1}{n!}\left(\frac{2c}{\delta a}\right)^n e^{-nax}, \quad x \geq 0,$$

so that

$$\Big|\sum_{n\geq 1} r_n(x,z)\Big| \leq \sum_{n\geq 1} \frac{1}{n!}\left(\frac{2c}{\delta a}\right)^n e^{-nax} = \exp\left(\frac{2c}{\delta a}e^{-ax}\right) - 1.$$

Since

$$|\chi(x,z)| = e^{-x\,\mathrm{Im}(z)}\Big|1 + \sum_{n\geq 1} r_n(x,z)\Big| \geq e^{-x\,\mathrm{Im}(z)}\Big\{1 - \Big|\sum_{n\geq 1} r_n(x,z)\Big|\Big\},$$

we obtain

$$|\chi(z)| \geq 2 - \exp\left(\frac{2c}{\delta a}\right).$$

In particular, $\chi(z)$ does not vanish if

$$2 - \exp\left(\frac{2c}{\delta a}\right) > 0,$$

i.e. if

$$\delta > \frac{2c}{a\ln(2)},$$

from which (*i*), (*ii*) and (*iii*) follow. □

Note that, under the hypotheses of theorem 3.2, if $\lambda = z^2$ is an eigenvalue of L_0 then z can only be located on the semi disk $\{z \in \mathbb{C} : |z| \leq \delta, \mathrm{Im}(z) > 0\}$ and, if q is real-valued, on the segment line $z = it,\ 0 < t \leq \delta$. Also, under the hypotheses of theorem 3.2, the resonances situated on $\{z \in \mathbb{C} : -a/3 < \mathrm{Im}(z) < 0\}$ must be inside the set $\{z \in \mathbb{C} : -a/3 < \mathrm{Im}(z) < 0, |z| \leq \delta\}$ and the spectral singularities $\lambda = z^2$ must satisfy $-\delta < z < \delta$.

We now show that a similar situation prevails in the case $\alpha \neq 0$.

Lemma 3.3. *Suppose that* (3.1) *holds and fix* $\delta > 2c/a$. *Then*

$$|s_n(x,z)| \le \frac{|z|}{n!}\big(\frac{2c}{|z|a}\big)^n e^{-nax}, \quad x \ge 0, \quad \operatorname{Im}(z) > -a/3, \quad n \ge 1$$

and the series (2.3) *converges absolutely and uniformly for* $x \ge 0$, $z \in \Lambda_{a,\delta}$.

Proof. From (2.2), (2.3) and (2.4), we have

$$\frac{d}{dx}\chi(x,z) = e^{izx}\Big(iz + \sum_{n\ge 1} s_n(x,z)\Big)$$

and

$$|s_n(x,z)| \le \frac{|z|}{2|z|}\int_x^{+\infty} |q(t)r_{n-1}(t,z)|\left(1 + e^{-2\operatorname{Im}(z)(t-x)}\right)dt, \quad n \ge 1.$$

Arguing as in lemma 3.1, we obtain the stated result. □

The bounds we obtain for $\alpha \in (0,\pi/2)\cup(\pi/2,\pi)$ are not as tight as the ones obtained in theorem 3.2, which is rather natural as, for $\alpha \in (0,\pi/2)\cup(\pi/2,\pi)$, it is possible to find resonances far below the real axis or large eigenvalues, depending on the value of α. We refer to the first example in the next section for an illustration of this phenomenon.

Theorem 3.4. *Suppose that* (3.1) *holds and let* δ *be such that*

$$\delta > \frac{2c}{a\ln(2)}.$$

Then (i), (ii) and (iii) of theorem 3.2 hold as they stand for the operator $\mathrm{L}_{\pi/2}$ *and (i), (ii) and (iii) of theorem 3.2 continue to hold for the operator* L_α, $\alpha \in (0,\pi/2)\cup(\pi/2,\pi)$, *provided we replace* δ *by* $\max\{\delta,\delta_\alpha\}$, *where*

$$\delta_\alpha = |\cot(\alpha)|\frac{\exp\big(\frac{2c}{\delta a}\big)}{2-\exp\big(\frac{2c}{\delta a}\big)}.$$

Proof. We first suppose that $\alpha = \pi/2$. According to (1.5), (2.3) and lemma 3.3 we have, for $z \in \Lambda_{a,\delta}$,

$$|\chi_{\pi/2}(z)| \ge |z| - |z|\Big\{\exp\big(\frac{2c}{\delta a}\big) - 1\Big\} = |z|\Big\{2 - \exp\big(\frac{2c}{\delta a}\big)\Big\}. \tag{3.2}$$

It follows that $\chi_{\pi/2}(z)$ cannot vanish inside $\Lambda_{a,\delta}$ if $\delta > 2c/a\ln(2)$, and the first part of the theorem is proved.
Suppose now that $\alpha \in (0,\pi/2)\cup(\pi/2,\pi)$. From (2.1) and lemma 3.1 we get

$$|\chi(z)| \le 1 + \sum_{n\ge1}|r_n(0,z)| \le \exp\left(\frac{2c}{\delta a}\right).$$

On the other hand, according to (1.5),

$$|\chi_\alpha(z)| \ge |\sin(\alpha)\chi_{\pi/2}(z)| - |\cos(\alpha)\chi(z)|$$

so that, with (3.2), we obtain

$$|\chi_\alpha(z)| \ge |z\sin(\alpha)|\big\{2 - \exp\big(\frac{2c}{\delta a}\big)\big\} - |\cos(\alpha)|\exp\big(\frac{2c}{\delta a}\big).$$

From the equality above, it is not hard to see that $\chi_\alpha(z) > 0$ for

$$|z| > |\cot(\alpha)| \frac{\exp\left(\frac{2c}{\delta a}\right)}{2 - \exp\left(\frac{2c}{\delta a}\right)},$$

from which the last part of the theorem follows. □

Let $\delta' = \max\{\delta, \delta_\alpha\}$. Under the hypotheses of theorem 3.4, the eigenvalues $\lambda = z^2$ must be such that $z \in \{z \in \mathbb{C} : |z| \le \delta'\}$, the resonances situated on $\{z \in \mathbb{C} : -a/3 < \mathrm{Im}(z) < 0\}$ must be inside the set $\{z \in \mathbb{C} : -a/3 < \mathrm{Im}(z) < 0, |z| \le \delta'\}$ and the spectral singularities $\lambda = z^2$ must satisfy $-\delta' < z < \delta'$.

4. Examples

The case $q \equiv 0$. Let $q \equiv 0$ in (1.1). Then the Jost solution is $\chi(x, z) = e^{izx}$ so that

$$\chi_\alpha(x, z) = \cos(\alpha)e^{izx} + iz\sin(\alpha)e^{izx}, \quad \alpha \in (0, \pi).$$

Hence the only zero of $\chi_\alpha(z)$ is

- $z = 0$ if $\alpha = \pi/2$
- $z_\alpha = i\cot(\alpha)$ if $\alpha \in (0, \pi/2) \cup (\pi/2, \pi)$.

If $\alpha \in (0, \pi/2)$ then $\mathrm{Im}(z_\alpha) > 0$, so that $\lambda_\alpha = -\cot^2(\alpha)$ is an eigenvalue and, if $\alpha \in (\pi/2, \pi)$, then $\mathrm{Im}(z_\alpha) < 0$ so that $z_\alpha = i\cot(\alpha)$ is a resonance.

If we suppose that α is strictly complex then

$$z_\alpha = -\frac{\sinh(2\,\mathrm{Im}(\alpha))}{\cosh(2\,\mathrm{Im}(\alpha)) - \cos(2\,\mathrm{Re}(\alpha))} + i\frac{\sin(2\,\mathrm{Re}(\alpha))}{\cosh(2\,\mathrm{Im}(\alpha)) - \cos(2\,\mathrm{Re}(\alpha))},$$

so that $\lambda_\alpha = z_\alpha^2$ is an eigenvalue if $\sin(2\,\mathrm{Re}(\alpha)) > 0$, and z_α is a resonance if $\sin(2\,\mathrm{Re}(\alpha)) < 0$ and $\lambda_\alpha = z_\alpha^2$ is a spectral singularity if $\sin(2\,\mathrm{Re}(\alpha)) = 0$.

The Jost-Bessel function. If we take $q(x) = be^{-dx}$ in (1.1), with $b,\ d \in \mathbb{C}$ and $\mathrm{Re}(d) > 0$, then it can be proved by induction [7] that, in the notation of (2.1),

$$\begin{aligned}\chi(x, z) &= e^{izx}\Big\{1 + \sum_{n\ge1} r_n(x, z)\Big\}\\ &= e^{izx}\Big\{1 + \sum_{n\ge1} \frac{(bd^{-2}e^{-dx})^n}{n!}\left(\frac{1}{(1 - 2iz/d)} \cdots \frac{1}{(n - 2iz/d)}\right)\Big\}.\end{aligned}$$

This formula for the Jost solution is independently confirmed in [2], where it is noted that when q is real valued, (1.1) is satisfied by the Bessel function

$$\mathrm{J}_{-2iz/d}\left\{(2id^{-1}\sqrt{b})e^{-dx/2}\right\},$$

which is in $\mathrm{L}^2(\mathbb{R}^+)$ for $\mathrm{Im}(z) > 0$ (see also [13, §4.14] and [14, §2.13]).
If $d > 0$ and $b > 0$, then as in [2] L_0 had no eigenvalues and also no antibound states in the segment line $z = it$, $-d/2 < t < 0$.

Taking $b = -1$ and $d = 1$, it was shown in [7], using methods we have not discussed in the present paper, that although L_0 has no eigenvalues, it does have a unique antibound state $z_0 = it_0$ such that $t_0 \in (-1/2,\ 0)$, more precisely $t_0 \in [-0.139,\ -0.112]$.
In order to compare the last of these examples with the results obtained in theorem 3.2, take $a = 1$ and $c = 1$ in theorem 3.2. Theorem 3.2 predicts that if $\delta \ge 2.9$, then

$\chi(z)$ has no zero inside the set $\{z \in \mathbb{C} : \operatorname{Im}(z) > -1/3, |z| > \delta\}$, so that the estimate obtained in the last example is consistent with the bound obtained in theorem 3.2.

Note that the bounds obtained in theorem 3.2 with $a = 1$ and $c = 1$ also apply, for example, to the complex valued potential

$$q(x) = \frac{x-i}{x+i} e^{(-1+2i)x}.$$

5. Jost function and Titchmarsh-Weyl function

We suppose in the first instance that $q \in \mathrm{L}(\mathbb{R}^+)$ is real valued and give a brief account of the relationship between the Jost function and the Titchmarsh-Weyl function, since the eigenvalues and more generally the spectrum of the operator L_α have traditionally been studied using the properties of the Titchmarsh-Weyl function $m_\alpha(\lambda)$. Let $\phi_\alpha(x, \lambda)$ be defined as above and let $\theta_\alpha(x, \lambda)$ be the solution of (1.1) satisfying

$$\theta_\alpha(0, \lambda) = \cos(\alpha), \quad \theta'_\alpha(0, \lambda) = \sin(\alpha).$$

Since Weyl's limit-point case applies at $+\infty$, it is known that there exists a unique linearly independent $\mathrm{L}^2(\mathbb{R}^+)$-solution ψ_α of (1.1) such that

$$\psi_\alpha(x, \lambda) = \theta_\alpha(x, \lambda) + m_\alpha \phi_\alpha(x, \lambda), \quad x \geq 0, \ \operatorname{Im} \lambda > 0,$$

which is known as the Weyl solution [13]. The function $m_\alpha(\lambda)$ is analytic in the upper half plane $\{\lambda \in \mathbb{C} : \ \operatorname{Im}(\lambda) > 0\}$ and satisfies

$$\operatorname{Im}(m_\alpha(\lambda)) > 0 \quad \text{for } \operatorname{Im}(\lambda) > 0,$$

so that $\lim_{\operatorname{Im}\lambda \to 0+} m_\alpha(\lambda)$ exists and is finite Lebesgue almost everywhere. The eigenvalues of L_α are the poles of m_α.

On the other hand, it is readily seen that

$$\chi(x, z) = \mathrm{W}_0(\chi, \phi_\alpha)\theta_\alpha(x, z^2) + \mathrm{W}_0(\theta_\alpha, \chi)\phi_\alpha(x, z^2), \quad \operatorname{Im}(z) > 0,$$

so that we have formally

$$\psi_\alpha(x, z^2) = \frac{1}{\mathrm{W}_0(\chi, \phi_\alpha)} \chi(x, z).$$

It follows that

$$m_\alpha(z^2) = \frac{\mathrm{W}_0(\theta_\alpha, \chi)}{\mathrm{W}_0(\chi, \phi_\alpha)} = \frac{\mathrm{W}_0(\theta_\alpha, \chi)}{\chi_\alpha(z)}, \quad \operatorname{Im}(z) > 0, \quad \operatorname{Re}(z) > 0 \tag{5.1}$$

and the poles of $m_\alpha(z^2)$ are the zeros of $\chi_\alpha(z)$. Since $\mathrm{W}_0(\theta_\alpha, \chi)$ and $\chi_\alpha(z)$ are analytic in the upper half plane $\{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$, we can analytically extend $m_\alpha(\lambda)$ using (5.1). The extended Titchmarsh-Weyl function is meromorphic on $\mathbb{C} \setminus [0, +\infty)$.

If $q \in \mathrm{L}(\mathbb{R}^+)$ is allowed to be complex valued and if $\operatorname{Im}(q) \leq 0$, a similar situation prevails [12] and we can construct a Titchmarsh-Weyl function which is analytic on $\{\lambda \in \mathbb{C} : \operatorname{Im}(\lambda) > 0\}$ and can be analytically extended to a function meromorphic on $\mathbb{C} \setminus [0, +\infty)$. For additional information and references on the relationship between the Jost solution and the Titchmarsh-Weyl function, we refer to [6].

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