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MULTIPLICITY AND SYMMETRY BREAKING FOR POSITIVE RADIAL SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS MODELLING MEMS ON ANNULAR DOMAINS

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Abstract. The use of electrostatic forces to provide actuation is a method of central importance in microelectromechanical system (MEMS) and in nano-electromechanical systems (NEMS). Here, we study the electrostatic deflection of an annular elastic membrane. We investigate the exact number of positive radial solutions and non-radially symmetric bifurcation for the model

$$-\Delta u = \frac{\lambda}{(1-u)^2} \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

where $\Omega = \{x \in \mathbb{R}^2 : \epsilon < |x| < 1\}$. The exact number of positive radial solutions maybe 0, 1, or 2 depending on λ. It will be shown that the upper branch of radial solutions has non-radially symmetric bifurcation at infinitely many $\lambda_N \in (0, \lambda^*)$. The proof of the multiplicity result relies on the characterization of the shape of the time-map. The proof of the bifurcation result relies on a well-known theorem due to Kielhöfer.

1. Introduction

In this paper, we shall study the multiplicity and symmetry breaking of positive radial solutions to the equation

$$-\Delta u = \frac{\lambda}{(1-u)^2} \quad \text{in } \Omega, \tag{1.1}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

where $\Omega = \{x \in \mathbb{R}^2 : \epsilon < |x| < 1\}$ is an annulus in $\mathbb{R}^2$, λ is a positive parameter and its meaning will become clear later in the paper.

This paper is motivated by the recent work of Pelesko, Bernstein and McCuan [9]. In [9], they showed that asymmetric solutions exist through numerical investigation. A bifurcation diagram was obtained. They conjectured that there are an infinite number of branches of asymmetric solutions intersecting the upper radially symmetric solution branch.

In this paper, we use shooting method and time map to show the exact multiplicity of radial solutions, see for example [3, 6, 7, 8, 11, 12]. A well-known bifurcation

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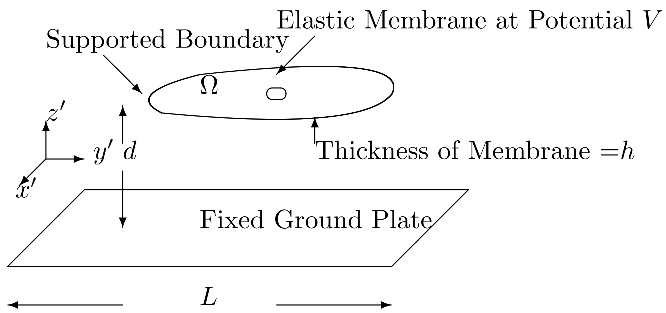

FIGURE 1. A basic electrostatically actuated elastic membrane. The prime coordinates indicate they have not yet been scaled.

theorem essentially due to H.Kielhöfer [4] is used to show the radial symmetry breaking result.

We can establish the following theorems:

Theorem 1.1. *There exists a λ^* such that the problem has no positive radial solution for $\lambda > \lambda^*$, one and only one radial solution for $\lambda = \lambda^*$ and exactly two radial solutions for $0 < \lambda < \lambda^*$.*

Theorem 1.2. *There exists infinitely many $\lambda_k \in (0, \lambda^*)$ such that the upper branch of radially symmetric solutions has a non-radially symmetric bifurcation at each λ_k, $k = 1, 2, \dots$.*

The paper is organized as follows. In section 2, we briefly describe the model proposed in [9]. For more information on this topic, refer to [1, 2]. In section 3, we show the existence results for small λ. In section 4, we obtain the multiplicity results. In section 5, we study the radial symmetry breaking problem and the conjecture in [9] is proved.

2. Formulation of the model

We model the device shown in Figure 1 which consists of an annular elastic membrane suspended above a rigid plate. The membrane is supported along the inner and outer boundaries. A voltage difference is applied across the device in order to cause deflection of the membrane. In particular, the upper surface of the membrane is held at potential V, while the ground plate is held at zero potential.

We shall notice the fact that most MEMS devices are of small aspect ratio, $d/L << 1$, and use thin components, $h/d << 1$. Here d is the distance between the membrane and the plate, L is the size of the plate and h is the thickness of membrane. We derive an approximate solution. For the completeness of the paper, we have reproduced the model following [9].

We assume the electrostatic potential satisfies Laplace's equation everywhere away from the membrane and the plate.

$$\Delta\phi = 0. \tag{2.1}$$

It also satisfies appropriate boundary conditions on the membrane.

$$\begin{aligned} \phi = V \quad &\text{on} \quad \text{elastic} \quad \text{plate}, \\ \phi = 0 \quad &\text{on} \quad \text{ground} \quad \text{plate}. \end{aligned}$$

We model the elastic membrane using the plate equation. In particular, the deflection u' of the membrane satisfies

$$\rho h \frac{\partial^2 u'}{\partial^2 t'} + a \frac{\partial u'}{\partial t'} - \mu \nabla_\perp^2 u' + D \nabla_\perp^4 u' = -\frac{\epsilon_0}{2} |\nabla \phi|^2.$$

Here ρ is the density of the membrane, h is the thickness, μ is the tension in the membrane, D is the flexural rigidity, and ϵ_0 is the permittivity of free space. $\nabla_\perp$ represents the differentiation with respect to x' and y'. The standard plate equation has been modified in two ways. First, a damping term has been added. The parameter a is the damping constant. Second, we have assumed a is proportional to velocity. We shall rescale the system and rewrite in dimensionless form. We rescale the electrostatic potential with the applied voltage, time with a damping timescale of the system, the x' and y' with a characteristic length of the device, and z' and u' with the size of the gap between the ground plate and the elastic membrane. We define

$$u = \frac{u'}{d}, \quad \phi = \frac{\phi}{V}, \quad x = \frac{x'}{L}, \quad y = \frac{y'}{L}, \quad z = \frac{z'}{d}, \quad t = \frac{\mu t'}{aL^2}. \tag{2.2}$$

In dimensionless form, we have

$$\epsilon^2 \left(\frac{\partial^2 \phi}{\partial^2 x^2} + \frac{\partial^2 \phi}{\partial^2 y^2} \right) + \frac{\partial^2 \phi}{\partial^2 z^2} = 0, \tag{2.3}$$

$$\phi = 0 \quad \text{on the ground plate,} \tag{2.4}$$

$$\phi = 1 \quad \text{on the membrane,} \tag{2.5}$$

$$\frac{1}{\alpha^2} \frac{\partial^2 u}{\partial^2 t} + \frac{\partial u}{\partial t} - \nabla_\perp^2 u + \delta \nabla_\perp^4 u = -\lambda \big[\epsilon^2 |\nabla_\perp \phi|^2 + (\frac{\partial \phi}{\partial z})^2 \big]. \tag{2.6}$$

Here ϕ is a dimensionless potential scaled with respect to voltage V, x and y are scaled with respect to the length of the ground pate L, z is scaled with respect to the gap size d. $\alpha = \frac{aL}{\sqrt{\rho h \mu}}$ is the inverse of the quality factor for the system. $\delta = \frac{D}{L^2 \mu}$ measures the relative importance of tension and rigidity. $\epsilon = \frac{d}{L}$ is the aspect ratio of the system. $\lambda = \epsilon_0 V^2 L^2 / 2T d^3$, where T is the tension in the membrane and ϵ_0 is the permittivity of free space. Note that λ is a dimensionless number which characterizes the relative strengths of electrostatic and mechanical forces in the system. As λ is proportional to the applied voltage, it serves as a convenient bifurcation parameter.We assume the displacement of the membrane u satisfies

$$\Delta u = \lambda \big[\delta^2 (\frac{\partial^2 \phi}{\partial^2 x^2} + \frac{\partial^2 \phi}{\partial^2 y^2}) + \frac{\partial^2 \phi}{\partial^2 z^2} \big],$$
$$u = 0 \quad \text{on the boundary.}$$

Assuming $d \ll L$, that is $\epsilon \ll 1$. Physically, this means that the lateral dimension of the device are large compared to the gap between the membrane and the ground plate. For many MEMS systems this is an excellent approximation. We exploit the small-aspect ratio by setting ϵ goes to zero in equation (2.3). This reduces the electrostatic problem to

$$\frac{\partial^2 \phi}{\partial z^2} = 0, \tag{2.7}$$

which we may solve to find the approximate potential,

$$\phi \approx Az + B.$$

We are primarily concerned with the field between the plates and hence apply the boundary condition on ϕ which is

$$\phi(x, y, u, t) = 1,$$
$$\phi(x, y, 0, t) = 0.$$

Hence

$$\phi \approx \frac{z}{u}.$$

Therefore, by sending ϵ goes to zero and use this approximate potential in equation (2.6), we find

$$\frac{1}{\alpha^2}\frac{\partial^2 u}{\partial^2 t} + \frac{\partial u}{\partial t} - \nabla_\perp^2 u + \delta \nabla_\perp^4 u = -\frac{\lambda}{u^2}. \tag{2.8}$$

We shall focus on the equilibrium state deflection. For convenience, we change variable $u \mapsto 1 - u$. The result is the following semi-linear elliptic equation for the displacement u:

$$-\Delta u = \frac{\lambda}{(1-u)^2} \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega.$$

3. Existence

In this section, we shall study the following semilinear elliptic equation with Dirichlet boundary condition.

$$-\Delta u = \frac{\lambda}{(u-1)^2} \quad \text{in } \Omega, \tag{3.1}$$
$$u = 0 \quad \text{on } \partial\Omega. \tag{3.2}$$

Theorem 3.1. *There exists a λ^* such that when $\lambda > \lambda^*$ there is no solution to* (3.1) *and* (3.2).

Proof. Let λ_1 be the lowest eigenvalue of

$$-\Delta u = \lambda u \quad \text{in } \Omega, \tag{3.3}$$
$$u = 0 \quad \text{on } \partial\Omega, \tag{3.4}$$

with u_1 the corresponding eigenfunction which can be chosen strictly positive on Ω. Multiplying (3.1) by u_1 and integrating yields

$$\int_\Omega -u\Delta u_1 = \lambda \int_\Omega \frac{u_1}{(1-u)^2}.$$

Or equivalently,

$$\lambda_1 \int_\Omega u u_1 = \lambda \int_\Omega \frac{u_1}{(1-u)^2}.$$

Since $\frac{1}{(1-u)^2} \geq \frac{27}{4} u$ for $0 \leq u < 1$, we have

$$\lambda_1 \int_\Omega u u_1 = \lambda \int_\Omega \frac{u_1}{(1-u)^2} \geq \frac{27\lambda}{4} \int_\Omega u_1 u.$$

Hence, $\lambda \leq \frac{\lambda_1}{27}$. This completes the proof. □

Next we shall obtain the existence result for small λ. We have the following theorem.

Theorem 3.2. *There exists a solution to* (3.1) *and* (3.2) *for some small* λ.

To prove this theorem, we should apply the method of upper and lower solution. We have the following definition.

Definition 3.3. A function $\bar{u} \in C^2(\Omega)$ is called an uppersolution of (3.1) and (3.2) if it satisfies the inequalities

$$-\Delta u \geq \frac{\lambda}{(u-1)^2} \quad \text{on } \Omega$$
$$u \geq 0 \quad \text{on } \partial\Omega.$$

Similarly, $\underline{u}$ is called a lower solution if it satisfies all the reversed inequalities.

The following two lemmas provide us with a proper choice of lower and upper solutions.

Lemma 3.4. *Any constant* $c < 0$ *is a lower solution.*

Lemma 3.5. $\bar{u} = \frac{1}{3}v_1$ *is an upper solution when* $\lambda \leq \frac{4}{27}\alpha_1 m$. *Here* α_1 *and* v_1 *is the first eigenvalue and eigenfunction for the problem*

$$-\Delta v = \alpha v \quad \text{in } \Omega', \tag{3.5}$$
$$v = 0 \quad \text{on } \partial\Omega', \tag{3.6}$$

where Ω' *is a proper domain with smooth boundary which contains* Ω *and has been chosen such that* $m \leq v_1 \leq 1$ *on* Ω *and* $m > 1/2$.

Proof. It is sufficient to show that

$$-\Delta \bar{u} \geq \frac{\lambda}{(1-\bar{u})^2} \quad \text{in } \Omega.$$

In fact,

$$-\Delta \bar{u} = -\frac{1}{3}\Delta v_1 = \frac{1}{3}\alpha_1 v_1 \geq \frac{1}{3}\alpha_1 m \geq \frac{\lambda}{3} \cdot \frac{27}{4} \geq \frac{\lambda}{(1-1/3v_1)^2} = \frac{\lambda}{(1-\bar{u})^2}.$$

This completes the proof. □

The existence result follows from the above two Lemmas.

4. Multiplicity

In this section we are concerned with the multiplicity of positive radial solutions. A radial solution $u = u(r)$ of (3.1) and (3.2) satisfies the following equations

$$u''(r) + \frac{1}{r}u'(r) + \frac{\lambda}{(1-u)^2} = 0, \quad r \in (\epsilon, 1), u(\epsilon) = u(1) = 0.$$

Let $s = -\ln r, w(s) = u(r)$, then $w(s)$ satisfies

$$w'' + \lambda e^{-2s}\frac{1}{(1-w)^2} = 0 \quad \text{in } (0, -\ln\epsilon),$$
$$w(0) = w(-\ln\epsilon) = 0.$$

Henceforth, we shall consider the following initial value problem

$$u''(r) + \lambda e^{-2r}\frac{1}{(1-u(r))^2} = 0 \quad \text{in } (0, -\ln\epsilon),$$
$$u(0) = 0 \quad \text{and} \quad u'(0) = p.$$

Let $u(\cdot) = u(\cdot, p, \lambda)$ be the solution and define

$$R(p, \lambda) = min\{R > 0 : u(R, p, \lambda) = 0\}.$$

We shall prove in the next lemma that $R(p, \lambda)$ is well defined for all p.

By the boundary condition, u has exactly one critical point, at which it takes the maximum value. We shall denote this critical point by $\tau(p, \lambda)$. Hence

$$u'(r) > 0 \quad \text{for} \quad r \in (0, \tau(p, \lambda)) \quad \text{and} \quad u'(r) < 0 \quad \text{for } r \in (\tau(p, \lambda), R(p, \lambda)).$$

Also note that

$$u(r) = pr + \lambda \int_0^r (s - r)e^{-2s} \frac{1}{(1 - u(s))^2} ds.$$

To prove our multiplicity result, we need to establish several useful lemmas.

Lemma 4.1. *$R(p, \lambda)$ is well defined.*

Proof. First we claim that it is indeed well defined for sufficiently large p and sufficiently small p. Suppose otherwise that $\lim_{r\to+\infty} u'(r) = 0$. Multiplying equation (4.5) by u' and integrating yields

$$\int_0^r u''(s)u'(s)ds = -\lambda \int_0^r \frac{e^{-2s}}{(1 - u(s))^2} u'(s)ds.$$

Hence

$$\frac{1}{2}u'(r)^2 - \frac{1}{2}p^2 = \lambda - \lambda\frac{e^{-2r}}{1 - u(r)} - 2\lambda \int_0^r \frac{e^{-2s}}{1 - u(s)} ds.$$

Let $h(r) = \int_0^r \frac{e^{-2s}}{1-u(s)} ds$, then we have

$$\lambda h'(r) + 2\lambda h(r) - \frac{1}{2}p^2 + \frac{1}{2}u'(r)^2 - \lambda = 0.$$

Notice that when r is sufficiently large,

$$\begin{aligned} h(r) &= \int_0^r \frac{e^{-2s}}{1 - u(s)} ds \\ &= -\frac{1}{\lambda} \int_0^r u''(s)(1 - u(s))ds \\ &= \frac{1}{\lambda}\Big[- (1 - u)u' + p - \int_0^r u'^2(s)ds\Big] \\ &\le \frac{p}{\lambda}. \end{aligned}$$

Hence for sufficiently large r,

$$\begin{aligned} \lambda h'(r) &= -2\lambda h(r) + \frac{1}{2}p^2 - \frac{1}{2}u'(r)^2 + \lambda \\ &\ge -2p + \frac{1}{2}p^2 - \frac{1}{2}u'(r)^2 + \lambda \\ &\ge c > 0 \end{aligned}$$

for some constant c and p sufficiently large or small. Therefore,

$$\frac{e^{-2r}}{1 - u(r)} \ge \frac{c}{\lambda} > 0$$

for sufficiently large r. It follows that, $\lim_{r\to+\infty} u(r) = 1$ for p sufficiently large or small. Applying L'Hopital's rule we have

$$\begin{aligned}
\lim_{r\to+\infty} h'(r) &= \lim_{r\to+\infty} \frac{e^{-2r}}{1-u(r)} \\
&= \lim_{r\to+\infty} \frac{2e^{-2r}}{u'(r)} \\
&= \lim_{r\to+\infty} \frac{-4e^{-2r}}{u''(r)} \\
&= \lim_{r\to+\infty} \frac{4e^{-2r}}{\frac{e^{-2r}}{\lambda(1-u)^2}} \\
&= \lim_{r\to+\infty} 4(1-u(r)) = 0.
\end{aligned}$$

This is a contradiction to the previous conclusion that $h'(r) \geq \frac{c}{\lambda} > 0$. Hence $R(p,\lambda)$ is well defined for p sufficiently large and small. By continuous dependence on parameters, $R(p,\lambda)$ is well defined for all p. This completes the proof. $\square$

Lemma 4.2.

$$\lim_{p\to 0+} R(p,\lambda) = \lim_{p\to 0+} \tau(p,\lambda) = 0.$$

Proof. Suppose otherwise, there exists a $\lambda > 0$, $\epsilon > 0$ and a sequence $p_k \to 0+$ such that

$$R_k \equiv R(p_k,\lambda) \geq \epsilon.$$

Since

$$\begin{aligned}
u(r,p_k) &= p_k r + \lambda \int_0^r (s-r)e^{-2s} \frac{1}{\big(1-u(s)\big)^2} ds \\
&\leq p_k r + \lambda \int_0^r (s-r)e^{-2s} ds \\
&= p_k r + \lambda \Big[-\frac{1}{2}e^{-2s}(s-r)|_0^r + \int_0^r \frac{1}{2}e^{-2s} ds \Big] \\
&= p_k r + \lambda \Big[-\frac{r}{2} - \frac{e^{-2r}}{4} + \frac{1}{4} \Big] \\
&< p_k r - \frac{\lambda r^2}{4},
\end{aligned}$$

thus $R_k < \frac{4p_k}{\lambda}$. Hence,

$$\begin{aligned}
p_k R_k &= -\lambda \int_0^{R_k} (s-R_k)e^{-2s} \frac{1}{(1-u(s))^2} ds \\
&\geq \lambda \int_0^{\epsilon} (\epsilon - s)e^{-2s} ds > 0.
\end{aligned}$$

This is a contradiction. Hence $\lim_{p\to 0+} R(p,\lambda) = 0$. It follows that $\lim_{p\to 0+} \tau(p,\lambda) = 0$. This completes the proof. $\square$

Lemma 4.3.

$$\lim_{p\to+\infty} R(p,\lambda) = \lim_{p\to+\infty} \tau(p,\lambda) = 0.$$

Proof. Suppose $\lim_{p\to\infty}\tau(p,\lambda)\neq 0$, then there exists a $\tau_0>0$ and a sequence $p_k\to+\infty$ with $u_k(r)\equiv u(r,p_k,\lambda)>0$ and $u_k'(r)>0$ in $(0,\tau_0)$.
Let $\bar\tau=\tau_0/2$, we claim

$$\limsup_{k\to+\infty} u_k(\bar\tau)=1.$$

Otherwise, there exists $\epsilon>0$ such that $0<u_k(\bar\tau)\le 1-\epsilon$. It follows that

$$\begin{aligned}u_k(\bar\tau)&=p_k\bar\tau+\lambda\int_0^{\bar\tau}(r-\bar\tau)e^{-2r}\frac{1}{(1-u_k(r))^2}dr\\&\ge p_k\bar\tau+\frac{\lambda}{\epsilon^2}\int_0^{\bar\tau}(r-\bar\tau)e^{-2r}dr\end{aligned}$$

which is impossible since $p_k\to+\infty$. Hence choosing a subsequence if necessary, we may assume

$$\lim_{k\to+\infty}u_k(\bar\tau)=1.$$

Note that u_k satisfies

$$u''(r)+\frac{\lambda e^{-2r}}{u_k(1-u_k)^2}u(r)=0\quad textin(\bar\tau,\tau_0).$$

Let

$$M_k=\inf\{\frac{1}{u_k(1-u_k)^2}:r\in(\bar\tau,\tau_0)\},$$

then

$$\lim_{k\to+\infty}M_k=\infty.$$

Note that $\lambda e^{-2r}\ge\lambda e^{-2\tau_0}$ in $(\bar\tau,\tau_0)$. Let v_k solves

$$v''(r)+\lambda e^{-2\tau_0}M_kv(r)=0\quad textin(\bar\tau,\tau_0).$$

It follows that v_k has at least two zeros in $(\bar\tau,\tau_0)$ when k is sufficiently large. By Sturm Comparison Principle, u_k has at least one zero in $(\bar\tau,\tau_0)$. But this is impossible. Hence

$$\lim_{p\to+\infty}\tau(p,\lambda)=0.$$

Finally, we prove $\lim_{p\to+\infty}R(p,\lambda)=0$. Otherwise, there exists a point $r_0>0$ and a sequence $p_k\to+\infty$ with

$$u_k(r)>0\quad\text{and}\quad u_k'(r)\le 0\quad textin(\tau_k,r_0)$$

where $u_k\equiv u(r,p_k,\lambda)$ and $\tau_k\equiv\tau(p_k,\lambda)$. Let $\bar r=\frac{r_0}{2}$, in view of previous lemma that $\lim_{p\to+\infty}\tau(p,\lambda)=0$, we may assume $\bar r>\tau_k$ for any k. We claim that

$$\limsup_{k\to+\infty}u_k(\bar r)<1.$$

Otherwise, by Sturm Comparison Principle again, u_k has zeros in $(\tau_k,\bar r)$ when k is sufficiently large which is impossible since $\tau_k\to 0$ as $k\to+\infty$.
Note that

$$u_k'(r)=-\int_{\tau_k}^{r}\frac{\lambda e^{-2s}}{(1-u_k(s))^2}ds,$$

and

$$\Big(\frac12 u'^2+\frac{\lambda e^{-2r}}{1-u(r)}\Big)'=-\frac{2\lambda e^{-2r}}{1-u(r)}.\tag{4.1}$$

Integrate equation (4.1) on $(\tau_k, \bar{r})$, we have

$$\frac{1}{2}u_k'(\bar{r})^2 = -\frac{\lambda e^{-2\bar{r}}}{1-u_k(\bar{r})} + \lambda\frac{e^{-2\tau_k}}{1-u_k(\tau_k)} - \int_{\tau_k}^{\bar{r}} \frac{2\lambda e^{-2s}}{1-u_k(s)}ds.$$

On the other hand, we have

$$\begin{aligned}\frac{1}{2}u_k'(r)^2 + \int_{\tau_k}^{\bar{r}} \frac{2\lambda e^{-2s}}{1-u_k(s)}ds &\le \frac{1}{2}u_k'(r)^2 + \int_{\tau_k}^{\bar{r}} \frac{2\lambda e^{-2s}}{(1-u_k(s))^2}ds \\ &\le \frac{1}{2}u_k'(\bar{r})^2 + 2|u_k'(\bar{r})|.\end{aligned}$$

Hence

$$-\frac{\lambda e^{-2\bar{r}}}{1-u_k(\bar{r})} + \lambda\frac{e^{-2\tau_k}}{1-u_k(\tau_k)} \le \frac{1}{2}u_k'(\bar{r})^2 + 2|u_k'(\bar{r})|. \tag{4.2}$$

Integrating equation (4.1) on $(0, \tau_k)$, we have

$$\frac{\lambda e^{-2\tau_k}}{1-u_k(\tau_k)} + \int_0^{\tau_k} \frac{2\lambda e^{-2s}}{1-u_k(s)}ds = \frac{1}{2}p_k^2 + \lambda.$$

Therefore,

$$\frac{\lambda e^{-2\tau_k}}{1-u_k(\tau_k)} \ge \frac{1}{2}(\frac{1}{2}p_k^2 + \lambda). \tag{4.3}$$

Combining inequalities (4.2) and (4.3), we have $u_k'(\bar{r}) \to -\infty$. Thus for $r > \bar{r}$, we have

$$u_k(r_0) < u_k(\bar{r}) + u_k'(\bar{r})(r_0 - \bar{r}) \to -\infty,$$

a contradiction to $u_k(r_0) > 0$. This completes the proof. $\square$

Lemma 4.4. *Define $\tilde{R} = \tilde{R}(\lambda) = sup\{R(p,\lambda), p > 0\}$. Then $\tilde{R}(\lambda)$ is strictly decreasing.*

Proof. Let $0 < \lambda_1 < \lambda_2$ and u_2 is a solution at λ_2 on $(0, \tilde{R}(\lambda_2))$. Let $v(s) = cu_2(r)$ with $r = s/c$ where c is some constant greater but close to 1. It's easy to see $v(0) = 0$ and $v(\tilde{R}(\lambda_2) + \epsilon_1) = 0$ for $\epsilon_1 = (c-1)\tilde{R}(\lambda_2)$. We note that

$$\begin{aligned}v'' + \lambda_1\frac{e^{-2s}}{(1-v(s))^2} &= \frac{1}{c}u_2''(r) + \lambda_1\frac{e^{-2s}}{(1-cu_2(r))^2} \\ &= -\frac{1}{c}\left(\lambda_2\frac{e^{-2r}}{(1-u_2(r))^2} - \lambda_1\frac{e^{-2s}}{(1-cu_2(r))^2}\right) \le 0\end{aligned}$$

when c is sufficient close to 1. Hence v is a lower solution for

$$\begin{gathered}v''(r) + \lambda_1\frac{e^{-2r}}{(1-v)^2} = 0, \\ v(0) = 0, \quad v(\tilde{R}(\lambda_2) + \epsilon_1) = 0.\end{gathered}$$

Hence $\tilde{R}(\lambda_1) \ge \tilde{R}(\lambda_2) + \epsilon_1$. Hence $\tilde{R}(\lambda)$ is strictly decreasing. This completes the proof. $\square$

Lemma 4.5. $\lim_{\lambda\to 0+}\tilde{R}(\lambda) = +\infty$, $\lim_{\lambda\to+\infty}\tilde{R}(\lambda) = 0$.

Proof. Suppose $\lim_{\lambda\to 0+}\tilde{R}(\lambda)\neq+\infty$, then there exists a number $R^*>0$ and a sequence $\lambda_k\to 0+$ with $\lim_{k\to+\infty}\tilde{R}(\lambda_k)=\lim_{k\to+\infty}\tilde{R}(\lambda_k,p_k)=R^*$. Let us write $u_k(r)=u(r,\lambda_k,p_k)$, then

$$\begin{aligned}
0&=u_k(R^*)\\
&=p_kR^*+\lambda_k\int_0^{R^*}(s-R^*)e^{-2s}\frac{1}{(1-u_k(s))^2}ds\\
&\geq p_kR^*+\frac{\lambda_k}{\epsilon^2}\int_0^{R^*}(s-R^*)e^{-2s}ds.
\end{aligned}$$

Hence $p_k\to 0+$ or $R^*=0$. But this contradicts the fact that $\lim_{p\to 0+}R(p,\lambda)=0$ and $R(\lambda)$ is strictly decreasing. Similarly we may prove the second statement. This completes the proof. □

Finally for any given λ, we study the shape of $R(p)$. Notice that $R(p)$ is determined by the implicit equation

$$u(R(p),p)=0. \tag{4.4}$$

Differentiating equation (4.4) with respect to p we get the following equations for the derivatives of R:

$$u_r(R(p),p)R'(p)+u_p(R(p),p)=0, \tag{4.5}$$

$$\begin{aligned}
&u_{rr}(R(p),p)R'(p)^2+2u_{rp}(R(p),p)R'(p)\\
&+u_r(R(p),p)R''(p)+u_{pp}(R(p),p)=0.
\end{aligned} \tag{4.6}$$

If we write $h(r,p)=u_p(r,p)$, $z(r,p)=u_{pp}(r,p)$ and $v(r,p)=u_r(r,p)$, then we can rewrite (4.5) as

$$v(R(p),p)R'(p)+h(R(p),p)=0. \tag{4.7}$$

Also notice that when $R'(p)=0$, from equation (4.6) we have

$$v(R(p),p)R''(p)+z(R(p),p)=0 \tag{4.8}$$

We have the following important Lemma.

Lemma 4.6. *For a given λ, if $R'(p)=0$, then $R''(p)<0$.*

Proof. Note that $h(r,p)$ satisfies the following initial value problem

$$h''+\frac{2\lambda e^{-2r}}{(1-u)^3}h(r,p)=0, \tag{4.9}$$

$$h(0,p)=0,\quad h'(0,p)=1. \tag{4.10}$$

If $R'(p)=0$, then equation (4.5) gives us $h(R(p),p)=0$.

We claim that $h(r,p)>0$ on $(0,R(p))$. Otherwise let $h(\xi(p),p)=0$ and $h>0$ on $(0,\xi(p))$. Note that v satisfies the following

$$v''+\frac{2\lambda e^{-2r}}{(1-u)^3}v-\frac{2\lambda e^{-2r}}{(1-u)^2}=0, \tag{4.11}$$

$$v(0,p)=p,\quad v'(0,p)=-\lambda. \tag{4.12}$$

Recall that $v(\tau(p),p)=0$. If $\xi(p)\geq\tau(p)$, then $v<0$ on $(\xi(p),R(p))$. By Sturm Comparison Theorem, v should have a zero on $(\xi(p),R(p))$ since $h(R(p),p)=0$. This is impossible.

If $\xi(p) < \tau(p)$, then $v < 0$ on $(\tau(p), R(p))$. Since $0 = v(\tau(p), p) > h(\tau(p), p)$, by Sturm Second Comparison Theorem, $v > h$ on $(\tau(p), R(p))$ which is impossible since h has to cross over v and reaches zero at $R(p)$.

Next we claim $z(R(p), p) < 0$. Note that

$$\begin{gathered} z'' + \frac{2\lambda e^{-2r}}{(1-u)^3} z + \frac{6\lambda e^{-2r}}{(1-u)^4} h^2 = 0, \\ z(0,p) = 0, \quad z'(0,p) = 0. \end{gathered} \tag{4.13}$$

We claim z is negative in some neighborhood of 0. Otherwise by observing equation (4.13), we have $z'' < 0$. It follows that $z' < 0$ in the neighborhood of 0 since $z'(0,p) = 0$. This contradicts the assumption.

Next we claim $z < 0$ in $(0, R(p)]$. Otherwise, let $z(r_1, p) = 0$ with $z < 0$ in $(0, r_1)$. Comparing equation (4.9) and equation (4.13), it follows that h must have a zero in $(0, r_1)$ which contradicts our previous statement. Hence $z(R(p), p) < 0$ and it follows from (4.8) that $R''(p) < 0$. $\square$

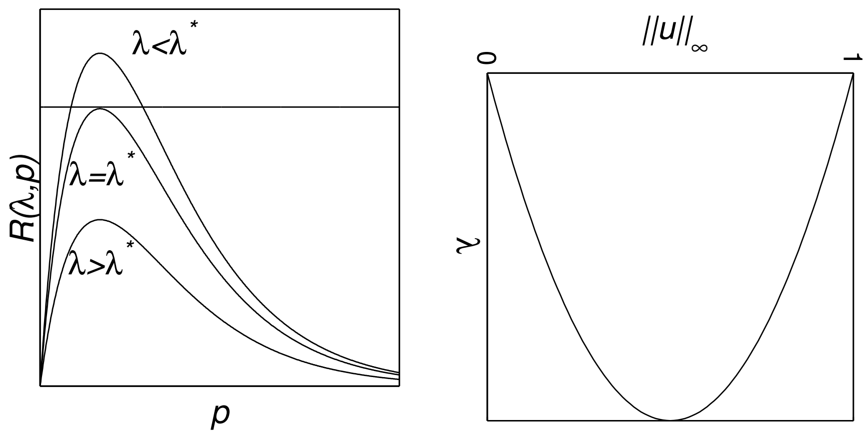

FIGURE 2. Time map diagram and bifurcation diagram

We are now in position to prove Theorem 1.1.

Proof of Theorem 1.1. In view of the above lemmas, we may obtain the timemap diagram as shown in Figure 2. From which we can easily conclude the theorem. In fact, for any given $\epsilon > 0$, $\exists\ \lambda^*$ such that $\tilde{R}(\lambda^*) = -\ln \epsilon$ and there is a unique p such that $R(\lambda^*, p) = -\ln \epsilon$, thus there exists a unique radial solution at $\lambda = \lambda^*$. For $\lambda < \lambda^*$, we can find p_1, p_2 such that $R(\lambda, p_1) = R(\lambda, p_2) = -\ln \epsilon$. The problem has two radial solutions in this case. For $\lambda > \lambda^*$, since $\tilde{R}(\lambda) < -\ln \epsilon$, there is no radial solution. This result is shown in Figure 2. $\square$

5. SYMMETRY BREAKING

In previous section, we studied the multiplicity of radial solutions. Our purpose in this section is to study how radial symmetry can be broken, that is, to describe the bifurcation of these radial solutions into non-radial solutions. The bifurcation problem has been studied by many authors, see [6, 7, 8, 11].

We shall consider two real Banach Spaces, $U \subset V$, as well as a nonlinear abstract operator

$$F: \quad R \times U \to V$$

of the form

$$F(\lambda, u) = L(\lambda)u + R(\lambda, u)$$

and the associated nonlinear abstract equation

$$F(\lambda, u) = 0$$

where the following assumptions are assumed to be satisfied:

- There exists $\lambda_0 \in R$ and $a, b \in R$, $\quad a < \lambda_0 < b$, such that $L(\lambda)$ is a linear operator from U to V for all $\lambda \in (a, b)$. Moreover, *exists* $r \geq 2$ such that the map $\lambda \to L(\lambda)$ is of class C^r and $L(\lambda_0)$ is a Fredholm operator of index zero.
- R is an operator of class C^r such that $R(\lambda, 0) = 0$ and $D_u R(\lambda, 0) = 0$ for each $\lambda \in (a, b)$.

Definition 5.1. $(\lambda_0, 0)$ is a bifurcation point from the curve of $(\lambda, 0)$ if there exists a sequence $(\lambda_n, u_n) \in (a, b) \times (U \setminus \{0\})$ such that $\lim_{n \to infty}(\lambda_n, u_n) = (\lambda_0, 0)$ and $F(\lambda_n, u_n) = 0$.

Definition 5.2. λ_0 is a nonlinear eigenvalue of $L(\lambda)$ if $(\lambda_0, 0)$ is a bifurcation point from the curve $(\lambda, 0)$ and $R(\lambda, u)$ satisfies the second assumption.

On other word, λ_0 is a nonlinear eigenvalue of $L(\lambda)$ if the fact that bifurcation occurs is exclusively based on the linear part.

Definition 5.3. We call zero a simple eigenvalue of $L(\lambda_0)$ if $N[L(\lambda_0)] \oplus R[L(\lambda_0)] = V$.

Definition 5.4. Define λ_0 as an eigenvalue of the pair (L_0, L_1) if zero is an eigenvalue of $L_0 - \lambda_0 L_1$.

Let $a(\lambda)$ denote the classical eigenvalue of the family $L(\lambda)$ perturbed from the zero eigenvalue of $L(\lambda_0)$. If zero is a simple eigenvalue of $L(\lambda_0)$, then $a'(\lambda_0) \neq 0$ if and only if zero is a simple eigenvalue of the pair (L_0, L_1). As we recall Crandall's theorem which states that if zero is a simple eigenvalue of the pair (L_0, L_1), then $(\lambda_0, 0)$ is a bifurcation point. In other word, $(\lambda_0, 0)$ is a bifurcation point if $a'(\lambda_0) \neq 0$. This condition is usually referred to as "transversality condition" or "nondegeneracy condition". We shall remove this condition by the following theorem essentially due to Kielhöfer.

Theorem 5.5. *Assume $U \subset V$ and zero is a simple eigenvalue of $L(\lambda_0)$. Then λ_0 is a nonlinear eigenvalue of $L(\lambda)$ if and only if $a(\lambda)$ changes sign as λ crosses λ_0.*

With the aid of this result, we now study the symmetry breaking problem. We shall consider the linearized problem about a given radial solution u

$$\Delta w + \frac{2\lambda}{(1-u)^3} w = 0.$$

We may write w in the spherical harmonic decomposition form:

$$w = \sum_{N=0}^{\infty} a_N(r) \Phi_N(\theta),$$

and a_N satisfies the equation:

$$a_N'' + \frac{1}{r}a_N' + \Big(\frac{2\lambda}{(1-u)^3} - \frac{N^2}{r^2}\Big)a_N = 0$$

together with the boundary conditions $a_N(1) = 0 = a_N(\epsilon)$.

If the above equation admits a nonzero solution $a_N \neq 0$ for some $N \geq 1$, then radial symmetry breaks. We consider the eigenvalue problem

$$a_N'' + \frac{1}{r}a_N' + \Big(\frac{2\lambda}{(1-u)^3} - \frac{N^2}{r^2}\Big)a_N = -\mu_{N,k}a_N.$$

In the context of our previous setting, we shall let $U = C_0^2(\epsilon, 1)$ and $V = C(\epsilon, 1)$. We have the following lemma.

Lemma 5.6. *If u is a radial solution on the upper branch, then for arbitrary positive integer N, $\mu_{N,1}(\lambda) < 0$ for λ sufficiently close to zero.*

Proof. The eigenvalue $\mu_{N,1}(\lambda)$ can be characterized as

$$\mu_{N,1} = \inf_{\phi\in C_0^2([\epsilon,1])}\Big\{\frac{\int_\epsilon^1 r(\phi'^2 - \frac{2\lambda}{(1-u)^3}\phi^2 + N^2r^{-2}\phi^2)dr}{\int_\epsilon^1 r\phi^2 dr}\Big\}.$$

If u is a positive radial solution, then

$$\int_\Omega |\bigtriangledown u|^2 = \lambda\int_\Omega \frac{u}{(1-u)^2}.$$

Since u is a solution on the upper branch, $||u||_\infty \to 1$ as $\lambda \to 0+$. Notice that for arbitrary $p > 0$, there exists $\alpha > 0$ such that

$$\frac{2u}{1-u} \geq p \quad \text{for } u \geq 1-\alpha.$$

We write

$$Q(u) = \int_\epsilon^1 r\Big(u'^2 - \frac{2\lambda}{(1-u)^3}u^2 + \frac{N^2}{r^2}u^2\Big).$$

Hence

$$\begin{aligned}
2\pi Q(u) &= \lambda\int_\epsilon^1 \Big(\frac{1}{(1-u)^2} - \frac{2u}{(1-u)^3}\Big)u + N^2\int_\epsilon^1 \frac{u^2}{r^2} \\
&= \int_\epsilon^1 |\bigtriangledown u|^2 - \lambda\int_\epsilon^1 \frac{2u}{(1-u)}\cdot\frac{1}{(1-u)^2}u + N^2\int_\epsilon^1\frac{u^2}{r^2} \\
&\leq (1-p)\int_\epsilon^1 |\bigtriangledown u|^2 + \frac{N^2}{\epsilon^2}\int_\epsilon^1 u^2 - \lambda\int_{u\leq 1-\alpha}\frac{2u}{(1-u)}\cdot\frac{1}{(1-u)^2}u \\
&\leq \Big(1 - p + \frac{N^2}{\epsilon^2\nu_1}\Big)\int_\epsilon^1 |\bigtriangledown u|^2 - M
\end{aligned}$$

for some constant $M > 0$ which is independent of λ. Hence for any given $N > 0$, $\mu_{N,1}(\lambda) < 0$ for λ sufficiently close to zero since $p > 0$ can always be chosen to be sufficiently large. This completes the proof. $\square$

Remark 5.7. It's easy to see that if u is an upper branch solution, then

$$\Big(\int_\Omega |\bigtriangledown u|^2\Big)^{1/2} \geq \sqrt{\frac{2\pi\epsilon}{1-\epsilon}}$$

as $\lambda \to 0$. In fact,

$$
\begin{aligned}
u(r) = \int_\epsilon^r u'(s)ds &\le (1-\epsilon)^{1/2}\Big(\int_\epsilon^1 (u'(s))^2 ds\Big)^{1/2} \\
&\le (1-\epsilon)^{1/2}\frac{1}{\sqrt{2\pi\epsilon}}\Big(\int_\Omega |\bigtriangledown u|^2\Big)^{1/2}
\end{aligned}
$$

We now prove the symmetry breaking result.

Proof of Theorem 1.2. Since $\mu_{0,1}(\lambda^*) = 0$, it follows that $\mu_{N,1}(\lambda^*) > 0$ for $N \ge 1$. By Lemma 5.6, for any $N \ge 1$ there exists $\lambda_N \in (0, \lambda^*)$ such that $\mu_{N,1}(\lambda_N) = 0$ and $\mu(\lambda)$ changes sign as λ crosses λ_N. Hence by Theorem 5.5, there is a bifurcation at λ_N where the radial symmetry breaks. The proof is completed. □

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