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ASYMPTOTIC BEHAVIOR FOR THE CENTERED-RAREFACTION APPEARANCE PROBLEM

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Abstract. We construct a uniform in time asymptotic behavior, describing the interaction of two isothermal shock waves with the same direction of motion. The main result is that any smooth regularization of the problem implies the realization of the stable scenario of interaction. In particular, the rarefaction wave appearance is described.

1. Introduction

Let us consider the gas dynamics system in the isothermal case

$$\begin{gathered}\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}=0,\quad x\in\mathbb{R}^1,\quad t>0,\\ \frac{\partial(\rho u)}{\partial t}+\frac{\partial}{\partial x}(\rho u^2+c_0^2\rho)=0.\end{gathered}\tag{1.1}$$

Let the initial data be two shock waves with the same direction of motion (to the right):

$$\begin{gathered}\rho\big|_{t=0}=\rho_0+e_1H(-x+x_1^0)+e_2H(-x+x_2^0),\\ u\big|_{t=0}=v_1H(-x+x_1^0)+v_2H(-x+x_2^0).\end{gathered}\tag{1.2}$$

Here $H(x)$ is the Heaviside function, $e_i=\rho_i-\rho_{i-1}>0$ and $v_i=u_i-u_{i-1}>0$, $i=1,2$, are the amplitudes of jumps, $u_0=0$, and ρ_i, u_i, $c_0>0$ are constants. For definiteness, we assume that $x_1^0>x_2^0$. The initial shock waves are assumed to be stable, so that

$$u_1=c_0\frac{e_1}{\sqrt{\rho_0\rho_1}},\quad u_2=u_1+c_0\frac{e_2}{\sqrt{\rho_1\rho_2}}.\tag{1.3}$$

The solution of problem (1.1), (1.2) seems to be well-known nowadays. Indeed, the standard procedure of "step-by-step" consideration before and after the interaction time instant $t=t^*$ shows that the solution is described by the two noninteracting shock waves for $t<t^*$, namely

$$\begin{gathered}\rho=\rho_0+e_1H\big(-x+\varphi_{10}(t)\big)+e_2H\big(-x+\varphi_{20}(t)\big),\\ u=v_1H\big(-x+\varphi_{10}(t)\big)+v_2H\big(-x+\varphi_{20}(t)\big),\end{gathered}\tag{1.4}$$

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where $\varphi_{i0} = \varphi_{i0_t} t + x_i^0$ are the phases of the shocks,

$$\varphi_{10_t} = c_0\sqrt{\frac{\rho_1}{\rho_0}}, \quad \varphi_{20_t} = u_1 + c_0\sqrt{\frac{\rho_2}{\rho_1}}. \tag{1.5}$$

Next, at the time instant t^* of the confluence, the initial conditions (1.2) are replaced by the shock wave with the amplitudes ρ_2 and u_2 of the jumps of ρ and u which are concentrated at the point $x^* := \varphi_{10}(t^*) = \varphi_{20}(t^*)$. Solving this Riemann problem we obtain that the solution for $t > t^*$ is a uniquely defined combination of a shock wave and a centered rarefaction (see, for example [1, 10]). Let us call this behavior of the solution the "stable scenario".

However, the uniqueness of weak solutions for hyperbolic systems of conservation laws has been proved (with additional conditions) only for sufficiently small amplitudes of shocks (see [1, 2, 9]). It is well–known that apart from the above mentioned solution, the Riemann problem admits a family of artificial solutions (see, for example, [9]). Therefore, the described construction can not be treated as a well-posed one for the case of arbitrary amplitudes of shocks.

It is clear that the weak point of this scheme is the consideration of shock waves as noninteracting ones for time close to t^*. Moreover, this conflicts with the physical sense of the problem since the actual gas dynamics include viscosity phenomena. Therefore, it is necessary to smooth the solution for time close to t^* and to consider the process of interaction in detail.

Whitham [11] was the first to solve a similar problem for the inviscid Burgers-Hopf equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0 \tag{1.6}$$

with the quadratic nonlinearity $f(u) = u^2$. Passing to the Burgers regularization and using the Hopf-Cole transformation, Whitham found the exact solution for the initial data similar to (1.2) and, as a result, established that the regularization implies the choice of a stable scenario of interaction. However, this procedure can be applied uniquely for the quadratic nonlinearity. A progress in this problem has been achieved only recently by Danilov and Shelkovich for equation (1.6) with convex nonlinearities ([3]; see also [4, 5]). Since it is impossible to find exact solutions in the general case, they constructed an asymptotic solution in the framework of the "weak asymptotic method" [3, 4, 5, 6, 7, 8]. The main point here is the treatment of the solution $u_\varepsilon(x,t)$ of the regularized problem as a $\mathcal{C}^\infty([0,T];\mathcal{C}^\infty(\mathbb{R}^1))$ mapping for ε a positive constant and a $\mathcal{C}([0,T];\mathcal{D}'(\mathbb{R}^1))$ mapping uniformly in $\varepsilon \in [0,1]$, where ε is a parameter of regularization. Respectively, a family $u_\varepsilon(t,x)$ is called an *asymptotic* mod $\mathcal{O}_{\mathcal{D}'}(\varepsilon)$ *solution* of equation (1.6) if the relation

$$\frac{d}{dt}\int_{-\infty}^{\infty} u_\varepsilon \psi dx - \int_{-\infty}^{\infty} f(u_\varepsilon)\frac{\partial \psi}{\partial x} dx = \mathcal{O}(\varepsilon)$$

holds for any test function $\psi = \psi(x)$. The main advantage of this approach is the possibility to describe the interaction of nonlinear waves by an ordinary differential equation. Let us note that this method allows also to describe interactions of solitons for non integrable problems [5, 8].

Our aim is a generalization of the weak asymptotic method for hyperbolic systems of conservation laws. Using system (1.1) as a simple but meaningful example we show that this tool easily allows to construct a uniform in time asymptotic solution. The case of shock waves with opposite directions of motion has been examined in the paper [7]. In the present paper we consider the case of the same direction of shock waves motion. It is necessary to note that the construction of a weak asymptotic solution for a system of conservation laws entails the appearance of a dynamical system (instead of an equation in the scalar case). Analysis of this system requires the use of the specifics of the original problem. This can be done for any particular problem and we indicate the way how to do this in the general case (see Conclusion).

Specifically for problem (1.1), (1.2), we consider the phenomena of a rarefaction wave appearance as the most interesting effect. That is why we point out this part of the work in the title of the paper. Note also that the origin of centered rarefaction is possible exclusively for systems. To consider rarefaction waves in scalar cases we should set such wave as the initial data or set discontinuous initial data with appropriate sign of the jump. Asymptotic behavior for the corresponding problems of weak discontinuities interaction for the inviscid Burgers-Hopf equation have been constructed recently in [4].

2. Construction of the asymptotic solution

2.1. **Definitions and statement of the main result.** Following the ideas sketched above we arrive at:

Definition 2.1. Sequences $\rho_\varepsilon(t,x)$ and $u_\varepsilon(t,x)$ are called a weak asymptotic mod $\mathcal{O}_{\mathcal{D}'}(\varepsilon)$ solution of system (1.1) if $\rho_\varepsilon(t,x)$ and $u_\varepsilon(t,x)$ belong to $\mathcal{C}^\infty([0,T]\times\mathbb{R}^1)$ for ε=constant>0 and to $\mathcal{C}(0,T;\mathcal{D}'(\mathbb{R}^1))$ uniformly in $\varepsilon\in[0,const]$ and if the relations

$$
\begin{gathered}
\frac{d}{dt}\int_{-\infty}^{\infty}\rho_\varepsilon\psi_1 dx-\int_{-\infty}^{\infty}\rho_\varepsilon u_\varepsilon\frac{\partial\psi_1}{\partial x}dx=\mathcal{O}(\varepsilon),\\
\frac{d}{dt}\int_{-\infty}^{\infty}\rho_\varepsilon u_\varepsilon\psi_2 dx-\int_{-\infty}^{\infty}(\rho_\varepsilon u_\varepsilon^2+c_0^2\rho_\varepsilon)\frac{\partial\psi_2}{\partial x}dx=\mathcal{O}(\varepsilon)
\end{gathered}
\tag{2.1}
$$

hold for any test function $\psi_i=\psi_i(x)\in\mathcal{D}(\mathbb{R})$, $i=1,2$.

It is necessary to note that any $\mathcal{O}(\varepsilon)$ diffusion (or diffusion–dispersion) regularization (see, for example [9]) of equations (1.1) implies the appearance of $\mathcal{O}(\varepsilon)$ corrections in relations (2.1).

To present the asymptotic solution, let us denote $\omega=\omega(\eta)\in\mathcal{C}^\infty(\mathbb{R}^1)$ an auxiliary function such that

$$
\lim_{\eta\to-\infty}\omega=0\quad\text{and}\quad\lim_{\eta\to\infty}\omega=1.
$$

For simplicity, we assume that ω tends to its limiting values at an exponential rate. Moreover, let

$$
\omega'_\eta>0,\quad\text{and}\quad\omega(\eta)+\omega(-\eta)=1. \tag{2.2}
$$

Obviously this implies that $\omega(\eta)-1/2$ is an odd function and $\omega((-x+\phi)/\varepsilon)$ tends to the Heaviside function $H(-x+\phi)$ as $\varepsilon\to0$.

Now, let us write the weak asymptotic solution for the problem (1.1), (1.2) in the following form:

$$\begin{aligned} \rho_\varepsilon &= \rho_0 + \sum_{i=1}^{2} E_i\omega_i + (R_3 - R)\omega_3 - (R_4 - R)\omega_4, \\ u_\varepsilon &= \sum_{i=1}^{2}(V_i\omega_i + W_i\omega_i') + U\{(x-\phi_4)\omega_4 - (x-\phi_3)\omega_3\}, \end{aligned} \tag{2.3}$$

where

$$\omega_i = \omega\big(\frac{-x+\phi_i}{\varepsilon}\big), \quad \omega_i' = \frac{d\omega(\eta)}{d\eta}\Big|_{\eta=(-x+\phi_i)/\varepsilon}, \quad R = \rho_2 e^{-\beta(x-\phi_3)/c_0}, \tag{2.4}$$

$$\begin{aligned} \phi_i &= \phi_{i0}(\tau,t) + \varepsilon\varphi_{i2}(\tau), \quad \phi_{i0} = \varphi_{i0}(t) + \psi_0(t)\varphi_{i1}(\tau) \quad \text{for } i = 1,2, \\ \phi_j &= \phi_{j0}(\tau,t) + \varepsilon\varphi_{j2}(\tau), \quad \phi_{j0} = \varphi_{j0}(t) + \psi_1(t)\varphi_{j1}(\tau) \quad \text{for } j = 3,4, \end{aligned} \tag{2.5}$$

$$\psi_0 = \varphi_{20} - \varphi_{10}, \quad \psi_1 = \varphi_{40} - \varphi_{30}, \quad R_j = R|_{x=\phi_j}, \quad \tau = \psi_0/\varepsilon\,. \tag{2.6}$$

The functions φ_{10} and φ_{20} are the phases (1.5) of noninteracting shock waves (1.4) (for $t < t^*$), $E_i = E_i(\tau)$, $V_i = V_i(\tau)$, $W_i = W_i(\tau)$, $\beta = \beta(\tau,t,\varepsilon) > 0$, $U = U(\tau,t,\varepsilon) > 0$, and φ_{k1}, φ_{k2}, $k = 1,\dots,4$, are smooth functions. We will assume that

$$E_i \to e_i,\ V_i \to v_i,\ W_i \to 0,\ \varphi_{i1} \to 0,\ \varphi_{j1} \to \overline{\varphi}_{j1} \quad \text{as } \tau \to -\infty, \tag{2.7}$$

$$E_i \to \overline{E}_i,\ V_i \to \overline{V}_i,\ W_i \to \overline{W}_i,\ \varphi_{i1} \to \overline{\varphi}_{i1},\ \varphi_{j1} \to 0 \quad \text{as } \tau \to +\infty, \tag{2.8}$$

$$\varphi_{k2} \to \overline{\varphi}_{k2,\pm} \quad \text{as } \tau \to \pm\infty. \tag{2.9}$$

Here $\overline{E}_i$, $\overline{V}_i$, $\overline{W}_i$, $\overline{\varphi}_{k1}$, $\overline{\varphi}_{k2,\pm}$ are constants, $i = 1,2$, $j = 3,4$, $k = 1,2,3,4$.

Moreover, we assume that

$$\phi_{30} - \phi_{40} \to 0 \quad \text{as } \tau \to -\infty, \quad \phi_{10} - \phi_{20} \to 0 \quad \text{as } \tau \to +\infty, \tag{2.10}$$

and set the geometric conditions for the phases of the regularized waves

$$\text{for} \quad t < t^*: \quad \phi_2 < \phi_1 < \phi_j \quad \mod \mathcal{O}(\varepsilon), \quad j = 3,4, \tag{2.11}$$

$$\text{for} \quad t > t^*: \quad \phi_3 < \phi_4 < \phi_i \quad \mod \mathcal{O}(\varepsilon), \quad i = 1,2. \tag{2.12}$$

The notation $\mod \mathcal{O}(\varepsilon)$ means here that we do not compare these phases when the distances between the paths are $\mathcal{O}(\varepsilon)$.

For the functions β and U we assume the convergence

$$\beta(\tau,t,\varepsilon) \to \overline{\beta}^{\pm}(t) + \mathcal{O}(\varepsilon), \quad U(\tau,t,\varepsilon) \to \overline{U}^{\pm}(t) + \mathcal{O}(\varepsilon) \quad \text{as } \tau \to \pm\infty \tag{2.13}$$

to some bounded by constants (in C-sense) smooth functions. However, we suppose the following relations hold:

$$\beta(0,t^*,\varepsilon) = \mathcal{O}(\varepsilon^{-1}), \quad U(0,t^*,\varepsilon) = \mathcal{O}(\varepsilon^{-1}). \tag{2.14}$$

We will assume also that all convergences mentioned above are of exponential rate.

Assumptions (2.7) and the first assumption (2.10) imply that the anzatz (2.3) describes the two noninteracting waves (1.4) before the interaction ($t < t^*$, so that for $\tau \to -\infty$ as $\varepsilon \to 0$). After the time instant of interaction ($t > t^*$, so that for $\tau \to +\infty$ as $\varepsilon \to 0$) anzatz (2.3) describes shock waves with amplitudes $\overline{E}_i$, $\overline{V}_i$ completed by weak discontinuities on the curves $x = \phi_3$ and $x = \phi_4$. In fact, we will prove that $\overline{\beta}^{+}(t) = \overline{U}^{+}(t) = 1/(t-t^*)$. Therefore, the terms $U(x-\phi_4)$ and $R - R_4$ describe in the limit as $\tau \to \infty$ a centered rarefaction concentrated between the curves $x = \phi_3$ and $x = \phi_4$. The phase corrections $\varepsilon\varphi_{k2}$ describe small shifts of the trajectories which appear as a result of the interaction. The terms $W_i\omega_i'$ describe soliton–type corrections concentrated at the time of the waves interaction.

Now let us formulate our main result.

Theorem 2.1. *Any smooth regularization of the problem* (1.1), (1.2) *implies the existence of a weak asymptotic solution which describes uniformly in time the stable scenario of the shock waves interaction.*

Remark 2.2. We choose a regularization ω of the initial data with additional properties (2.2) since these assumptions allow to simplify the analysis. However, it is clear that these assumptions have only a technical nature, and a similar to Theorem 2.1 result holds for any general regularization (see, for example, [5]).

In what follows we will use the notation:

Definition 2.2. *A sequence $f(t,x,\varepsilon)$ is said to be of the value $\mathcal{O}_{\mathcal{D}'}(\varepsilon^k)$ if the relation*

$$\int_{-\infty}^{\infty} f(x,t,\varepsilon)\psi(x)dx = \mathcal{O}(\varepsilon^k) \tag{2.15}$$

holds for any test function $\psi = \psi(x) \in \mathcal{D}(\mathbb{R})$.

2.2. **Preliminary calculations.** To determine the asymptotic statement (2.3) we should calculate weak expansions of ρ_ε and of the products $\rho_\varepsilon u_\varepsilon$, $\rho_\varepsilon u_\varepsilon^2$. Almost trivial calculations show that

$$\rho_\varepsilon = \rho_0 + \sum_{i=1}^{2} E_i H_i + (R_3 - R)H_3 - (R_4 - R)H_4 + \mathcal{O}_{\mathcal{D}'}(\varepsilon^2). \tag{2.16}$$

Here and in what follows we have

$$H_i = H(-x + \phi_i).$$

Next, products $\omega_i\omega_j$ appear in the formulas for $\rho_\varepsilon u_\varepsilon$ and $\rho_\varepsilon u_\varepsilon^2$. For example,

$$\begin{aligned}
\rho_\varepsilon u_\varepsilon &= \sum_{i=1}^{2} V_i(\rho_0\omega_i + E_i\omega_i^2) + (E_1V_2 + E_2V_1)\omega_1\omega_2 \\
&\quad + U\sum_{j=3}^{4}(-1)^j(x-\phi_j)(\rho_0\omega_j - (-1)^j(R_j - R)\omega_j^2) \\
&\quad + U\big((x-\phi_4)(R_3-R) - (x-\phi_3)(R_4-R)\big)\omega_3\omega_4 \\
&\quad + \sum_{i=1}^{2}\sum_{j=3}^{4}(-1)^j\big(UE_i(x-\phi_j) - V_i(R_j-R)\big)\omega_i\omega_j + \sum_{i=1}^{2}\big\{\rho_0 + E_i\omega_i \\
&\quad + (R_3-R)\omega_3 - (R_4-R)\omega_4\big\}W_i\omega_i' + W_1E_2\omega_1'\omega_2 + W_2E_1\omega_1\omega_2'.
\end{aligned} \tag{2.17}$$

Lemma 2.1. *Under the assumptions mentioned above the following relations hold*

$$\omega_i^k = H_i + \varepsilon\, d_k\, \delta_i + \mathcal{O}_{\mathcal{D}'}(\varepsilon^2), \tag{2.18}$$

$$\omega_i^k\omega_j^\ell = B_{k\ell}(\sigma_{ji})H_i + B_{\ell k}(\sigma_{ij})H_j - \varepsilon\{C_{k\ell}(\sigma_{ji})\delta_i + C_{\ell k}(\sigma_{ij})\delta_j\} + \mathcal{O}_{\mathcal{D}'}(\varepsilon^2), \tag{2.19}$$

$$\omega_j\omega_i' = \varepsilon B_{11}(\sigma_{ji})\delta_i + \mathcal{O}_{\mathcal{D}'}(\varepsilon^2), \tag{2.20}$$

where $k,\ell \geqslant 1$, d_k are some constants, $d_1 = 0$ and

$$
\begin{aligned}
B_{k\ell}(\sigma_{ji}) &= k\int_{-\infty}^{\infty} \omega^{k-1}(\eta)\omega'(\eta)\omega^{\ell}(\sigma_{ji}+\eta)d\eta,\\
C_{k\ell}(\sigma_{ji}) &= k\int_{-\infty}^{\infty} \eta\,\omega^{k-1}(\eta)\omega'(\eta)\omega^{\ell}(\sigma_{ji}+\eta)d\eta.
\end{aligned}
\tag{2.21}
$$

Here and in what follows

$$
\sigma_{ji} := \frac{\phi_j-\phi_i}{\varepsilon}, \quad d_2 := \int_{-\infty}^{\infty}\omega(\eta)\omega(-\eta)d\eta, \quad \delta_i = \delta(-x+\phi_i), \tag{2.22}
$$

where δ is the Dirac delta-function.

Proof of Lemma 2.1. Relation (2.18) is almost obvious. Let us note only that the equality $d_1 = 0$ is a direct consequence of the equality in (2.2). Furthermore, considering the left-hand side of relation (2.19) in the weak sense we obtain the following:

$$
\begin{aligned}
I &:= \int_{-\infty}^{\infty}\omega^k\big(\frac{-x+\phi_i}{\varepsilon}\big)\omega^{\ell}\big(\frac{-x+\phi_j}{\varepsilon}\big)\psi(x)dx\\
&= \int_{-\infty}^{\infty}\omega^k\big(\frac{-x+\phi_i}{\varepsilon}\big)\omega^{\ell}\big(\frac{-x+\phi_j}{\varepsilon}\big)\frac{d}{dx}\int_{-\infty}^{x}\psi(x')dx'\,dx\\
&= -\int_{-\infty}^{\infty}\psi_0(x)\big\{\omega^{\ell}\big(\frac{-x+\phi_j}{\varepsilon}\big)\frac{\partial}{\partial x}\omega^k\big(\frac{-x+\phi_i}{\varepsilon}\big)+\\
&\quad+\omega^k\big(\frac{-x+\phi_i}{\varepsilon}\big)\frac{\partial}{\partial x}\omega^{\ell}\big(\frac{-x+\phi_j}{\varepsilon}\big)\big\}dx\\
&= k\int_{-\infty}^{\infty}\omega^{k-1}(\eta)\omega'(\eta)\omega^{\ell}(\sigma_{ji}+\eta)\psi_0(\phi_i-\varepsilon\eta)d\eta+\\
&\quad+\ell\int_{-\infty}^{\infty}\omega^{\ell-1}(\eta)\omega'(\eta)\omega^{k}(\sigma_{ij}+\eta)\psi_0(\phi_j-\varepsilon\eta)d\eta,
\end{aligned}
\tag{2.23}
$$

where $\psi \in \mathcal{D}(\mathbb{R}^1)$, $\psi_0(x) = \int_{-\infty}^{x}\psi(x')dx'$ and we took into account the exponential rate of vanishing of the product $\omega_i\omega_j\psi_0$ as $x \to \pm\infty$. Now applying the Taylor expansion and using the notation (2.21), (2.22) we can rewrite the right-hand side in (2.23) in the form

$$
I = B_{k\ell}(\sigma_{ji})\psi_0(\phi_i) + B_{\ell k}(\sigma_{ij})\psi_0(\phi_j) - \varepsilon C_{k\ell}(\sigma_{ji})\psi(\phi_i) - \varepsilon C_{\ell k}(\sigma_{ij})\psi(\phi_j) + \mathcal{O}(\varepsilon^2).
$$

Since $\psi_0(\phi_i) = \int_{-\infty}^{\infty} H(-x+\phi_i)\psi(x)dx$, we obtain relation (2.19). Furthermore,

$$
\int_{-\infty}^{\infty}\omega_j\omega_i'\psi(x)dx = \varepsilon\int_{-\infty}^{\infty}\omega(\eta+\sigma_{ji})\omega'(\eta)\psi(\phi_i-\varepsilon\eta)d\eta = \varepsilon B_{11}(\sigma_{ji})\psi(\phi_i) + \mathcal{O}(\varepsilon^2),
$$

as was to be proved. □

A simple analysis of the integrals in (2.21) implies the statement

Lemma 2.2. *The convolutions $B_{k\ell}$ and $C_{k\ell}$ exist and have the following properties:*

$$
\begin{gathered}
B_{k\ell}(\sigma_{ij}) + B_{\ell k}(\sigma_{ji}) = 1,\\
\lim_{\sigma\to-\infty} B_{k\ell}(\sigma) = \lim_{\sigma\to-\infty} C_{k\ell}(\sigma) = 0, \quad \lim_{\sigma\to+\infty} B_{k\ell} = 1 \quad \text{for } k,\ell \geqslant 1,
\end{gathered}
\tag{2.24}
$$

$$
\lim_{\sigma\to+\infty} C_{1\ell} = 0, \quad \lim_{\sigma\to+\infty} C_{2\ell} = d_2, \quad \text{for } \ell \geqslant 1. \tag{2.25}
$$

Applying the statement of Lemma 2.1 we obtain that the weak asymptotic behavior of the right-hand side in formula (2.17) has the form

$$\rho_\varepsilon u_\varepsilon = \sum_{i=1}^{4} M_i H_i + \varepsilon \sum_{i=1}^{2} N_i \delta_i + \mathcal{O}_{\mathcal{D}'}(\varepsilon^2), \tag{2.26}$$

where for $i = 1, 2$:

$$\begin{gathered} M_i = M_{i0} + \sum_{j=3}^{4} (-1)^j \big(U E_i (x - \phi_j) - V_i (R_j - R) \big) B_{11}(\sigma_{ji}), \\ M_{i0} = V_i(\rho_0 + E_i) + (E_1 V_2 + E_2 V_1) B_{11}(\sigma_{\bar{i}i}), \end{gathered} \tag{2.27}$$

$$\begin{aligned} N_i =& d_2 V_i E_i - (E_1 V_2 + E_2 V_1) C_{11}(\sigma_{\bar{i}i}) + W_i \big\{ \rho_0 + E_i/2 \\ &+ W_i E_{\bar{i}} B_{11}(\sigma_{\bar{i}i}) + (R_3 - R) B_{11}(\sigma_{3i}) - (R_4 - R) B_{11}(\sigma_{4i}) \big\}, \\ &\bar{i} = 2 \text{ for } i = 1, \ \bar{i} = 1 \text{ for } i = 2; \end{aligned} \tag{2.28}$$

for $i = 3, 4$,

$$\begin{aligned} M_i =& (-1)^i \Big\{ \rho_0 U (x - \phi_i) + \sum_{j=1}^{2} \Big(U E_j (x - \phi_i) - V_j (R_i - R) \Big) B_{11}(\sigma_{ji}) \Big\} \\ &- U (x - \phi_i)(R_i - R) \\ &+ U \Big((x - \phi_4)(R_3 - R) + (x - \phi_3)(R_4 - R) \Big) B_{11}(\sigma_{\bar{i}i}), \\ &\bar{i} = 4 \quad \text{for } i = 3, \quad \bar{i} = 3 \quad \text{for } i = 4. \end{aligned} \tag{2.29}$$

Now let us calculate the time derivatives of ρ_ε and $\rho_\varepsilon u_\varepsilon$. Since

$$\frac{d\tau(t,\varepsilon)}{dt} = \frac{\psi_{0_t}}{\varepsilon}, \quad \psi_{0_t} = \varphi_{20_t} - \varphi_{10_t}, \tag{2.30}$$

to obtain the precision $\mathcal{O}(\varepsilon)$ in the right-hand side of relations (2.1) we should take into account the terms of order $\mathcal{O}_{\mathcal{D}'}(\varepsilon)$ in (2.16) and (2.26). At the same time, the phase derivatives do not include $\mathcal{O}(1/\varepsilon)$ terms since

$$\frac{d\phi_{i0}}{dt} = \varphi_{i0_t} + \psi_{0_t} \varphi_{i1} + \frac{\psi_0}{\varepsilon} \psi_{0_t} \varphi_{i1}' = \varphi_{i0_t} + \psi_{0_t} (\tau \varphi_{i1})', \quad i = 1, 2 \tag{2.31}$$

and similarly

$$\frac{d\phi_{j0}}{dt} = \varphi_{j0_t} + \alpha (\tau \varphi_{j1})', \quad j = 3, 4. \tag{2.32}$$

Here and in what follows the apostrophe denotes terms of value $\mathcal{O}(1)$ of the derivative with respect to τ (or, what is the same, terms $\mathcal{O}(1/\varepsilon)$ of the derivative with respect to t) and

$$\alpha = \psi_{1_t} / \psi_{0_t}. \tag{2.33}$$

Thus, using formula (2.16), the obvious equalities $(R_j - R)\delta_j = 0$, $j = 3, 4$ and notation (2.31) we find that

$$\begin{aligned} \frac{\partial \rho_\varepsilon}{\partial t} =& \frac{\psi_{0_t}}{\varepsilon} \Big\{ \sum_{i=1}^{2} E_i' H_i - R' H_3 + (R' - R_4') H_4 \Big\} \\ &+ \sum_{i=1}^{2} \frac{d\phi_i}{dt} E_i \delta_i - \frac{dR}{dt} H_3 + \frac{d}{dt}(R - R_4) H_4 + \mathcal{O}_{\mathcal{D}'}(\varepsilon). \end{aligned} \tag{2.34}$$

Here and in what follows df/dt means all terms of value $\mathcal{O}(1)$ of the partial derivative $\partial f(x,t,\tau(t,\varepsilon))/\partial t$.

Next, we need to use the following statement:

Lemma 2.3. *Let $S = S(\tau)$ be a function from the Schwartz space and let a function $\phi_k = \phi_k(\tau,\varepsilon) \in \mathcal{C}^\infty$ have the representation*

$$\phi_k = x^* + \varepsilon\chi_k,$$

where x^* is a constant and $\chi_k = \chi_k(\tau,\varepsilon)$ is a slowly increasing function. Then

$$SH(-x+\phi_k) = SH(-x+x^*) + \varepsilon S\chi_k\delta(x-x^*) + \mathcal{O}_{\mathcal{D}'}(\varepsilon^2). \tag{2.35}$$

Moreover,

$$S\delta(x-x^*) = S\delta(x-\phi_k) + \mathcal{O}_{\mathcal{D}'}(\varepsilon) = S\delta(x-\phi_{k0}) + \mathcal{O}_{\mathcal{D}'}(\varepsilon). \tag{2.36}$$

At the same time

$$H(-x+\phi_k) = H(-x+\phi_{k0}) + \varepsilon\varphi_{k2}\delta(x-\phi_{k0}) + \mathcal{O}_{\mathcal{D}'}(\varepsilon^2), \tag{2.37}$$

if $\phi_k = \phi_{k0} + \varepsilon\varphi_{k2}$ and $\varphi_{k2} = \varphi_{k2}(\tau)$ is a uniformly bounded function.

Proof. For any test function $\psi = \psi(x)$ we have

$$\begin{aligned}
&S\langle H(-x+\phi_k), \psi(x)\rangle \\
&= S\Big\{\int_{-\infty}^{x^*} + \int_{x^*}^{x^*+\varepsilon\chi_k}\Big\}\psi(x)dx \\
&= S\langle H(-x+x^*), \psi(x)\rangle + \varepsilon S\Big(\chi_k\psi(x^*) + \int_0^{\chi_k}\big(\psi(x^*+\varepsilon\eta) - \psi(x^*)\big)d\eta\Big).
\end{aligned}$$

The last integral is bounded by $\varepsilon c\chi_k$, where c is a constant. However, the product $\chi_k(\tau)S(\tau)$ remains bounded by a constant uniformly in τ. This implies relation (2.35). Next,

$$\begin{aligned}
&S\langle\delta(x-x^*), \psi(x)\rangle \\
&= S\psi(x^* \pm \varepsilon\chi_k) \\
&= S\langle\delta(x-\phi_k), \psi(x)\rangle + \mathcal{O}(\varepsilon) \\
&= S\langle\delta(x-\phi_{k0}), \psi(x)\rangle + \varepsilon S\varphi_{k2}\psi'(\phi_{k0} + \varepsilon\theta\varphi_{k2}) + \mathcal{O}(\varepsilon), \quad \theta \in [0,1].
\end{aligned}$$

Thus, we obtain both equalities (2.36). To prove relation (2.37) is enough to take into account the boundedness of the function φ_{k2}. Let us take notes that the second equality in (2.36) holds for any smooth function S if the difference $\phi_k - \phi_{k0}$ is a function of the value $\mathcal{O}(\varepsilon)$. This completes the proof. $\square$

The relations mentioned in Lemma 2.3 allow to simplify the right-hand side in formula (2.34). Indeed, denoting by x^*, t^* the point and the time instant of intersection of the paths $x = \varphi_{i0}(t), i = 1, \dots, 4$, we obtain the equality

$$\tau = \frac{\psi_0(t)}{\varepsilon} = \psi_{0_t}\frac{t-t^*}{\varepsilon}. \tag{2.38}$$

Thus, for the functions of the form (2.5) we have

$$\begin{gathered}
\phi_i = x^* + \varepsilon\tau\Big(\frac{\varphi_{i0_t}}{\psi_{0_t}} + \varphi_{i1}(\tau)\Big) + \varepsilon\varphi_{i2} =: x^* + \varepsilon\chi_i(\tau), \quad i = 1, 2, \\
\phi_j = x^* + \varepsilon\tau\alpha\Big(\frac{\varphi_{j0_t}}{\psi_{1_t}} + \varphi_{j1}(\tau)\Big) + \varepsilon\varphi_{j2} =: x^* + \varepsilon\chi_j(\tau), \quad j = 3, 4.
\end{gathered}$$

Here and in the sequel $\chi_k = \chi_{k0} + \varphi_{k2}$, $k = 1, 2, 3, 4$,

$$\chi_{i0} = \tau\Big(\frac{\varphi_{i0_t}}{\psi_{0_t}} + \varphi_{i1}\Big), \quad i = 1, 2, \quad \chi_{j0} = \alpha\tau\Big(\frac{\varphi_{j0_t}}{\psi_{1_t}} + \varphi_{j1}\Big), \quad j = 3, 4. \tag{2.39}$$

The assumptions for E_i, ϕ_i, and β allow to use the statement of Lemma 2.3. Therefore,

$$\begin{aligned}
&\frac{1}{\varepsilon}\Big\{\sum_{i=1}^{2} E_i' H_i - R' H_3 + (R' - R_4')H_4\Big\} \\
&= \frac{1}{\varepsilon}(\sum_{i=1}^{2} E_i - R_4)' H(x - x^*) \\
&\quad + \big(\sum_{i=1}^{2} \chi_i E_i' - \chi_4 R_4' + (\chi_4 - \chi_3)R'\big)\delta(x - x^*) + \mathcal{O}_{\mathcal{D}'}(\varepsilon).
\end{aligned} \tag{2.40}$$

Furthermore, let us define the function β in the following form:

$$\beta = \{t - t^* + \varepsilon g_1(\tau)/\psi_{0_t}\}^{-1} = \psi_{0_t}\{\varepsilon(\tau + g_1(\tau))\}^{-1}. \tag{2.41}$$

Here g_1 is assumed to be a smooth function such that with an exponential rate

$$g_1 \to 0 \quad \text{as } \tau \to +\infty \quad \text{and} \quad g_1 \to \bar{g}_1|\tau| \quad \text{as } \tau \to -\infty, \tag{2.42}$$

where $\bar{g}_1$ is a constant. Then

$$\beta(\phi_4 - \phi_3) = \psi_{0_t}\frac{\chi_4 - \chi_3}{\tau + g_1}. \tag{2.43}$$

Respectively, R_4 does not depend on t, whereas the derivative R' does not include any term of the value $\mathcal{O}(1)$. Therefore, formulas (2.34), (2.40), and (2.43) imply the relation

$$\begin{aligned}
\frac{\partial\rho_\varepsilon}{\partial t} &= \frac{\psi_{0_t}}{\varepsilon}(E_1 + E_2 - R_4)' H(-x + x^*) + \frac{dR}{dt}(H_4 - H_3) \\
&\quad + \sum_{i=1}^{2}\frac{d\phi_i}{dt}E_i\delta_i + \psi_{0_t}\{\sum_{i=1}^{2}\chi_i E_i' - \chi_4 R_4'\}\delta(x - x^*) + \mathcal{O}_{\mathcal{D}'}(\varepsilon).
\end{aligned} \tag{2.44}$$

Observe now that

$$\frac{d\phi_i}{dt} = \frac{d\phi_{i0}}{dt} + \psi_{0_t}\varphi_{i2}'$$

and the derivative φ_{i2}' is assumed to be a function from the Schwartz space. Applying the statement of Lemma 2.3 again we can transform formula (2.44) to the final form

$$\begin{aligned}
\frac{\partial\rho_\varepsilon}{\partial t} &= \frac{\psi_{0_t}}{\varepsilon}(E_1 + E_2 - R_4)' H(-x + x^*) + \frac{dR}{dt}(H_4 - H_3) \\
&\quad + \sum_{i=1}^{2}\frac{d\phi_{i0}}{dt}E_i\delta_i + \psi_{0_t}\{\sum_{i=1}^{2}(E_i\varphi_{i2})' + L_1\}\delta(x - x^*) + \mathcal{O}_{\mathcal{D}'}(\varepsilon),
\end{aligned} \tag{2.45}$$

where

$$L_1 = \sum_{i=1}^{2}\chi_i E_i' - \chi_4 R_4'. \tag{2.46}$$

Carrying out similar calculations for the time derivative of $\rho_\varepsilon u_\varepsilon$ we obtain the formula

$$\begin{aligned}\frac{\partial\,\rho_\varepsilon u_\varepsilon}{\partial\,t} &= \frac{\psi_{0_t}}{\varepsilon}\sum_{i=1}^{4} M_i' H(-x+x^*) + \sum_{i=1}^{2}\frac{d\phi_i}{dt} M_i\delta_i \\ &\quad + \sum_{i=1}^{4}\frac{dM_i}{dt} H_i + L_2\delta(x-x^*) + \mathcal{O}_{\mathcal{D}'}(\varepsilon),\end{aligned} \tag{2.47}$$

where notation (2.27)-(2.29) has been used and

$$L_2 = \psi_{0_t}\big\{\sum_{i=1}^{2} N_i' + \sum_{i=1}^{4} M_i'\chi_i\big\}. \tag{2.48}$$

The explicit formulas (2.27), (2.29) and simple algebra imply the identity

$$\sum_{i=1}^{4} M_i = (\rho_0+\rho_2+E_1+E_2-R_4)\big(V_1+V_2+U(\phi_3-\phi_4)\big). \tag{2.49}$$

Since the spatial derivatives of ρ_ε, $\rho_\varepsilon u_\varepsilon$ and $\rho_\varepsilon u_\varepsilon^2$ do not include any term of value $\mathcal{O}_{\mathcal{D}'}(\varepsilon^{-1})$, the definition of the asymptotic solution and formulas (2.44), (2.47) and (2.49) require the following equalities:

$$\frac{\partial}{\partial\tau}(E_1+E_2-R_4) = 0, \quad \frac{\partial}{\partial\tau}\big(V_1+V_2+U(\phi_3-\phi_4)\big) = 0. \tag{2.50}$$

Firstly, let us define the function U in the form similar to (2.41), namely

$$U = \{t-t^*+\varepsilon g_2(\tau)/\psi_{0_t}\}^{-1} = \psi_{0_t}\{\varepsilon(\tau+g_2(\tau))\}^{-1}, \tag{2.51}$$

where g_2 is a smooth function with the same properties as g_1. Then the product $U(\phi_3-\phi_4)$ does not depend on t. Next, to obtain the constants of integration of equations (2.50) it is enough to consider the limits of the expressions as $\tau\to-\infty$ and to use the first assumptions in (2.7) and (2.10). Therefore, we obtain the identities

$$\rho_0+E_1+E_2 = R_4, \quad V_1+V_2+U(\phi_3-\phi_4) = u_2. \tag{2.52}$$

Actually, these identities require the continuity of the solution on the curve $x=\phi_4$ that separates the shock waves and the centered rarefaction after the interaction.

Now let us note that $M_i - M_{i0}$, $i=1,2$, are soliton-type functions with respect to τ. Indeed, $M_i - M_{i0} \to 0$ as $\tau\to\infty$ since $B_{11}(\sigma_{ji}) \to 0$ as $\tau\to\infty$ and $i=1,2$, $j=3,4$. For $\tau\to-\infty$ the convolutions $B_{11}(\sigma_{ji})\to 1$. So under the first assumption (2.10)

$$M_i - M_{i0} \to UE_i(\phi_3-\phi_4) + V_i(R_3-R_4) \to 0, \quad i=1,2.$$

Therefore, we can apply the statement of Lemma 2.3 and rewrite formula (2.47) in the form

$$\frac{\partial\rho_\varepsilon u_\varepsilon}{\partial t} = \sum_{i=1}^{2}\frac{d\phi_i}{dt} M_{i0}\delta_i + \sum_{i=1}^{4}\frac{dM_i}{dt} H_i + L_3\delta(x-x^*) + \mathcal{O}_{\mathcal{D}'}(\varepsilon), \tag{2.53}$$

where

$$L_3 = \sum_{i=1}^{2}\frac{d\phi_i}{dt}(M_i - M_{i0}) + L_2, \tag{2.54}$$

and L_2 is defined in (2.48), and the equalities (2.52) have been took into account.

Let us transform the term $\sum_{i=1}^4 H_i dM_i/dt$. Since $M_{i0} = M_{i0}(\tau)$, $i = 1, 2$, we can apply the statement of Lemma 2.3 again and obtain the relation

$$\begin{aligned}\sum_{i=1}^{4} \frac{dM_i}{dt} H_i &= \sum_{j=3}^{4} \frac{dM_j}{dt} H_j + \sum_{i=1}^{2} \frac{dM_i}{dt} H(-x + x^*) + \mathcal{O}_{\mathcal{D}'}(\varepsilon) \\ &= \sum_{i=1}^{3} \frac{dM_i}{dt} H_3 + \frac{dM_4}{dt} H_4 + \mathcal{O}_{\mathcal{D}'}(\varepsilon).\end{aligned}$$

Furthermore, since the equalities (2.49) and (2.52) imply the identity

$$\sum_{i=1}^{3} \frac{dM_i}{dt} = -\frac{dM_4}{dt},$$

we derive the relation

$$\sum_{i=1}^{4} \frac{dM_i}{dt} H_i = \frac{dM_4}{dt}(H_4 - H_3) + \mathcal{O}_{\mathcal{D}'}(\varepsilon). \tag{2.55}$$

Next, the explicit formula (2.29) for $i = 4$ and the first equality (2.52) allow to rewrite M_4 as follows

$$\begin{aligned}M_4 = R\big(V_1 + V_2 + U(x - \phi_4)\big) &- \big(R(V_1 + V_2) + U(x - \phi_4)(E_1 + E_2)\big)B_{11}(\sigma_{41}) \\ &+ U\Big((x - \phi_4)(R_3 - R) + (x - \phi_3)(R_4 - R)\Big)B_{11}(\sigma_{34}) \\ &- \big(UE_2(x - \phi_4) + V_2 R\big)\big(B_{11}(\sigma_{42}) - B_{11}(\sigma_{41})\big) \\ &- R_4\big(V_1 B_{11}(\sigma_{14}) + V_2 B_{11}(\sigma_{24})\big).\end{aligned}$$

The last term here does not depend on t. Furthermore, under the geometric assumptions (2.11), (2.12) the function $B_{11}(\sigma_{42}) - B_{11}(\sigma_{41})$ belongs to the Schwarz space. Next, $d\varphi_{k2}/dt$ belong to the Schwarz space also whereas $\phi_k - \phi_{k0}$ have the value of $\mathcal{O}(\varepsilon)$, $k = 1, \dots, 4$. This and the statement of Lemma 2.3 imply the equality

$$\frac{dM_4}{dt}(H_4 - H_3) = \frac{dM_{40}}{dt}(H_4 - H_3) + L_4 \delta(x - x^*) + \mathcal{O}_{\mathcal{D}'}(\varepsilon), \tag{2.56}$$

where

$$\begin{aligned}M_{40} &= R\big(V_1 + V_2 + U(x - \phi_{40})\big) + M_{401} B_{11}(\sigma_{34}) \\ &\quad - \{R(V_1 + V_2) + U(E_1 + E_2)(x - \phi_{40})\}B_{11}(\sigma_{41}), \\ M_{401} &= U\big((x - \phi_{40})(R_3 - R) + (x - \phi_{30})(R_4 - R)\big)\end{aligned} \tag{2.57}$$

and

$$\begin{aligned}L_4 = -\frac{d}{dt}\big\{&\big(UE_2(x - \phi_4) + V_2 R\big)\big(B_{11}(\sigma_{42}) - B_{11}(\sigma_{41})\big) \\ &+ \varepsilon\varphi_{42} U\big(R - (E_1 + E_2)B_{11}(\sigma_{41}) + (R_3 - R)B_{11}(\sigma_{34})\big) \\ &+ \varepsilon\varphi_{32} U(R_4 - R)B_{11}(\sigma_{34})\big\}.\end{aligned} \tag{2.58}$$

Next, by similar reasons we derive the equality

$$\sum_{i=1}^{2} \frac{d\phi_i}{dt} M_{i0}\delta_i = \sum_{i=1}^{2} \frac{d\phi_{i0}}{dt} M_{i0}\delta_i + \psi_{0_t} \sum_{i=1}^{2} \frac{d\varphi_{i2}}{d\tau} M_{i0}\delta(x - x^*) + \mathcal{O}_{\mathcal{D}'}(\varepsilon). \tag{2.59}$$

Combining (2.55) - (2.59) we rewrite formula (2.53) in the resulting form

$$\frac{\partial \rho_\varepsilon u_\varepsilon}{\partial t} = \sum_{i=1}^{2} \frac{d\phi_{i0}}{dt} M_{i0}\delta_i + \frac{dM_{40}}{dt}(H_4 - H_3) + L_5\delta(x-x^*) + \mathcal{O}_{\mathcal{D}'}(\varepsilon), \tag{2.60}$$

where

$$L_5 = L_3 + L_4 + \psi_{0_t} \sum_{i=1}^{2} \frac{d\varphi_{i2}}{d\tau} M_{i0}, \tag{2.61}$$

and L_3, L_4 are presented in (2.54), (2.58).

Now let us pass to the calculation of spatial derivatives. Carrying out similar as above analysis we obtain the statement

Lemma 2.4. *Under the assumptions mentioned above the following relations hold*

$$\begin{aligned}
\frac{\partial\,\rho_\varepsilon}{\partial\,x} &= -\sum_{i=1}^{2} E_i\delta_i + \frac{\partial\,R}{\partial\,x}(H_4 - H_3) + \mathcal{O}_{\mathcal{D}'}(\varepsilon),\\
\frac{\partial\,\rho_\varepsilon u_\varepsilon}{\partial\,x} &= -\sum_{i=1}^{2} M_{i0}\delta_i + \frac{\partial\,M_{40}}{\partial\,x}(H_4 - H_3) + M_*\delta(x-x^*) + \mathcal{O}_{\mathcal{D}'}(\varepsilon),\\
\frac{\partial\,\rho_\varepsilon u_\varepsilon^2}{\partial\,x} &= -\sum_{i=1}^{2} K_i\delta_i + \frac{\partial\,K_4}{\partial\,x}(H_4 - H_3) + K_*\delta(x-x^*) + \mathcal{O}_{\mathcal{D}'}(\varepsilon),
\end{aligned} \tag{2.62}$$

where

$$\begin{aligned}
K_i =& V_i^2(\rho_0 + E_i) + 2\rho_0 V_1 V_2 B_{11}(\sigma_{\bar{i}i}) + V_1(V_1E_2 + 2V_2E_1)B_{\bar{i}i}(\sigma_{\bar{i}i})\\
&+ V_2(V_2E_1 + 2V_1E_2)B_{i\bar{i}}(\sigma_{\bar{i}i}), \quad i = 1,2,\\
K_4 =& R\big(U(x-\varphi_{40}) + V_1 + V_2\big)^2 B_{11}(\sigma_{14})\\
&- U^2(x-\varphi_{40})\big((E_1 + E_2 - R)(x-\varphi_{40})B_{11}(\sigma_{41})\\
&+ 2\rho_0(x-\varphi_{30})B_{11}(\sigma_{34})\big)
\end{aligned}$$

and K_*, M_* are smooth functions from the Schwarz space.

We do not present the explicit formulas of K_* and M_* since they are huge. It is important only that K_* and M_* depend on φ_{i2}, $i = 1,2$ via convolutions $B_{kl}(\sigma_{ij})$ only since $\sigma_{ij} = (\phi_{i0} - \phi_{j0})/\varepsilon + \varphi_{i2} - \varphi_{j2}$.

Substituting the expressions (2.45), (2.60) and (2.62) into the definition (2.1), we derive our main relations for obtaining of the asymptotic solution parameters

$$\begin{aligned}
&\sum_{i=1}^{2}\Big(\frac{d\phi_{i0}}{dt}E_i - M_{i0}\Big)\delta_i + \Big(\frac{dR}{dt} + \frac{\partial\,M_{40}}{\partial\,x}\Big)(H_4 - H_3)\\
&\qquad + \Big(\psi_{0_t}\sum_{i=1}^{2}(E_i\varphi_{i2})' + L_1 + M_*\Big)\delta(x-x^*) = \mathcal{O}_{\mathcal{D}'}(\varepsilon),
\end{aligned} \tag{2.63}$$

$$\begin{aligned}
&\sum_{i=1}^{2}\Big(\frac{d\phi_{i0}}{dt}M_{i0} - K_i - c_0^2E_i\Big)\delta_i + \Big(\frac{dM_{40}}{dt} + \frac{\partial\,K_4}{\partial\,x} + c_0^2\frac{\partial\,R}{\partial\,x}\Big)(H_4 - H_3)\\
&\qquad + (L_5 + K_*)\delta(x-x^*) = \mathcal{O}_{\mathcal{D}'}(\varepsilon).
\end{aligned} \tag{2.64}$$

3. Analysis of the shock wave dynamics

Let us consider the system obtained by setting equal to zero the coefficients of the functions δ_i

$$\frac{d\phi_{i0}}{dt}E_i = V_i(\rho_0 + E_i) + (E_1V_2 + E_2V_1)B_{11}(\sigma_{\bar{i}i}), \tag{3.1}$$

$$\begin{aligned}&\frac{d\phi_{i0}}{dt}(V_i(\rho_0 + E_i) + (E_1V_2 + E_2V_1)B_{11}(\sigma_{\bar{i}i}))\\ &= V_i^2(\rho_0 + E_i) + 2\rho_0V_1V_2B_{11}(\sigma_{\bar{i}i}) + V_1(V_1E_2 + 2V_2E_1)B_{\bar{i}i}(\sigma_{\bar{i}i})\\ &\quad + V_2(V_2E_1 + 2V_1E_2)B_{i\bar{i}}(\sigma_{\bar{i}i}) + c_0^2E_i, \quad i = 1, 2.\end{aligned} \tag{3.2}$$

Lemma 3.1. *Let the assumptions mentioned above hold. Then the second assumption* (2.10) *holds and system* (3.1), (3.2) *describes the confluence of the shock waves for* $\tau \to \infty$.

Proof. Let us denote

$$z_i = V_i/E_i, \quad i = 1, 2 \tag{3.3}$$

and divide equations (3.1) by E_i. Obviously, we obtain

$$\frac{d\phi_{i0}}{dt} = z_i(\rho_0 + E_i) + E_{\bar{i}}(z_1 + z_2)B_{11}(\sigma_{\bar{i}i}), \quad i = 1, 2. \tag{3.4}$$

Subtracting equation (3.4) for $i = 1$ from (3.4) for $i = 2$ and changing the coefficient of the $\delta(x-x^*)$ function in (2.63) we derive the following equation for the difference $\sigma_{21} = (\phi_2 - \phi_1)/\varepsilon$

$$\psi_{0_t}\frac{d\sigma_{21}}{d\tau} = z_2\Big\{\rho_0\Big(1 - \frac{z_1}{z_2}\Big) + (E_1 + E_2)\Big(1 - \Big(1 + \frac{z_1}{z_2}\Big)B_{11}(\sigma_{21})\Big)\Big\}, \tag{3.5}$$

since

$$\frac{d}{d\,t}(\phi_2 - \phi_1) = \psi_{0_t}\frac{d\sigma_{21}}{d\tau}.$$

Note that the assumptions (2.7) require the scattering–type "initial" datum

$$\sigma_{21}/\tau|_{\tau\to-\infty} \to 1. \tag{3.6}$$

Next, using notation (3.3) and equations (3.1) again, we can rewrite equations (3.2) in the form

$$\begin{aligned}&\big(z_i(\rho_0 + E_i) + E_{\bar{i}}(z_1 + z_2)B_{11}(\sigma_{\bar{i}i})\big)^2\\ &= c_0^2 + E_iz_i^2(\rho_0 + E_i) + 2\rho_0z_1z_2E_{\bar{i}}B_{11}(\sigma_{\bar{i}i}) + E_{\bar{i}}^2(z_{\bar{i}}^2 + 2z_1z_2)B_{12}(\sigma_{\bar{i}i})\\ &\quad + E_1E_2(z_i^2 + 2z_1z_2)B_{21}(\sigma_{\bar{i}i}), \quad i = 1, 2.\end{aligned} \tag{3.7}$$

Therefore, treating E_i as known coefficients we obtain two algebraic equations for the functions z_1 and z_2. To solve them let us subtract one equation (3.7) from another one. Obviously, we obtain a homogeneous quadratic equation. Solving this equation and choosing the sign using the first two assumptions (2.7), we pass to the equality

$$z_1 = \mathcal{L}z_2, \tag{3.8}$$

where

$$
\begin{aligned}
\mathcal{L} &= \big(\sqrt{b^2+a_1a_2}-b\big)/a_1,\\
a_j &= \big(\rho_0+E_j+E_{\bar j}B_{11}(\sigma_{\bar j j})\big)^2-E_j\big(\rho_0+E_jB_{11}(\sigma_{j\bar j})^2+\sum_{i=1}^2 E_iB_{21}(\sigma_{\bar j j})\big),\\
b &=\big(E_1+E_2B_{11}(\sigma_{21})\big)^2-\big(E_2+E_1B_{11}(\sigma_{12})\big)^2+E_1E_2(1-2B_{11}(\sigma_{21}))\\
&\quad -(E_1+E_2)\big(E_1B_{21}(\sigma_{21})-E_2B_{21}(\sigma_{12})\big).
\end{aligned}
$$

A simple analysis of the convolutions $B_{ij}(\sigma)$ and the Hölder inequality result in the statement

Lemma 3.2. *The function $B_{11}^2(\sigma)-B_{12}(\sigma)$ is even and negative. The inequality*

$$
2B_{11}(\sigma)-B_{21}(\sigma)>0
$$

holds uniformly in σ.

Therefore,

$$
B_{11}^2(\sigma_{j\bar j})+B_{21}(\sigma_{\bar j j})=1+B_{11}^2(\sigma_{j\bar j})-B_{12}(\sigma_{j\bar j})\leqslant 1.
$$

This implies the inequality

$$
\begin{aligned}
a_j =&\rho_0\big(\rho_0+E_j+2E_{\bar j}B_{11}(\sigma_{\bar j j})\big)+E_1E_2\big(2B_{11}(\sigma_{\bar j j})^2-B_{21}(\sigma_{\bar j j})\big)\\
&+E_j^2\big(1-B_{11}^2(\sigma_{j\bar j})-B_{21}(\sigma_{\bar j j})\big)>0,\quad j=1,2,
\end{aligned}
$$

and the existence of the function $\mathcal{L}$.

Now we can write out the solutions z_j of equations (3.7) as functions of E_1, E_2, and σ_{21},

$$
z_1=c_0\mathcal{L}\mathcal{M},\quad z_2=c_0\mathcal{M}, \tag{3.9}
$$

where

$$
\begin{aligned}
\mathcal{M} &= \{\alpha_1\mathcal{L}^2+2\alpha_2\mathcal{L}+\alpha_3\}^{-1/2},\\
\alpha_1 &=\rho_0\big(\rho_0+E_1+2E_2B_{11}(\sigma_{21})\big)\\
&\quad +E_1E_2\big(2B_{11}(\sigma_{21})-B_{21}(\sigma_{21})\big)+E_2^2B_{11}^2(\sigma_{21}),\\
\alpha_2 &= E_1E_2\big(B_{11}(\sigma_{21})-B_{21}(\sigma_{21})\big)+E_2^2\big(B_{11}^2(\sigma_{21})-B_{12}(\sigma_{21})\big),\\
\alpha_3 &= E_2^2\big(B_{11}^2(\sigma_{21})-B_{12}(\sigma_{21})\big).
\end{aligned}
$$

Furthermore, considering $\mathcal{L}$ and $\mathcal{M}$ as functions of σ_{21} and using the properties of the convolutions $B_{ij}(\sigma_{21})$, one can prove the following statement.

Lemma 3.3. *Let $E_i(\tau)>0$ uniformly in τ. Then the function $\mathcal{L}=\mathcal{L}(\sigma_{21})$ decreases from the value $\mathcal{L}_{-\infty}>1$ to the value $\mathcal{L}_\infty<1$ as σ_{21} goes from $-\infty$ to ∞ and $\mathcal{L}|_{\sigma_{21}=0}=1$, where*

$$
\mathcal{L}_{-\infty}=\frac{\sqrt{\rho_0+E_1+E_2}}{\sqrt{\rho_0}},\quad \mathcal{L}_\infty=\frac{\sqrt{\rho_0}}{\sqrt{\rho_0+E_1+E_2}}.
$$

Conversely, the function $\mathcal{M}=\mathcal{M}(\sigma_{21})$ increases from the value $\mathcal{M}_{-\infty}$ to the value $\mathcal{M}_\infty$ as σ_{21} goes from $-\infty$ to ∞, where

$$
\mathcal{M}_{-\infty}=\big((\rho_0+E_1)(\rho_0+\sum_{i=1}^2 E_i)\big)^{-\frac12},\ \mathcal{M}_\infty=\frac{\sqrt{\rho_0+E_1+E_2}}{\sqrt{\rho_0\big(\rho_0E_1+(\rho_0+E_2)^2\big)}}.
$$

At the point $\sigma_{21}=0$, $\mathcal{M}=\mathcal{M}_0$, where

$$\mathcal{M}_0=\big(\rho_0(\rho_0+\sum_{i=1}^{2}E_i)\big)^{-\frac{1}{2}}. \tag{3.10}$$

Using the first two assumptions (2.7) we find now the limiting values $\bar{\mathcal{M}}$ and $\bar{z}_i$ of $\mathcal{M}$ and z_i

$$\bar{\mathcal{M}}=1/\sqrt{\rho_1\rho_2},\quad \bar{z}_1=c_0/\sqrt{\rho_0\rho_1},\quad \bar{z}_2=c_0/\sqrt{\rho_1\rho_2}\quad \text{as } \sigma_{21}\to-\infty. \tag{3.11}$$

Therefore, we obtain the limiting value of the right-hand side $F=F(\tau,\sigma_{21})$ of equation (3.5), namely

$$F\to\bar{F}=c_0(\sqrt{\rho_2}-\sqrt{\rho_0})/\sqrt{\rho_1}\quad \text{as } \sigma_{21}\to-\infty. \tag{3.12}$$

According to formulae (1.3), (1.5) and (2.30), the limiting right-hand side (3.12) is equal to ψ_{0t}. So that

$$\sigma_{21}\to\tau\quad \text{as } \tau\to-\infty, \tag{3.13}$$

which justifies the existence of a solution with the property (3.6).

Next, $z_1=z_2$ at the point $\sigma_{21}=0$. Thus, $F=0$ at this point since $B_{11}(0)=1/2$. Moreover, close to this point

$$F=\sigma_{21}F'(\tau,0),\quad F'(\tau,0)<0.$$

Therefore, the value $\sigma_{21}=0$ can be reached only as $\tau\to\infty$. Furthermore, stabilization of E_i requires the vanishing of the derivative F'_τ as $\tau\to\infty$. Respectively, one can prove that all derivatives of F vanish as $\tau\to\infty$. Hence,

$$\sigma_{21}\to 0\quad \text{as } \tau\to\infty. \tag{3.14}$$

It remains to show that system (3.1), (3.2) coincides with the standard Rankine-Hugoniot conditions as $\tau\to\pm\infty$. Indeed, let $\tau\to-\infty$. Since $\bar{z}_i=e_i/v_i$, the last two equalities in (3.11) present the standard formulas (1.3) for the velocities u_1 and u_2. It is obvious now that formulas (3.4) define in this limit the same values of φ_{i0_t} as by standard formulas (1.5).

Let $\tau\to\infty$. Using the limiting value (3.10) of $\mathcal{M}$ it is easy to derive the following relations:

$$z_i\to\overline{z}=:\frac{c_0}{\sqrt{\rho_0(\rho_0+\overline{E}_1+\overline{E}_2)}},\quad \frac{d\phi_{i0}}{dt}\to\overline{\phi}_{0_t}=:c_0\sqrt{\frac{\rho_0+\overline{E}_1+\overline{E}_2}{\rho_0}}, \tag{3.15}$$

$$V_1+V_2\to\overline{V}_1+\overline{V}_2=:c_0\frac{\overline{E}_1+\overline{E}_2}{\sqrt{\rho_0(\rho_0+\overline{E}_1+\overline{E}_2)}}. \tag{3.16}$$

Therefore, we obtain again the standard formulas but for an alone shock wave. This completes the proof of Lemma 3.1. □

The statement of this lemma describes qualitatively the behavior of the system (3.1), (3.2) solution and justifies our main assumptions (2.10). However, actually we do not need to solve this system exactly. Indeed, according to formulas (3.3), (3.15) and (3.16) we can specify

$$V_i(\tau)=v_i+(\overline{V}_i-v_i)\zeta_1(\tau),\quad \varphi_{i1}(\tau)=\overline{\varphi}_{i1}\xi_i(\tau),\quad i=1,2, \tag{3.17}$$

where

$$\overline{V}_i=\overline{z}\overline{E}_i,\quad \overline{\varphi}_{i1}=(\overline{\phi}_{0_t}-\varphi_{i0_t})/\psi_{0_t}, \tag{3.18}$$

and $\zeta_1(\tau)$, $\xi_i(\tau)$ are smooth functions that increase monotonically from zero to unit as τ goes from $-\infty$ to ∞. Obviously, functions (3.17), (3.18) satisfy the corresponding assumptions (2.7), (2.8) and (2.10). Moreover, the analysis drawn above imply that the differences of the left and right hand sides of equalities (3.1), (3.2) become soliton-type functions after substitution V_i and φ_{i1} of the form (3.17), (3.18). Therefore, according to the statement of Lemma 2.3 these differences can be supplemented to the coefficients of the $\delta(x-x^*)$ functions in relations (2.63) and (2.64).

So we define V_i and φ_{i1}, $i=1,2$, by formulas (3.17), (3.18). Consequently, we obtain the equalities

$$\Big(\frac{d\phi_{i0}}{dt}E_i - M_{i0}\Big)\Big|_* \overset{\text{def}}{=} G_{i1}, \quad \Big(\frac{d\phi_{i0}}{dt}M_{i0} - K_i - c_0^2E_i\Big)\Big|_* := G_{i2}, \tag{3.19}$$

where the stars mean the substitution of V_j, φ_{j1}, $j=1,2$, mentioned above and $G_{ik}=G_{ik}(\tau)$ are soliton-type functions. As for the amplitudes E_i, they remain indeterminate on this stage.

4. Analysis of the centered rarefaction

Considering the centered rarefaction terms in (2.63) and (2.64) we will use the statement of Lemma 2.3 again. However, we can not neglect soliton type terms now, since the functions U and β are not bounded uniformly in t as $\varepsilon\to 0$. Hence, we should take into account some soliton terms as the coefficients of $\delta(x-x^*)$ functions.

Let us start with some algebraic transformations.

Lemma 4.1. *Let functions U, β and ϕ_j, $j=3,4$ be such that*

$$\frac{dR}{dt} + \frac{\partial M_{40}}{\partial x} = F_1, \tag{4.1}$$

where F_1 is a soliton-type function. Then

$$\begin{aligned}
&\frac{dM_{40}}{dt} + \frac{\partial K_4}{\partial x} + c_0^2\frac{\partial R}{\partial x}\\
&= R\Big\{\big(\frac{dU}{dt}+U^2\big)(x-\phi_4) - U\big(\frac{d\phi_4}{dt} - V_1 - V_2\big)\Big\} + c_0^2\frac{\partial R}{\partial x}\\
&\quad - \Big\{(E_1+E_2)\Big(\big(\frac{dU}{dt}+U^2\big)(x-\phi_4) - U\frac{d\phi_4}{dt}\Big)\\
&\quad + RU(V_1+V_2)\Big\}B_{11}(\sigma_{41}) + F_2 = 0.
\end{aligned}\tag{4.2}$$

Here F_2 is the soliton-type function defined by

$$\begin{aligned}
F_2 = &-\Big\{U(2x-\phi_3-\phi_4)\Big(U\big(E_1+E_2-R+2\rho_0-2\frac{\partial R}{\partial x}(x-\phi_4)\big)\\
&- N_2B_{11}(\sigma_{34})\Big) + U^2(x-\phi_4)(R_3+R_4-2R) + \frac{dM_{401}}{dt}\Big\}B_{11}(\sigma_{34})\\
&+ F_1\{V_1+V_2+U\big((x-\phi_4)B_{11}(\sigma_{43}) - (x-\phi_3)B_{11}(\sigma_{34})\big)\},
\end{aligned}$$

and N_2, M_{40}, M_{401} are defined in formulas (2.27), (2.57).

To prove this statement it is sufficient to eliminate the term dR/dt from the derivative dM_{40}/dt using the equality (4.1).

Let us analyze equations (4.1) and (4.2) beforehand. To this aim, consider equation (4.1) on the curve $x = \phi_3$. Neglecting soliton - type functions we obtain the equation

$$\frac{d\phi_{30}}{dt} + c_0\frac{U}{\beta} - U(\phi_3 - \phi_4) - (V_1 + V_2)B_{11}(\sigma_{14}) - c_0\frac{U}{\beta\rho_2}\sum_{i=1}^{2} E_i = 0. \tag{4.3}$$

Considering the left-hand side of equation (4.3) for $\tau \to \pm\infty$ and taking into account the geometric assumptions (2.11), (2.12), we derive

$$\frac{d\varphi_{30}}{dt} + c_0\frac{\overline{U}^+}{\overline{\beta}^+} - \overline{U}^+(\varphi_{30} - \varphi_{40}) - (\overline{V}_1 + \overline{V}_2) = 0 \quad \text{as } \tau \to \infty, \tag{4.4}$$

$$\frac{d\overline{\phi}_{30}}{dt} + c_0\frac{\rho_0}{\rho_2}\frac{\overline{U}^-}{\overline{\beta}^-} = 0 \quad \text{as } \tau \to -\infty. \tag{4.5}$$

Next, consider equation (4.2) on the curve $x = \phi_4$. Neglecting soliton - type functions we obtain

$$\Big(1 - \frac{E_1 + E_2}{R_4}B_{11}(\sigma_{41})\Big)\frac{d\phi_{40}}{dt} + c_0\frac{\beta}{U} - (V_1 + V_2)B_{11}(\sigma_{14}) = 0. \tag{4.6}$$

Therefore,

$$\frac{d\varphi_{40}}{dt} + c_0\frac{\overline{\beta}^+}{\overline{U}^+} - (\overline{V}_1 + \overline{V}_2) = 0 \quad \text{as } \tau \to \infty, \tag{4.7}$$

$$\frac{d\overline{\phi}_{40}}{dt} + c_0\frac{\rho_2}{\rho_0}\frac{\overline{\beta}^-}{\overline{U}^-} = 0 \quad \text{as } \tau \to -\infty. \tag{4.8}$$

Now let us consider equations (4.1) and (4.2) on the curves $x = \phi_4$ and $x = \phi_3$ respectively. Neglecting soliton - type functions again and using the equalities (4.3), (4.6) we derive the following equalitites:

$$\begin{gathered}(\phi_{30} - \phi_{40})\Big\{\frac{d\beta}{dt} + U\beta\big(1 - 2B_{11}(\sigma_{34})\big)\Big\} = 0,\\ (\phi_{30} - \phi_{40})\Big\{\frac{dU}{dt} + U^2\Big\} = 0.\end{gathered} \tag{4.9}$$

This implies the desired equations for the limiting functions

$$\frac{d\overline{\beta}^+}{dt} + \overline{U}^+\overline{\beta}^+ = 0, \quad \frac{d\overline{U}^+}{dt} + \overline{U}^{+2} = 0 \quad \text{as } \tau \to \infty. \tag{4.10}$$

Therefore,

$$\overline{\beta}^+ = \overline{U}^+ = (t - t^*)^{-1} \quad \text{as} \quad \tau \to \infty, \tag{4.11}$$

which coincides with formulas (2.41), (2.42) and (2.51). Moreover, formulas (2.52) and (4.11) allow to rewrite equations (4.4) and (4.7) as follows:

$$\frac{d\varphi_{30}}{dt} = u_2 - c_0, \quad \frac{d\varphi_{40}}{dt} = \overline{V}_1 + \overline{V}_2 - c_0 \quad \text{as } \tau \to \infty. \tag{4.12}$$

It is clear now that φ_{30} and φ_{40} are the standard characteristics which go to the left in the media with the velocities $u = u_2$ and $u = \overline{V}_1 + \overline{V}_2$ respectively.

Now let us consider the situation as $\tau \to -\infty$. Equations (4.5) and (4.8) imply the congruence of the curves $x = \overline{\phi}_{30}$ and $x = \overline{\phi}_{40}$ if and only if

$$\Big(\frac{\overline{U}^{-}}{\overline{\beta}^{-}}\Big)^2 = \Big(\frac{\rho_2}{\rho_0}\Big)^2.$$

Extracting the root and taking into account geometric assumption (2.11) we obtain the condition

$$\frac{U}{\beta} \to \frac{\rho_2}{\rho_0} \quad \text{as } \tau \to -\infty. \tag{4.13}$$

At the same time, equations (4.9) do not prescribe anything as $\tau \to -\infty$. Therefore, we can try to define the functions β and U in an arbitrary manner preserving the properties (4.11) and (4.13) to within $\mathcal{O}(\varepsilon)$. To this aim we use the representation (2.41) and (2.51) setting

$$g_1 = (1+\rho_2)\sqrt{\tau^2+\gamma_1^2}\,\zeta_3(-\tau), \quad g_2 = (1+\rho_0)\sqrt{\tau^2+\gamma_2^2}\,\zeta_4(-\tau), \tag{4.14}$$

where $\gamma_i > 0$ are arbitrary constants, $\zeta_i(\tau) > 0$ are smooth functions such that $\zeta_i(\tau) \to 0$ as $\tau \to -\infty$ and $\zeta_i(\tau) \to 1$ as $\tau \to \infty$, $i = 3, 4$. Moreover, let

$$\zeta_3(0) = (1+\rho_2)^{-1}, \quad \zeta_4(0) = (1+\rho_0)^{-1}. \tag{4.15}$$

It is easy to check that this choice of g_i guarantees the fulfillment of both the inequalities $\beta^{-1}(\tau) \geqslant \text{const} > 0$ and $U^{-1}(\tau) \geqslant \text{const} > 0$ for any $\tau \in \mathbb{R}^1$, and the limiting relations (4.11), (4.13).

This representation of β, U and formulas (4.5), (4.8) have as consequence the equations

$$\frac{d\overline{\phi}_{30}}{dt} = \frac{d\overline{\phi}_{40}}{dt} = -c_0 \quad \text{as } \tau \to -\infty. \tag{4.16}$$

Obviously we obtain that $\overline{\phi}_{i0}$, $i = 3, 4$, are the standard characteristics which go to the left in the media with the velocity $u = 0$.

Completing the preliminary analysis we define the functions ϕ_{i0}, $i = 3, 4$, by formulas (2.5), where φ_{i0} are determined in (4.12) and

$$\varphi_{31} = -\frac{u_2}{\psi_{1_t}}\xi_3(-\tau), \quad \varphi_{41} = -\frac{\overline{V}_1 + \overline{V}_2}{\psi_{1_t}}\xi_4(-\tau). \tag{4.17}$$

Here $\xi_i(\tau)$, $i = 3, 4$, are smooth functions which increase from zero to unit.

Now we need to justify the choice (4.12), (4.14) and (4.17). To this aim let us substitute these formulas into the left-hand side of equality (4.1) and denote the result of substitution by F_1.

Lemma 4.2. *Under the choice* (4.12), (4.14) *and* (4.17) *the following relation holds:*

$$F_1(H_4 - H_3) = F_1^* \delta(x - x^*) + \mathcal{O}_{\mathcal{D}'}(\varepsilon), \tag{4.18}$$

where $F_1^* = F_1^*(\tau, \varepsilon)$ *is a function from the Schwarz space.*

Proof. Obviously,

$$F_1(H_4 - H_3) = f_1(H_4 - H_3) + \mathcal{O}_{\mathcal{D}'}(\varepsilon), \;\; f_1 = F_1\big(\omega\big(\frac{x-\phi_4}{\varepsilon}\big) - \omega\big(\frac{x-\phi_3}{\varepsilon}\big)\big).$$

Next, using the representation (2.4) of R we rewrite F_1 in the form

$$\begin{aligned} F_1 = &-\frac{R}{c_0}\Big\{\frac{d}{dt}\big(\beta(x-\phi_3)\big)+\beta(V_1+V_2)B_{11}(\sigma_{14})+\beta U(x-\phi_4)\Big\}\\ &+RU-U(E_1+E_2)B_{11}(\sigma_{41})\\ &+U\Big\{R(2x-\phi_3-\phi_4)\frac{\beta}{c_0}+R_3+R_4-2R\Big\}B_{11}(\sigma_{34}). \end{aligned} \tag{4.19}$$

Let us consider F_1 for $\tau >> 1$. Since the convolutions $B_{11}(\sigma_{41})$, $B_{11}(\sigma_{34})$ vanish we derive the relation for $\tau\to\infty$,

$$F_1=-R\frac{\overline{\beta}^{\,+}}{c_0}\Big\{\frac{(x-\varphi_{30})}{\overline{\beta}^{\,+}}\frac{d\overline{\beta}^{\,+}}{dt}-\frac{d\varphi_{30}}{dt}+\overline{U}^{\,+}(x-\varphi_{40})+\overline{V}_1+\overline{V}_2-c_0\Big\}+\dots,$$

where the dots mean vanishing terms and terms of value $\mathcal{O}(\varepsilon)$. Since $\beta\to U\to (t-t^*)^{-1}$ as $\tau\to\infty$, it is clear now that formulas (4.12) and (4.14) imply the vanishing of F_1 as $\tau\to\infty$.

Next, considering F_1 for $\tau\ll -1$ we rewrite the equality (4.19) as follows:

$$F_1=-(x-\phi_3)\frac{R}{c_0}\frac{d\beta}{dt}+(R-\rho_2)\frac{\beta}{c_0}\frac{d\phi_{30}}{dt}+F_{11},$$

where

$$\begin{aligned} F_{11}=&\beta\Big(\frac{\rho_2}{c_0}\frac{d\phi_3}{dt}+\rho_0\frac{U}{\beta}\Big)+\beta U(\phi_{40}-\phi_{30})\frac{R}{2c_0}\\ &+\frac{1}{2}U\big(\rho_2+R_4-2(\rho_0+E_1+E_2)\big)+\dots, \end{aligned}$$

and the dots denote small terms as $\tau\to-\infty$. Under the choice (4.14) and (4.17) the function F_{11} vanishes. Furthermore,

$$|x-\phi_3|\big(\omega\big(\frac{x-\phi_4}{\varepsilon}\big)-\omega\big(\frac{x-\phi_3}{\varepsilon}\big)\big)\leqslant|\phi_{40}-\phi_{30}|\to 0\quad\text{as } \tau\to-\infty.$$

Estimating $R-\rho_2$ similarly we obtain that $f_1\to 0$ as $\tau\to-\infty$. Thus, we can apply the statement of Lemma 2.3 and obtain relation (4.18), where $F_1^*=\varepsilon(\chi_4-\chi_3)f_1|_{x=x^*}$.

It remains to estimate F_1^* for $|\tau|\leqslant$ const. Obviously, to this aim it is enough to estimate the maximum values of β and $(x^*-\phi_3)d\beta/dt$. Formulas (2.41), (4.14) and (4.15) imply the inequalities

$$\varepsilon\beta\leqslant\frac{\psi_{0_t}}{\gamma_1},\quad \varepsilon\Big|(x^*-\phi_3)\frac{d\beta}{dt}\Big|=|\varepsilon^2\beta^2(1+g_{1\tau}')\chi_3|\leqslant\text{const}.$$

This completes the proof of Lemma 4.2. □

Considering in the same way the coefficients of the Heaviside functions in relation (2.64) we obtain the similar to Lemma 4.2 statement

Lemma 4.3. *Under the choice* (4.12), (4.14) *and* (4.17) *the following relation holds:*

$$\Big\{\frac{dM_{40}}{dt}+\frac{\partial K_4}{\partial x}+c_0^2\frac{\partial R}{\partial x}\Big\}(H_4-H_3)=F_3\delta(x-x^*)+\mathcal{O}_{\mathcal{D}'}(\varepsilon), \tag{4.20}$$

where F_3 is a function from the Schwarz space.

Therefore, we define the functions β, U, and ϕ_{j0}, $j = 3, 4$, in accordance with formulas (4.12), (4.14), (4.17) and supplement the corresponding corrections to the coefficients of $\delta(x - x^*)$ functions in relations (2.63), (2.64). We stress that the limiting values $\overline{V}_i$ remain undefined and conditions (4.15) remain unique ones for the functions ζ_3 and ζ_4.

5. Completion of the construction

Now we can complete the construction of the shock waves and the centered rarefaction defining the functions E_i, ξ_i, and ζ_j. To this aim let us consider the continuity conditions (2.52). For the exact solutions $V_i = V_i(E_1, E_2, \tau)$ of the Rankine–Hugoniot conditions (3.1), (3.2) (see formulas (3.3), (3.9)) system (2.52) implies the appearance of a transcendental algebraic equation for E_i. However, we can simplify the procedure using the arbitrariness in representation of the functions E_i, V_i, g_i and ϕ_j. Indeed, taking into account the exact formulas (2.4), (2.6) for R_4 we pass to the algebraic equations

$$\begin{gathered} \rho_0 + E_1 + E_2 = \rho_2 e^{-\frac{\beta}{c_0}(\phi_4 - \phi_3)}, \\ V_1 + V_2 = u_2 + U(\phi_4 - \phi_3). \end{gathered} \tag{5.1}$$

Let us eliminate the term $\phi_4 - \phi_3$ from this system. Then we obtain the equation

$$\rho_0 + E_1 + E_2 = \rho_2 e^{-\frac{\beta}{c_0 U}(V_1 + V_2 - u_2)}. \tag{5.2}$$

Lemma 5.1. *Under the choice* (3.17), (4.12), (4.14) *and* (4.17) *there exist monotonic functions* E_i, $i = 1, 2$, *which satisfy equation* (5.2) *and*

$$E_i \to e_i \quad \text{as } \tau \to -\infty, \quad E_1 + E_2 \to \rho_0(M_0^2 - 1) \quad \text{as } \tau \to \infty. \tag{5.3}$$

Here the Mach number M_0 *is the solution of the equation*

$$\rho_0 M^2 = \rho_2 e^{-\left\{M - \frac{1}{M} - \frac{u_2}{c_0}\right\}} \tag{5.4}$$

and satisfies the inequalities

$$\sqrt{\frac{\rho_1}{\rho_0}} < M_0 < \sqrt{\frac{\rho_2}{\rho_0}}. \tag{5.5}$$

Proof. First of all consider equation (5.2) in the limit as $\tau \to \infty$. Representation (3.17) of V_i and formula (3.16) imply the dependence of the right-hand side of equation (5.2) on $\overline{E}_1 + \overline{E}_2$. Let us use the standard notation M_0 for the Mach number calculated before the shock wave front. In the case under consideration

$$M_0 = \frac{\sqrt{\rho_0 + \overline{E}_1 + \overline{E}_2}}{\sqrt{\rho_0}}.$$

Since $\overline{V}_1 + \overline{V}_2 = c_0(M_0 + 1/M_0)$ and $\beta/U \to 1$ as $\tau \to \infty$, we see that equation (5.2) takes the form (5.4) in the limit. The left-hand side of this equation is the monotonically increasing function whereas the right-hand side monotonically decreases with respect to $M \geqslant 1$. This implies the existence of a unique root $M = M_0 > 1$ of the equation since $\rho_2 > \rho_0$ and $u_2 > 0$. Respectively, we derive the desired limiting amplitudes

$$\overline{E}_1 + \overline{E}_2 = \rho_0(M_0^2 - 1). \tag{5.6}$$

Note also that inequality $M_0 > 1$ is the stability condition for shock waves.

Next, let us set $M = M^* := \sqrt{\rho_1/\rho_0}$. Direct calculations and the exact formula (1.3) of u_2 show that

$$M^* - \frac{1}{M^*} - \frac{u_2}{c_0} = -\frac{e_2}{\sqrt{\rho_1\rho_2}}.$$

Thus, the right-hand side of (5.4) is greater than ρ_2 whereas $\rho_0 {M^*}^2 = \rho_1 < \rho_2$. Therefore, $M^* < M_0$ and we derive the first inequality in (5.5). To prove the second inequality in (5.5) consider the number $M_* > 0$ such that

$$M_* - \frac{1}{M_*} = \frac{u_2}{c_0}.$$

Solving this equation and using formula (1.3) again we deduce M_* as a function with respect to ρ_0, ρ_1, and ρ_2. Next, simple algebra shows that $\rho_0 M_*^2 < \rho_2$. Thus, $M_0 > M_*$. Therefore,

$$\overline{V}_1 + \overline{V}_2 = c_0(M_0 - \frac{1}{M_0}) > u_2. \tag{5.7}$$

This implies the desired inequality

$$\rho_0 M_0^2 = \rho_0 + \overline{E}_1 + \overline{E}_2 < \rho_2. \tag{5.8}$$

We stress that inequalities (5.7) and (5.8) guarantee the stability of the limiting rarefaction wave.

Furthermore, in accordance with (2.7) we have

$$\rho_0 + E_1 + E_2 \to \rho_0 + e_1 + e_2 = \rho_2, \quad V_1 + V_2 \to v_1 + v_2 = u_2 \quad \text{as } \tau \to -\infty.$$

This implies the consistency of equation (5.2) and assumptions (2.7).

Now we can use the arbitrariness of the functions E_i, V_i, g_i and ϕ_j. We set

$$\gamma_1 = \theta\gamma_2, \quad \theta > 1 \tag{5.9}$$

and fix a monotonic function ζ_4. Then there exists a function ζ_3 such that the ratio β/U monotonically increases from ρ_0/ρ_2 to 1. Let us fix a monotonic function ζ_1 in representation (3.17) of V_i and define the amplitudes E_i as follows

$$E_i = e_i + (\overline{E}_i - e_i)\zeta_2(\tau), \quad i = 1, 2, \tag{5.10}$$

where $0 < \overline{E}_i < e_i$ are numbers such that equality (5.6) holds. Then formulas (5.6), (5.7) and equation (5.2) allow to define the function ζ_2, namely

$$\Big(1 - \frac{\rho_0}{\rho_2} M_0^2\Big)\zeta_2(\tau) = 1 - \exp\Big(- \frac{\beta}{U}\big(\frac{M_0^2 - 1}{M_0} - \frac{u_2}{c_0}\big)\zeta_1(\tau)\Big). \tag{5.11}$$

It is easy to check that ζ_2 is the smooth function which increases monotonically from zero to unit. This completes the proof of Lemma 5.1. $\square$

Furthermore, the second equation in (5.1) allows to define the difference $\phi_4 - \phi_3$. Indeed, this equation and (3.17), (5.7) imply the equality

$$\phi_4 - \phi_3 = U^{-1}(V_1 + V_2 - u_2) = U^{-1}\big(c_0(M_0 - \frac{1}{M_0}) - u_2\big)\zeta_1(\tau). \tag{5.12}$$

Thus, the difference $\phi_4 - \phi_3$ is positive and tends to zero as $\tau \to -\infty$. Moreover, in accordance with (2.5), (4.12), (4.17) we have

$$\phi_4 - \phi_3 = \big((\overline{V}_1 + \overline{V}_2)(1 - \xi_4(-\tau)) - u_2(1 - \xi_3(-\tau))\big)(t - t^*) + \varepsilon(\varphi_{42} - \varphi_{32}). \tag{5.13}$$

The function U^{-1} varies rapidly in a small neighborhood of the point $t = t^*$ whereas $U^{-1} \sim t - t^*$ and $U^{-1} \sim \rho_0|t - t^*|$ for $t - t^* \sim \pm\varepsilon^{1-\gamma}$, $\gamma > 0$, respectively. It is clear now that we can set $\varphi_{32} = -\varphi_{42}$ and choose φ_{42} as a soliton–type function. The equality

$$(\phi_4 - \phi_3)|_{t=t^*} = \varepsilon\gamma_2\big(c_0(M_0 - \frac{1}{M_0}) - u_2\big)\zeta_1(0) \tag{5.14}$$

shows that the amplitude of φ_{42} is positive and proportional to γ_2.

The last step of the construction is consideration of the coefficients of $\delta(x - x^*)$ functions in the relations (2.63) and (2.64). Taking into account the statements of Lemmas 4.2, 4.3 and formulas (3.19) we obtain the equations

$$\begin{gathered}\psi_{0_t}(\sum_{i=1}^{2} E_i\varphi_{i2})' + L_1 + M_* + F_1^* + \sum_{i=1}^{2} G_{i1} = 0,\\ L_5 + K_* + F_3 + \sum_{i=1}^{2} G_{i2} = 0.\end{gathered} \tag{5.15}$$

Setting $\varphi_{12} = \varphi_{22}$, $W_1 = W_2$ and using formulas (2.46), (2.61) we rewrite equations (5.15) in the standard form

$$\sum_{i=1}^{2} E_i \frac{d\varphi_{12}}{d\tau} = f_1(\tau, \varphi_{12}), \quad \frac{d(aW_1)}{d\tau} = f_2(\tau), \tag{5.16}$$

where

$$\begin{aligned} a =& 2\rho_0 + \frac{1}{2}\sum_{i=1}^{2} E_i\big(1 + 2B_{11}(\sigma_{i\bar{i}})\big)\\ &+ (R_3 - R_*)\sum_{i=1}^{2} B_{11}(\sigma_{3i}) + (R_* - R_4)\sum_{i=1}^{2} B_{11}(\sigma_{4i}),\end{aligned} \tag{5.17}$$

and f_i are soliton-type functions.

Lemma 5.2. *There exist such numbers γ_2, $\overline{E}_2$ and functions ζ_1, ξ_i, $i = 1, 2, 3$, that equations* (5.16) *have uniformly bounded in τ solutions.*

Proof. The solvability of the first equation (5.16) is obvious since $E_1 + E_2 \geqslant \text{const} > 0$ uniformly in τ. We specify φ_{12} by the condition $\varphi_{12}|_{\tau=0} = 0$. Note that the choice $\varphi_{12} = \varphi_{22}$ implies the independence σ_{21} of this correction and allows to write out the difference $\phi_1 - \phi_2$ in the form

$$\phi_1 - \phi_2 = \big\{\psi_{0_t}(1 - \xi_2) + (\overline{\phi}_{0_t} - \varphi_{10_t})(\xi_2 - \xi_1)\big\}(t^* - t). \tag{5.18}$$

Let $\xi_2(\tau) \geqslant \xi_1(\tau)$. Then $(t^* - t)(\phi_1 - \phi_2) > 0$ accordingly to (5.18) and the first inequality in (5.5).

To prove the inequality $a \geqslant \text{const} > 0$ let us denote τ_i^*, $i = 1, 2$, the time instants of the intersections of the line $x = x^*$ and the curves $x = \phi_3$ and $x = \phi_4$ respectively. Using the explicit formula (2.4) of R it is easy to check that the inequalities $R_3 \geqslant R_* \geqslant R_4$ hold for $\tau \in [\tau_1^*, \tau_2^*]$, where $R_* = R|_{x=x^*}$. Thus, all terms in (5.17) are positive for such τ.

Furthermore, let us note the fulfillment of the relations

$$\phi_3|_{\tau=0} = x^* - \varepsilon\varphi_{42}(0) < x^* < x^* + \varepsilon\varphi_{42}(0) = \phi_4|_{\tau=0}.$$

Hence, $\phi_3|_{\tau=0} \to -0$ and $\phi_4|_{\tau=0} \to +0$ as $\gamma_2 \to 0$, whereas $\phi_1|_{\tau=0} = x^*$ and this position does not depend on γ_2. Therefore, there exist such number γ_2 and functions ξ_1, ξ_3 that the inequalities

$$\phi_1|_{\tau=\tau_1^*} \leqslant x^*, \quad \phi_1|_{\tau=\tau_2^*} \geqslant x^* \tag{5.19}$$

hold for any fixed ζ_1. Then the formulas for ϕ_i, $i = 1,\dots,4$ and (5.18) imply the fulfillment of more precise version of geometric conditions (2.11) and (2.12)

$$\begin{gathered}\phi_2 < \phi_1 < \phi_3 < \phi_4 \quad \text{for } \tau_1^* > \tau > -T,\\ \phi_3 < \phi_4 < \phi_1 < \phi_2 \quad \text{for } \tau_2^* < \tau < T,\end{gathered} \tag{5.20}$$

where $T > \max\{-\tau_1^*, \tau_2^*\}$ is a constant and $\phi_2 - \phi_1 \to +0$, $\phi_4 - \phi_3 \to +0$ as $\tau \to \pm\infty$ respectively.

Let $\tau \in (\tau_2^*, T)$. Then $R_3 > R_4 > R_*$. Simple transformations and the first equality (2.52) allow to rewrite formula (5.17) as follows:

$$\begin{aligned} a =&\rho_0\big(\frac{1}{2} + B_{11}(\sigma_{21})\big) + (R_3 - R_*)\sum_{i=1}^{2} B_{11}(\sigma_{3i}) + R_*\sum_{i=1}^{2} B_{11}(\sigma_{4i})\\ &+ E_2\big(1 - 2B_{11}(\sigma_{12})\big) + R_4\big(\frac{1}{2} + B_{11}(\sigma_{12}) - \sum_{i=1}^{2} B_{11}(\sigma_{4i})\big).\end{aligned} \tag{5.21}$$

The second inequalities in (5.20) imply that $B_{11}(\sigma_{12}) < 1/2$ and $B_{11}(\sigma_{41}) < 1/2$. Moreover, $B_{11}(\sigma_{12}) > B_{11}(\sigma_{42})$ since $\sigma_{42} = \sigma_{41} + \sigma_{12}$. Hence, $a \geqslant \text{const} > 0$ for such τ. For $\tau \geqslant T$ and sufficiently large T we obtain the desired estimate $a \sim \rho_0 + R_4 > 0$ since $B_{11}(\sigma_{12}) \sim B_{11}(\sigma_{21}) \sim 1/2$ and $B_{11}(\sigma_{ji}) \sim 0$, $j = 3, 4$, $i = 1, 2$.

Let $\tau \in (-T, \tau_1^*)$. Then $R_* > R_3 > R_4$. Using the first equality (2.52) again, we rewrite formula (5.17) in the form

$$\begin{aligned} a =&\rho_0\big(\frac{1}{2} + B_{11}(\sigma_{21})\big) + R_*\sum_{i=1}^{2}\big(B_{11}(\sigma_{i3}) - B_{11}(\sigma_{i4})\big)\\ &+ R_4\big(B_{11}(\sigma_{12}) + \sum_{i=1}^{2} B_{11}(\sigma_{i4}) - \frac{1}{2}\big)\\ &+ \rho_2\big(1 - \sum_{i=1}^{2} B_{11}(\sigma_{i3})\big) - E_2\big(1 - 2B_{11}(\sigma_{21})\big).\end{aligned}$$

The location (5.20) of the phases ϕ_i implies the inequalities $B_{11}(\sigma_{i3}) > B_{11}(\sigma_{i4})$ and $B_{11}(\sigma_{12}) > 1/2$. Thus,

$$a \geqslant \frac{1}{2}\rho_0 + \rho_2\big(1 - 2B_{11}(\sigma_{13})\big) - E_2\big(1 - 2B_{11}(\sigma_{21})\big).$$

Since $\rho_2 > E_2(\tau) > 0$ uniformly in τ, the last estimate can be transformed to the following form:

$$a \geqslant \frac{1}{2}\rho_0 + 2E_2\big(B_{11}(\sigma_{21}) - B_{11}(\sigma_{13})\big).$$

Now we use the remained arbitrariness in the choice of $\overline{E}_2$ and ζ_1. Accordingly to (5.11), $\zeta_2(\tau_1^*)$ will be an arbitrarily small number if $\zeta_1(\tau_1^*) \ll 1$. Therefore, we can choose $\overline{E}_2$ and $\zeta_1(\tau_1^*)$ such that the inequality

$$E_2\big|B_{11}(\sigma_{21}) - B_{11}(\sigma_{13})\big|\Big|_{\tau=\tau_1^*} \leqslant \frac{1}{8}\rho_0 \tag{5.22}$$

holds. The trajectories ϕ_1 and ϕ_2 of the shock waves go to the right and have different velocities, whereas the characteristic ϕ_3 goes to the left. So the distances $\phi_1-\phi_2$ and $\phi_3-\phi_1$ increase with time when τ decreases from τ_1^* to $-T$. Therefore, the convolutions $B_{11}(\sigma_{21})$ and $B_{11}(\sigma_{13})$ vanish. The function E_2 increases for such τ. However, we can specify $\zeta_1(\tau)$ such that the rate of increasing of E_2 will be smaller as the rate of decreasing of $B_{11}(\sigma_{21})$ and $B_{11}(\sigma_{13})$. Obviously, the estimate similar to (5.22) holds for any $\tau \in (-T, \tau_1^*)$ under this choice. Consequently, we obtain the desired estimate $a \geqslant \text{const} > 0$. Evidently, $a \geqslant \rho_0/2$ for $\tau \leqslant -T$ and sufficiently large T. It remains to specify W_1 by the condition $W_1 \to 0$ as $\tau \to -\infty$. This completes the proof of Lemma 5.2 and Theorem 2.1. □

Conclusion. Let us indicate a way of generalization of the presented above construction for strictly hyperbolic systems of conservation laws. Let, for simplicity, the initial data be a superposition of two stable shock waves with constant amplitudes. Obviously, to construct a uniform in time asymptotic solution we need to involve into anzatz regularizations of all possible shock waves, contact discontinuities and centered rarefaction. More or less simple algebra allows to represent the nonlinearity again as a linear combination of these regularizations (see, for example, Subsection 2.2). Therefore, to determine parameters of the anzatz we derive a dynamical system with scattering–type initial data. The proof of the existence of a special solution of this problem is the main obstacle to the construction. For example, the direct substitution of the anzatz (2.3) into the gas dynamics equations does not allow to understand for problem (1.1), (1.2) almost anything. Many simplifications based on the statement of Lemma 2.3 allow to transform the corresponding dynamical system to more reasonable form. However, the resulting system remains very complicated. Another example is the interaction of shock waves with opposite directions of motion [7]. We can prove there the solvability of the corresponding scattering–type problem for a special case of initial amplitudes, whereas it remains unclear till now, how to prove the same in the general case. It is clear that the similar problem for general systems of conservation laws will be an insuperable obstacle. So the approach proposed in the present paper can be the main tool to avoid this difficulty. Indeed, changing the desired solution of the dynamical system to appropriate approximation, we pass to a simpler existence problem for small corrections. We guess that this problem can be solved in the general case similarly to the example considered above.

As specifically for rarefaction wave in the problem (1.1), (1.2), we observe a soliton–type deformation of the trajectories close to the interaction point as the most unexpected result. This phenomenon shows that the mechanism of the rarefaction wave appearance is realized via formation of a regularization of a step–type function with negative amplitude a little bit before the shock waves pasting together.

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