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GREEN'S FUNCTION AND EXISTENCE OF SOLUTIONS FOR A FUNCTIONAL FOCAL DIFFERENTIAL EQUATION

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Abstract. We determine Green's function for a third-order three-point boundary-value problem of focal type and determine conditions on the coefficients and boundary points to ensure its positivity. We then apply this in the determination of the existence of positive solutions to a related higher-order functional differential equation.

1. Finding Green's function

Since at least the time of Chazy's attempt [5] to completely classify all third-order differential equations of certain form, analysts have been fascinated by the study of third-order differential equations in the pure sense, but also in the applied sense, as in Gamba and Jüngel [8]. Here we will be concerned initially with a certain class of third-order differential equations, namely the homogeneous three-point mixed boundary-value type given by

$$x'''(t) = 0, \quad t_1 \le t \le t_3 \tag{1.1}$$

$$\begin{gathered} \alpha x(t_1) - \beta x'(t_1) = 0 \\ \gamma x(t_2) + \delta x'(t_2) = 0 \\ x''(t_3) = 0. \end{gathered} \tag{1.2}$$

Here we assume

- (*i*) $t_1 < t_2 < t_3$ are real numbers;
- (*ii*) $\alpha, \beta, \gamma \ge 0$;
- (*iii*) $k := \alpha\delta + \beta\gamma + \alpha\gamma(t_2 - t_1) \neq 0$;
- (*iv*) $\delta > \max\{\gamma(t_3 - t_2), \frac{k(t_3-t_1)^2}{2(t_2-t_1)[\alpha(t_3-t_1)+\beta]} - \frac{\gamma}{2}(t_2 - t_1)\}$; see Lemma 1.1 and Theorem 2.1.

This is a generalization of the third-order, three-point, right-focal boundary value problem found in [1, 2, 3, 9], where $\alpha = \delta = 1$ and $\beta = \gamma = 0$. We prove the existence of and find an explicit formula for Green's function associated with (1.1), (1.2); for more on Green's functions and their applications, see [12].

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Lemma 1.1. *The number k satisfies*

$$k = \alpha\delta + \beta\gamma + \alpha\gamma(t_2 - t_1) \neq 0 \tag{1.3}$$

if and only if the boundary value problem (1.1), (1.2) *has only the trivial solution.*

Proof. A general solution of (1.1) is $x(t) = k_1 t^2 + k_2 t + k_3$. The condition at $t = t_3$ implies that $k_1 = 0$. The mixed boundary conditions at t_1 and t_2 lead to the two equations

$$\beta k_2 - \alpha(t_1 k_2 + k_3) = 0$$
$$(t_2\gamma + \delta)k_2 + \gamma k_3 = 0.$$

The determinant of the coefficients for this system is k. It follows that $k_2 = k_3 = 0$ if and only if $k \neq 0$. This implies the given boundary value problem (1.1), (1.2) has only the trivial solution if and only if $k \neq 0$. □

Theorem 1.2. *Assume for k given in* (1.3) *that $k > 0$. Then Green's function for the homogeneous problem* (1.1) *satisfying the boundary conditions* (1.2) *is given via*

$$g(t,s) = \begin{cases} s \in [t_1, t_2] & : \begin{cases} u_1(t,s) & : t \leq s \\ v_1(t,s) & : t \geq s \end{cases} \\ s \in [t_2, t_3] & : \begin{cases} u_2(t,s) & : t \leq s \\ v_2(t,s) & : t \geq s \end{cases} \end{cases} \tag{1.4}$$

for $t, s \in [t_1, t_3]$, where

$$\begin{aligned} u_1(t,s) &:= \frac{1}{k}(s - t_1)[\alpha(t - t_1) + \beta]\left[\delta + \frac{\gamma}{2}(2t_2 - t_1 - s)\right] - \frac{1}{2}(t - t_1)^2, \\ v_1(t,s) &:= u_1(t,s) + \frac{1}{2}(t - s)^2 = \frac{1}{2k}(s - t_1)[\alpha(s - t_1) + 2\beta][\gamma(t_2 - t) + \delta], \\ u_2(t,s) &:= \frac{1}{k}[\alpha(t - t_1) + \beta]\left[\delta(t_2 - t_1) + \frac{\gamma}{2}(t_2 - t_1)^2\right] - \frac{1}{2}(t - t_1)^2, \\ v_2(t,s) &:= u_2(t,s) + \frac{1}{2}(t - s)^2. \end{aligned}$$

Proof. Note that $g(t,s)$ is well defined for all $(t,s) \in [t_1, t_3] \times [t_1, t_3]$. First, check that g satisfies the boundary conditions (1.2). For convenience we note that

$$\frac{\partial}{\partial t} g(t,s) = \begin{cases} s \in [t_1, t_2] & : \begin{cases} \frac{\alpha}{k}(s - t_1)\left[\delta + \frac{\gamma}{2}(2t_2 - t_1 - s)\right] - t + t_1 \\ \frac{\alpha}{k}(s - t_1)\left[\delta + \frac{\gamma}{2}(2t_2 - t_1 - s)\right] - s + t_1 \end{cases} \\ s \in [t_2, t_3] & : \begin{cases} \frac{\alpha}{k}\left[\delta(t_2 - t_1) + \frac{\gamma}{2}(t_2 - t_1)^2\right] - t + t_1 \\ \frac{\alpha}{k}\left[\delta(t_2 - t_1) + \frac{\gamma}{2}(t_2 - t_1)^2\right] - s + t_1 \end{cases} \end{cases}$$

and

$$\frac{\partial^2}{\partial t^2} g(t,s) = \begin{cases} s \in [t_1, t_2] & : \begin{cases} -1 & : t < s \\ 0 & : t > s \end{cases} \\ s \in [t_2, t_3] & : \begin{cases} -1 & : t < s \\ 0 & : t > s, \end{cases} \end{cases}$$

for fixed s; in the rest of this proof we will employ the shorthand g' and g'' for these two expressions.

For $t = t_1$ and $s \in [t_1, t_2]$:

$$\begin{aligned}\alpha g(t_1,s) - \beta g'(t_1,s) &= \alpha u_1(t_1,s) - \beta u_1'(t_1,s)\\ &= \frac{\alpha\beta}{k}(s-t_1)\Big[\delta + \frac{\gamma}{2}(2t_2 - t_1 - s)\Big]\\ &\quad - \frac{\beta\alpha}{k}(s-t_1)\Big[\delta + \frac{\gamma}{2}(2t_2 - t_1 - s)\Big]\\ &= 0.\end{aligned}$$

For $t = t_1$ and $s \in [t_2, t_3]$:

$$\begin{aligned}\alpha g(t_1,s) - \beta g'(t_1,s) &= \alpha u_2(t_1,s) - \beta u_2'(t_1,s)\\ &= \frac{\alpha\beta}{k}\Big[\delta(t_2-t_1) + \frac{\gamma}{2}(t_2-t_1)^2\Big]\\ &\quad - \frac{\beta\alpha}{k}\Big[\delta(t_2-t_1) + \frac{\gamma}{2}(t_2-t_1)^2\Big]\\ &= 0.\end{aligned}$$

For $t = t_2$ and $s \in [t_1, t_2]$:

$$\begin{aligned}\gamma g(t_2,s) + \delta g'(t_2,s) &= \gamma v_1(t_2,s) + \delta v_1'(t_2,s)\\ &= \frac{\gamma}{2}[(t_2-s)^2 - (t_2-t_1)^2]\\ &\quad + \frac{\gamma}{k}(s-t_1)[\alpha(t_2-t_1)+\beta]\Big[\delta + \frac{\gamma}{2}(2t_2-t_1-s)\Big]\\ &\quad + \delta(t_1-s) + \frac{\delta\alpha}{k}(s-t_1)\Big[\delta + \frac{\gamma}{2}(2t_2-t_1-s)\Big]\\ &= \frac{\gamma}{2}(2t_2-t_1-s)(t_1-s) + \delta(t_1-s)\\ &\quad + (s-t_1)\Big[\delta + \frac{\gamma}{2}(2t_2-t_1-s)\Big]\frac{k}{k}\\ &= 0.\end{aligned}$$

For $t = t_2$ and $s \in [t_2, t_3]$:

$$\begin{aligned}\gamma g(t_2,s) + \delta g'(t_2,s) &= \gamma u_2(t_2,s) + \delta u_2'(t_2,s)\\ &= \frac{\gamma}{k}[\alpha(t_2-t_1)+\beta]\Big[\delta(t_2-t_1) + \frac{\gamma}{2}(t_2-t_1)^2\Big]\\ &\quad - \frac{\gamma}{2}(t_2-t_1)^2 + \delta(t_1-t_2)\\ &\quad + \frac{\delta\alpha}{k}\Big[\delta(t_2-t_1) + \frac{\gamma}{2}(t_2-t_1)^2\Big]\\ &= 0.\end{aligned}$$

Finally, differentiating the expression $g'(t,s)$ with respect to t shows that $g''(t_3,s) = 0$ for any $s \in [t_1,t_3]$. Thus, g as in (1.4) satisfies the boundary conditions (1.2).

Now, for any function f continuous on $[t_1,t_3]$, define

$$x(t) := \int_{t_1}^{t_3} g(t,s)f(s)ds.$$

As shown above, this x satisfies the boundary conditions (1.2) via g. We will show that $x'''(t) = f(t)$. Note that for $t \in [t_1, t_2]$,

$$\begin{aligned} x''(t) &= \Big(\int_{t_1}^{t_2} + \int_{t_2}^{t_3}\Big) g''(t,s)f(s)ds \\ &= \Big(\int_{t_1}^{t} + \int_{t}^{t_2}\Big) g''(t,s)f(s)ds + \int_{t_2}^{t_3}(-1)f(s)ds \\ &= \int_{t_1}^{t}(0)f(s)ds + \int_{t}^{t_2}(-1)f(s)ds - \int_{t_2}^{t_3} f(s)ds \\ &= \int_{t_3}^{t} f(s)ds, \end{aligned}$$

so that $x'''(t) = f(t)$ using the Fundamental Theorem of Calculus. Likewise for $t \in [t_2, t_3]$,

$$x''(t) = \int_{t_1}^{t_2}(0)f(s)ds + \int_{t_2}^{t}(0)f(s)ds - \int_{t}^{t_3} f(s)ds$$

again implies that $x'''(t) = f(t)$. Therefore g as given in (1.4) is Green's function for (1.1), (1.2). □

2. Positivity of Green's function

Theorem 2.1. *Assume $k > 0$. If*

$$\delta > \max\Big\{\gamma(t_3 - t_2),\ \ \frac{k(t_3 - t_1)^2}{2(t_2 - t_1)[\alpha(t_3 - t_1) + \beta]} - \frac{\gamma}{2}(t_2 - t_1)\Big\},$$

then Green's function as given in (1.4) *satisfies $g(t, s) > 0$ on $(t_1, t_3] \times (t_1, t_3]$.*

Proof. Note that $g(t, t_1) = 0$ for all $t \in [t_1, t_3]$. We proceed by cases on the two branches of Green's function (1.4).
Case I: Let $s \in (t_1, t_2]$. Then

$$g(t_1, s) = u_1(t_1, s) = \frac{\beta}{k}(s - t_1)\Big[\delta + \frac{\gamma}{2}(2t_2 - t_1 - s)\Big] \geq 0,$$

and

$$\frac{\partial}{\partial t}g(t, s) = \frac{\partial}{\partial t}u_1(t, s) = \frac{\alpha}{k}(s - t_1)\Big[\delta + \frac{\gamma}{2}(2t_2 - t_1 - s)\Big] - t + t_1 \geq 0$$

for $t \in [t_1, \tau(s)]$; here

$$\tau(s) := \frac{\alpha}{k}(s - t_1)\Big[\delta + \frac{\gamma}{2}(2t_2 - t_1 - s)\Big] + t_1 \leq s \qquad (2.1)$$

for $s \in (t_1, t_2]$, since $\tau(s) = s$ only if $s = t_1 - \frac{2\beta}{\alpha}$ or $s = t_1$, and $\beta \geq 0$. For $t \geq s$,

$$\frac{\partial}{\partial t}g(t, s) = \frac{\partial}{\partial t}v_1(t, s) = \tau(s) - s < 0,$$

so that g is increasing in t on $[t_1, \tau(s)]$, decreasing in t on $[\tau(s), t_3]$, τ as defined in (2.1). It follows that $g(t,s) > 0$ on $(t_1, t_3] \times (t_1, t_2]$ if $g(t_3, s) > 0$ for these s:

$$\begin{aligned} g(t_3, s) &= v_1(t_3, s) \\ &= \frac{1}{k}(s - t_1)[\alpha(t_3 - t_1) + \beta]\Big[\delta + \frac{\gamma}{2}(2t_2 - t_1 - s)\Big] \\ &\quad - \frac{1}{2}(t_3 - t_1)^2 + \frac{1}{2}(t_3 - s)^2 \\ &= \frac{1}{2k}(s - t_1)[\alpha(s - t_1) + 2\beta][\delta + \gamma(t_2 - t_3)]. \end{aligned}$$

From this expression we see that

$$\frac{\partial}{\partial s} v_1(t_3, s) = \frac{1}{k}[\alpha(s - t_1) + \beta][\delta + \gamma(t_2 - t_3)] > 0$$

if $\delta > \gamma(t_3 - t_2)$; this is the first condition on δ mentioned in the theorem. Since $v_1(t_3, t_1) = 0$ and $v_1(t_3, s)$ is increasing in s, $v_1(t_3, s) > 0$ for all $s \in (t_1, t_2]$. Consequently, $g(t,s) > 0$ for $(t,s) \in (t_1, t_3] \times (t_1, t_2]$. In addition, we note for later use that

$$0 < g(t,s) \le g(\tau(s), s)$$

for $(t,s) \in (t_1, t_3] \times (t_1, t_2]$.

Case II: Now let $s \in [t_2, t_3]$. For $t \le s$,

$$\begin{aligned} \frac{\partial}{\partial t} g(t,s) &= \frac{\partial}{\partial t} u_2(t,s) \\ &= \frac{\alpha}{k}\Big[\delta(t_2 - t_1) + \frac{\gamma}{2}(t_2 - t_1)^2\Big] + t_1 - t \\ &= \tau(t_2) - t \ge 0 \end{aligned}$$

if $t \le \tau(t_2)$, τ as in (2.1). Note that $\tau(t_2) < t_2 \le s$ here. As a result we have that

$$0 \le \frac{\beta}{k}\Big[\delta(t_2 - t_1) + \frac{\gamma}{2}(t_2 - t_1)^2\Big] = u_2(t_1, s) \le g(t,s)$$

for all $(t,s) \in [t_1, \tau(t_2)] \times [t_2, t_3]$. For $t \in [\tau(t_2), s]$, g is then decreasing in t, and for $t \ge s$,

$$\begin{aligned} \frac{\partial}{\partial t} g(t,s) &= \frac{\partial}{\partial t} v_2(t,s) \\ &= \frac{\alpha}{k}\Big[\delta(t_2 - t_1) + \frac{\gamma}{2}(t_2 - t_1)^2\Big] + t_1 - s \\ &= \tau(t_2) - s < 0 \end{aligned}$$

as mentioned previously. Therefore g is increasing in t on $[t_1, \tau(t_2)]$ and decreasing in t on $[\tau(t_2), t_3]$, with a maximum at $g(\tau(t_2), s)$. Again we check to see that $g(t_3, s) > 0$ for $s \in [t_2, t_3]$:

$$\begin{aligned} g(t_3, s) &= v_2(t_3, s) \\ &= \frac{1}{k}[\alpha(t_3 - t_1) + \beta]\Big[\delta(t_2 - t_1) + \frac{\gamma}{2}(t_2 - t_1)^2\Big] \\ &\quad - \frac{1}{2}(t_3 - t_1)^2 + \frac{1}{2}(t_3 - s)^2. \end{aligned}$$

As a function of s we have

$$\frac{\partial}{\partial s} g(t_3, s) = \frac{\partial}{\partial s} v_2(t_3, s) = s - t_3 \le 0$$

for $s \in [t_2, t_3]$; in other words, $g(t_3, t_3) \le g(t_3, s)$ for these s. To ensure that

$$g(t_3,t_3) = \frac{1}{k}[\alpha(t_3-t_1)+\beta]\Big[\delta(t_2-t_1)+\frac{\gamma}{2}(t_2-t_1)^2\Big] - \frac{1}{2}(t_3-t_1)^2$$

is positive, take

$$\delta > \frac{k(t_3-t_1)^2}{2(t_2-t_1)[\alpha(t_3-t_1)+\beta]} - \frac{\gamma}{2}(t_2-t_1);$$

this is the second condition on δ mentioned in the theorem. (The fraction in this last expression is a finite real number, since by (1.3) α and β cannot both be zero.) $\square$

3. Functional focal problem

Letting $\gamma = 0$ and $\delta = 1$, we now apply Green's function and its properties from the first two sections to an investigation of the existence of positive solutions to the higher-order, three-point functional problem

$$x^{(n)}(t) = f(t, x(t+\theta)), \quad t_1 \le t \le t_3, \quad -\tau \le \theta \le 0 \tag{3.1}$$

$$\begin{gathered} x^{(i)}(t_1) = 0, \quad 0 \le i \le n-4, \quad n \ge 4 \\ \alpha x^{(n-3)}(t) - \beta x^{(n-2)}(t) = \sigma(t), \quad t_1 - \tau \le t \le t_1 \\ x^{(n-2)}(t_2) = x^{(n-1)}(t_3) = 0. \end{gathered} \tag{3.2}$$

Here we assume

- (i) $t_1 < t_2 < t_3$;
- (ii) $\alpha, \beta > 0$, $\quad t_3 - t_1 \ge \tau \ge 0$, and $\theta \in [-\tau, 0]$ is constant;
- (iii) $\sigma : [t_1 - \tau, t_1] \to \mathbb{R}$ is continuous with $\sigma(t_1) = 0$;
- (iv) $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous and nonnegative for $x \ge 0$.

For the rest of this paper we also have the assumptions

(A1) $G(t, s)$ is Green's function for the differential equation

$$u^{(n)}(t) = 0, \;\; t \in (t_1, t_3)$$

subject to the boundary conditions (3.2) with $\tau = 0$.

(A2) $g(t, s)$ is Green's function for the differential equation

$$u'''(t) = 0, \;\; t \in (t_1, t_3)$$

subject to the boundary conditions

$$\begin{gathered} \alpha u(t_1) - \beta u'(t_1) = 0 \\ u'(t_2) = u''(t_3) = 0 \end{gathered}$$

for α, β as in (ii).

(A3) $\|y\|_{[u,v]} := \sup_{u \le x \le v} |y^{(n-3)}(x)|$.

(A4) For $\Xi := \{s \in [t_1, t_3] : t_1 \le s + \theta \le t_3\}$, the set

$$\Xi_h := \{s \in \Xi : t_2 - h \le s + \theta \le t_2 + h\}$$

has nonzero measure for some $h \in (0, t_3 - t_2)$.

The corresponding Green's function for the homogeneous problem $u'''(t) = 0$ satisfying the boundary conditions (3.3) is given in (1.4), rewritten here for convenience as

$$g(t,s) = \begin{cases} s \in [t_1,t_2] & : \begin{cases} \frac{1}{2}(t-t_1)(2s-t-t_1) + \frac{\beta}{\alpha}(s-t_1) & : t \le s \\ \frac{1}{2}(s-t_1)^2 + \frac{\beta}{\alpha}(s-t_1) & : s \le t \end{cases} \\ s \in [t_2,t_3] & : \begin{cases} \frac{1}{2}(t-t_1)(2t_2-t-t_1) + \frac{\beta}{\alpha}(t_2-t_1) \\ \frac{1}{2}(t-t_1)(2t_2-t-t_1) + \frac{\beta}{\alpha}(t_2-t_1) + \frac{1}{2}(t-s)^2 \end{cases} \end{cases} \tag{3.3}$$

Remark 3.1. As in Theorem 2.1, if

$$\frac{\beta}{\alpha}(t_2-t_1) > \frac{1}{2}(t_3-t_1)(t_3+t_1-2t_2),$$

then $g(t,s) > 0$ for all $t \in (t_1,t_3]$, $s \in (t_1,t_3]$. Note that if the boundary points satisfy

$$t_3 - t_2 < t_2 - t_1, \tag{3.4}$$

then the above inequality holds for any choice of $\alpha, \beta > 0$. Thus throughout this section we assume that (3.4) holds. Moreover, as in [1, Lemma 3] or [9, Lemma 1], we have the following boundedness result.

Lemma 3.2. *For all* $t, s \in [t_1,t_3]$,

$$\ell(t) g(t_2,s) \le g(t,s) \le g(t_2,s) \tag{3.5}$$

where

$$\ell(t) := \frac{\alpha(t-t_1)(2t_2-t-t_1) + 2\beta(t_2-t_1)}{\alpha(t_2-t_1)^2 + 2\beta(t_2-t_1)}. \tag{3.6}$$

Remark 3.3. The following discussion is similar to that found in [11] for a two-point problem on the unit interval. If x is a solution of (3.1), (3.2), it can be written as

$$x(t) = \begin{cases} x(-\tau;t) & t_1 - \tau \le t \le t_1 \\ \int_{t_1}^{t_3} G(t,s) f(s, x(s+\theta)) ds & t_1 \le t \le t_3 \end{cases}$$

where $x(-\tau;t)$ satisfies

$$x^{(n-3)}(-\tau;t) = e^{\frac{\alpha}{\beta}(t-t_1)} x^{(n-3)}(t_1) + \frac{1}{\beta} \int_t^{t_1} e^{\frac{\alpha}{\beta}(t-s)} \sigma(s) ds$$

for $t \in [t_1 - \tau, t_1]$.

Now assume that u_0 is the solution of (3.1), (3.2) with $f \equiv 0$. Then u_0 satisfies

$$u_0^{(n-3)}(t) = \begin{cases} \frac{1}{\beta} \int_t^{t_1} e^{\frac{\alpha}{\beta}(t-s)} \sigma(s) ds & t_1 - \tau \le t \le t_1 \\ 0 & t_1 \le t \le t_3. \end{cases} \tag{3.7}$$

If x is any solution of (3.1), (3.2) set $u(t) := x(t) - u_0(t)$. Then $u(t) \equiv x(t)$ on $[t_1,t_3]$, and u satisfies

$$u^{(n-3)}(t) = \begin{cases} e^{\frac{\alpha}{\beta}(t-t_1)} u^{(n-3)}(t_1) & t_1 - \tau \le t \le t_1 \\ \int_{t_1}^{t_3} g(t,s) f(s, u(s+\theta) + u_0(s+\theta)) ds & t_1 \le t \le t_3. \end{cases}$$

But this implies

$$u(t) = \begin{cases} \left(\frac{\beta}{\alpha}\right)^{n-3} e^{\frac{\alpha}{\beta}(t-t_1)} u^{(n-3)}(t_1) & t_1 - \tau \le t \le t_1 \\ \int_{t_1}^{t_3} G(t,s) f(s, u(s+\theta) + u_0(s+\theta)) ds & t_1 \le t \le t_3. \end{cases}$$

4. Existence of at Least One Positive Solution

As mentioned in the previous section, assume $(i)-(iv)$ and $(A1)-(A4)$ hold. We are concerned with proving the existence of positive solutions of the higher-order nonlinear boundary value problem (3.1), (3.2); for related work on the existence of positive solutions, see [6, 7, 10]. In light of the above discussion in Remark 3.3, the solutions of (3.1), (3.2) can be found using the fixed points of the operator $\mathcal{A}$ with domain $C^{n-3}[t_1-\tau, t_3]$ defined by

$$\mathcal{A}u(t) = \begin{cases} \left(\frac{\beta}{\alpha}\right)^{n-3} e^{\frac{\alpha}{\beta}(t-t_1)} u^{(n-3)}(t_1) & t_1-\tau \le t \le t_1 \\ \int_{t_1}^{t_3} G(t,s) f(s, u(s+\theta)+u_0(s+\theta))ds & t_1 \le t \le t_3. \end{cases}$$

If $u=\mathcal{A}u$, then a solution x of (3.1), (3.2) is given by $x=u+u_0$, where u_0 satisfies (3.7).

Remark 4.1. In the following discussion we will need an $h\in(0,t_3-t_2)$ to satisfy $(A4)$; note that

$$\ell(t_2+h)=\ell(t_2-h)=\frac{\alpha(t_2+h-t_1)(t_2-h-t_1)+2\beta(t_2-t_1)}{\alpha(t_2-t_1)^2+2\beta(t_2-t_1)} \tag{4.1}$$

for all $h\in(0,t_3-t_2)$, where ℓ is given in (3.6), and $\ell(t)\ge\ell(t_2+h)$ for all $t\in[t_2-h,t_2+h]$. Moreover, let $k,m>0$ such that

$$\begin{aligned} k^{-1} &:= \int_{t_1}^{t_3} g(t_2,s)ds \\ &= \frac{1}{6}(t_2-t_1)^2(3t_3-2t_2-t_1)+\frac{\beta}{2\alpha}(t_2-t_1)(2t_3-t_2-t_1) \end{aligned}$$

and

$$m^{-1} := \int_{\Xi_h} g(t_2,s)\,ds. \tag{4.2}$$

Finally, set

$$M_0 := \|u_0\|_{[t_1-\tau,t_3]} \tag{4.3}$$

for u_0 as in (3.7).

We will employ the following fixed point theorem due to Krasnoselskii [13].

Theorem 4.2. *Let E be a Banach space, $P\subseteq E$ be a cone, and suppose that Ω_1, Ω_2 are bounded open balls of E centered at the origin with $\overline{\Omega}_1\subset\Omega_2$. Suppose further that $\mathcal{A}: P\cap(\overline{\Omega}_2\setminus\Omega_1)\to P$ is a completely continuous operator such that either*

(i) *$\|\mathcal{A}u\|\le\|u\|$, $u\in P\cap\partial\Omega_1$ and $\|\mathcal{A}u\|\ge\|u\|$, $u\in P\cap\partial\Omega_2$, or*
(ii) *$\|\mathcal{A}u\|\ge\|u\|$, $u\in P\cap\partial\Omega_1$ and $\|\mathcal{A}u\|\le\|u\|$, $u\in P\cap\partial\Omega_2$*

holds. Then $\mathcal{A}$ has a fixed point in $P\cap(\overline{\Omega}_2\setminus\Omega_1)$.

Theorem 4.3. *Assume $(i)-(iv)$ and $(A1)-(A4)$ hold. Let k,m,M_0 be as in (4.2), (4.2), (4.3), respectively, and suppose the following conditions are satisfied.*

(C1) *There exists a $p>0$ such that $f(t,w)\le kp$ for $t\in[t_1,t_3]$ and $0\le\|w\|\le p+M_0$.*
(C2) *There exists a $q>p$ such that $f(t,w)\ge mq$ for $t\in\Xi_h$ and $q\ell(t_2+h)\le\|w\|\le q$, for $h\in(0,t_3-t_2)$ and Ξ_h as in $(A4)$.*

Then system (3.1), (3.2) has a positive solution x such that $\|x\|_{[t_1-\tau,t_3]}$ lies between $\max\{0,p-M_0\}$ and $q+M_0$.

Proof. Many of the techniques employed here are as in [10, 11]. Let $\mathcal{B}$ denote the Banach space $C^{n-3}[t_1-\tau, t_3]$ with the norm

$$\|u\|_{[t_1-\tau,t_3]} = \sup_{t\in[t_1-\tau,t_3]} |u^{(n-3)}(t)|.$$

Define the cone $\mathcal{P} \subset \mathcal{B}$ by

$$\mathcal{P} = \{u \in \mathcal{B} : \min_{t\in[t_2-h,t_2+h]} u^{(n-3)}(t) \ge \ell(t_2+h)\|u\|_{[t_1-\tau,t_3]}\}.$$

Consider the mapping $\mathcal{A} : \mathcal{P} \to \mathcal{B}$ via

$$\mathcal{A}u(t) = \begin{cases} \big(\frac{\beta}{\alpha}\big)^{n-3} e^{\frac{\alpha}{\beta}(t-t_1)} u^{(n-3)}(t_1) & t_1-\tau \le t \le t_1 \\ \int_{t_1}^{t_3} G(t,s) f(s, u(s+\theta)+u_0(s+\theta))ds & t_1 \le t \le t_3. \end{cases}$$

Then

$$(\mathcal{A}u)^{(n-3)}(t) = \begin{cases} e^{\frac{\alpha}{\beta}(t-t_1)} \int_{t_1}^{t_3} g(t_1,s) f(s, u(s+\theta)+u_0(s+\theta))ds \\ \int_{t_1}^{t_3} g(t,s) f(s, u(s+\theta)+u_0(s+\theta))ds, \end{cases} \tag{4.4}$$

so that $(\mathcal{A}u)^{(n-3)}(t) \le (\mathcal{A}u)^{(n-3)}(t_1)$ for $t_1-\tau \le t \le t_1$. In other words, $\|\mathcal{A}u\|_{[t_1-\tau,t_3]} = \|\mathcal{A}u\|_{[t_1,t_3]}$. It follows for $h \in (0, t_3-t_2)$ and $t \in [t_2-h, t_2+h]$ that

$$\begin{aligned} (\mathcal{A}u)^{(n-3)}(t) &= \int_{t_1}^{t_3} g(t,s) f(s, u(s+\theta)+u_0(s+\theta))ds \\ &\ge \ell(t) \int_{t_1}^{t_3} g(t_2,s) f(s, u(s+\theta)+u_0(s+\theta))ds \\ &\ge \ell(t_2+h)\|\mathcal{A}u\|_{[t_1-\tau,t_3]} \end{aligned} \tag{4.5}$$

by properties of Green's function (3.5), so that $\mathcal{A} : \mathcal{P} \to \mathcal{P}$.

For $0 < p < q$ as in the statement of the theorem, define open sets

$$\Omega_p = \{u \in \mathcal{B} : \|u\|_{[t_1-\tau,t_3]} < p\}, \quad \Omega_q = \{u \in \mathcal{B} : \|u\|_{[t_1-\tau,t_3]} < q\};$$

then $0 \in \Omega_p \subset \Omega_q$. If $u \in \mathcal{P} \cap \partial\Omega_p$, then $\|u\| = p$ and

$$|u^{(n-3)}(t) + u_0^{(n-3)}(t)| \le p + M_0 \tag{4.6}$$

for all $t \in [t_1, t_3]$. As a result,

$$\begin{aligned} \|\mathcal{A}u\| &= \int_{t_1}^{t_3} g(t_2,s) f(s, u(s+\theta)+u_0(s+\theta))ds \\ &\le kp \int_{t_1}^{t_3} g(t_2,s)\, ds = p = \|u\| \end{aligned}$$

using $(C1)$ and (4.2). Thus, $\|\mathcal{A}u\| \le \|u\|$ for $u \in \mathcal{P} \cap \partial\Omega_p$.

Similarly, let $u \in \mathcal{P} \cap \partial\Omega_q$, so that $\|u\| = q$. Then for $s \in \Xi_h$,

$$u^{(n-3)}(s+\theta) \ge \min_{t\in[t_2-h,t_2+h]} u^{(n-3)}(t) \ge \|u\|\ell(t_2+h)$$

for all $h \in (0, t_3-t_2)$ and $\ell(\cdot)$ as in (4.1). Consequently,

$$q\ell(t_2+h) \le u^{(n-3)}(s+\theta) + u_0^{(n-3)}(s+\theta) \le q \tag{4.7}$$

for $s \in \Xi_h$, since $u_0^{(n-3)} \equiv 0$ on $[t_1, t_3]$. It follows that

$$\begin{aligned}\|\mathcal{A}u\| &= \int_{t_1}^{t_3} g(t_2, s) f(s, u(s+\theta) + u_0(s+\theta)) ds \\ &\geq \int_{\Xi_h} g(t_2, s) f(s, u(s+\theta) + u_0(s+\theta)) ds \\ &\geq mq \int_{\Xi_h} g(t_2, s) ds = q = \|u\|\end{aligned}$$

by $(C2)$ and (4.2). Consequently, $\|\mathcal{A}u\| \geq \|u\|$ for $u \in \mathcal{P} \cap \partial\Omega_q$. By Theorem 4.2, $\mathcal{A}$ has a fixed point $u \in \mathcal{P} \cap (\overline{\Omega}_q \setminus \Omega_p)$ with

$$p \leq \|u\| \leq q.$$

We conclude that a positive solution of (3.1), (3.2) is $x = u + u_0$ for u_0 satisfying (3.7), such that $p - M_0 \leq \|x\| \leq q + M_0$, for M_0 as in (4.3). □

5. Existence of at Least Two Positive Solutions

In this section we prove the existence of at least two positive solutions to (3.1), (3.2), again under certain restrictions on the nonlinearity f. The following lemma, pertinent to the discussion that follows, in easily proven using the branches of Green's function (3.3).

Lemma 5.1. *Let $h \in (0, t_3 - t_2)$. Then $g(t_2 + h, s) \geq g(t_2 - h, s)$ for all $s \in [t_1, t_3]$.*

The following is the Avery-Henderson Fixed Point Theorem [4], that we will employ to prove the existence of two solutions. Notationally, the cone $\mathcal{P}$ has subsets of the form $P(\chi, c) := \{u \in \mathcal{P} : \chi(u) < c\}$ for a given functional χ.

Theorem 5.2. *Let $\mathcal{P}$ be a cone in a real Banach space $\mathcal{B}$. Let η and χ be increasing, nonnegative continuous functionals on $\mathcal{P}$. Let ψ be a nonnegative continuous functional on $\mathcal{P}$ with $\psi(0) = 0$ such that, for some positive constants c and M,*

$$\chi(u) \leq \psi(u) \leq \eta(u) \text{ and } ||u|| \leq M\chi(u), \quad \forall u \in \overline{P(\chi, c)}.$$

Suppose that there exist positive numbers a and b with $a < b < c$ such that

$$\psi(\lambda u) \leq \lambda\psi(u), \quad \text{for all } 0 \leq \lambda \leq 1 \text{ and } u \in \partial P(\psi, b).$$

Suppose $A : \overline{P(\chi, c)} \to \mathcal{P}$ is a completely continuous operator satisfying

(i) *$\chi(Au) > c$ for all $u \in \partial P(\chi, c)$;*
(ii) *$\psi(Au) < b$ for all $u \in \partial P(\psi, b)$;*
(iii) *$P(\eta, a) \neq \emptyset$ and $\eta(Au) > a$ for all $u \in \partial P(\eta, a)$.*

Then A has at least two fixed points u_1 and u_2 such that

$$a < \eta(u_1) \text{ with } \psi(u_1) < b \quad \text{and} \quad b < \psi(u_2) \text{ with } \chi(u_2) < c.$$

Again let $\mathcal{B}$ denote the Banach space $C^{n-3}[t_1 - \tau, t_3]$ with the norm

$$\|u\|_{[t_1-\tau, t_3]} = \sup_{t \in [t_1-\tau, t_3]} |u^{(n-3)}(t)|.$$

Define the cone $\mathcal{P} \subset \mathcal{B}$ by

$$\begin{aligned}\mathcal{P} = \Big\{ u \in \mathcal{B} : \ & u^{(n-3)} \text{ is nondecreasing on } [t_1, t_2], \\ & u^{(n-3)} \text{ is nonincreasing on } [t_2, t_3]; \\ & u \text{ is nonnegative valued on } [t_1, t_3]; \\ & u^{(n-3)}(t_2 + h) \geq u^{(n-3)}(t_2 - h), \text{ and} \\ & \min_{t \in [t_2 - h, t_2 + h]} u^{(n-3)}(t) \geq \ell(t_2 + h) \|u\|_{[t_1 - \tau, t_3]} \Big\}.\end{aligned} \tag{5.1}$$

Finally, let the nonnegative increasing continuous functionals χ, ψ, and η be defined on the cone $\mathcal{P}$ by

$$\begin{aligned}\chi(u) &= \min_{t \in [t_2 - h, t_2 + h]} u^{(n-3)}(t) = u^{(n-3)}(t_2 - h), \\ \psi(u) &= \max_{t \in [t_1, t_2 - h] \cup [t_2 + h, t_3]} u^{(n-3)}(t) = u^{(n-3)}(t_2 + h), \\ \eta(u) &= \max_{t \in [t_2 - h, t_2 + h]} u^{(n-3)}(t) = u^{(n-3)}(t_2).\end{aligned}$$

Observe that, for each $u \in \mathcal{P}$,

$$\chi(u) \leq \psi(u) \leq \eta(u), \tag{5.2}$$

$$\|u\| = u^{(n-3)}(t_2) \leq \frac{1}{\ell(t_2 + h)} u^{(n-3)}(t_2) = \frac{1}{\ell(t_2 + h)} \eta(u), \tag{5.3}$$

$$\|u\| \leq \frac{1}{\ell(t_2 + h)} u^{(n-3)}(t_2 - h) = \frac{1}{\ell(t_2 + h)} \chi(u) \leq \frac{1}{\ell(t_2 + h)} \psi(u). \tag{5.4}$$

Theorem 5.3. *Assume (i) – (iv) and (A1)-(A4) hold. Let $\ell(t_2 + h)$, m, and M_0 be as in* (4.1), (4.2), *and* (4.3), *respectively. Suppose there exist positive numbers a, b, and c such that $0 < a < b < c$, and suppose a continuous function f satisfies the following conditions:*

- *(i) $f(s, w) \geq 0$ for all $s \in [t_1, t_3]$ and $\|w\| \in \Big[0, \frac{c}{\ell(t_2+h)} + M_0\Big]$,*
- *(ii) $f(s, w) > am$ for all $s \in \Xi_h$ and $\|w\| \in [a, \frac{a}{\ell(t_2+h)} + M_0]$,*
- *(iii) $f(s, w) < \frac{b}{\int_{t_1}^{t_3} g(t_2+h, s) ds}$ for all $s \in [t_1, t_3]$ and $\|w\| \in [0, \frac{b}{\ell(t_2+h)} + M_0]$,*
- *(iv) $f(s, w) > \frac{cm}{\ell(t_2+h)}$ for $s \in \Xi_h$ and $\|w\| \in [c, \frac{c}{\ell(t_2+h)} + M_0]$.*

Then, the higher-order boundary value problem (3.1), (3.2)*, has at least two positive solutions x_1 and x_2 such that*

$$\max_{t \in [t_2 - h, t_2 + h]} x_1^{(n-3)}(t) > a \quad \textit{with} \quad \max_{t \in [t_1, t_2 - h] \cup [t_2 + h, t_3]} x_1^{(n-3)}(t) < b,$$

and

$$\max_{t \in [t_1, t_2 - h] \cup [t_2 + h, t_3]} x_2^{(n-3)}(t) > b \quad \textit{with} \quad \min_{t \in [t_2 - h, t_2 + h]} x_2^{(n-3)}(t) < c.$$

Proof. As in the previous section, the solutions of (3.1), (3.2) can be found from the fixed points of the operator $\mathcal{A}$, defined by

$$\mathcal{A}u(t) = \begin{cases} \left(\frac{\beta}{\alpha}\right)^{n-3} e^{\frac{\alpha}{\beta}(t - t_1)} u^{(n-3)}(t_1) & t_1 - \tau \leq t \leq t_1 \\ \int_{t_1}^{t_3} G(t, s) f(s, u(s + \theta) + u_0(s + \theta)) ds & t_1 \leq t \leq t_3, \end{cases}$$

where u_0 satisfies (3.7). Note that if $u \in \mathcal{P}$, then $\mathcal{A}u(t) \geq 0$ on $[t_1, t_3]$. Using the properties of g in (3.3), (4.4) implies that $(\mathcal{A}u)^{(n-3)}$ is nondecreasing on $[t_1, t_2]$ and nonincreasing on $[t_2, t_3]$. ¿From Lemma 5.1 it follows that

$$(\mathcal{A}u)^{(n-3)}(t_2 + h) \geq (\mathcal{A}u)^{(n-3)}(t_2 - h),$$

and

$$\min_{t \in [t_2 - h, t_2 + h]} (\mathcal{A}u)^{(n-3)}(t) \geq \ell(t_2 + h) \|\mathcal{A}u\|_{[t_1 - \tau, t_3]}$$

as in (4.5). Therefore $\mathcal{A} : \mathcal{P} \to \mathcal{P}$. For any $u \in \mathcal{P}$, (5.2) and (5.4) imply that

$$\chi(u) \leq \psi(u) \leq \eta(u),$$
$$\|u\| \leq \frac{1}{\ell(t_2 + h)} \chi(u).$$

It is clear that $\psi(0) = 0$, and for all $u \in \mathcal{P}$, $\lambda \in [0, 1]$ we have

$$\begin{aligned}
\psi(\lambda u) &= \max_{t \in [t_1, t_2 - h] \cup [t_2 + h, t_3]} (\lambda u)^{(n-3)}(t) \\
&= \lambda \max_{t \in [t_1, t_2 - h] \cup [t_2 + h, t_3]} u^{(n-3)}(t) = \lambda \psi(u).
\end{aligned}$$

Since $0 \in \mathcal{P}$ and $a > 0$, $P(\eta, a) \neq \emptyset$.

In the following claims, we verify the remaining conditions of Theorem 5.2.

Claim 1. If $u \in \partial P(\eta, a)$, then $\eta(\mathcal{A}u) > a$: Note that $u \in \partial P(\eta, a)$ and (5.3) yield $a = \|u\| \leq \frac{a}{\ell(t_2+h)}$. By hypothesis *(ii)*,

$$\begin{aligned}
\eta(\mathcal{A}u) &= \max_{t \in [t_2 - h, t_2 + h]} \int_{t_1}^{t_3} g(t, s) f(s, u(s + \theta) + u_0(s + \theta)) ds \\
&= \int_{t_1}^{t_3} g(t_2, s) f(s, u(s + \theta) + u_0(s + \theta)) ds \\
&\geq \int_{\Xi_h} g(t_2, s) f(s, u(s + \theta) + u_0(s + \theta)) ds \\
&> am \int_{\Xi_h} g(t_2, s) ds = a.
\end{aligned}$$

Claim 2. If $u \in \partial P(\psi, b)$, then $\psi(\mathcal{A}u) < b$: In this case $u \in \partial P(\psi, b)$ implies that $b \leq \|u\| \leq \frac{b}{\ell(t_2+h)}$ by (5.4), so that $\|u + u_0\| \leq \frac{b}{\ell(t_2+h)} + M_0$. We then get

$$\begin{aligned}
\psi(\mathcal{A}u) &= \max_{t \in [t_1, t_2 - h] \cup [t_2 + h, t_3]} \int_{t_1}^{t_3} g(t, s) f(s, u(s + \theta) + u_0(s + \theta)) ds \\
&= \int_{t_1}^{t_3} g(t_2 + h, s) f(s, u(s + \theta) + u_0(s + \theta)) ds \\
&< \frac{b}{\int_{t_1}^{t_3} g(t_2 + h, s) ds} \int_{t_1}^{t_3} g(t_2 + h, s) ds = b
\end{aligned}$$

by hypothesis *(iii)*.

Claim 3. If $u \in \partial P(\chi, c)$, then $\chi(\mathcal{A}u) > c$: Since $u \in \partial P(\chi, c)$, from (5.4) we have that $\min_{t\in[t_2-h,t_2+h]} u^{(n-3)}(t) = c$ and $c \le \|u\| \le \frac{c}{\ell(t_2+h)}$. Thus,

$$\begin{aligned}
\chi(\mathcal{A}u) &= \min_{t\in[t_2-h,t_2+h]} \int_{t_1}^{t_3} g(t,s)f(s,u(s+\theta)+u_0(s+\theta))ds \\
&\ge \min_{t\in[t_2-h,t_2+h]} \ell(t) \int_{t_1}^{t_3} g(t_2,s)f(s,u(s+\theta)+u_0(s+\theta))ds \\
&\ge \ell(t_2+h) \int_{\Xi_h} g(t_2,s)f(s,u(s+\theta)+u_0(s+\theta))ds \\
&> \ell(t_2+h)\frac{cm}{\ell(t_2+h)} \int_{\Xi_h} g(t_2,s)ds = c
\end{aligned}$$

by hypothesis (iv), using arguments as in Claim 1. Therefore the hypotheses of Theorem 5.2 are satisfied and there exist at least two positive fixed points u_1 and u_2 of $\mathcal{A}$ in $\overline{P(\chi, c)}$. Thus, the higher-order boundary value problem (3.1), (3.2), has at least two positive solutions x_1 and x_2 such that

$$\begin{aligned}
a < \eta(x_1) \quad &\text{with} \quad \psi(x_1) < b\,, \\
b < \psi(x_2) \quad &\text{with} \quad \chi(x_2) < c
\end{aligned}$$

since $x \equiv u$ on $[t_1, t_3]$ as shown in Remark 3.3. ☐

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