Electronic Journal of Differential Equations, Vol. 2006(2006), No. 15, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

A LIOUVILLE THEOREM FOR F-HARMONIC MAPS WITH FINITE F-ENERGY

M'HAMED KASSI

Abstract. Let (M, g) be a m-dimensional complete Riemannian manifold with a pole, and (N, h) a Riemannian manifold. Let $F : \mathbb{R}^+ \to \mathbb{R}^+$ be a strictly increasing C^2 function such that $F(0) = 0$ and $d_F := \sup(tF'(t)(F(t))^{-1}) < \infty$. We show that if $d_F < m/2$, then every F-harmonic map $u : M \to N$ with finite F-energy (i.e a local extremal of $E_F(u) := \int_M F(|du|^2/2)dV_g$ and $E_F(u)$ is finite) is a constant map provided that the radial curvature of M satisfies a pinching condition depending to d_F.

1. Introduction and statement of result

Let (M, g) and (N, h) be two Riemannian manifolds and F be a given C^2 function $F : \mathbb{R}^+ \to \mathbb{R}^+$. Then, a map $u : M \to N$ of class C^2 is said to be F-harmonic if for every compact K of M, the map u is extremal of F-energy:

$$E_F(u) := \int_K F(\frac{|du|^2}{2})dV_g.$$

In a normal coordinate system, the tension field associated with $E_F(u)$ by the Euler-Lagrange equations is

$$\tau_F(u) := \sum_{i=1}^{m}(\nabla_{e_i}(F'(\frac{|du|^2}{2})du))e_i = F'(\frac{|du|^2}{2})\tau(u) + du.\Big\{\operatorname{grad}\big(F'(\frac{|du|^2}{2})\big)\Big\}$$

where $\tau(u)$ is the usual tension field of u defined by

$$\tau(u)_k = \Delta_M u^k + \sum_{\beta,\gamma;i,j}^{n;m} {}^N\Gamma^k_{\alpha\gamma}(u)g^{ij}\frac{\partial u^\beta}{\partial x_i}\frac{\partial u^\gamma}{\partial x_j}, \quad k = 1, \dots, n\,.$$

Then, the map u is F-harmonic if $\tau_F(u) = 0$. For further properties of F-harmonic maps, we refer the reader to [1, 2]. For the particular case of $F(t) = t$, the Liouville problem for harmonic maps with finite energy have been studied in [4, 6, 7, 8, 9]. While for $F(t) = \frac{2}{p}t^{p/2}$, with $p \geq 2$, this is the problem of p-harmonic maps with finite p-energy (corollary 1.2. If $F(t) = \sqrt{1+2t} - 1$ corresponding to the minimal graph (corollary 1.3). In this paper, we study the same problem for F-harmonic maps with finite F-energy without condition on the curvature for the

2000 *Mathematics Subject Classification.* 58E20, 53C21, 58J05.
Key words and phrases. F-harmonic maps; Liouville propriety; Stokes formula; comparison theorem.

Submitted March 24, 2005. Published January 31, 2006.

target manifold. We assume that F is strictly increasing, $F(0)=0$, and $d_F = \sup \frac{tF'(t)}{F(t)} < \infty$, "the degree of F". For x in M, we set $r(x) = d_g(x, x_0)$.

Theorem 1.1. *Let (M,g) be a m-dimensional complete Riemannian manifold, $m>2$, with a pole x_0, and let (N,h) be a Riemannian manifold. If $d_F < m/2$, then every F-harmonic map of M into N with finite F-energy is constant provided that the radial curvature K_r of M satisfies one of the following two conditions:*

(i) $-\alpha^2 \le K_r \le -\beta^2$ *with* $\alpha>0, \beta>0$ *and* $1+(m-1)\beta - 2d_F\alpha > 0$

(ii) $-\frac{\alpha}{1+r^2} \le K_r \le \frac{\beta}{1+r^2}$ *with* $\alpha \ge 0$ *and* $\beta \in [0, \frac{1}{4}]$ *such that* $2+(m-1)(1+\sqrt{1-4\beta}) - 2d_F(1+\sqrt{1+4\alpha}) > 0$.

Furthermore, we have the following corollaries.

Corollary 1.2. *Let (M,g) and (N,h) be as in the theorem. Then, every C^2 p-harmonic map of M into N with finite p-energy, for $p<m$, is constant.*

Corollary 1.3. *Let (M,g) and (N,h) be as in the theorem. Then, for $m>2$, every C^2 map u of M into N, with finite energy, solution of*

$$\frac{\tau(u)}{\sqrt{1+|du|^2}} + du.\Big\{\mathrm{grad}\big(\frac{1}{\sqrt{1+|du|^2}}\big)\Big\} = 0$$

is constant.

For $m=2$, the statement of the theorem is false in general. In fact, for the case (i), there exist holomorphic maps of the hyperbolic disc with finite energy [9]. While for the case (ii) there exist holomorphic maps of $\mathbb{C}$ into $\mathbb{P}^1$ with finite energy [8].

2. Proof of Theorem 1.1

Let X and Y be two vector fields on M. It is well-known [3, 6], that the stress-energy for harmonic maps is

$$S_u := \frac{|du|^2}{2}\langle X,Y\rangle_g - \langle du(X), du(Y)\rangle_h$$

and satisfies

$$(\operatorname{div} S_u)(X) = -\langle \tau(u), du(X)\rangle_h.$$

Following [2], we define the stress-energy of F-harmonic maps by

$$S_{F,u}(X,Y) := F(\frac{|du|^2}{2})\langle X,Y\rangle_g - F'(\frac{|du|^2}{2})\langle du(X), du(Y)\rangle_h\,.$$

When $F(t) := t$ we have $S_{F,u} := S_u$. Also $(\operatorname{div} S_{F,u})(X) = -\langle \tau_F(u), du(X)\rangle_h$ thanks to the following lemma.

Lemma 2.1. *For every vector field X on M, we have*

$$(\operatorname{div} S_{F,u})(X) = -\langle \tau_F(u), du(X)\rangle_h, \tag{2.1}$$

$$\begin{aligned} &\operatorname{div}(F(\frac{|du|^2}{2})X) \\ &= \operatorname{div}(F'(\frac{|du|^2}{2})\langle du(X), du(e_i)\rangle_h e_i) - \langle \tau_F(u), du(X)\rangle_h + [S_{F,u}, X], \end{aligned} \tag{2.2}$$

where

$$[S_{F,u},X](x)=\sum_{i,j=1}^{m}\Big(F(\frac{|du|^2}{2})\delta_{ij}-F'(\frac{|du|^2}{2})\langle du(e_i),du(e_j)\rangle_h\Big)\langle\nabla_{e_i}X,e_j\rangle_g\,.$$

In particular, if u is F-harmonic and $D\subset\subset M$ is a C^1 boundary domain, then we have

$$\int_{\partial D}S_{F,u}(X,\nu)d\sigma_g=\int_D[S_{F,u},X]dV_g$$

where ν is the normal to ∂D.

Proof. Let $x\in M$. Chose a normal coordinate system such that at x. $g_{ij}(x)=\delta_{ij}$ $dg(x)=0$, where $(e_1,\dots,e_m)$ being a normal basis, we have $\nabla_{e_j}e_k=0$ for all j,k and

$$\begin{aligned}
&(\operatorname{div}S_{F,u})(X)\\
&=\sum_{i=1}^{m}\Big\{\nabla_{e_i}S_{F,u}(e_i,X)-S_{F,u}(e_i,\nabla_{e_i}X)-S_{F,u}(\nabla_{e_i}e_i,X)\Big\}\\
&=\sum_{i=1}^{m}\Big\{\nabla_{e_i}\big(F(\frac{|du|^2}{2})\langle e_i,X\rangle-\langle F'(\frac{|du|^2}{2})du(e_i),du(X)\rangle\big)-F(\frac{|du|^2}{2})\langle e_i,\nabla_{e_i}X\rangle\\
&\quad+F'(\frac{|du|^2}{2})\langle du(e_i),du(\nabla_{e_i}X)\rangle-S_{F,u}(\nabla_{e_i}e_i,X)\Big\}\\
&=\sum_{i=1}^{m}\Big\{\nabla_{e_i}\big(F(\frac{|du|^2}{2})\langle e_i,X\rangle\big)\\
&\quad-\nabla_{e_i}\big(\langle F'(\frac{|du|^2}{2})du(e_i),du(X)\rangle\big)-F(\frac{|du|^2}{2})\langle e_i,\nabla_{e_i}X\rangle\\
&\quad+F'(\frac{|du|^2}{2})\langle du(e_i),du(\nabla_{e_i}X)\rangle-S_{F,u}(\nabla_{e_i}e_i,X)\Big\}\\
&=\sum_{i=1}^{m}\Big\{\big(\sum_{j=1}^{m}F'(\frac{|du|^2}{2})\langle\nabla_{e_i}(du(e_j)),du(e_j)\rangle\big)\langle e_i,X\rangle\\
&\quad+F(\frac{|du|^2}{2})\nabla_{e_i}\langle e_i,X\rangle-\langle\nabla_{e_i}(F'(\frac{|du|^2}{2})du(e_i)),du(X)\rangle\\
&\quad-F'(\frac{|du|^2}{2})\langle du(e_i),\nabla_{e_i}(du(X))\rangle\\
&\quad-F(\frac{|du|^2}{2})\langle e_i,\nabla_{e_i}X\rangle+F'(\frac{|du|^2}{2})\langle du(e_i),du(\nabla_{e_i}X)\rangle\\
&\quad-S_{F,u}(\nabla_{e_i}e_i,X)\Big\}\,.
\end{aligned}$$

Thus

$$\begin{aligned}
(\operatorname{div}S_{F,u})(X)=&\sum_{i,j=1}^{m}\Big\{F'(\frac{|du|^2}{2})\langle\nabla_{e_i}(du(e_j)),du(e_j)\rangle X_i\Big\}\\
&+\sum_{i=1}^{m}\Big\{F(\frac{|du|^2}{2})\langle\nabla_{e_i}e_i,X\rangle+F(\frac{|du|^2}{2})\langle e_i,\nabla_{e_i}X\rangle\\
&-\langle\nabla_{e_i}(F'(\frac{|du|^2}{2})du(e_i)),du(X)\rangle
\end{aligned}$$

$$\begin{aligned}
&- F'(\frac{|du|^2}{2})\langle du(e_i), \nabla_{e_i}(du(X))\rangle - F(\frac{|du|^2}{2})\langle e_i, \nabla_{e_i}X\rangle \\
&+ F'(\frac{|du|^2}{2})\langle du(e_i), du(\nabla_{e_i}X)\rangle - S_{F,u}(\nabla_{e_i}e_i, X)\Big\} \\
=& \sum_{i,j=1}^{m} \Big\{F'(\frac{|du|^2}{2})\langle X_i \nabla_{e_i}(du(e_j)), du(e_j)\rangle\Big\} \\
&- \sum_{i=1}^{m} \Big\{F'(\frac{|du|^2}{2})\langle du(e_i), \nabla_{e_i}(du(X))\rangle \\
&+ F'(\frac{|du|^2}{2})\langle du(e_i), du(\nabla_{e_i}X)\rangle + F(\frac{|du|^2}{2})\langle \nabla_{e_i}e_i, X\rangle \\
&+ F(\frac{|du|^2}{2})\langle e_i, \nabla_{e_i}X\rangle - F(\frac{|du|^2}{2})\langle e_i, \nabla_{e_i}X\rangle \\
&- \langle \nabla_{e_i}(F'(\frac{|du|^2}{2})du(e_i)), du(X)\rangle - S_{F,u}(\nabla_{e_i}e_i, X)\Big\}.
\end{aligned}$$

Since $\nabla_{e_i}e_i = 0$, with $(\nabla_{e_i}du)(X) = \nabla_{e_i}(du(X)) - du(\nabla_{e_i}X)$ and by symmetry $(\nabla_{e_i}du)(X) = (\nabla_X du)(e_i)$, we have

$$\begin{aligned}
\operatorname{div}(S_{F,u})(X) =& \sum_{j=1}^{m} \Big\{F'(\frac{|du|^2}{2})\langle \nabla_X(du(e_j), du(e_j)\rangle\Big\} \\
&- \sum_{i=1}^{m} \Big\{F'(\frac{|du|^2}{2})\langle du(e_i), \nabla_{e_i}(du(X)) - du(\nabla_{e_i}X)\rangle \\
&- \langle \nabla_{e_i}(F'(\frac{|du|^2}{2})du(e_i)), du(X)\rangle\Big\}.
\end{aligned}$$

Finally,

$$\operatorname{div}(S_{F,u})(X) = -\langle \tau_F(u), du(X)\rangle.$$

Also

$$\begin{aligned}
\operatorname{div}(F(\frac{|du|^2}{2})X) &= \sum_{i=1}^{m} \langle \nabla_{e_i}(F(\frac{|du|^2}{2})X), e_i\rangle \\
&= \sum_{i=1}^{m} \Big\{\langle \nabla_{e_i}(F(\frac{|du|^2}{2}))X, e_i\rangle + F(\frac{|du|^2}{2})\langle \nabla_{e_i}X, e_i\rangle\Big\} \\
&= \nabla_X F(\frac{|du|^2}{2}) + \sum_{i=1}^{m} F(\frac{|du|^2}{2})\langle \nabla_{e_i}X, e_i\rangle.
\end{aligned}$$

Then, by straightforward computation, we obtain

$$\begin{aligned}
\nabla_X F(\frac{|du|^2}{2}) &= \sum_{i=1}^{m} \frac{1}{2} F'(\frac{|du|^2}{2})\nabla_X\langle du(e_i), du(e_i)\rangle \\
&= \sum_{i=1}^{m} F'(\frac{|du|^2}{2})\langle \nabla_X(du(e_i)), du(e_i)\rangle \\
&= \sum_{i=1}^{m} F'(\frac{|du|^2}{2})\langle (\nabla_X du)(e_i) + du(\nabla_X e_i), du(e_i)\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^m F'(\frac{|du|^2}{2})\langle(\nabla_X du)(e_i), du(e_i)\rangle \\
&= \sum_{i=1}^m F'(\frac{|du|^2}{2})\langle(\nabla_{e_i} du)(X), du(e_i)\rangle \quad \text{(by symmetry)} \\
&= \sum_{i=1}^m \Big\{\langle\nabla_{e_i}(du(X)), F'(\frac{|du|^2}{2})du(e_i)\rangle \\
&\quad - F'(\frac{|du|^2}{2})\langle du(\nabla_{e_i}X), du(e_i)\rangle\Big\}
\end{aligned}
$$

Thus

$$
\begin{aligned}
\nabla_X F(\frac{|du|^2}{2}) &= \sum_{i=1}^m \Big\{\nabla_{e_i}\langle du(X), F'(\frac{|du|^2}{2})du(e_i)\rangle \\
&\quad - \langle du(X), \nabla_{e_i}(F'(\frac{|du|^2}{2})du(e_i))\rangle \\
&\quad - F'(\frac{|du|^2}{2}))\rangle du(\nabla_{e_i}X), du(e_i)\rangle\Big\} \\
&= \sum_{i=1}^m \Big\{\operatorname{div}\big(F'(\frac{|du|^2}{2}))\rangle du(X), du(e_i)\rangle e_i\big) \\
&\quad + \langle du(X), -\nabla_{e_i}(F'(\frac{|du|^2}{2})du(e_i))\rangle \\
&\quad - F'(\frac{|du|^2}{2})\langle du(\nabla_{e_i}X), du(e_i)\rangle\Big\} \\
&= \sum_{i=1}^m \Big\{\operatorname{div}\big(F'(\frac{|du|^2}{2})\langle du(X), du(e_i)\rangle e_i\big)\Big\} \\
&\quad - \langle du(X), \tau_F(u)\rangle - \sum_{i=1}^m F'(\frac{|du|^2}{2})\langle du(\nabla_{e_i}X), du(e_i)\rangle
\end{aligned}
$$

Thus

$$
\begin{aligned}
\operatorname{div}(F(\frac{|du|^2}{2})X) &= \sum_{i=1}^m \Big\{\operatorname{div}\big(F'(\frac{|du|^2}{2})\langle du(X), du(e_i)\rangle e_i\big)\Big\} \\
&\quad - \langle du(X), \tau_F(u)\rangle + [S_{F,u}, X]
\end{aligned}
$$

with

$$
[S_{F,u}, X] = \sum_{i,j=1}^m \Big(F(\frac{|du|^2}{2})\delta_{ij} - F'(\frac{|du|^2}{2})\langle du(e_i), du(e_j)\rangle_h\Big)\langle\nabla_{e_i}X, e_j\rangle_g
$$

because $\nabla_{e_i}X = \langle\nabla_{e_i}X, e_j\rangle e_j$. If $D \subset\subset M$ is a C^1 boundary domain, we get by the use of Stokes formula

$$
\begin{aligned}
&\int_D (\operatorname{div} S_{F,u})(X) + \int_D [S_{F,u}, X] \\
&= \int_D \operatorname{div}(F(\frac{|du|^2}{2})X) - \int_D \sum_{i=1}^m \operatorname{div}\big(F'(\frac{|du|^2}{2}) < du(X), du(e_i) > e_i\big)
\end{aligned}
$$

$$= \int_{\partial D} F(\frac{|du|^2}{2})\langle X, \nu\rangle - \int_{\partial D} F'(\frac{|du|^2}{2})\langle du(X), du(\nu)\rangle\,.$$

Thus, if u is F-harmonic:

$$\int_{\partial D} \Big(F(\frac{|du|^2}{2})\langle X, \nu\rangle - F'(\frac{|du|^2}{2})\langle du(X), du(\nu)\rangle\Big) = \int_D [S_{F,u}, X].$$

This completes the proof. □

Lemma 2.2. *Let $u : M \to N$ be a F-harmonic with finite F-energy and X a vector field on M such that $|X| \le \phi(r)$ for $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying*

$$\int_1^{+\infty} \frac{dt}{\phi(t)} = +\infty.$$

Then there exists an increasing strictly sequence (R_n) such that

$$\lim_{n\to\infty} \int_{B(x_0,R_n)} [S_{F,u}, X] dV_g = 0.$$

Proof. Since $tF'(t) \le d_F F(t)$ we have

$$\begin{aligned}
&\Big|\int_{B(x_0,R)} [S_{F,u}, X]\Big| \\
&\le \Big|\int_{\partial B(x_0,R)} F(\frac{|du|^2}{2})\langle X, \nu\rangle\Big| + \Big|\int_{\partial B(x_0,R)} F'(\frac{|du|^2}{2})\rangle du(X), du(\nu)\rangle\Big| \\
&\le \int_{\partial B(x_0,R)} F(\frac{|du|^2}{2})|\langle X, \nu\rangle| + \int_{\partial B(x_0,R)} F'(\frac{|du|^2}{2})|\rangle du(X), du(\nu)\rangle| \\
&\le (1 + 2d_F) \int_{\partial B(x_0,R)} F(\frac{|du|^2}{2})|X|\,.
\end{aligned}$$

By the Co-area formula and $|X| \le \phi(r(x))$,

$$\begin{aligned}
\int_0^\infty \frac{1}{\phi(t)}\Big(\int_{\partial B(x_0,t)} F(\frac{|du|^2}{2})|X|\Big)dt &= \int_M \frac{|X||\nabla r|}{\phi(r)} F(\frac{|du|^2}{2}) \\
&\le \int_M F(\frac{|du|^2}{2}) < \infty
\end{aligned}$$

Since $\int_1^\infty \frac{dt}{\phi(t)} = \infty$, there exists a increasing strictly sequence (R_n) such that $\lim_{n\to\infty} \int_{\partial B(x_0,R_n)} F(\frac{|du|^2}{2})|X| = 0$. Hence

$$\lim_{n\to\infty} \int_{B(x_0,R_n)} [S_{F,u}, X] dV_g = 0.$$

This completes the proof of Lemma 2.2. □

For the theorem, it suffices to choose X satisfying Lemma 2.2 and the condition $[S_{F,u}, X] \ge cF(|du|^2/2)$ where $c > 0$ is a constant. For that we take $X = r\nabla r$ and using the comparison theorem of the Hessian [5].

Theorem 2.3 (Comparison theorem)**.** *Let (M, g) be a complete Riemannian manifold with a pole x_0 and k_1, k_2 be two continuous functions on $\mathbb{R}^+$ such that*

$k_2(r) \le K_r \le k_1(r)$, where K_r is the radial curvature of M, i.e., the sectional curvature of the tangent planes containing the radial vector ∇r. Also, let J_i $(i = 1, 2)$ be the solution of classical Jacobi equation

$$J_i'' + k_i J_i = 0; \quad J_i(0) = 0 \quad \text{and} \quad J_i'(0) = 1.$$

Then, if $J_1 > 0$ on $\mathbb{R}^+$, we have on $M \setminus \{x_0\}$

$$\frac{J_1'(r)}{J_1(r)}(g - dr \otimes dr) \le \operatorname{Hess}(r) \le \frac{J_2'(r)}{J_2(r)}(g - dr \otimes dr)\,.$$

Case (i) of Theorem 2.3: With $k_1(r) = -\beta^2$ and $k_2(r) = -\alpha^2$, we have

$$\beta \coth(\beta r)(g - dr \otimes dr) \le \operatorname{Hess}(r) \le \alpha \coth(\alpha r)(g - dr \otimes dr)\,.$$

Case (ii) of Theorem 2.3: With $k_1(r) = \frac{\beta}{r^2}$ and $k_2(r) = -\frac{\alpha}{r^2}$, and the fact that on $M \setminus \{x_0\}$,

$$-\frac{\alpha}{r^2} \le -\frac{\alpha}{1+r^2} \le K_r \le \frac{\beta}{1+r^2} \le \frac{\beta}{r^2}$$

we have

$$\Big(\frac{1+\sqrt{1-4\beta}}{2r}\Big)(g - dr \otimes dr) \le \operatorname{Hess}(r) \le \Big(\frac{1+\sqrt{1+4\alpha}}{2r}\Big)(g - dr \otimes dr)\,.$$

Lemma 2.4. *Under hypothesis of Theorem 2.3, in case (1), we have*

$$[S_{F,u}, X] \ge (1 + (m-1)\beta - 2d_F\alpha)F(\frac{|du|^2}{2})$$

and in case (ii),

$$[S_{F,u}, X] \ge \frac{1}{2}(2 + (m-1)(1+\sqrt{1-4\beta}) - 2d_F(1+\sqrt{1+4\alpha})F(\frac{|du|^2}{2}).$$

Proof. First note that

$$[S_{F,u}, X] = \sum_{i,j=1}^{m} \Big(F(\frac{|du|^2}{2})\delta_{ij} - F'(\frac{|du|^2}{2})\langle du(e_i), du(e_j)\rangle_h\Big) < \nabla_{e_i} X, e_j\rangle_g,$$

where $(e_1, \dots, e_{m-1}, \frac{\partial}{\partial r})$ with $e_m = \frac{\partial}{\partial r}$, being a normal basis on $B(x_0, R)$. Then, since $X = r\frac{\partial}{\partial r}$, it follows that $\nabla_{\frac{\partial}{\partial r}} X = \frac{\partial}{\partial r}$ and so we get

$$\begin{gathered}
\langle \nabla_{\frac{\partial}{\partial r}} X, \frac{\partial}{\partial r}\rangle_g = 1\,, \\
\langle \nabla_{e_i} X, e_i\rangle_g = r \operatorname{Hess}(r)(e_i, e_i), \quad \text{for } i = 1, \dots, m-1, \\
\nabla_{e_i} X = \sum_{j=1}^{m-1} r \operatorname{Hess}(r)(e_i, e_j)e_j, \quad \text{for } i = 1, \dots, m-1.
\end{gathered}$$

Therefore,

$$\begin{aligned}
[S_{F,u}, X] = F(\frac{|du|^2}{2})(1 + &\sum_{i=1}^{m-1} r \operatorname{Hess}(r)(e_i, e_i)) \\
&- \sum_{i,j=1}^{m-1} F'(\frac{|du|^2}{2})\langle du(e_i), du(e_j)\rangle_h \langle \nabla_{e_i} X, e_j\rangle_g \\
&- F'(\frac{|du|^2}{2})\langle du(\frac{\partial}{\partial r}), du(\frac{\partial}{\partial r})\rangle_h \langle \nabla_{\frac{\partial}{\partial r}} X, \frac{\partial}{\partial r}\rangle_g
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{j=1}^{m-1} F'(\frac{|du|^2}{2})\langle du(\frac{\partial}{\partial r}), du(e_j)\rangle_h \langle \nabla_{\frac{\partial}{\partial r}} X, e_j\rangle_g \\
&- \sum_{i=1}^{m-1} F'(\frac{|du|^2}{2})\langle du(e_i), du(\frac{\partial}{\partial r})\rangle_h \langle \nabla_{e_i} X, \frac{\partial}{\partial r}\rangle_g \\
&= F(\frac{|du|^2}{2})(1 + \sum_{i=1}^{m-1} r \operatorname{Hess}(r)(e_i, e_i)) \\
&- \sum_{i,j=1}^{m-1} F'(\frac{|du|^2}{2})\langle du(e_i), du(e_j)\rangle r \operatorname{Hess}(r)(e_i, e_j) \\
&- F'(\frac{|du|^2}{2})\langle du(\frac{\partial}{\partial r}), du(\frac{\partial}{\partial r})\rangle
\end{aligned}
$$

For the case (i), we have

$$
\begin{aligned}
[S_{F,u}, X] &\geq F(\frac{|du|^2}{2}) + (m-1)(\beta r)\coth(\beta r) F(\frac{|du|^2}{2}) \\
&\quad - F'(\frac{|du|^2}{2})|du|^2 (\alpha r)\coth(\alpha r) \\
&\quad + F'(\frac{|du|^2}{2})((\alpha r)\coth(\alpha r) - 1)\langle du(\frac{\partial}{\partial r}), du(\frac{\partial}{\partial r})\rangle \\
&\geq F(\frac{|du|^2}{2}) + F(\frac{|du|^2}{2})((m-1)(\beta r)\coth(\beta r) - 2d_F(\alpha r)\coth(\alpha r)) \\
&\geq F(\frac{|du|^2}{2}) + F(\frac{|du|^2}{2}) r\coth(\beta r)((m-1)\beta - 2d_F \alpha \frac{\coth(\alpha r)}{\coth(\beta r)}).
\end{aligned}
$$

Since the function $\coth(x)$ is decreasing and, $x\coth(x)$ is bounded below by a positive constant in $\mathbb{R}^+$, we have

$$
[S_{F,u}, X] \geq (1 + (m-1)\beta - 2d_F\alpha) F(\frac{|du|^2}{2})
$$

For the case (ii), we have

$$
\begin{aligned}
[S_{F,u}, X] &\geq F(\frac{|du|^2}{2}) + (m-1)aF(\frac{|du|^2}{2}) - bF'(\frac{|du|^2}{2})|du|^2 \\
&\quad + (b-1)F'(\frac{|du|^2}{2})\langle du(\frac{\partial}{\partial r}), du(\frac{\partial}{\partial r})\rangle \\
&\geq (1 + (m-1)a - 2d_F b) F(\frac{|du|^2}{2}),
\end{aligned}
$$

where we have set

$$
a = \frac{1 + \sqrt{1 - 4\beta}}{2} \quad \text{and} \quad b = \frac{1 + \sqrt{1 + 4\alpha}}{2} \geq 1.
$$

□

Acknowledgements. I would like to express my gratitude to Professor S. Asserda for his valuable suggestions and warm encouragement. Also I thank the anonymous referee for many valuable comments.

References

[1] Ara M, *Instability and nonexistence theorems for F-harmonic maps*, Illinois Jour. Math. vol 15, No 2 (2001), 657-679.

[2] Ara M, *Geometry of F-harmonic maps*, Kodai Math. 22 (1999), 243-263.

[3] Baird P., Eells J., *A conservation law for harmonic maps*, Geom.Sympos. (Utrecht, 1980), Lecture Notes in Math, vol. 894, Springer-Verlag, Berlin and New York, 1980, pp. 1-25.

[4] Escobar J.F, Freire A., Min-Oo M., *L^2 Vanishing theorems in positive curvature*, Indiana University Mathematics Journal, vol 42, No 4 (1993), 1545-1554.

[5] Greene R. E, Wu H., *Function Theory on Manifolds Which Posses a Pole*, Lecture Notes in Math. 699 (1979), Springer-Verlag.

[6] Karcher H., Wood J., *Non-existence results and growth properties for harmonic maps and forms*, J. Reine Angew. Math. 353 (1984), 165-180.

[7] Kasue A., *A note on L^2 harmonic forms on a complete manifold*, Tokyo J. Math. vol 17, No 2 (1994), 455-465.

[8] Schoen R., Yau S.T, *harmonic maps and the topology of stable hypersurface and manifolds of nonnegative Ricci curvature*, Comm. Math. Helv. 39 (1976), 333-341.

[9] Takegoshi K., *A non-existence theorem for pluriharmonic maps of finite energy*, Math. Zeit. 192 (1986), 21-27.

M'hamed Kassi

Equipe d'Analyse Complexe, Laboratoire d'Analyse Fonctionnelle, Harmonique et Complexe, Département de Mathématiques, Faculté des Sciences, Université Ibn Tofail, Kénitra, Maroc

E-mail address: mhamedkassi@yahoo.fr