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EXISTENCE OF POSITIVE SOLUTIONS OF p-LAPLACIAN FUNCTIONAL DIFFERENTIAL EQUATIONS

CHANG-XIU SONG

Abstract. In this paper, the author studies the boundary value problems of p-Laplacian functional differential equation. Sufficient conditions for the existence of positive solutions are established by using a fixed point theorem in cones.

1. Introduction

For p-Laplace equations there are have many results published, but most of them are about ordinary differential equations; see for example [2, 5, 10, 11, 13] and references therein. As pointed out in [3, 12], the study of boundary value problems (BVP) of functional differential equations (FDE) is of significance since it arises and has applications in variational problems in control theory and other areas of applied mathematics. In this paper, we shall investigate the existence of positive solutions for p-Laplacian problem

$$\begin{gathered}(\phi_p(y'))' + r(t)f(y^t) = 0,\ 0 < t < 1,\\ \alpha y(t) - \beta y'(t) = \xi(t),\ -\tau \le t \le 0,\\ \gamma y(t) + \delta y'(t) = \eta(t),\ 1 \le t \le 1+a,\end{gathered} \tag{1.1}$$

where $y^t(\theta) = y(t+\theta)$, $\theta \in [-\tau, a]$, $\tau \ge 0$, $a \ge 0$ are constants satisfying $0 \le \tau + a < 1$; $\phi_p(u)$ is the p-Laplacian operator, i.e., $\phi_p(u) = |u|^{p-2}u$, $p > 1$, $(\phi_p)^{-1}(u) = \phi_q(u)$, $\frac{1}{p} + \frac{1}{q} = 1$.

For the situation that $\tau = a = 0$, BVP (1.1) becomes the two-point BVP and has been investigated in [6, 7, 8]. Furthermore, for $\tau = a = 0$ and $p = 2$, BVP (1.1) has been studied in [1, 9]. In this paper, we shall generalize the above-mentioned equations. Our results include four subcases: $\tau = a = 0$ (two point BVP); $\tau > 0$, $a = 0$ (BVP of retarded FDE); $\tau = 0$, $a > 0$ (BVP of advanced FDE); $\tau > 0$, $a > 0$ (BVP of mixed FDE).

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Let $C = C([-\tau, a], \mathbb{R})$ be the Banach space of continuous functions from $[-\tau, a]$ into $\mathbb{R}$ with the norm $\|\varphi\|_C = \sup_{-\tau\le\theta\le a} |\varphi(\theta)|$ and

$$C^+ = \{\varphi \in C : \varphi(\theta) \ge 0,\ \theta \in [-\tau, a]\}.$$

Define

$$E = \{t \in [0, 1] : 0 \le t + \theta \le 1,\ -\tau \le \theta \le a\} = [\tau, 1 - a].$$

Noting that from the assumption that $0 \le \tau + a < 1$, we conclude $\operatorname{meas} E \ne 0$.

We shall assume the following conditions:

(H1) $f(\varphi)$ is a nonnegative continuous function defined on C^+.

(H2) $r(t)$ is a nonnegative measurable function defined on $[0, 1]$, and satisfies

$$0 < \int_E \phi_q[\int_\tau^t r(u)du]dt < \phi_q[\int_0^1 r(u)du] < +\infty.$$

(H3) $\alpha, \beta, \gamma, \delta \ge 0$, $\rho := \gamma\beta + \alpha\gamma + \alpha\delta > 0$.

(H4) $\xi(t)$ and $\eta(t)$ are continuous functions defined, respectively, on $[-\tau, 0]$ and $[1, b]$, where $b = 1 + a$, $\eta(1) = 0$; $\xi(t) \ge 0$ if $\beta = 0$; $\int_t^0 e^{-(\alpha/\beta)s}\xi(s)ds \ge 0$, $\xi(0) \ge 0$ if $\beta > 0$; $\eta(t) \ge 0$ if $\delta = 0$; $\int_1^t e^{(\gamma/\delta)s}\eta(s)ds \ge 0$ if $\delta > 0$.

Lemma 1.1. ([4]) *Assume that X is a Banach space and $K \subset X$ is a cone in X; Ω_1, Ω_2 are open subsets of X, and $0 \in \overline{\Omega}_1 \subset \Omega_2$. Furthermore, let $\Phi : K \bigcap(\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator satisfying one of the following conditions:*

(i) *$\|\Phi(x)\| \le \|x\|$, for all $x \in K \bigcap \partial\Omega_1$; $\|\Phi(x)\| \ge \|x\|$, for all $x \in K \bigcap \partial\Omega_2$;*

(ii) *$\|\Phi(x)\| \le \|x\|$, for all $x \in K \bigcap \partial\Omega_2$; $\|\Phi(x)\| \ge \|x\|$, for all $x \in K \bigcap \partial\Omega_1$.*

Then there is a fixed point of Φ in $K \bigcap(\overline{\Omega}_2 \setminus \Omega_1)$.

2. Main Results

Firstly, we give the definitions of solution and positive solution.

Definition 2.1. We say a function $y(t)$ is a solution (1.1) if:

(1) $y(t)$ is continuous on $[0, b]$.

(2) $y(t) = y(-\tau; t)$ for $t \in [-\tau, 0]$, where $y(-\tau; t) : [-\tau, 0] \to [0, +\infty)$ is defined as

$$y(-\tau; t) = \begin{cases} e^{(\alpha/\beta)t}(\frac{1}{\beta}\int_t^0 e^{-(\alpha/\beta)s}\xi(s)ds + y(0)), & \beta > 0, \\ \frac{1}{\alpha}\xi(t), & \beta = 0, \end{cases}$$

and $y(-\tau; t)$ satisfies $\alpha y(-\tau; 0) = \alpha y(0) = \xi(0)$ if $\beta > 0$.

(3) $y(t) = y(b; t)$ for $t \in [1, b]$, where $y(b; t) : [1, b] \to [0, +\infty)$ is defined as

$$y(b; t) = \begin{cases} e^{-(\gamma/\delta)t}(\frac{1}{\delta}\int_1^t e^{(\gamma/\delta)s}\eta(s)ds + e^{(\gamma/\delta)}y(1)), & \delta > 0, \\ \frac{1}{\gamma}\eta(t), & \delta = 0. \end{cases}$$

(4) $(\phi_p(y'))' = -r(t)f(y^t)$ for $t \in (0, 1)$ almost everywhere.

Furthermore, a solution $y(t)$ of (1.1) is called as a positive solution if $y(t) > 0$ for $t \in (0, 1)$.

In what follows, we shall show the results in this paper only for the case $\beta > 0$, $\delta = 0$, since the other situations could be discussed similarly. Suppose that $y(t)$ is a solution of (1.1), then it can be written as

$$y(t) = \begin{cases} e^{(\alpha/\beta)t}(\frac{1}{\beta}\int_t^0 e^{-(\alpha/\beta)s}\xi(s)ds + y(0)), & -\tau \le t \le 0, \\ \int_t^1 \phi_q[\int_0^s r(u)f(y^u)du]ds, & 0 \le t \le 1, \\ \frac{1}{\gamma}\eta(t), & 1 \le t \le b. \end{cases} \tag{2.1}$$

Suppose that $x_0(t)$ is the solution of (1.1) with $f \equiv 0$, then it can be expressed as

$$x_0(t) = \begin{cases} \frac{1}{\beta}e^{(\alpha/\beta)t}\int_t^0 e^{-(\alpha/\beta)s}\xi(s)ds, & -\tau \le t \le 0, \\ 0, & 0 \le t \le 1, \\ \frac{1}{\gamma}\eta(t), & 1 \le t \le b. \end{cases}$$

If $y(t)$ is a solution of BVP (1.1) and $x(t) = y(t) - x_0(t)$, noting that $x(t) = y(t)$ for $0 \le t \le 1$, then we have from (2.1) that

$$x(t) = \begin{cases} e^{(\alpha/\beta)t}x(0), & -\tau \le t \le 0, \\ \int_t^1 \phi_q[\int_0^s r(u)f(x^u + x_0^u)du]ds, & 0 \le t \le 1, \\ 0, & 1 \le t \le b. \end{cases} \tag{2.2}$$

By (H2), we can choose a $\sigma \in (0, \min\{\frac{1}{4}, \frac{1-a-\tau}{2}\})$ such that

$$T := \int_{E_\sigma} \phi_q\big[\int_{\sigma+\tau}^t r(u)du\big]dt > 0,$$

where $E_\sigma := \{t \in E; \sigma \le t + \theta \le 1 - \sigma \text{ for } -\tau \le \theta \le a\} = [\sigma + \tau, 1 - \sigma - a] \subset E$. Note that when $t \in E_\sigma$, we have $x_0^t = 0 \in C$.

Let K be a cone in the Banach space $X = C[-\tau, b]$ defined by

$$K = \{x \in C[-\tau, b] : x(t) \ge g(t)\|x\|,\ t \in [-\tau, b]\},$$

where $\|x\| := \sup\{|x(t)| : -\tau \le t \le b\}$ and

$$g(t) := \begin{cases} e^{(\alpha/\beta)t}, & -\tau \le t \le 0, \\ 1 - t, & 0 \le t \le 1, \\ 0, & 1 \le t \le b. \end{cases}$$

Define $\Phi : K \to C[-\tau, b]$ as

$$(\Phi x)(t) = \begin{cases} e^{(\alpha/\beta)t}\int_0^1 \phi_q[\int_0^s r(u)f(x^u + x_0^u)du]ds, & -\tau \le t \le 0, \\ \int_t^1 \phi_q[\int_0^s r(u)f(x^u + x_0^u)du]ds, & 0 \le t \le 1, \\ 0, & 1 \le t \le b. \end{cases} \tag{2.3}$$

Under assumptions (H1)-(H4), BVP (1.1) has a solution if and only if $\Phi x(t) = x(t)$.

We define $\|x\|_{[0,1]} = \sup\{|x(t)| : 0 \le t \le 1\}$, then we have $\|\Phi x\| = \|\Phi x\|_{[0,1]}$. It follows from (2.3) that

$$\|\Phi x\|_{[0,1]} = \int_0^1 \phi_q[\int_0^s r(u)f(x^u + x_0^u)du]ds \le \phi_q[\int_0^1 r(u)f(x^u + x_0^u)du]. \tag{2.4}$$

In the following, we express $T_\alpha = \{x \in C[0, b] : \|x\| < \alpha\}$.

Lemma 2.2. *With the above notation, $\Phi(K) \subset K$.*

Proof. For $-\tau \leq t \leq 0$, one has $0 \leq (\Phi x)(t) \leq (\Phi x)(0)$. Thus we get $\|\Phi x\| = \|\Phi x\|_{[0,1]}$ and $(\Phi x)(t) \geq e^{(\alpha/\beta)t}\|\Phi x\|$.

For $0 \leq t \leq 1$, $x \in K$, we obtain from (2.3) and (2.4) that $(\Phi x)(t) \geq 0, (\Phi x)(1) = 0$, and

$$\|\Phi x\| = (\Phi x)(0) = \int_0^1 \phi_q[\int_0^s r(u)f(x^u + x_0^u)du]ds.$$

Let $U(t) = (\Phi x)(t) - (1-t)\|\Phi x\|$, then $U(0) = U(1) = 0$, $U''(t) \leq 0$, for all $t \in [0,1]$. So $U(t) \geq 0$, for all $t \in [0,1]$, i.e., $(\Phi x)(t) \geq (1-t)\|\Phi x\|$ for $t \in [0,1]$. For $1 \leq t \leq b$, $x \in K$, one has $(\Phi x)(t) = 0 = g(t)$. Furthermore, for $-\tau \leq t \leq b$, $x \in K$, we have that $(\Phi x)(t) \geq g(t)\|\Phi x\|$, that is $\Phi(K) \subset K$. $\square$

Lemma 2.3. *The function $\Phi : K \to K$ is completely continuous.*

Proof. We can obtain the continuity of Φ from the continuity of f. In fact, suppose that x_n, $x \in K$ and $\|x_n - x\| \to 0$ as $n \to \infty$, then we get

$$\|x_n^u - x^u\| = \sup_{-\tau \leq \theta \leq a} |x_n(u+\theta) - x(u+\theta)| \to 0, \ u \in [0,1].$$

Thus, for $t \in [-\tau, b]$ we have from (2.3) that

$$|(\Phi x_n)(t) - (\Phi x)(t)| \leq \max_{0 \leq u \leq 1} |f^{q-1}(x_n^u + x_0^u) - f^{q-1}(x^u + x_0^u)|\phi_q[\int_0^1 r(u)du],$$

This implies that $\|\Phi x_n - \Phi x\| \to 0$ as $n \to \infty$.

Now let $A \subset K$ be a bounded subset of K and $M > 0$ is the constant such that $\|x\| \leq M$ for $x \in A$. Define a set S as

$$S = \{\varphi \in C^+; \|\varphi\|_C \leq M\}.$$

Then, we have

$$\|\Phi x\| \leq \max_{\varphi \in S} f^{q-1}(\varphi + \max_{u \in [0,1]} x_0^u) \int_0^1 \phi_q[\int_0^s r(u)du]ds < \infty,$$

which implies the boundedness of $\Phi(A)$. It is easy to see that $\Phi x \in C^1[-\tau, 1] \bigcap C[-\tau, b]$.

Furthermore, we have for $-\tau \leq t \leq 0$,

$$\begin{aligned} (\Phi x)'(t) &= \frac{\alpha}{\beta} e^{(\alpha/\beta)t} \int_0^1 \phi_q[\int_0^s r(u)f(x^u + x_0^u)du]ds \\ &\leq \frac{\alpha}{\beta}\|\Phi x\| \leq \frac{\alpha}{\beta} \max_{\varphi \in S} f^{q-1}(\varphi + \max_{u \in [0,1]} x_0^u)\phi_q[\int_0^1 r(u)du] =: L_1; \end{aligned}$$

for $0 \leq t \leq 1$,

$$\begin{aligned} 0 \leq |(\Phi x)'(t)| &= \phi_q[\int_0^t r(u)f(x^u + x_0^u)du] \\ &\leq \phi_q[\int_0^1 r(u)f(x^u + x_0^u)du] \\ &\leq \max_{\varphi \in S} f^{q-1}(\varphi + \max_{u \in [0,1]} x_0^u)\phi_q[\int_0^1 r(u)du] =: L_2. \end{aligned}$$

For $1 \leq t \leq b$, we have that $(\Phi x)'(t) = 0$. Thus, suppose that $x \in A$. $\forall\, \varepsilon > 0$, let $\delta = \frac{\varepsilon}{\max\{L_1, L_2\}}$, for $t_1, t_2 \in [-\tau, b]$, $|t_1 - t_2| < \delta$, one has

$$|(\Phi x)(t_1) - (\Phi x)(t_2)| \leq \max\{L_1, L_2\}|t_1 - t_2| < \varepsilon.$$

That is to say that $\Phi : K \to K$ is completely continuous. $\square$

Let

$$C^* = \{\varphi \in C^+ : \ 0 < \sigma\|\varphi\|_C \leq \varphi(\theta), \theta \in [-\tau, a]\};$$

$$\lambda = [\int_0^1 \phi_q[\int_0^s r(u)du]ds]^{-1};$$

$$\mu = [\sigma \int_{E_\sigma} \phi_q[\int_{\sigma+\tau}^s r(u)du]ds]^{-1} = \frac{1}{\sigma T};$$

$$\nu = [2(1 + \|x_0\|)\phi_q[\int_0^1 r(u)du]]^{-1}.$$

For the next theorem we set the condition

(H5)

$$\lim_{\varphi\in C^*, \|\varphi\|_C \downarrow 0} \frac{f(\varphi)}{\|\varphi\|_C^{p-1}} > \mu^{p-1}, \quad \lim_{\|\varphi\|_C \uparrow \infty} \frac{f(\varphi)}{\|\varphi\|_C^{p-1}} < \nu^{p-1}.$$

Theorem 2.4. *Under assumption (H5), BVP* (1.1) *has at least a positive solution.*

Proof. From assumption (H5), there exists a $\rho_1 > 0$ such that

$$f(\varphi) \geq (\mu\|\varphi\|_C)^{p-1}, \quad \varphi \in C^*, \quad \|\varphi\|_C \leq \rho_1.$$

For $x \in K$ and $\|x\| = \rho_1$, we have $\|x^u\|_C \leq \rho_1$ for $u \in [0, 1]$. Furthermore, we have $x^u \in C^*$ for $u \in E_\sigma$ and

$$\|x^u\|_C \geq \sigma\|x\| = \sigma\rho_1, \quad u \in E_\sigma. \tag{2.5}$$

For $u \in E_\sigma$, we have $x_0^u = 0$. Thus, we obtain

$$\begin{aligned}
\|\Phi x\| &= \int_0^1 \phi_q[\int_0^s r(u)f(x^u + x_0^u)du]ds \\
&\geq \int_{E_\sigma} \phi_q[\int_{\sigma+\tau}^s r(u)f(x^u)du]ds \\
&\geq \mu \int_{E_\sigma} \phi_q[\int_{\sigma+\tau}^s r(u)\|x^u\|_C^{p-1}du]ds \\
&\geq \mu\sigma\rho_1 \int_{E_\sigma} \phi_q[\int_{\sigma+\tau}^s r(u)du]ds \\
&= \mu\sigma\rho_1 T \\
&\geq \rho_1 = \|x\|,
\end{aligned} \tag{2.6}$$

which implies that

$$\|\Phi x\| = \|\Phi x\|_{[0,1]} \geq \|x\|, \quad \forall\, x \in K \bigcap \partial\Omega_1,$$

where $\Omega_1 = T_{\rho_1}$. On the other hand, since $\lim_{\|\varphi\|_C \uparrow \infty} \frac{f(\varphi)}{\|\varphi\|_C^{p-1}} < \nu^{p-1}$, there exists $N > \rho_1$, such that

$$f(\varphi) \leq (\nu\|\varphi\|_C)^{p-1}, \quad \varphi \in C^+, \|\varphi\|_C > N.$$

Choose a positive constant ρ_2 such that

$$\rho_2 > 1 + \max\{f^{q-1}(\varphi) : 0 \leq \|\varphi\|_C \leq N + \|x_0\|\}\phi_q[\int_0^1 r(u)du].$$

For $x \in K$, $\|x\| = \rho_2$, we have, from the facts: $x_0(t) \geq 0$, $x(t) \geq 0$ for $t \in [-\tau, b]$, that for $u \in [0, 1]$,

$$\|x^u + x_0^u\|_C \geq \|x^u\|_C > N, \text{ if } \|x^u\|_C > N,$$
$$\|x^u + x_0^u\|_C \leq \|x^u\|_C + \|x_0^u\|_C \leq N + \|x_0\|, \text{ if } \|x^u\|_C \leq N.$$

We have

$$\begin{aligned}
\|\Phi x\| =& \int_0^1 \phi_q[\int_0^s r(u) f(x^u + x_0^u) du] ds \\
\leq& \int_0^1 \phi_q[\int_0^1 r(u) f(x^u + x_0^u) du] ds \\
=& \phi_q[\int_0^1 r(u) f(x^u + x_0^u) du] \\
=& \phi_q[\int_{\|x^u\|_C > N} r(u) f(x^u + x_0^u) du + \int_{0 \leq \|x^u\|_C \leq N} r(u) f(x^u + x_0^u) du] \\
\leq& \max\{\nu \|x^u + x_0^u\|_C, \max\{f^{q-1}(\varphi) : \ 0 \leq \|\varphi\|_C \leq N + \|x_0\|\}\} \phi_q[\int_0^1 r(u) du] \\
\leq& \max\{\nu \|x + x_0\|, \max\{f^{q-1}(\varphi) : \ 0 \leq \|\varphi\|_C \leq N + \|x_0\|\}\} \phi_q[\int_0^1 r(u) du] \\
\leq& \max\{\frac{1}{2}\|x\| + \frac{1}{2}, \ \max\{f^{q-1}(\varphi) \phi_q[\int_0^1 r(u) du] : \ 0 \leq \|\varphi\|_C \leq N + \|x_0\|\}\} \\
<& \|x\| = \rho_2,
\end{aligned}$$

which implies that

$$\|\Phi x\| = \|\Phi x\|_{[0,1]} < \|x\|, \forall \ x \in K \bigcap \partial\Omega_2,$$

where $\Omega_2 = T_{\rho_2}$. By the second part of Lemma 1.1, it follows that Φ has a fixed point $x \in K \bigcap (\overline{\Omega}_2 \setminus \Omega_1)$ such that

$$0 < \rho_1 \leq \|x\| = \|x\|_{[0,1]} \leq \rho_2.$$

Suppose that $x(t)$ is the fixed point of Φ in $K \bigcap (\overline{\Omega}_2 \setminus \Omega_1)$, then

$$x(t) = \begin{cases} e^{(\alpha/\beta)t} x(0), & -\tau \leq t \leq 0, \\ \int_t^1 \phi_q[\int_0^s r(u) f(x^u + x_0^u) du] ds, & 0 \leq t \leq 1, \\ 0, & 1 \leq t \leq b. \end{cases}$$

Let $y(t) = x(t) + x_0(t)$. By the facts $0 < \rho_1 \leq \|x\| = \|x\|_{[0,1]} \leq \rho_2$, $x(t) \in K$ and $x_0(t) \geq 0$, we conclude that $y(t)$ is a positive solution of BVP (1.1). $\square$

In what follows, we shall consider the existence of multiple positive solutions for BVP (1.1).

For the next theorem we have the following hypotheses:

(H6)

$$\lim_{\varphi \in C^*, \|\varphi\|_C \downarrow 0} \frac{f(\varphi)}{\|\varphi\|_C^{p-1}} > \mu^{p-1}; \quad \lim_{\varphi \in C^*, \|\varphi\|_C \uparrow \infty} \frac{f(\varphi)}{\|\varphi\|_C^{p-1}} > \mu^{p-1};$$

(H7) There exists a $p_1 > 0$ such that for all $0 \le \|\varphi\|_C \le p_1 + p_0$, one has $f(\varphi) \le (\lambda p_1)^{p-1}$, where
$$p_0 = \max\{\max_{-\tau \le t \le 0} \frac{1}{\beta} e^{\frac{\alpha}{\beta}t} \int_t^0 e^{-\frac{\alpha}{\beta}s}\xi(s)ds, \max_{1 \le t \le b} \frac{1}{\gamma}\eta(t)\}.$$

Theorem 2.5. *Under assumptions (H6)-(H7), BVP* (1.1) *has at least two positive solutions* $y_1,\ y_2$ *such that* $0 < \|y_1\|_{[0,1]} < p_1 < \|y_2\|_{[0,1]}$.

Proof. (H6), there exists a $r : 0 < r < p_1$ such that
$$f(\varphi) \ge (\mu\|\varphi\|_C)^{p-1}, \quad \|\varphi\|_C \le r, \varphi \in C^*,$$
For $x \in K,\ \|x\| = r$, similar to (2.5) one has $x^u \in C^*$ and
$$r \ge \|x^u\|_C \ge \sigma\|x\| = \sigma r, \quad u \in E_\sigma.$$
Also we obtain an analogous inequality $\|\Phi(x)\| \ge r = \|x\|$, which implies $\|\Phi x\| = \|\Phi x\|_{[0,1]} \ge \|x\|$, for all $x \in K \bigcap \partial T_r$.

On the other hand, from (H6) there exists a $R > p_1$ such that
$$f(\varphi) \ge (\mu\|\varphi\|_C)^{p-1}, \quad \|\varphi\|_C \ge \sigma R,\ \varphi \in C^*.$$
For $x \in K,\ \|x\| = R$, we have
$$x^u \in C^* \quad \text{and } \|x^u\|_C \ge \sigma\|x\| = \sigma R \quad \text{for } u \in E_\sigma.$$
Furthermore, $\|\Phi(x)\| \ge R = \|x\|$, which implies that $\|\Phi x\| \ge \|x\|$ for $\forall\ x \in K \bigcap \partial T_R$.

Now, by (H_7), for all $x \in \partial T_{p_1}$, one has
$$\begin{aligned}\|\Phi x\| &= \int_0^1 \phi_q[\int_0^s r(u)f(x^u + x_0^u)du]ds \\ &\le \lambda p_1 \int_0^1 \phi_q[\int_0^s r(u)du]ds = p_1 = \|x\|.\end{aligned}$$
According to Lemma 1.1, it follows that Φ has two fixed points x_1, x_2 such that $x_1 \in K \bigcap \overline{T}_{p_1} \setminus T_r,\ x_2 \in K \bigcap \overline{T}_R \setminus T_{p_1}$, that is $0 < \|x_1\| < p_1 < \|x_2\|$. Since $x_i (i = 1, 2) \in K$, we have $x_i(t) > 0$, for all $t \in (0, 1)$, $i = 1, 2$. Let $y_1 = x_1 + x_0$, $y_2 = x_2 + x_0$, then y_1, y_2 are positive solutions of BVP (1.1) satisfying $0 < \|y_1\|_{[0,1]} < p_1 < \|y_2\|_{[0,1]}$. □

The following Corollaries are obvious.

Corollary 2.6. *Under the conditions*
$$\lim_{\varphi \in C^*, \|\varphi\|_C \downarrow 0} \frac{f(\varphi)}{\|\varphi\|_C^{p-1}} = +\infty, \quad \lim_{\|\varphi\|_C \uparrow \infty} \frac{f(\varphi)}{\|\varphi\|_C^{p-1}} = 0,$$
BVP (1.1) *has at least a positive solution.*

Corollary 2.7. *Assume the conditions:*
$$\lim_{\varphi \in C^*, \|\varphi\|_C \downarrow 0} \frac{f(\varphi)}{\|\varphi\|_C^{p-1}} = +\infty; \quad \lim_{\varphi \in C^*, \|\varphi\|_C \uparrow \infty} \frac{f(\varphi)}{\|\varphi\|_C^{p-1}} = +\infty;$$
and that there exists a $p_1 > 0$ *such that for all* $0 \le \|\varphi\|_C \le p_1 + p_0$, *one has* $f(\varphi) \le (\lambda p_1)^{p-1}$, *where*
$$p_0 = \max\{\max_{-\tau \le t \le 0} \frac{1}{\beta} e^{\frac{\alpha}{\beta}t} \int_t^0 e^{-\frac{\alpha}{\beta}s}\xi(s)ds, \max_{1 \le t \le b} \frac{1}{\gamma}\eta(t)\}.$$

Then BVP (1.1) *has at least two positive solutions* y_1, y_2 *such that* $0 < \|y_1\|_{[0,1]} < p_1 < \|y_2\|_{[0,1]}$.

3. Examples

Example 3.1. Consider the boundary-value problem

$$\begin{gathered}(|y'|^{p-2}y')' + r(t)y^{\frac{1}{2}}(t-\frac{1}{3}) = 0, \quad 0<t<1,\\ y(t) = \xi(t), \quad -\frac{1}{3} \le t \le 0, \quad y(1)=0.\end{gathered} \tag{3.1}$$

where $\alpha = \gamma = 1$, $\beta = \delta = 0$, $r > 0$ is a constant, $\xi(t)$ is continuous on $[-\frac{1}{3}, 0]$, $\xi(t) > 0$, $\tau = \frac{1}{3}$, $a = 0$, $E = [\frac{1}{3}, 1]$, $p > \frac{3}{2}$, and $f(\varphi) = \varphi^{\frac{1}{2}}(-\frac{1}{3})$. If $\|\varphi\|_C \to +\infty$ we have

$$\frac{f(\varphi)}{\|\varphi\|_C^{p-1}} = \frac{\varphi^{1/2}(-1/3)}{\|\varphi\|_C^{p-1}} \le \frac{\|\varphi\|_C^{\frac{1}{2}}}{\|\varphi\|_C^{p-1}} = \|\varphi\|_C^{\frac{3-2p}{2}} \to 0.$$

That is to say that $\lim_{\|\varphi\|_C \uparrow \infty} \frac{f(\varphi)}{\|\varphi\|_C^{p-1}} = 0$ holds. On the other hand, suppose that $\varphi \in C^*$, then $\varphi(\theta) \ge \sigma\|\varphi\|_C$, thus, if $\|\varphi\|_C \to 0$ we get

$$\frac{f(\varphi)}{\|\varphi\|_C^{p-1}} = \frac{\varphi^{1/2}(-1/3)}{\|\varphi\|_C^{p-1}} \ge \frac{\sigma^{\frac{1}{2}}\|\varphi\|_C^{\frac{1}{2}}}{\|\varphi\|_C^{p-1}} = \sigma^{\frac{1}{2}}\|\varphi\|_C^{\frac{3-2P}{2}} \to +\infty.$$

That is to say that $\lim_{\varphi \in C^*, \|\varphi\|_C \downarrow 0} \frac{f(\varphi)}{\|\varphi\|_C^{p-1}} = +\infty$ holds. According to Corollary 2.6, it follows that BVP (3.1) has at least a positive solution $y(t)$.

Example 3.2. Consider the boundary-value problem

$$\begin{gathered}(|y'|^{p-2}y')' + r[y^{\frac{1}{9}}(t-\frac{1}{3}) + y^{\frac{1}{3}}(t-\frac{1}{3})] = 0,\ 0<t<1,\\ y(t) = \xi(t),\ -\frac{1}{3} \le t \le 0,\ y(1) = 0.\end{gathered} \tag{3.2}$$

where $\alpha = \gamma = 1$, $\beta = \delta = 0$, r is a positive constant, $\xi(t)$ is continuous on $[-\frac{1}{3}, 0]$, $\xi(t) > 0$, $m_0 = \max_{-\frac{1}{3} \le t \le 0} \xi(t) \ne 0$, $\frac{10}{9} < p = \frac{7}{6} < \frac{4}{3}$, $\frac{1}{p} + \frac{1}{q} = 1$ and

$$f(\varphi) = \varphi^{1/9}(-\frac{1}{3}) + \varphi^{1/3}(-\frac{1}{3}).$$

Here, $\tau = \frac{1}{3}$, $a = 0$, $E = [\frac{1}{3}, 1]$.

Suppose that $\varphi \in C^*$, then $\varphi(\theta) \ge \sigma\|\varphi\|_C$, thus, if $\|\varphi\|_C \to 0$ or $\|\varphi\|_C \to +\infty$ we get

$$\begin{aligned}\frac{f(\varphi)}{\|\varphi\|_C^{p-1}} &= \frac{\varphi^{1/9}(-1/3) + \varphi^{1/3}(-1/3)}{\|\varphi\|_C^{p-1}}\\ &\ge \frac{\sigma^{\frac{1}{9}}\|\varphi\|_C^{\frac{1}{9}} + \sigma^{\frac{1}{3}}\|\varphi\|_C^{\frac{1}{3}}}{\|\varphi\|_C^{p-1}}\\ &= \sigma^{\frac{1}{9}}\|\varphi\|_C^{\frac{10-9p}{9}} + \sigma^{\frac{1}{3}}\|\varphi\|_C^{\frac{4-3p}{3}} \to +\infty.\end{aligned}$$

We deduce that

$$\lambda = \Big[\int_0^1 \phi_q(\int_0^s r du) ds\Big]^{-1} = \frac{q}{r^{q-1}},$$

then for all $m > 0$ and $0 \le \|\varphi\|_C \le m + m_0$, one has

$$0 \le f(\varphi) \le (m+m_0)^{\frac{1}{9}} + (m+m_0)^{\frac{1}{3}} = (m+m_0)^{\frac{1}{9}}(m^{1-p} + \frac{(m+m_0)^{\frac{2}{9}}}{m^{p-1}})m^{p-1}.$$

Define $H(m) = (m+m_0)^{1/9}(m^{1-p} + \frac{(m+m_0)^{2/9}}{m^{p-1}})$, then

$$\lim_{m\to 0} H(m) = +\infty, \quad \lim_{m\to+\infty} H(m) = +\infty. \tag{3.3}$$

Suppose that r and m_0 satisfy

$$(2m_0)^{\frac{1}{9}}(m_0^{-1/6} + 2^{2/9}m_0^{1/18}) < \frac{q^{p-1}}{r},$$

then $H(m_0) = (2m_0)^{1/9}(m_0^{-1/6} + 2^{2/9}m_0^{1/18}) < \lambda^{p-1} = \frac{q^{p-1}}{r}$ holds. By the continuity of $H(m)$ and (3.3), we can find an $m > 0$ (for example $m = m_0$) such that $f(\varphi) \le H(m)m^{p-1} < (\lambda m)^{p-1}$ for $0 \le \|\varphi\|_C \le m + m_0$. By Corollary 2.7, we know that BVP (3.2) has at least two positive solutions.

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Chang-Xiu Song

School of Mathematical Sciences, South China of Normal University, Guangzhou, 510631, China;

School of Applied Mathematics, Guangdong University of Technology, Guangzhou, 510006, China

E-mail address: scx168@sohu.com