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BOUNDS AND CRITICAL PARAMETERS FOR A CLASS OF NON-LOCAL PROBLEMS

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Abstract. A non-local elliptic equation, for which comparison methods are applicable, associated with Robin boundary conditions is considered. Upper and lower solutions for this problem are obtained by solving algebraic equations. These upper and lower solutions are used to obtain analytical bounds for the critical (blow-up) parameter of the problem. Numerical results are presented for the slab, cylindrical and spherical geometries. The results are compared with the existing ones in the literature.

1. Introduction

The non-local problem

$$u_t = \nabla^2 u + \frac{\lambda f(u)}{\big(\int_\Omega f(u)\,dx\big)^p}, \quad x \in \Omega \subset \mathbb{R}^N,\ N \geq 1,\ t > 0, \tag{1.1}$$

$$\frac{\partial u(x,t)}{\partial \nu} + \beta\, u(x,t) = 0, \quad x \in \partial\Omega,\ t > 0, \tag{1.2}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.3}$$

where $0 < \beta < \infty$ and $p > 0$ is connected with a variety of applications. In particular for $p = 2$ problem (1.1)-(1.3) describes the operation of a device is flowed by an electric current, e.g. thermistors, fuse wires, electric arcs and fluorescent lights [12, 13], resulting Ohmic heating, with Newtonian cooling imposed on the boundary. In the case of a nonlinear conductor problem (1.1)-(1.3) with $p > 1$, can be derived to describe the thermo-electric flow in the conductor, [11]. Besides, for $p = 1$ the same model can describe phenomena associated with the occurrence of shear bands in metals being deformed under high strain rates [2]-[4], in the theory of gravitational equilibrium of polytropic stars [10], in the investigation of the fully turbulent behaviour of real flows, using invariant measures for the Euler equation [5], in modelling aggregation of cells via interaction with a chemical substance (chemotaxis), [15].

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The key-problem for the study of (1.1)-(1.3) is the corresponding steady-state problem

$$\nabla^2 w + \mu f(w) = 0, \quad x \in \Omega, \tag{1.4}$$

$$\frac{\partial w(x)}{\partial \nu} + \beta\, w(x) = 0, \quad x \in \partial\Omega, \tag{1.5}$$

where $\mu = \lambda / \big(\int_\Omega f(w)\, dx \big)^p$. The existence of a critical parameter $0 < \lambda^* < \infty$ such that problem (1.4)-(1.5) has at least one solution for $0 < \lambda < \lambda^*$ and no solution for $\lambda > \lambda^*$, indicates the ocurence of a singular behaviour of the solution of the time-dependent problem (1.1)-(1.3) above this critical value. More precisely, the phenomenon of finite-time blow-up, i.e. $\|u(\cdot,t)\|_\infty \to \infty$ as $t \to t^* < \infty$, occurs for $\lambda > \lambda^*$, see for example [2, 7, 9, 12, 13, 14]. Also from the point of view of applications it is very important to derive some estimates of the blow-up time. But the most useful (upper, lower, asymptotical) estimates of blow-up time are provided in terms of λ^*, see [7, 8]. Consequently, either the determination of the critical parameter λ^*, when it is possible, or the computation of some upper and lower estimates become very important.

Some times the computation of λ^* is rather simple, see for example [12, 13, 14], where λ^* is calculated for Dirichlet boundary conditions in the one-dimensional and two-dimensional radial symmetric cases but only for $p = 2$. In higher dimensions and asymetric cases, the proof of the existence of λ^* it is not so easy even considering some special functions f, [2, 6]. Moreover in [2], where the steady-state problem (1.4)-(1.5) is studied in detail, some estimates of λ^* are derived covering mainly the Dirichlet boundary conditions, while for the Robin problem only the existence of λ^* is obtained. Some upper estimates for the Robin problem, when f is a deceasing function, have been obtained in [7]. It is worth noting that for the Neumann problem we have $\lambda^* = 0$, i.e. the steady-state problem (1.4)-(1.5) has no solutions for every $\lambda > 0$. This a direct consequence of the maximum principle.

Here, we investigate the two special cases $f(s) = e^{-s}$ and $f(s) = (1+s)^{-q}, q > 0$, dealing only with the Robin problem. First, we derive some lower and upper estimates of λ^* for a general domain Ω and then focusing on some special geometries we improve these estimates by using proper approximations. These estimates improve those obtained in [7, 13, 14], at least for the geometries we checked. Our approach is based on comparison arguments, that can be applied for problem (1.4)-(1.5) only when f is decreasing, and it is quite similar to the approach used in [1].

2. General results

Let now write the steady-state problem in the form

$$\nabla^2 w + \frac{\lambda}{h(w)} f(w) = 0, \quad x \in \Omega, \tag{2.1}$$

$$\frac{\partial w}{\partial n} + \beta w = 0, \quad x \in \partial\Omega, \tag{2.2}$$

where $h(w) = \big(\int_\Omega f(w) dx \big)^p$.

When f is a decreasing function, we have a variant of the comparison results that apply to more usual elliptic problems, [12]. So in this case we can define the notion of lower and upper solutions.

Definition 2.1. A function ϕ is a lower solution of (2.1)-(2.2) if it satisfies
$$P(\phi) := \nabla^2\phi + \frac{\lambda}{h(\phi)} f(\phi) \geq 0,$$
$$B(\phi) := \frac{\partial\phi}{\partial n} + \beta\phi \leq 0.$$
Analogously z is an upper solution of (2.1)-(2.2) if $P(z) \leq 0$ and $B(z) \geq 0$.

Let ψ be the solution of the problem
$$\nabla^2\psi = -1, \quad x \in \Omega, \tag{2.3}$$
$$\frac{\partial\psi}{\partial n} + \beta\psi = 0, \quad x \in \partial\Omega, \tag{2.4}$$
and $M = \max_{x\in\Omega} \psi(x) > 0$, $m = \min_{x\in\Omega} \psi(x) > 0$, then we infer the following result.

Now we provide a method to construct upper and lower solutions to problem (2.1)-(2.2).

Proposition 2.2. *Let $f_1(s)$ and $f_2(s)$, be such that $f_1(s) \leq h(s\psi) \leq f_2(s)$. Let k and c, respectively, be the solutions to*
$$-k + \frac{\lambda}{f_1(k)} f(km) = 0, \tag{2.5}$$
$$-c + \frac{\lambda}{f_2(c)} f(cM) = 0. \tag{2.6}$$
Then $z = k\psi$ and $\phi = c\psi$ are upper and lower solutions of (2.1)-(2.2), respectively.

Proof. From the definition of z and ϕ we have
$$P(k\psi) = -k + \frac{\lambda}{h(k\psi)} f(k\psi) \leq -k + \frac{\lambda}{f_1(k)} f(k\,m) = 0,$$
$$P(c\psi) = -c + \frac{\lambda}{h(c\psi)} f(c\psi) \geq -c + \frac{\lambda}{f_2(c)} f(c\,M) = 0,$$
and the result is obtained, since also $B(z) = B(\phi) = 0$. □

In the following we present some results that will be used through this paper.

Proposition 2.3. *Consider the equation $\frac{1}{\lambda} = g(k)$, where $g(k)$ is differentiable for $k > 0$, and $g(k) \to \infty$, as $k \to \infty$. Let k^s be the largest solution (if any) of $g'(k) = 0$, and $\frac{1}{\lambda^s} = g(k^s)$. Then*
(a) for $\lambda > \lambda^s$, and $k > k^s$, we have $\frac{1}{\lambda} \leq g(k)$, and
(b) for $\lambda \leq \lambda^s$, the equation $\frac{1}{\lambda} = g(k)$, has at least one solution.

The proof of the above proposition is straight forward using the fact that $g(k)$ is increasing for $k > k^s$.

Theorem 2.4. *Let f be a positive decreasing C^1-function and p_{cr} an exponent (if any) such that*
$$\lim_{k\to\infty} \frac{f(kM)}{k f^{p_{cr}}(km)} = \infty, \tag{2.7}$$
then for every $p > p_{cr}$, there exists $\lambda^* > 0$ such that problem (2.1)-(2.2), has at least one solution for $0 < \lambda < \lambda^*$ and no solution for $\lambda > \lambda^*$.

Proof. Recalling that $M = \max_{x\in\Omega}\psi(x) > 0$ and $m = \min_{x\in\Omega}\psi(x) > 0$, then using the definition of ψ we obtain

$$P(k\psi) \geq -k + \frac{\lambda f(kM)}{|\Omega|^p f^p(km)}.$$

Considering the function

$$g(k) = \frac{f(kM)}{|\Omega|^p k f^p(km)},$$

we infer that $g(k) \to \infty$ as $k \to 0+$, since f is positive and that $g(k) \to \infty$ as $k \to \infty$ for every $p > p_{cr}$ using (2.7). The latter with $g(k) \to \infty$ as $k \to \infty$ implies that there exists at least one solution of the equation $g'(k) = 0$. Let k_0 be the largest solution of this equation and set $\lambda_0 = 1/g(k_0)$. Now if we consider $\lambda > \lambda_0$ we obtain, in view of Proposition 2.3, that $1/\lambda \leq g(k)$ and so $P(k\psi) \geq 0$ for every $k > k_0$. Therefore for every $\lambda > \lambda_0$ we are able to construct an unbounded lower solution $k\psi$ and so the steady-state solution does not exist for $\lambda > \lambda_0$, hence $\lambda^* \leq \lambda_0$. This completes the proof. □

Remark 2.5. Regarding the critical exponent p_{cr} of Theorem 2.4 there should be $p_{cr} > 1$, otherwise

$$\lim_{k\to\infty} \frac{f(kM)}{k f^p(km)} \leq \lim_{k\to\infty} \frac{f^{1-p}(kM)}{k} = 0.$$

Remark 2.6. Note that a critical exponent p_{cr} of Theorem 2.4, exists when $-\log f(s)$ does not grow at infinity faster than algebraically, i.e. $-f'(s)/f(s) \lesssim \theta\, s^q$, $q > 0$, as $s \to \infty$, where θ is a positive constant.

In the following sections, we determine this critical exponent p_{cr} in the two special cases $f(s) = e^{-s}$ and $f(s) = (1+s)^{-q}$, $q > 0$, which provides us with some upper estimates of λ^*.

3. The exponential case

3.1. **Bounds for a general domain.** First we give a general upper bound for λ^* under the condition $pm > M$. Namely, under this condition we have

$$h(k\psi) = \Big(\int_\Omega e^{-k\psi(x)}dx\Big)^p \leq \big(e^{-km}|\Omega|\big)^p = e^{-pkm}|\Omega|^p,$$

hence

$$P(k\psi) \geq -k + \frac{\lambda}{|\Omega|^p} e^{k(pm-M)}.$$

We set

$$g(k) = \frac{e^{k(pm-M)}}{k|\Omega|^p},$$

then $g(k) \to \infty$ as $k \to \infty$ under the condition $pm > M$. The unique solution of the equation $g'(k) = 0$ is

$$k_0 = \frac{1}{pm - M}, \tag{3.1}$$

so if we consider $\lambda > \lambda_0$ where

$$\lambda_0 = \frac{1}{g(k_0)} = \frac{|\Omega|^p}{(pm-M)e}, \tag{3.2}$$

we obtain, in view of Proposition 2.3, $P(k\psi) \geq 0$ for every $k > k_0$. Therefore, we can construct an unbounded lower solution $k\psi$ to (2.1)-(2.2) and so the steady-state solution w does not exist for $\lambda > \lambda_0$ and then Theorem 2.4 implies that λ_0 is an upper bound for λ^*.

Next we find a function $f_1(k)$, see Proposition 2.2, using linear approximation and then we obtain a lower bound for λ^*. We approximate $f(w) = e^{-w}$ by a linear function as follows

$$e^{-k\psi} \geq e^{-k\,M}(1 + k\,M - k\psi),$$

and therefore,

$$h(k\psi) \geq e^{-pk\,M}\Big[(1 + k\,M)\int_\Omega dx - k\int_\Omega \psi dx\Big]^p = e^{-pk\,M}[(1 + k\,M)|\Omega| - kR]^p,$$

where $|\Omega| = \int_\Omega dx$ and $R = \int_\Omega \psi dx$. Hence

$$P(k\psi) \leq -k + \lambda \frac{e^{k(pM-m)}}{[k(M|\Omega| - R) + |\Omega|]^p}.$$

Let us now consider the function

$$g(k) = \frac{e^{k(pM-m)}}{k[k(M|\Omega| - R) + |\Omega|]^p},$$

then $g(k) \to \infty$ as $k \to \infty$ provided that $p > m/M$. $g(k)$ is also differentiable and the only positive solution k^0 of the equation $g'(k) = 0$ is given by

$$k^0 = \frac{\delta(1+p) - \gamma|\Omega| + \sqrt{[\gamma|\Omega| - \delta(1+p)]^2 + 4\gamma\delta|\Omega|}}{2\gamma\delta}, \tag{3.3}$$

where $\gamma = pM - m$ and $\delta = M|\Omega| - R$. For $\lambda \leq \lambda^0$ where

$$\lambda^0 = \frac{1}{g(k^0)} = k^0\left(\delta k^0 + |\Omega|\right)^p e^{-\gamma k^0}, \tag{3.4}$$

we derive, in view of Proposition 2.3, that $P(k^0\psi) \leq 0$ thus $k^0\psi$ is a bounded upper solution to (2.1)-(2.2) and so is the steady-state solution w. Hence λ^0 is a lower bound for λ^*.

3.2. **Bounds for the slab.** We consider the problem (2.1)-(2.2) in the slab, $-1 \leq x \leq 1$, and hence the boundary conditions reduced to

$$-w'(-1) + \beta w(-1) = 0,$$
$$w'(1) + \beta w(1) = 0.$$

For $\beta = 1$, we have $\psi(x) = \frac{1}{2}(3 - x^2)$, and $1 = m \leq \psi(x) \leq \frac{3}{2} = M$. For $p > \frac{3}{2}$, the condition $pm > M$ is satisfied and an upper bound $\lambda_0 = \frac{2^p}{(p-\frac{3}{2})e}$ for λ^* is obtained using equations (3.1) and (3.2). Also, a lower bound

$$\lambda^0 = k^0\big(\frac{1}{3}k^0 + 2\big)^p e^{-(\frac{3}{2}p-1)k^0}, \quad \text{where} \quad k^0 = \frac{7 - 8p + 3\sqrt{(\frac{7-8p}{3})^2 + 4p - \frac{8}{3}}}{3p - 2},$$

is obtained using (3.3) and (3.4). If $p = 2$, then $\lambda^0 = 0.874\cdots \leq \lambda^* \leq \lambda_0 = \frac{8}{e}$. To derive better bounds we start with

$$\int_{-1}^1 e^{\frac{k}{2}x^2}\,dx = 2\int_0^1 e^{\frac{k}{2}x^2}\,dx \leq 2\int_0^1 e^{\frac{k}{2}x}\,dx = \frac{4}{k}(e^{\frac{k}{2}} - 1),$$

hence
$$\int_{-1}^{1} e^{-k\psi(x)}\,dx = e^{-\frac{3}{2}k}\int_{-1}^{1} e^{\frac{k}{2}x^2}\,dx \le \frac{4}{k}(e^{\frac{k}{2}}-1)e^{-\frac{3}{2}k},$$
and
$$\frac{1}{h(k\psi)} = \frac{1}{\left(\int_{-1}^{1} e^{-k\psi(x)}\,dx\right)^p} \ge \frac{k^p e^{\frac{3}{2}kp}}{4^p(e^{\frac{k}{2}}-1)^p}.$$
Therefore,
$$P(k\psi) = -k+\lambda\frac{e^{-k\psi}}{h(k\psi)} \ge -k+\lambda\frac{k^p e^{\frac{3}{2}k(p-1)}}{4^p(e^{\frac{k}{2}}-1)^p}.$$
An upper bound λ_0 for λ^* is obtained by
$$\lambda_0 = \frac{1}{g_1(k_0)},$$
where k_0 is the largest solution of the equation $g_1'(k)=0$ with
$$g_1(k) = \frac{k^{p-1}e^{\frac{3}{2}k(p-1)}}{4^p(e^{\frac{k}{2}}-1)^p}.$$
Such k_0 exists for $p>\frac{3}{2}$, since $\lim_{k\to 0^+} g_1(k) = \lim_{k\to\infty} g_1(k) = \infty$. To derive a better lower bound we use the Maclaurin series
$$e^{\frac{k}{2}x^2} = \sum_{i=0}^{n}\frac{k^i}{2^i i!}x^{2i} + E_n,$$
where the error $E_n(x,\xi)\equiv E_n = e^{\xi}\frac{k^{n+1}}{2^{n+1}(n+1)!}x^{2n+2}$ and $0\le\xi\le\frac{k}{2}$. Hence
$$E_n \ge \frac{k^{n+1}}{2^{n+1}(n+1)!}x^{2n+2},$$
and
$$\begin{aligned}\int_{-1}^{1} e^{\frac{k}{2}x^2}dx &\ge \sum_{i=0}^{n}\frac{k^i}{2^i i!}\int_{-1}^{1}x^{2i}dx + \frac{k^{n+1}}{2^{n+1}(n+1)!}\int_{-1}^{1}x^{2n+2}dx\\ &= 2\sum_{i=0}^{n+1}\frac{k^i}{2^i i!}\frac{1}{2i+1}.\end{aligned}$$
Let
$$\alpha(n,k)\equiv\alpha = 2\sum_{i=0}^{n+1}\frac{k^i}{2^i i!(2i+1)},$$
then
$$\int_{-1}^{1} e^{-k\psi(x)}dx = e^{-\frac{3}{2}k}\int_{-1}^{1} e^{\frac{k}{2}x^2}dx \ge \alpha e^{-\frac{3}{2}k}$$
and
$$\frac{1}{h(k\psi)} = \frac{1}{\left(\int_{-1}^{1} e^{-k\psi(x)}dx\right)^p} \le \frac{e^{\frac{3}{2}kp}}{\alpha^p}.$$
Therefore,
$$P(k\psi) = -k+\frac{\lambda}{h(k\psi)}e^{-k\psi} \le -k+\frac{\lambda}{\alpha^p}e^{k(\frac{3}{2}p-1)}.$$

Then a lower bound λ^0 for λ^* is provided by

$$\lambda^0 = \frac{1}{g_2(k^0)},$$

where k^0 is the largest solution of the equation $g_2'(k) = 0$ with

$$g_2(k) = \frac{e^{k(\frac{3}{2}p-1)}}{k\alpha^p}.$$

For $p > 2/3$, it is clear that such k^0 exists since $\lim_{k\to 0^+} g_2(k) = \lim_{k\to\infty} g_2(k) = \infty$.

Remark 3.1. Using the above method we obtain, see also the next two subsections, that the critical exponent is $p_{cr} = 3/2$, although it is known that for $N = 1, 2$, λ^* is bounded for every $p > 1$. In other words the optimal critical exponent is $p^* = 1$, see [2].

Remark 3.2. For the slab and general β, we have $\psi(x) = -\frac{x^2}{2} + \frac{1}{\beta} + \frac{1}{2}$, with $m = \frac{1}{\beta} \le \psi \le M = \frac{1}{\beta} + \frac{1}{2}$. Then using equation (3.2) we have $\lambda_0 = \frac{|\Omega|^p}{(pm-M)e} = \frac{2^p}{\left[\frac{1}{\beta}(p-1)-\frac{1}{2}\right]e}$, provided that $p > \frac{\beta}{2} + 1$. Now, for $\beta \to 0$, we have $\lambda_0 \to 0$ and so $\lambda^* \to 0$ as well. This implies that the problem with Neumann boundary conditions has no solution regardless the value of λ, which is in agreement with what is already known, see comments in the introduction.

3.3. **Bounds for the circular cylinder.** For the cylindrical geometry where the Laplacian operator depends only on the radial, we have $\nabla^2 w = w_{rr} + \frac{1}{r}w_r$, $0 < r < 1$, and $\psi(r) = \frac{1}{4}(3 - r^2)$, satisfies

$$\psi_{rr} + \frac{1}{r}\psi_r = -1, \psi_r(0) = 0, \psi_r(1) + \psi(1) = 0.$$

By substituting $\psi(r) = \frac{1}{4}(3 - r^2)$ in the expression of $h(k\psi)$, we have

$$h(k\psi) = \Big(2\pi \int_0^1 re^{-\frac{k}{4}(3-r^2)}\,dr\Big)^p = 4^p\pi^p k^{-p} e^{-\frac{3pk}{4}}(e^{\frac{k}{4}} - 1)^p.$$

Since $\frac{1}{2} \le \psi \le \frac{3}{4}$, we obtain

$$-k + \lambda\big(\frac{1}{4\pi}\big)^p k^p e^{\frac{3k}{4}(p-1)}(e^{\frac{k}{4}} - 1)^{-p} \le P(k\psi) \le -k + \lambda\big(\frac{1}{4\pi}\big)^p k^p e^{\frac{k}{4}(3p-2)}(e^{\frac{k}{4}} - 1)^{-p}.$$

So, in view of Proposition 2.3, an upper bound of the critical parameter λ^* is provided by $\lambda_0 = 1/g_1(k_0)$, where

$$g_1(k) = (\frac{1}{4\pi})^p k^{p-1} e^{\frac{3k}{4}(p-1)}(e^{\frac{k}{4}} - 1)^{-p}$$

and k_0 is the largest solution of the equation $g_1'(k) = 0$. Such a solution exists since $\lim_{k\to 0+} g(k) = \lim_{k\to\infty} g(k) = \infty$ provided that $p > 3/2$. This means that the critical exponent for the existence of λ^* using this method, is $p_{cr} = 3/2$.

Analogously, a lower estimate of λ^* is obtained by $\lambda^0 = 1/g_1(k^0)$, where

$$g_2(k) = (\frac{1}{4\pi})^p k^{p-1} e^{\frac{k}{4}(3p-2)}(e^{\frac{k}{4}} - 1)^{-p}$$

and k^0 is the largest solution of the equation $g_2'(k) = 0$, which exists for $p > 1$.

3.4. **Bounds for the unit sphere.** For the spherical geometry where the Laplacian operator depends again only on the radial, we have $\nabla^2 w = w_{rr} + \frac{2}{r} w_r$, $0 < r < 1$. So $\psi(r) = \frac{1}{2}(1 - \frac{r^2}{3})$ satisfies

$$\psi_{rr} + \frac{2}{r}\psi_r = -1, \quad \psi_r(0) = 0, \quad \psi_r(1) + \psi(1) = 0.$$

We have

$$h(k\psi) = \Big(4\pi \int_0^1 r^2 e^{-\frac{k}{2}(1-\frac{r^2}{3})}\, dr\Big)^p = (4\pi)^p e^{-\frac{kp}{2}} \Big(\int_0^1 r^2 e^{\frac{kr^2}{6}}\, dr\Big)^p.$$

Since $0 \le r \le 1$, we derive

$$\frac{2}{k}(e^{\frac{k}{6}} - 1) = \int_0^1 r^2 e^{\frac{kr^3}{6}}\, dr \le \int_0^1 r^2 e^{\frac{kr^2}{6}}\, dr \le \int_0^1 r e^{\frac{kr^2}{6}}\, dr = \frac{3}{k}(e^{\frac{k}{6}} - 1),$$

and hence

$$(4\pi)^p e^{-\frac{kp}{2}} \frac{2}{k}(e^{\frac{k}{6}} - 1) \le h(k\psi) \le (4\pi)^p e^{-\frac{kp}{2}} \frac{3}{k}(e^{\frac{k}{6}} - 1).$$

Since $\frac{1}{3} \le \psi \le \frac{1}{2}$, we obtain

$$-k + \frac{e^{\frac{k}{2}(p-1)} k^p}{(4\pi)^p 3^p (e^{\frac{k}{6}} - 1)^p} \le P(k\psi) \le -k + \frac{e^{k(\frac{p}{2}-\frac{1}{3})} k^p}{(4\pi)^p 2^p (e^{\frac{k}{6}} - 1)^p}.$$

Then an upper bound λ_0 for λ^* is provided by

$$\lambda_0 = \frac{1}{g_1(k_0)},$$

where k_0 is the largest solution of the equation $g_1'(k) = 0$, with

$$g_1(k) = \frac{k^{p-1} e^{\frac{k}{2}(p-1)}}{12^p \pi^p (e^{\frac{k}{6}} - 1)^p}.$$

A lower bound λ^0 for λ^* is provided by

$$\lambda^0 = \frac{1}{g_2(k^0)},$$

where k^0 is the largest solution of the equation $g_2'(k) = 0$, with

$$g_2(k) = \frac{k^{p-1} e^{k(\frac{p}{2}-\frac{1}{3})}}{8^p \pi^p (e^{\frac{k}{6}} - 1)^p}.$$

One can see that $g_1(k)$ and $g_2(k)$ approach infinity as k does provided that $p > 1$ and $p > \frac{3}{2}$ respectively. It is also clear that $g_1(k)$ and $g_2(k)$ approach infinity as $k \to 0^+$. Thus the critical exponent is again $p_{cr} = 3/2$.

Table 1, presents the value of λ^0 and λ_0 for different values of p. We can see that the values of λ^0 and λ_0 increase with p for the cylindrical and spherical geometries, and so does λ^*. The same result is obtained for the slab geometry, except at $p = 3$, where the upper bound λ_0 decreases.

TABLE 1. The upper and lower estimates of λ^* for $f(w) = e^{-w}$, and different values of p

	Slab		Cylinder		Sphere	
p	λ^0	λ_0	λ^0	λ_0	λ^0	λ_0
2	0.890257	1.503823	4.886952	7.421067	13.031873	19.789512
3	0.984718	1.316214	8.330318	10.202723	29.618908	36.276347
4	1.361582	1.687503	17.964361	20.547267	85.164379	97.409264
5	2.081067	2.484099	42.968411	47.511459	271.602796	300.319344
6	3.368082	3.930889	108.984747	118.097328	918.521650	995.322337

4. The power-law case

4.1. Bounds for a general domain. Another important case from the point of view of applications is the power-law case i.e. when $f(s) = (1 + s)^{-q}$, $q > 0$. In this case the steady-state problem has the form

$$\nabla^2 w + \frac{\lambda}{h(w)(1+w)^q} = 0, \quad x \in \Omega, \tag{4.1}$$

$$\frac{\partial w}{\partial n} + \beta w = 0, \quad x \in \partial\Omega, \tag{4.2}$$

where $h(w) = \Big(\int_\Omega \frac{1}{(1+w)^q} dx\Big)^p$, $p > 0$.

First, we find some conditions should be satisfied by p and q and give some bounds of λ^* for a general domain Ω under these conditions. Then we provide some more accurate estimates of λ^* for some special geometries.

We consider again potential upper and lower solutions to problem (4.1)-(4.2) of the form $k\psi$, where ψ is the solution of the problem (2.3)-(2.4). Then we have

$$P(k\psi) = \nabla^2(k\psi) + \frac{\lambda\,(1 + k\,\psi)^{-q}}{\Big[\int_\Omega (1 + k\psi(x))^{-q}\,dx\Big]^p} \geq -k + \lambda(1 + k\,M)^{-q}(1 + k\,m)^{p\,q}|\Omega|^{-p},$$

recalling that $M = \max_{x\in\Omega} \psi(x) > 0$ and $m = \min_{x\in\Omega} \psi(x) > 0$ for $0 < \beta < \infty$. Let $g_1(k) = |\Omega|^{-p} k^{-1}\,(1+k\,M)^{-q}(1+k\,m)^{p\,q}$ then $\lim_{k\to 0+} g_1(k) = \lim_{k\to\infty} g_1(k) = \infty$ provided that $p > (1+q)/q$. So the equation $g_1'(k) = 0$ has at least one solution for $k > 0$. Let k_0 be the largest solution of this equation then if we consider $\lambda > \lambda_0$ where

$$\lambda_0 = \frac{1}{g_1(k_0)} = |\Omega|^p\,k_0\,(1 + k_0\,M)^q(1 + k_0\,m)^{-pq},$$

we obtain, in view of Proposition 2.3, $P(k\psi) \geq 0$ for every $k > k_0$. That is, we can construct an arbitrary large (for any $k > k_0$) lower solution of problem (4.1)-(4.2), for $\lambda > \lambda_0$. Hence, in view of Theorem 2.4, we derive an upper estimate for λ^* of the form

$$\lambda^* \leq \lambda_0 = |\Omega|^p\,k_0\,(1 + k_0\,M)^q\,(1 + k_0\,m)^{-p\,q}, \tag{4.3}$$

and we conclude that $p_{cr} = (q + 1)/q$, which coincides with the optimal critical exponent existing in this case, see [2].

To obtain a lower estimate of λ^* we should construct an upper solution of the steady-state problem (4.1)-(4.2). Namely, we have

$$P(k\psi) = \nabla^2(k\psi) + \frac{\lambda(1+k\psi)^{-q}}{\Big[\int_\Omega (1+k\psi(x))^{-q}\,dx\Big]^p} \le -k + \lambda(1+k\,m)^{-q}(1+k\,M)^{p\,q}|\Omega|^{-p}.$$

We consider the function $g_2(k) = |\Omega|^{-p}k^{-1}\,(1+k\,m)^{-q}(1+k\,M)^{p\,q}$, then it can be proved that the equation $g_2'(k) = 0$ has at least one solution for $k > 0$ under again the condition $p > (q+1)/q$. If k^0 is the largest solution of this equation, then regarding

$$\lambda^0 = \frac{1}{g_2(k^0)} = |\Omega|^p\, k^0\,(1+k^0\,m)^q(1+k^0\,M)^{-p\,q},$$

we derive that $P(k^0\psi) \le 0$. Thus for $\lambda = \lambda^0$, $k^0\psi$ is an upper solution of problem (4.1)-(4.2) which is bounded and so the steady state w is. This implies that λ^0 should be a lower bound for the critical parameter λ^*.

Remark 4.1. It can be observed that the critical exponent $p_{cr} = (q+1)/q \to 1$ as $q \to \infty$, for every $N \ge 1$. This agrees with the observation in [2] for the one-dimensional case. Thus if f decreases faster than any power (such as $f(s) = e^{-s}$) then $p_{cr} = 1$ and so we recover the optimal critical exponent existing in this case, see also Remark 3.1.

Remark 4.2. We can derive upper and lower estimates of the critical parameter λ^*, using the same arguments as above, for every function $f(s)$ such that $-\log f(s)$ grows at infinity at most algebraically, see also Remark 2.6.

Remark 4.3. Our method gives an upper estimate for λ^* even if $\int_0^\infty (1+s)^{-q}ds = \infty$ i.e. when $0 < q \le 1$. However, the methods used in [2, 7] provide an upper estimate only if $f(s)$ satisfies $\int_0^\infty f(s)\,ds < \infty$, see below.

4.2. **Bounds for the slab.** For the slab geometry, calculating $h(k\psi)$ instead of estimating it from above and below, we can improve the estimates obtained in the previous subsection. In this case $\psi(x) = \frac{3}{2}(1-x^2)$, and hence

$$h(k\psi) = \Big(\int_{-1}^1 \frac{1}{[1+\frac{k}{2}(3-x^2)]^q}\,dx\Big)^p.$$

By substituting $x = \sqrt{\frac{2}{k}+3}\sin(r)$, we end up with

$$h(k\psi) = 2^{p(q+1)}k^{-\frac{p}{2}}\,(2+3k)^{\frac{p}{2}-pq}\,J^p(k,q),$$

where $J(k,q) = \int_0^{\sin^{-1}\big(\frac{1}{\sqrt{\frac{2}{k}+3}}\big)} \sec^{2q-1}(r)dr$. Now,

$$-k + \frac{\lambda}{h(k\psi)(1+\frac{3}{2}k)^q} \le P(k\psi) \le -k + \frac{\lambda}{h(k\psi)(1+k)^q},$$

or

$$-k + \lambda\frac{k^{p/2}(2+3k)^{pq-p/2}}{2^{p(q+1)}(1+\frac{3}{2}k)^q J^p(k,q)} \le P(k\psi) \le -k + \lambda\frac{k^{p/2}(2+3k)^{pq-p/2}}{2^{p(q+1)}(1+k)^q J^p(k,q)}.$$

Then an upper bound for λ^* is provided by the relation $\lambda_0 = 1/g_1(k_0)$, where

$$g_1(k) = \frac{k^{p/2-1}(2+3k)^{pq-p/2}}{2^{p(q+1)}(1+\frac{3}{2}k)^q J^p(k,q)}$$

and k_0 is the largest solution of the equation $g_1'(k) = 0$.

Now k_0 exists since $g_1(k) \to \infty$ as $k \to 0+$ and $g_1(k) \sim Bk^{p/2-1}J^{-p}(k,q) \sim \Gamma k^{-1} \to \infty$ as $k \to \infty$ provided that $p > (q+1)/q$, see Lemmas 4.5 and 4.7 below.

Similarly a lower bound λ^0 for λ^* is obtained by $\lambda^0 = 1/g_2(k^0)$, where

$$g_2(k) = \frac{k^{p/2-1}(2+3k)^{pq-p/2}}{2^{p(q+1)}(1+k)^q J^p(k,q)}$$

and k^0 is the largest solution of the equation $g_2'(k) = 0$. The existence of k^0 is again guaranteed by the satisfaction of the conditions $\lim_{k\to 0+} g_2(k) = \lim_{k\to\infty} g_2(k) = \infty$ for $p > (q+1)/q$.

In the following we present some properties of the function $J(k,q)$, which help us in evaluating λ^0 and λ_0, and have been used in proving that $\lim_{k\to 0^+} g_1(k) = \lim_{k\to 0^+} g_2(k) = \infty$.

Proposition 4.4. *The function $J(k,q)$ satisfies the recursion relation*

$$J(k,q) = \frac{1}{2(q-1)}\Big[\frac{(\frac{2}{k}+3)^{q-\frac{3}{2}}}{(\frac{2}{k}+2)^{q-1}} + (2q-3)J(k,q-1)\Big], \quad q > 1. \tag{4.4}$$

Proof. Using the relation $\int \sec^n(x)dx = \frac{\sec^{n-1}(x)\sin(x)}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}(x)dx$, we have

$$\begin{aligned}
J(k,q) &= \int_0^\alpha \sec^{2q-1}(r)dr = \frac{\sec^{2q-2}(r)\sin(r)}{2q-2}\Big|_0^\alpha + \frac{2q-3}{2q-2}\int_0^\alpha \sec^{2q-3}(r)dr \\
&= \frac{1}{2(q-1)}\left[\sec^{2q-2}(\alpha)\sin(\alpha) + (2q-3)J(k,q-1)\right]
\end{aligned}$$

and the result is obtained using the facts that $\sec(\alpha) = \sqrt{\frac{\frac{2}{k}+3}{\frac{2}{k}+2}}$, and $\sin(\alpha) = \frac{1}{\sqrt{\frac{2}{k}+3}}$. □

Lemma 4.5. *If $2q-1 \in \mathbb{N}_+$, then the function $J(k,q)$ has a finite limit as $k \to \infty$, and $\lim_{k\to 0^+} J(k,q) = 0$.*

Proof. Since, $J(k,\frac{3}{2}) = \frac{1}{\sqrt{\frac{2}{k}+2}}$, has a finite limit as $k \to \infty$, the result can be obtained using induction and the recursion relation (4.4). The second statement is proved using similar arguments. □

Lemma 4.6. *If $2q-1 \in \mathbb{N}_+$, then $\frac{dJ(k,q)}{dk} \sim \frac{A}{\sqrt{k}}$ as $k \to 0^+$, for some positive constant A.*

Proof. We have $J(k,q) = \int_0^{\sin^{-1}\left(\frac{1}{\sqrt{\frac{2}{k}+3}}\right)} \sec^{2q-1}(r)dr$. Hence

$$\begin{aligned}
\frac{dJ(k,q)}{dk} &= \sec\left[\sin^{-1}\left(\frac{1}{\sqrt{\frac{2}{k}+3}}\right)\right]^{2q-1} \frac{d}{dk}\left[\sin^{-1}\left(\frac{1}{\sqrt{\frac{2}{k}+3}}\right)\right] \\
&= \left(\frac{2+3k}{2+2k}\right)^{q-\frac{1}{2}} \frac{1}{(2+3k)\sqrt{k}\sqrt{2+2k}} \\
&= \frac{(2+3k)^{q-\frac{3}{2}}}{2^q\sqrt{k}(1+k)^q}
\end{aligned}$$

and the result is obtained. □

Lemma 4.7. *For $p > 0$ and $2q - 1 \in \mathbb{N}_+$, we have*
$$\frac{k^{p/2-1}}{\big(J(p,q)\big)^p} \to \infty, \quad \text{as } k \to 0^+.$$

Proof. It is clear that the result is true for $p \le 2$. For $p > 2$ using L'Hospital rule and Lemma 4.6 we derive as $k \to 0^+$,
$$\frac{k^{p/2-1}}{\big(J(p,q)\big)^p} \sim \frac{(p/2-1)k^{p/2-2}}{p\big(J(p,q)\big)^{p-1}J'(p,q)} \sim \frac{(p/2-1)k^{p/2-3/2}}{p\big(J(p,q)\big)^{p-1}A},$$
and hence the result for $2 < p \le 3$. Applying L'Hopital rule and differentiating $n-2$ times, we obtain the result for $n-1 < p \le n$. $\square$

4.3. **Bounds for the circular cylinder.** Recalling that $\psi(r) = \frac{1}{4}(3 - r^2)$, and $\frac{1}{2} \le \psi \le \frac{3}{4}$, we have
$$\begin{aligned} h(k\psi) &= \Big(2\pi \int_0^1 \frac{r}{[1 + \frac{k}{4}(3 - r^2)]^q}\,dr\Big)^p \\ &= \Big(\frac{4\pi}{k(q-1)}\big[(1 + \frac{k}{2})^{1-q} - (1 + \frac{3}{4}k)^{1-q}\big]\Big)^p. \end{aligned}$$
Hence
$$\begin{aligned} &-k + \Big(\frac{4\pi}{k(q-1)}\Big[\big(1 + \frac{k}{2}\big)^{1-q} - \big(1 + \frac{3}{4}k\big)^{1-q}\Big]\Big)^{-p}\frac{\lambda}{(1 + \frac{3}{4}k)^q} \\ &\le P(k\psi) \le \\ &-k + \Big(\frac{4\pi}{k(q-1)}\Big[\big(1 + \frac{k}{2}\big)^{1-q} - \big(1 + \frac{3}{4}k\big)^{1-q}\Big]\Big)^{-p}\frac{\lambda}{(1 + \frac{k}{2})^q} \end{aligned}$$
Thus an upper bound of λ^* is obtained by $\lambda_0 = 1/g_1(k_0)$, where
$$g_1(k) = \Big(\frac{4\pi}{k(q-1)}\Big[\big(1 + \frac{k}{2}\big)^{1-q} - \big(1 + \frac{3}{4}k\big)^{1-q}\Big]\Big)^{-p}k^{-1}\big(1 + \frac{3}{4}k\big)^{-q}$$
and k_0 is the largest solution of the equation $g_1'(k) = 0$. Note that $\lim_{k\to 0+} g_1(k) = \lim_{k\to\infty} g_1(k) = \infty$ for $p > (q+1)/q$.

Also a lower bound of λ^* is provided by $\lambda^0 = 1/g_2(k^0)$, where
$$g_2(k) = \Big(\frac{4\pi}{k(q-1)}\Big[\big(1 + \frac{k}{2}\big)^{1-q} - \big(1 + \frac{3}{4}k\big)^{1-q}\Big]\Big)^{-p}k^{-1}\big(1 + \frac{k}{2}\big)^{-q}$$
and k^0 is the largest solution of the equation $g_2'(k) = 0$.

Again we have $\lim_{k\to 0+} g_2(k) = \lim_{k\to\infty} g_2(k) = \infty$ for $p > (q+1)/q$.

4.4. **Bounds for the unit sphere.** Recalling that $\psi(r) = \frac{1}{2}(1 - \frac{r^2}{3})$, and substituting $r = \sqrt{6/k + 3}\sin(x)$ in $h(k\psi)$ we have
$$\begin{aligned} h(k\psi) &= \Big(4\pi \int_0^1 \frac{r^2}{[1 + \frac{k}{2}(1 - \frac{r^2}{3})]^q}\,dr\Big)^p \\ &= \Big[\pi\sqrt{27}\frac{2^{q+1}}{k^q}\big(1 + \frac{2}{k}\big)^{3/2-q}H(k,q)\Big]^p, \end{aligned}$$

where $H(k,q)=\int_0^{\gamma}[\sec^{2q-1}(r)-\sec^{2q-3}(r)]dr$, and $\gamma=\sin^{-1}\big(\frac{1}{\sqrt{\frac{6}{k}+2}}\big)$. It is noted that $H(k,q)$ satisfies similar properties to those of $J(k,q)$. Since $\frac{1}{3}\le\psi\le\frac{1}{2}$, we have

$$-k+\frac{\lambda}{h(k\psi)\left(1+\frac{1}{2}k\right)^q}\le P(k\psi)\le -k+\frac{\lambda}{h(k\psi)\left(1+\frac{1}{3}k\right)^q}.$$

Then an upper bound λ_0 for λ^* is provided by

$$\lambda_0=\frac{1}{g_1(k_0)},$$

where k_0 is the largest solution of the equation $g_1'(k)=0$, with

$$g_1(k)=\frac{k^{pq-1}(1+\frac{k}{2})^{-q}}{\pi^p 27^{p/2}2^{p(q+1)}(1+\frac{2}{k})^{p(3/2-q)}H^p(k,q)}.$$

A lower bound λ^0 for λ^* is provided by

$$\lambda^0=\frac{1}{g_2(k^0)},$$

where k^0 is the largest solution of the equation $g_2'(k)=0$, with

$$g_2(k)=\frac{k^{pq-1}(1+\frac{k}{3})^{-q}}{\pi^p 27^{p/2}2^{p(q+1)}(1+\frac{2}{k})^{p(3/2-q)}H^p(k,q)}.$$

One can see that $g_1(k)$ and $g_2(k)$ approach infinity as k does provided that $p>(q+1)/q$.

Table 2, presents the values of λ^0 and λ_0 for $p=2$ and different values of q. One can see that the values of λ^0 and λ_0 are decreasing with q, and so is λ^*. This seems sensible since as q grows the function $f(s)=(1+s)^{-q}$ decreases faster and so a steady state ceases to exist for smaller values of λ. Also, for the same values of p and q we have $\lambda_s^*\le\lambda_c^*\le\lambda_{sp}^*$, where λ_s^*,λ_c^*, and λ_{sp}^*, denotes the critical parameter in the slab, cylindrical and spherical geometries, respectively.

Table 2. The upper and lower estimates of λ^* for $f(w)=\frac{1}{(1+w)^q}$, $p=2$, and different values of q.

	Slab		Cylinder		Sphere	
q	λ^0	λ_0	λ^0	λ_0	λ^0	λ_0
$\frac{3}{2}$	0.908333	1.336943	5.045209	7.558562	14.482507	21.912864
2	0.594493	0.869110	3.289868	4.934802	9.428617	14.358698
$\frac{5}{2}$	0.443868	0.646653	2.451625	3.682343	7.020242	10.741934
3	0.354617	0.515538	1.956356	2.941853	5.599051	8.598215
$\frac{7}{2}$	0.295409	0.428856	1.628417	2.451090	4.658795	7.174475

5. Numerical Results

In this section we compare our estimates with the existing ones in the literature. For sake of simplicity, as in the previous sections, we assume that $\beta=1$.

In [7] an upper estimate for λ^* has been obtained in the case where $f(s)$ is a decreasing function such that $\int_0^\infty f(s)\,ds < \infty$. More precisely this upper estimate has the form

$$\lambda^* \le \tilde{\lambda} = \frac{\mu_1\,|\Omega|^{p-1}}{m_{pr}}, \tag{5.1}$$

where μ_1 is the principal eigenvalue of $-\Delta$ for Robin boundary conditions while m_{pr} is the minimum of the corresponding positive normalized eigenfunction Φ so that $\int_\Omega \Phi(x)\,dx = 1$.

For the slab geometry the principal eigenvalue is $\mu_1 = 0.740175$, while the normalized corresponding eigenfunction is $\Phi(x) = 0.567457\cos(0.860334\,x)$ and so $m_{pr} = 0.370086$.

For the cylindrical geometry the principal eigenvalue is $\mu_1 = 1.576993$ and the normalized eigenfunction has the form

$$\Phi(r) = \frac{J_0(\sqrt{1.576993}\,r)}{2.764919} = \frac{J_0(1.255783\,r)}{2.764919},$$

where $J_0(r)$ is the Bessel function of first kind and so $m_{pr} = \frac{J_0(1.255783)}{2.764919} = 0.232538$.

For the spherical geometry we obtain that $\mu_1 = \frac{\pi^2}{4}$ and

$$\Phi(r) = \frac{\pi\sin(\frac{\pi}{2}\,r)}{16\,r},$$

hence $m_{pr} = \frac{\pi}{16}$.

¿From Tables 1, 2 and 3 it is easily seen that the upper estimate λ_0 of λ^* is more accurate than the upper estimate $\tilde{\lambda}$ obtained by (5.1) for any of the three considered geometries.

Table 3. The upper estimate $\tilde{\lambda}$ of λ^* for general decreasing f with $\int_0^\infty f(s)\,ds < \infty$, and different values of p.

	Slab	Cylinder	Sphere
p	$\tilde{\lambda}$	$\tilde{\lambda}$	$\tilde{\lambda}$
2	4.000016	21.305204	52.637890
3	8.000032	66.932273	220.489078
4	16.000064	210.273939	923.582493
5	32.000129	660.595063	3868.693300
6	64.000259	2075.320598	16205.144599

In [13] for the slab geometry and for a general decreasing f with $\int_0^\infty f(s)ds < \infty$ the upper estimate $\widehat{\lambda} = 8$ is obtained when $p = 2$. From Tables 1,2 it can be observed that the upper estimate λ_0 is again more accurate. Also in [14], under the same conditions on f, for the cylindrical geometry, it is proved that $\lambda^* < 8\pi^2$. Again from the above tables it is obvious that the upper estimate λ_0 is significantly smaller than $8\pi^2$ in both of the considered cases, exponential and power-law case.

Conclusion. For $p > p_{cr}$, there exists a critical parameter λ^* such that problem (2.1)-(2.2) has at least one solution for $\lambda < \lambda^*$ and no solution for $\lambda > \lambda^*$. Since for $\lambda > \lambda^*$ the solution of time-dependent problem (1.1)-(1.2) performs finite time blow-up, the determination of λ^* becomes very important. But in most of the cases the determination of λ^* is not possible and so upper and lower estimates of λ^* are very important.

In this paper we investigate the two special cases $f(s) = e^{-s}$ and $f(s) = (1+s)^{-q}$, $q > 0$, and we construct some upper and lower solutions of problem (2.1)-(2.2) of special form. Using these upper and lower solutions we obtain general upper and lower estimates of the critical parameter λ^*. Furthermore, our arguments permit to determine an upper bound of the critical exponent p_{cr} and provide the proof of the existence of λ^* as well.

In each case, we focus on the slab, the cylindrical and the spherical geometries and using some special approximations we improve the bounds obtained for a general domain Ω. Our estimates for these three geometries improve the existing ones in the literature, see [7, 13, 14].

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