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GEVREY PROBLEM FOR PARABOLIC EQUATIONS WITH CHANGING TIME DIRECTION

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ABSTRACT. This article concerns parabolic equations with changing time direction and Gevrey's boundary condition. Using expansion series and biorthogonal systems, we prove the existence of classical solutions.

1. Introduction and Main Result

This paper is devoted to the study of existence of classical solution of the parabolic problem

$$\frac{\partial u}{\partial t} - \operatorname{sign}(x)\frac{\partial^2 u}{\partial x^2} = 0 \tag{1.1}$$

in a domain $\Omega = \{(x,t) : x \in (-1;0) \cup (0;1), t \in (0;T)\}$ with Gevrey boundary conditions (see [1, 5])

$$\begin{gathered} u(-1;t) = u(1;t) = 0, \\ u(x;0) = u_0(x), \quad t \in (0;1), \\ u(x;T) = u_T(x), \quad t \in (-1;0); \end{gathered} \tag{1.2}$$

Also with sewing conditions (see [2])

$$\begin{gathered} u(-0;t) = u(+0;t), \\ \frac{\partial u}{\partial x}(-0;t) = -\frac{\partial u}{\partial x}(+0;t). \end{gathered} \tag{1.3}$$

Note that (1.1) is a parabolic equation with changing time direction. The boundary-value problems with such sewing conditions appear when modelling, for example, process of interaction between two reciprocal flows with mutual permeating, or when designing certain heat exchangers. Furthermore, the results obtained in the linear case, may be used for investigating nonlinear problems with changing time direction. Such problems arise in supersonic dynamics, boundary layer theory [3].

The main aim of the paper is to prove the existence of a classical solution. We call a function $u : [-1,1] \times [0;T] \to \mathbb{R}$ a classical solution of the problem (1.1)-(1.2)-(1.3) if $u \in C^2(\Omega)$ and satisfies to equation (1.1) in Ω, Gevrey condition (1.2) and sewing conditions (1.3) at 0.

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To state our main result, we need some preliminary constructions. First using separation of variables technique we obtain the following problem for $X(x)$, where $u(x,t) = X(x)T(t)$,

$$\operatorname{sign}(x)\frac{X''}{X} = \frac{T'}{T} = \mu, \tag{1.4}$$

$$\operatorname{sign}(x)X'' = \mu X. \tag{1.5}$$

$$\left.\begin{aligned} X(-1) = X(1) = 0, \\ X|_{x=+0} = X|_{x=-0}, \\ \frac{\partial X}{\partial x}|_{x=+0} = -\frac{\partial X}{\partial x}|_{x=-0}. \end{aligned}\right\} \tag{1.6}$$

Consider the set of twice differentiable functions for $x \in (-1,0)$ and $x \in (0,1)$ satisfying boundary conditions (1.2) and sewing conditions(1.3). Denote by W_x the completion of this set with respect to the norm

$$\|h\|^2 = \int_{-1}^0 \Big(h^2 + h_x^2 + h_{xx}^2\Big)dx + \int_0^1 \Big(h^2 + h_x^2 + h_{xx}^2\Big)dx.$$

It can be shown that W_x is a Hilbert space. Now introduce the operator $L_x : W_x \to L_2(-1,1)$ defined by $L_x X = \operatorname{sign}(x)X''$ for every $X \in W_x$. It is easy to see that L_x is symmetric. Hence there exists denumerable set of eigenvalues, both positive and negative. The corresponding eigenfunctions are: For $\mu = \lambda^2 > 0$,

$$X_\lambda = \begin{cases} \frac{\sin\lambda(x+1)}{\sin\lambda}, & x < 0 \\ \frac{\sinh\lambda(1-x)}{\sinh\lambda}, & x > 0, \end{cases} \tag{1.7}$$

For $\mu = -\lambda^2 < 0$,

$$X_\lambda = \begin{cases} \frac{\sinh\lambda(x+1)}{\sinh\lambda}, & x < 0 \\ \frac{\sin\lambda(1-x)}{\sin\lambda}, & x > 0, \end{cases} \tag{1.8}$$

In both cases eigenvalues are the solutions of the equation

$$\tan\lambda = \tanh\lambda. \tag{1.9}$$

In addition, there exists an eigenfunction for $\mu = 0$,

$$X_0 = \begin{cases} \sqrt{\frac{3}{2}}(x+1), & x < 0 \\ \sqrt{\frac{3}{2}}(1-x), & x > 0. \end{cases} \tag{1.10}$$

Functions (1.7)–(1.10) form orthogonal complete basis in W_x. As it follows from (1.9), the negative eigenvalues $\mu_k = -\lambda_k^2$ satisfy asymptotic relations

$$\lambda_n \sim -\frac{\pi}{4} - \pi n, \quad n \in \mathbb{N}$$

and the positive eigenvalues $\mu_k = \lambda_k^2$ satisfy asymptotic relations

$$\lambda_n \sim \frac{\pi}{4} + \pi n, \quad n \in \mathbb{N}\,.$$

We will seek solutions of the problem (1.1)–(1.3) in the form

$$u(x,t) = \sum_{k=0}^\infty \Big(A_k e^{-\lambda_k^2 t}\frac{\sinh\lambda_k(x+1)}{\sinh\lambda_k} + B_k e^{-\lambda_k^2(T-t)}\frac{\sin\lambda_k(x+1)}{\sin\lambda_k}\Big) + C(x+1), \tag{1.11}$$

for $x < 0$, and

$$u(x,t) = \sum_{k=0}^{\infty} \Big(A_k e^{-\lambda_k^2 t} \frac{\sin \lambda_k (x-1)}{\sin \lambda_k} + B_k e^{-\lambda_k^2 (T-t)} \frac{\sinh \lambda_k (x-1)}{\sinh \lambda_k} \Big) + C(1-x), \tag{1.12}$$

for $x > 0$.

Theroem 1.1. *Assume $u_0, u_T \in L_2(0;1)$. Then there exists unique classical solution $u(x,t)$ of the problem (1.1)-(1.2)-(1.3). Furthermore, there exists unique collection of coefficients $A_k, B_k \in l_2$ such that the solution $u(x,t)$ on the set Ω express by the series expansion (1.11) and (1.12), respectively. If $x < 0, t = T$ then the sum of series (1.11) is $u(x,T) = u_T$, if $x > 0, t = 0$ then the sum of series (1.12) is $u(x,0) = u_0$. Moreover, if $u_0, u_T \in C_0^2[0;1]$, then (1.11) and (1.12) converge absolutely in rectangular $\Omega = (-1;0) \times (0;T) \cup (0;1) \times (0;T)$.*

2. Proof of the main result

First, we substitute initial data (1.6) into (1.11) and (1.12). Hence we obtain that for $t = T, x < 0$,

$$u(x,T) = \sum_{k=0}^{\infty} \Big(A_k e^{-\lambda_k^2 T} \frac{\sinh \lambda_k (x+1)}{\sinh \lambda_k} + B_k \frac{\sin \lambda_k (x+1)}{\sin \lambda_k} \Big),$$

and for $t = 0, x > 0$

$$u(x,0) = \sum_{k=0}^{\infty} \Big(A_k \frac{\sin \lambda_k (1-x)}{\sin \lambda_k} + B_k e^{-\lambda_k^2 T} \frac{\sinh \lambda_k (1-x)}{\sinh \lambda_k} \Big).$$

Now we make the change of variables setting $x = y - 1$ for $x < 0$ and $x = 1 - y$ for $x > 0$. Then the foregoing equations will take the form

$$u_0(y) = \sum_{k=1}^{\infty} \Big(A_k \frac{\sin \lambda_k y}{\sin \lambda_k} + B_k e^{-\lambda_k^2 T} \frac{\sinh \lambda_k y}{\sinh \lambda_k} \Big) + Cy, \tag{2.1}$$

$$u_T(y) = \sum_{k=1}^{\infty} \Big(A_k e^{-\lambda_k^2 T} \frac{\sinh \lambda_k y}{\sinh \lambda_k} + B_k \frac{\sin \lambda_k y}{\sin \lambda_k} \Big) + Cy, \tag{2.2}$$

where

$$\begin{gathered} u_0(y) = u_0(1-x), \\ u_T(y) = u_T(1+x). \end{gathered}$$

Adding and subtracting (2.1) and (2.2) term by term we obtain

$$u_0 + u_T - 2Cy = \sum_{k=1}^{\infty} (A_k + B_k) \Big(\frac{\sin \lambda_k y}{\sin \lambda_k} + e^{-\lambda_k^2 T} \frac{\sinh \lambda_k y}{\sinh \lambda_k} \Big), \tag{2.3}$$

$$u_0 - u_T = \sum_{k=1}^{\infty} (A_k - B_k) \Big(\frac{\sin \lambda_k y}{\sin \lambda_k} - e^{-\lambda_k^2 T} \frac{\sinh \lambda_k y}{\sinh \lambda_k} \Big). \tag{2.4}$$

It is easy to see that both these equations represent the expansion of given functions with respect to

$$\alpha_k = \Big(\frac{\sin \lambda_k y}{\sin \lambda_k} + e^{-\lambda_k^2 T} \frac{\sinh \lambda_k y}{\sinh \lambda_k} \Big) \tag{2.5}$$

and

$$\beta_k = \Big(\frac{\sin\lambda_k y}{\sin\lambda_k} - e^{-\lambda_k^2 T}\frac{\sinh\lambda_k y}{\sinh\lambda_k}\Big). \tag{2.6}$$

Suppose that there exist biorthogonal systems $\{\psi_n\}$, $\{\omega_n\}$, $n = 1,\dots\infty$ in $L_2(0,1)$, such that

$$(\alpha_k;\psi_n) = \delta_{kn},$$
$$(\beta_k;\omega_n) = \delta_{kn},$$

where δ_{kn} is the delta Kronecker, and $(\cdot;\cdot)$ is a scalar product in $L_2(0,1)$. Then coefficients A_k and B_k could be found for arbitrary left sides of (2.3) and (2.4). It means that the following equalities will satisfy:

$$\int_0^1 \psi_n\Big(\frac{\sin\lambda_k y}{\sin\lambda_k} + e^{-\lambda_k^2 T}\frac{\sinh\lambda_k y}{\sinh\lambda_k}\Big)dy = \delta_{kn},$$
$$\int_0^1 \omega_n\Big(\frac{\sin\lambda_k y}{\sin\lambda_k} - e^{-\lambda_k^2 T}\frac{\sinh\lambda_k y}{\sinh\lambda_k}\Big)dy = \delta_{kn}.$$

Now we prove the existence of biorthogonal systems.

Proposition 2.1. *Let $\{h_n\}$, be a biogthogonal system with $\{\sigma_n\}$, $n = 1,\dots\infty$ and $\{h_n\}$ uniformly bounded. Then there exists a biorthogonal system for $\{\sigma_n + \tau_n\}$, provided that for all k,*

$$\sum_{n=1}^{\infty} |(\tau_k, h_n)|^2 < \delta < 1; \tag{2.7}$$

and

$$\sum_{n=1}^{\infty} |\tau_n| < \infty. \tag{2.8}$$

Proof. We look for a biorthogonal system in the form

$$\tilde h_n = h_n + \sum_{i=1}^{\infty} b_{in} h_i.$$

Then

$$\delta_{kn} = (\sigma_k + \tau_k; h_n + \sum_{i=1}^{\infty} b_{in}h_i) = \delta_{kn} + (\tau_k; h_n) + b_{kn} + \sum_{i=1}^{\infty} b_{in}(\tau_k; h_i).$$

Let us denote A the matrix whose elements are $(\tau_k; h_n)$ and B the matrix with elements b_{kn}. Then we obtain the equation

$$A + B + BA = 0, \quad \text{or} \quad B(E + A) = -A$$

where E is an identity matrix. Note that under condition (2.7) this equation is resolvable as there exists an inverse matrix

$$(E + A)^{-1} = E - A + A^2 - A^3 + \dots$$

Hence,

$$B = -A + A^2 - A^3 + \dots,$$

Furthermore,

$$\|B\| \le \frac{\|A\|}{1 - \|A\|}.$$

Thus B is the matrix of certain linear bounded operator on the space l_2, that is why the collection $\{b_{ik}\}$, $i = 1, 2, \dots$ belongs l_2 for all k. As $|h_i|$ is uniformly bounded and (2.8) is fulfilled, then all series $\sum_{i=1}^{\infty} b_{in} h_i$ converge. □

Proposition 2.2. *There exists $T_0 > 0$ such that for arbitrary $T > T_0$ there exists biorthogonal systems for systems of functions α_k and β_k .*

Proof. As follows from [4], for the system

$$\sigma_n = \frac{\sin(\frac{\pi}{4} + \pi n)x}{\sin(\frac{\pi}{4} + \pi n)} \tag{2.9}$$

there exists a biorthogonal system $\{h_n\}$; moreover the functions $\{h_n\}$ are uniformly bounded, for example by the constant 10.

The difference τ_k of α_k and (2.9) (note that for β_k all reasonings are the same) consists of two terms

$$\begin{gathered}\tau_k^{(1)} = \frac{\sin(\frac{\pi}{4} + \pi k)x}{\sin(\frac{\pi}{4} + \pi k)} - \frac{\sin \lambda_k x}{\sin \lambda_k};\\ \tau_k^{(2)} = e^{-\lambda_k^2 T} \frac{\sinh \lambda_k x}{\sinh \lambda_k}.\end{gathered}$$

By virtue of Lagrange theorem for $\tau_k^{(1)}$,

$$|\tau_k^{(1)}| \le \sqrt{2} \cos \lambda_k (\lambda_k - \frac{\pi}{4} - \pi k).$$

Taking into account that λ_k is a root of the equation $\tan \lambda = \tanh \lambda$, we obtain

$$\begin{aligned}|\tau_k^{(1)}| &\le \sqrt{2} \cos \lambda_k |\tanh \lambda_k - 1| \cdot \frac{1}{\cos^2 \lambda_k}\\ &\le 2|\tanh \lambda_k - 1| = \frac{2}{\cosh \lambda_k (\cosh \lambda_k + \sinh \lambda_k)},\end{aligned}$$

hence, norms $\tau_k^{(1)}$ are rapidly decreasing (at the rate e^{-Ck^2}). Thus, for $\tau_k^{(1)}$ condition (2.8) holds. In order to prove (2.7) we need to estimate sum of the series. Direct calculation shows that even first root $\lambda_1 \approx 3,9266$ differ from $\frac{5\pi}{4} \approx 3,9270$ less than by $0,5 \cdot 10^{-3}$, then the sum of squares $\tau_k^{(1)}$ can be bounded above by geometric progression

$$\sum_{k=1}^{\infty} 10^{-6} e^{-2\pi k} = \frac{10^{-6}}{1 - e^{-2\pi}} \le 10^{-5}.$$

The validity of (2.8) for $\tau_k^{(2)}$ is obvious for all positive T, and in order to estimate the sum of squared norms we calculate at first

$$\int_0^1 \Big(\frac{\sinh \lambda_k y}{\sinh \lambda_k}\Big)^2 dy = \frac{1}{2\lambda_k \tan \lambda_k} - \frac{1}{2 \sinh^2 \lambda_k}.$$

Hence the series of sums of the squared norms is majorized by

$$\sum_{k=1}^{\infty} \frac{e^{10^{-3} - (\frac{\pi}{2} + 2\pi k) T^2}}{\lambda_k}.$$

Direct calculations show that the sum of this series for $T = 0,15$ is $1,25 \cdot 10^{-3}$. From the estimate

$$\begin{aligned}\sum_{k=1}^{\infty} |(\tau_k, h_n)|^2 &\le \sum_{k=1}^{\infty} |\tau_k|^2 \cdot |h_n|^2 \\ &\le \sum_{k=1}^{\infty} 100|\tau_k^{(1)} + \tau_k^{(2)}|^2 \\ &\le \sum_{k=1}^{\infty} 200(|\tau_k^{(1)}|^2 + |\tau_k^{(2)}|^2) < 0,3\end{aligned}$$

we can see that if $T_0 = 0,15$ it is sufficient for concluding the proof. ☐

Proposition 2.3. *The following two systems*

$$\Big\{X_0^+ = \sqrt{3}y,\ X_k^+ = \frac{\sin\lambda_k y}{\sin\lambda_k} + \frac{\sinh\lambda_k y}{\sinh\lambda_k}\Big\}, \tag{2.10}$$

$$\Big\{X_k^- = \frac{\sin\lambda_k y}{\sin\lambda_k} - \frac{\sinh\lambda_k y}{\sinh\lambda_k}\Big\}, \tag{2.11}$$

$k = 1\dots\infty$, *are complete and orthogonal in* $L_2(0;1)$.

Proof. The first system is a complete system of eigenfunctions of operator $L_4 u = u^{IV}$ in the space of smooth functions, satisfying boundary conditions

$$u(0) = u''(0) = u''(1) = u'''(1) = 0.$$

It is easy to see if we integrate by parts that this operator is symmetric in this space, then eigenfunctions are orthogonal. As this system consists of all eigenfunctions then it is complete. Indeed, if we denote a^4, an eigenvalue of L_4, then all eigenfunctions, corresponding a^4, take the form

$$A\sin ax + B\cos ax + C\sinh ax + D\cosh ax.$$

Since $u(0) = u''(0) = 0$, it follows that $B = D = 0$. From the boundary condition $u''(1) = 0$ we obtain the equation

$$-Aa^2\sin a + Ca^2\sinh a = 0.$$

It turned out that

$$A = \frac{Z}{\sin a}, \quad C = \frac{Z}{\sinh a},$$

where Z is certain nontrivial parameter and without loss of generality we may let $Z = 1$. It follows from $u'''(1) = 0$, that

$$-Aa^3\cos a + Ca^3\cosh a = 0.$$

Substituting the expression obtained for A and C, into this equation, we have the relation

$$\tan a = \tanh a,$$

which must be valid for all eigenvalues. Using precisely the same reasoning, we prove the statement for second system; the only difference is that the boundary conditions are in the form

$$u(0) = u(1) = u'(1) = u''(0) = 0.$$

It should be noted that

$$\mu_0^2 = |X_0^+| = 1; \quad \mu_k^2 = |X_k^+| = |X_k^-| = \frac{1}{2\sin^2\lambda_k} - \frac{1}{2\sinh^2\lambda_k}. \tag{2.12}$$

By (1.9) and the asymptotic behavior of eigenvalues, the norms of the vectors tend to unit as $k \to \infty$. □

Proposition 2.4. *The biorthogonal systems for* (2.5)– (2.6)*, are unique.*

Proof. The validity of this statement will follow immediately from the completeness of systems α_k and β_k. We prove the completeness only for the first system as the proof for the second is similar. Assume that the system α_k is not complete.Then there exists a vector Z such that for all k

$$(\alpha_k, Z) = 0.$$

Since the system of vectors X_k^+ is complete, we may expand vector Z with respect to this system:

$$Z = \sum_{i=1}^{\infty} z_i X_i.$$

We denote here

$$X_k = \frac{X_k^+}{\mu_k},$$

which form an orthonormal system. Also denote

$$\alpha_k^* = \frac{\alpha_k}{\mu_k},$$

$$\gamma_k = X_k - \alpha_k^* = \frac{1}{\mu_k}(1 - e^{-\lambda_k^2 T})\frac{\sinh\lambda_k y}{\sinh\lambda_k}.$$

Then for all k,

$$(X_k - \gamma_k; \sum_{i=1}^{\infty} z_i X_i) = 0,$$

$$z_k - \sum_{i=1}^{\infty} z_i(\gamma_k; X_i) = 0.$$

Hence $\{z_k\}$ are eigenvectors of the matrix with elements $M_{ki} = (\gamma_k; X_i)$, and corresponding eigenvalue $+1$. The operator associated with this matrix is

$$M : X_k \longmapsto \gamma_k.$$

Then the operator

$$E - M : X_k \longmapsto \alpha_k^*,$$

and $\{z_k\}$ is and eigenvector of this operator, corresponding to eigenvalue 0. In other words,

$$\sum_{k=1}^{\infty} z_k \alpha_k^* = 0.$$

Existence of such liner combination with not only trivial z_k contradicts with the fact of linearly independence of α_k. □

Proposition 2.5. *Assume u_0, u_T are in $C_0^2[0;1]$. Then the coefficients A_k and B_k belong l_1.*

Proof. To prove this statement note that

$$\alpha_k = \frac{1}{2}((X_k^+ + X_k^-) + e^{-\lambda_k^2 T}(X_k^+ - X_k^-)),$$
$$\beta_k = \frac{1}{2}((X_k^+ + X_k^-) - e^{-\lambda_k^2 T}(X_k^+ - X_k^-)).$$

The outcome of these formulas is that if both series of coefficients when expanding with respect to X_k^+ and X_k^- converge absolutely, then both series of coefficients when expanding with respect to α_k and β_k also converge absolutely.

Let a f be a function in $C_0^2[0;1]$. Then there exists an expansion with respect to X_k^-,

$$f = \sum_{k=1}^{\infty} c_k X_k^-,$$

where the coefficients can be calculated using the orthogonality by formulas:

$$c_k = \frac{1}{\mu_k^2} \int_0^1 f \cdot X_k^- dx.$$

Since the second derivative of f is square integrable, there exists similar expansion with respect to X_k^+,

$$f'' = \sum_{k=0}^{\infty} d_k X_k^+$$

where the coefficients are

$$d_k = \frac{1}{\mu_k^2} \int_0^1 f'' \cdot X_k^+ dx.$$

Note that if we differentiate the system X_k^- twice, it will turn into the system $-\lambda_k^2 X_k^+$. So that, if we integrate twice and take into account that f and f' vanish in the boundary points of segment $[0;1]$, we derive the relation

$$d_k = -\lambda_k^2 c_k,$$

which implies the inclusion $c_k \in l_1$.

The coefficient C of (2.3) may be found from the boundary condition if $y = 1$,

$$C = \frac{1}{2}(u_0 + u_T - \sum_{k=1}^{\infty} (A_k + B_k)(1 + e^{-\lambda_k^2 T})).$$

To complete the proof of the theorem we need only to show that functional series (1.11)–(1.12) and the series of their first and second derivatives with respect to x and t converge uniformly. For $0 < t_1 < t_2 < T$ convergence on a segment $[t_1; t_2]$ is a consequence of majoring criterion for convergence. Since $[t_1$ and $t_2]$ are arbitrary the series converge uniformly for every interior point of the interval $(0; T)$. The proof of the theorem is complete. $\square$

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