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COMPLEX OSCILLATION OF ENTIRE SOLUTIONS OF HIGHER-ORDER LINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we investigate higher-order linear differential equations with entire coefficients of iterated order. We improve and extend a result of Belaïdi and Hamouda by using the estimates for the logarithmic derivative of a transcendental meromorphic function due to Gundersen and the Winman-Valiron theory. We also consider the nonhomogeneous linear differential equations by using the basic method and some lemmas from the present and other three authors.

1. Introduction and main results

In this paper, we shall assume that the reader is familiar with the fundamental results and the standard notation of the Nevanlinna value distribution theory of meromorphic functions (see [14],[10]). The term "meromorphic function" will mean meromorphic in the whole complex plane $\mathbb{C}$.

For $k \geq 2$, we consider a linear differential equation

$$f^{(k)} + A_{k-1} f^{(k-1)} + \dots + A_0 f = 0, \tag{1.1}$$

where $A_0, \dots, A_{k-1}$ are entire functions with $A_0 \not\equiv 0$. It is well known that all solutions of (1.1) are entire functions, and if some of the coefficients of (1.1) are transcendental, then (1.1) has at least one solution with order $\sigma(f) = \infty$.

Thus the question which arises is: What conditions on $A_0, \dots, A_{k-1}$ will guarantee that every solution $f \not\equiv 0$ of (1.1) has infinite order?

For the above question, there are many results for second order linear differential equations (see for example [7, 9, 6, 4]). In 2002, Belaïdi and Hamouda considered the higher order linear differential equations and obtained the following result.

Theorem 1.1 (Belaïdi and Hamouda [2])**.** *Suppose that there exist a positive number μ, and a sequence of points $(z_j)_{j\in\mathbb{N}}$ with $\lim_{j\to\infty} z_j = \infty$, and two real numbers*

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$\alpha, \beta (0 \le \beta < \alpha)$ such that

$$|A_0(z_j)| \ge \exp\{\alpha |z_j|^{\mu}\},$$
$$|A_n(z_j)| \le \exp\{\beta |z_j|^{\mu}\} \quad (n = 1, \dots, k-1),$$

as $j \to \infty$. Then every solution $f \not\equiv 0$ of (1.1) has infinite order.

Now there exists another question: For so many solutions of infinite order, how to describe precisely the properties of growth of solutions of infinite order of (1.1)?

In this paper, we improve and extend Theorem 1.1 by making use of the concept of iterated order. Let us define inductively, for $r \in [0, +\infty), \exp^{[1]} r = e^r$ and $\exp^{[n+1]} r = \exp(\exp^{[n]} r), n \in \mathbb{N}$. For all r sufficiently large, we define $\log^{[1]} r = \log r$ and $\log^{[n+1]} r = \log(\log^{[n]} r), n \in \mathbb{N}$. We also denote $\exp^{[0]} r = r = \log^{[0]} r$, $\log^{[-1]} r = \exp^{[1]} r$ and $\exp^{[-1]} r = \log^{[1]} r$. We recall the following definitions (see [12, 13, 3]).

Definition 1.2. The iterated p-order $\sigma_p(f)$ of a meromorphic function $f(z)$ is defined by

$$\sigma_p(f) = \limsup_{r \to \infty} \frac{\log^{[p]} T(r, f)}{\log r} \quad (p \in \mathbb{N}).$$

Remark 1.3. (1). If $p = 1$, then we denote $\sigma_1(f) = \sigma(f)$; (2). If $p = 2$. then we denote by $\sigma_2(f)$ the so-called hyper order (see [15]); (3). If $f(z)$ is an entire function, then

$$\sigma_p(f) = \limsup_{r \to \infty} \frac{\log^{[p+1]} M(r, f)}{\log r}.$$

Definition 1.4. The growth index of the iterated order of a meromorphic function $f(z)$ is defined by

$$i(f) = \begin{cases} 0 & \text{if } f \text{ is rational,} \\ \min\{n \in \mathbb{N} : \sigma_n(f) < \infty\} & \text{if } f \text{ is transcendental and} \\ & \sigma_n(f) < \infty \text{ for some } n \in \mathbb{N}, \\ \infty & \text{if } \sigma_n(f) = \infty \text{ for all } n \in \mathbb{N}. \end{cases}$$

Similarly, we can define the iterated lower order $\mu_p(f)$ of a meromorphic function $f(z)$ and the growth index $i_\mu(f)$ of $\mu_p(f)$.

Definition 1.5. The iterated convergence exponent of the sequence of a-points $(a \in \mathbb{C} \cup \{\infty\})$ is defined by

$$\lambda_n(f - a) = \lambda_n(f, a) = \limsup_{r \to \infty} \frac{\log^{[n]} N(r, \frac{1}{f-a})}{\log r} \quad (n \in \mathbb{N}),$$

and $\overline{\lambda}_n(f-a)$, the iterated convergence exponent of the sequence of distinct a-points is defined by

$$\overline{\lambda}_n(f - a) = \overline{\lambda}_n(f, a) = \limsup_{r \to \infty} \frac{\log^{[n]} \overline{N}(r, \frac{1}{f-a})}{\log r} \quad (n \in \mathbb{N}).$$

Remark 1.6. (1). $\lambda_1(f - a) = \lambda(f - a)$; (2). $\overline{\lambda}_1(f - a) = \overline{\lambda}(f - a)$.

Definition 1.7. The growth index of the iterated convergence exponent of the sequence of a-points of a meromorphic function $f(z)$ with iterated order is defined by

$$i_\lambda(f-a)=i_\lambda(f,a)=\begin{cases}0 & \text{if } n(r,\frac{1}{f-a})=O(\log r),\\ \min\{n\in\mathbb{N}:\lambda_n(f)<\infty\} & \text{if } \lambda_n(f-a)<\infty \text{ for}\\ & \text{some } n\in\mathbb{N},\\ \infty & \text{if } \lambda_n(f-a)=\infty \text{ for}\\ & \text{all } n\in\mathbb{N}.\end{cases}$$

Similarly, we can define the growth index $i_{\overline{\lambda}}(f-a)$ of $\overline{\lambda}_p(f-a)$.

Now we show one of our main result, which improves and extends Theorem 1.1.

Theorem 1.8. *Suppose there exist a sequence of points* $(z_j)_{j\in\mathbb{N}}$ *with* $\lim_{j\to\infty} z_j=\infty$*, and two real numbers* α,β $(0\le\beta<\alpha)$ *such that for any* $\varepsilon>0$,

$$|A_0(z_j)|\ge \exp^{[p]}\{\alpha|z_j|^{\sigma-\varepsilon}\} \tag{1.2}$$

and for $n=1,\dots,k-1$,

$$|A_n(z_j)|\le \exp^{[p]}\{\beta|z_j|^{\sigma-\varepsilon}\} \quad (), \tag{1.3}$$

as $j\to\infty$*, where* p *is a positive integer number, and* σ *satisfies* $\sigma_p(A_n)\le\sigma_p(A_0)=\sigma$ $(n=1,2,\dots,k-1)$*. Then every solution* $f\not\equiv 0$ *of* (1.1) *is of infinite order with* $i(f)=p+1$ *and* $\sigma_{p+1}(f)=\sigma_p(A_0)$.

When $p=1$ in Theorem 1.8, we obtain the following corollary.

Corollary 1.9. *Suppose there exist a sequence of points* $(z_j)_{j\in\mathbb{N}}$ *with* $\lim_{j\to\infty} z_j=\infty$*, and two real numbers* $\alpha,\beta(0\le\beta<\alpha)$ *such that for any given* $\varepsilon>0$,

$$|A_0(z_j)|\ge \exp\{\alpha|z_j|^{\sigma-\varepsilon}\}$$

and for $n=1,\dots,k-1$,

$$|A_n(z_j)|\le \exp\{\beta|z_j|^{\sigma-\varepsilon}\}$$

as $j\to\infty$*, where* σ *satisfies* $\sigma(A_n)\le\sigma(A_0)=\sigma<\infty\,(n=1,2,\dots,k-1)$*. Then every solution* $f\not\equiv 0$ *of* (1.1) *satisfies* $\sigma(f)=\infty$ *and* $\sigma_2(f)=\sigma(A_0)$.

From Theorem 1.8, we can also deduce the following results.

Corollary 1.10 (Kinnunen [12])**.** *Let* $A_0,A_1,\dots,A_{k-1}$ *be entire functions such that* $i(A_0)=p(0<p<\infty)$ *and* $\sigma_p(A_0)=\sigma$*. If either* $\max\{i(A_j):j=1,\dots,k-1\}<p$ *or* $\max\{\sigma_p(A_j):j=1,\dots,k-1\}<\sigma_p(A_0)$*. Then all solutions* $f\not\equiv 0$ *of* (1.1) *satisfy* $i(f)=p+1$ *and* $\sigma_{p+1}(f)=\sigma_p(A_0)$.

Corollary 1.11 (Chen, Yang [5])**.** *Let* $A_0,A_1,\dots,A_{k-1}$ *be entire functions satisfying*

$$\max\{\sigma(A_j),(j=1,\dots,k-1)\}<\sigma(A_0)<\infty,$$

then all solutions $f\not\equiv 0$ *of* (1.1) *satisfy* $\sigma_2(f)=\sigma(A_0)$.

Considering nonhomogeneous linear differential equations

$$f^{(k)}+A_{k-1}f^{(k-1)}+\dots+A_0f=F \tag{1.4}$$

corresponding to (1.1), we obtain the following results of the iterated order and iterated convergence exponent of zeros of solutions of (1.1).

Theorem 1.12. *Assume that $A_0, \dots, A_{k-1}$ satisfy the hypotheses of Theorem 1.8. Let $F \not\equiv 0$ be an entire function with $i(F) = q$, where $q \in \mathbb{N}$. Then*

(1) *If either $q > p+1$, or $q = p+1$ while $\sigma_q(F) > \sigma_p(A_0)$, then all solutions f of (1.4) satisfy $i(f) = i(F) = q$ and $\sigma_q(f) = \sigma_q(F)$.*
(2) *If either $q < p+1$, or $q = p+1$ while $\sigma_q(F) < \sigma_p(A_0)$, then all solutions $f(z)$ of (1.4) satisfy $i(f) = i_\lambda(f) = i_{\overline{\lambda}}(f) = p+1$ and $\sigma_{p+1}(f) = \lambda_{p+1}(f) = \overline{\lambda}_{p+1}(f) = \sigma_p(A_0)$, with at most one exception.*

The methods and tactics in the proof of Theorem 1.1 are mainly the estimate for the logarithmic derivative of a transcendental meromorphic function of finite order due to Gundersen (see [8]). In this paper, To prove Theorem 1.8, on the one hand, we mainly use the estimate for the logarithmic derivative of a transcendental meromorphic function of finite or infinite order due to Gundersen (see [8]); on the other hand, we make mainly use of the Winman-Valiron theory [11]. We prove Theorem 1.12 by making use of the basic method of the nonhomogeneous linear differential equations, Lemmas 2.6 and 2.7 due to the present author and other three authors (see [3]).

2. Lemmas

Lemma 2.1 (Winman-Valiron [11]). *Let $f(z)$ be a transcendental entire function, δ is a constant such that $0 < \delta < \frac{1}{8}$, and let z be a point with $|z| = r$ at which $|f(z)| > M(r,f) \cdot \nu_f(r)^{-\frac{1}{8}+\delta}$, where $\nu_f(r)$ denote the central index of f, then the estimation*

$$\frac{f^{(n)}(z)}{f(z)} = \big(\frac{\nu_f(r)}{z}\big)^n \big(1 + \eta_k(z)\big) \quad (n \in \mathbb{N}), \tag{2.1}$$

holds for all $|z| = r$ outside a subset E of finite logarithmic measure, where

$$\eta_k(z) = O\Big((\nu_f(r))^{-\frac{1}{8}+\delta}\Big).$$

Lemma 2.2 (Kinnunen [12, Remark 1.3]). *If f is a meromorphic function with $i(f) = p \geq 1$, then $\sigma_p(f) = \sigma_p(f')$.*

Lemma 2.3 (Bank [1]). *Let $g : [0,\infty) \to \mathbb{R}$ and $h : [0,\infty) \to \mathbb{R}$ be monotone nondecreasing functions such that $g(r) \leq h(r)$ outside of an exceptional set E_2 of finite linear measure. Then for any $\alpha > 1$, there exists r_0 such that $g(r) \leq h(\alpha r)$ for all $r > r_0$.*

Lemma 2.4 (Gundersen [8]). *Let f be a transcendental meromorphic function of finite order σ. Let $\varepsilon > 0$ be a constant, and k and j be integers satisfying $k > j \geq 0$. Then the following two statements hold:*

(a) *There exists a set $E_1 \subset (1,\infty)$ which has finite logarithmic measure, such that for all z satisfying $|z| \notin E_1 \bigcup [0,1]$, we have*

$$\Big|\frac{f^{(k)}(z)}{f^{(j)}(z)}\Big| \leq |z|^{(k-j)(\sigma-1+\varepsilon)}. \tag{2.2}$$

(b) *There exists a set $E_2 \subset [0,2\pi)$ which has linear measure zero, such that if $\theta \in [0,2\pi) - E_2$, then there is a constant $R = R(\theta) > 0$ such that (2.2) holds for all z satisfying $\arg z = \theta$ and $R \leq |z|$.*

Lemma 2.5 (Gundersen [8]). *Let f be a transcendental meromorphic function. Let $\alpha > 1$ be a constant, and k and j be integers satisfying $k > j \geq 0$. Then the following two statements hold:*

(a) *There exists a set $E_1 \subset (1, \infty)$ which has finite logarithmic measure, and a constant $C > 0$, such that for all z satisfying $|z| \notin E_1 \bigcup [0,1]$, we have (with $r = |z|$)*

$$\Big|\frac{f^{(k)}(z)}{f^{(j)}(z)}\Big| \leq C\Big[\frac{T(\alpha r, f)}{r}(\log r)^{\alpha} \log T(\alpha r, f)\Big]^{k-j}. \tag{2.3}$$

(b) *There exists a set $E_2 \subset [0, 2\pi)$ which has linear measure zero, such that if $\theta \in [0, 2\pi) - E_2$, then there is a constant $R = R(\theta) > 0$ such that (2.3) holds for all z satisfying $\arg z = \theta$ and $R \leq |z|$.*

Lemma 2.6 (Cao, Chen, Zheng and Tu [3]). *Let $g(z)$ be an entire function of finite iterated order with $i(g) = m, i_\mu(g) = n, \sigma_m(g) = \sigma, \mu_n(g) = \mu$ (where $m, n \in \mathbb{N}$), and $\nu_g(r)$ denote the central index of g, then*

$$\limsup_{r\to\infty} \frac{\log^{[m]} \nu_g(r)}{\log r} = \sigma_m(g) = \sigma,$$
$$\liminf_{r\to\infty} \frac{\log^{[n]} \nu_g(r)}{\log r} = \mu_n(g) = \mu.$$

Lemma 2.7 (Cao, Chen, Zheng and Tu [3]). *Let $f(z)$ be a meromorphic solution of the differential equation*

$$f^{(k)} + B_{k-1} f^{(k-1)} + \cdots + B_0 f = F,$$

where $B_0, \dots, B_{k-1}, F \not\equiv 0$ are meromorphic functions, such that
(1) $\max\{i(F), i(B_j)(j = 0, \dots, k-1)\} < i(f) := p$, $(0 < p < \infty)$; or
(2) $\max\{\sigma_p(F), \sigma_p(B_j)(j = 0, \dots, k-1)\} < \sigma_p(f)$. Then $i_{\overline{\lambda}}(f) = i_\lambda(f) = i(f) = p$ and $\overline{\lambda}_p(f) = \lambda_p(f) = \sigma_p(f)$.

3. Proof of main results

Proof of Theorem 1.8. Let $f \not\equiv 0$ be a solution of (1.1). Then we have

$$|A_0(z)| \leq \Big|\frac{f^{(k)}(z)}{f(z)}\Big| + |A_{k-1}(z)|\Big|\frac{f^{(k-1)}(z)}{f(z)}\Big| + \cdots + |A_1(z)|\Big|\frac{f'(z)}{f(z)}\Big|. \tag{3.1}$$

When $p = 1$, from Theorem 1.1, we have $\sigma(f) = \infty$. when $p > 1$, from Lemma 2.4 (a) and using similar discussion as to the proof of Theorem 1.1 (see [2, page 242-243]), one can also deduce that $\sigma(f) = \infty$.

By Lemma 2.5 (a), There exist a set $E_1 \subset (1, \infty)$ which has finite logarithmic measure, and positive constants c and M such that for all z_j satisfying $|z_j| \notin E_1 \bigcup [0,1]$, we have (with $r_j = |z_j|$),

$$\Big|\frac{f^{(t)}(z_j)}{f(z_j)}\Big| \leq M \cdot \big[r_j^c \cdot T(2r_j, f)\big]^{2k} \quad (t = 1, \dots, k-1). \tag{3.2}$$

Substituting (1.2), (1.3) and (3.2) into (3.1), we obtain

$$\exp^{[p]}\{\alpha |z_j|^{\sigma-\varepsilon}\} \leq |A_0(z_j)| \leq M \cdot \big[r_j^c \cdot T(2r_j, f)\big]^{2k} \cdot k \cdot \exp^{[p]}\{\beta |z_j|^{\sigma-\varepsilon}\}$$

as $z_j \to \infty$, $|z_j| \notin E_1 \bigcup [0,1]$. Since $\alpha > \beta \geq 0$, we get that

$$\exp\big((\alpha-\beta)|z_j|^{\sigma-\varepsilon}\big) \leq M\left[r_j^c T(2r_j,f)\right]^{2k} k,$$

for $p = 1$,

$$\exp\Big((1-\gamma)\exp^{[p-1]}\{\alpha|z_j|^{\sigma-\varepsilon}\}\Big) \leq M\left[r_j^c T(2r_j,f)\right]^{2k} k,$$

for $p > 1$, as $z_j \to \infty$, $|z_j| \notin E_1 \bigcup [0,1]$, where γ $(0 < \gamma < 1)$ is a real number. Hence from Definition 1.2, Definition 1.4 and Lemma 2.3 , we can deduce that $i(f) \geq p+1$ and

$$\sigma_{p+1}(f) + \varepsilon \geq \sigma = \sigma_p(A_0). \tag{3.3}$$

On the other hand, by Lemma 2.1, there exists a set $E_2 \subset \{1, +\infty\}$ with finite logarithmic measure, when $|z| = r \notin [0,1] \bigcup E_2$, and $|f(z)| = M(r,f)$, we have

$$\frac{f^{(m)}(z)}{f(z)} = \big(\frac{\nu_f(r)}{z}\big)^m (1+o(1)) \quad (m = 1,\dots,k). \tag{3.4}$$

Since $\sigma = \sigma_p(A_0) \geq \sigma_p(A_n), (n = 1,2,\dots,k-1)$, then for any given $\varepsilon(>0)$ and sufficiently large r, and by Definition 1.2, we get

$$|A_i(z)| \leq \exp^{[p]}\{r^{\sigma+\varepsilon}\} \quad (i = 0,\dots,k-1). \tag{3.5}$$

Substituting (3.4) and (3.5) into (1.1), we get

$$\begin{aligned} &\big(\frac{\nu_f(r)}{|z|}\big)^k |1+o(1)| \\ &\leq \big(\frac{\nu_f(r)}{|z|}\big)^{k-1} |1+o(1)| (|A_{k-1}| + \dots + |A_0|) \\ &\leq k\big(\frac{\nu_f(r)}{|z|}\big)^{k-1} |1+o(1)| \exp^{[p]}\{r^{\sigma+\varepsilon}\}, \quad r \notin [0,1] \bigcup E_2. \end{aligned} \tag{3.6}$$

By Lemma 2.3, Lemma 2.6 and (3.6), we obtain $i(f) \leq p+1$ and

$$\sigma_{p+1}(f) = \limsup_{r\to\infty} \frac{\log^{[p+1]} \nu_f(r)}{\log r} \leq \sigma + \varepsilon. \tag{3.7}$$

Since ε is arbitrary, we get from (3.3) and (3.7) that Theorem 1.8 holds. □

Proof of Theorem 1.12. We can assume that $\{f_1,\dots,f_k\}$ is an entire solution base of (1.1). By Theorem 1.8, we know that $i(f_j) = p+1$ and $\sigma_{p+1}(f_j) = \sigma_p(A_0)$. Thus any solution of (1.4) has the form

$$f(z) = B_1 f_1 + B_2 f_2 + \dots + B_k f_k, \tag{3.8}$$

where $B_1,\dots,B_k$ are suitable entire functions satisfying

$$B_j' = F \cdot G_j(f_1,\dots,f_k) \cdot W(f_1,\dots,f_k)^{-1} \quad (j = 1,\dots,k), \tag{3.9}$$

where G_j $(f_1, \dots, f_k)$ is differential polynomials in $f_1, \dots, f_k$ and their derivatives, and $W(f_1, \dots, f_k)$ is the Wronskian of $f_1,\dots,f_k$. By Lemma 2.2 and the above-mentioned, we obtain

$$i(f) \leq \max\{p+1, q\}. \tag{3.10}$$

(1) If either $q > p+1$, or $q = p+1$ while $\sigma_q(F) > \sigma_p(A_0)$, it follows from (3.8)-(3.10) and Eq.(1.4) that $i(f) = i(F) = q$ and $\sigma_q(f) = \sigma_q(F)$.

(2) If either $q < p+1$,or $q = p+1$ while $\sigma_q(F) < \sigma_p(A_0)$, it follows from (3.8) -(3.10) and Eq.(1.4) that either $i(f) < p+1$, or $i(f) = p+1$ and $\sigma_{p+1}(f) \leq \sigma_p(A_0)$.

Now we assert that all solutions f of (1.4) satisfy $i(f) = p + 1$ and $\sigma_{p+1}(f) = \sigma_p(A_0)$ with at most one exception. In fact, if there exist two entire solutions g_1, g_2 of (1.4) which satisfy either $i(g_j) < p + 1$ or $\sigma_{p+1}(g_j) < \sigma_p(A_0), (j = 1, 2)$, then $g = g_1 - g_2$ is a solution of (1.1) which satisfies either $i(g) = i(g_1 - g_2) < p + 1$ or $\sigma_{p+1}(g) = \sigma_{p+1}(g_1 - g_2) < \sigma_p(A_0)$. But $g = g_1 - g_2$ is a solution of (1.1) satisfying $i(g) = i(g_1 - g_2) = p + 1$ and $\sigma_{p+1}(g) = \sigma_{p+1}(g_1 - g_2) = \sigma_p(A_0)$ by Theorem 1.8. This is a contradiction.

By Lemma 2.7, we know that all solutions f of (1.4) with $i(f) = p + 1$ and $\sigma_{p+1}(f) = \sigma_p(A_0)$ satisfy $i_{\overline{\lambda}}(f) = i_\lambda(f) = i(f) = p + 1$ and $\sigma_{p+1}(f) = \overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f)$. Therefore, we deduce that Theorem 1.12 holds. $\square$

Proofs of Corollaries 1.10 and 1.11. Let $\max_{1\le n\le k-1}\{\sigma_p(A_n)\} = b < \sigma_p(A_0) = \sigma$. Then for any given ε, $0 < \varepsilon < (\sigma - b)/2$, by Definition 1.2, we have

$$|A_0(z)| > \exp^{[p]}\{|z|^{\sigma-\varepsilon}\},$$
$$|A_n(z)| < \exp^{[p]}\{|z|^{b+\varepsilon}\} \le \exp^{[p]}\{\delta|z|^{\sigma-\varepsilon}\} \quad (n = 1, \dots, k-1),$$

for sufficiently large $|z|$, where δ $(0 < \delta < 1)$ is a real number. By making use of the above two inequalities and Theorem 1.8, we get that Corollary 1.10 follows. Corollary 1.11 is just a special case of Corollary 1.10 when $p = 1$ $\square$

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