Electronic Journal of Differential Equations, Vol. 2006(2006), No. 96, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

EXISTENCE, MULTIPLICITY AND INFINITE SOLVABILITY OF POSITIVE SOLUTIONS FOR p-LAPLACIAN DYNAMIC EQUATIONS ON TIME SCALES

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Abstract. In this paper, by using Guo-Krasnosel'skii fixed point theorem in cones, we study the existence, multiplicity and infinite solvability of positive solutions for the following three-point boundary value problems for p-Laplacian dynamic equations on time scales

$$[\Phi_p(u^{\triangle}(t))]^{\nabla} + a(t)f(t,u(t)) = 0, \quad t \in [0,T]_{\mathbf{T}},$$
$$u(0) - B_0(u^{\triangle}(\eta)) = 0, \quad u^{\triangle}(T) = 0.$$

By multiplicity we mean the existence of arbitrary number of solutions.

1. Introduction

Let $\mathbb{T}$ be a closed nonempty subset of $\mathbb{R}$, and let $\mathbb{T}$ have the subspace topology inherited from the Euclidean topology on $\mathbb{R}$. In some of the current literature, $\mathbb{T}$ is called a time scale (or measure chain). For notation, we shall use the convention that, for each interval J of $\mathbb{R}$,

$$J_{\mathbb{T}} = J \cap \mathbb{T}.$$

The theory of dynamic equations on time scales has become a new important mathematical branch (see, for example [1, 8, 16]) since it was initiated by Hilger [14]. At the same time, boundary-value problems (BVPs) for scalar dynamic equations on time scales has received considerable attention [2, 3, 4, 5, 6, 9, 10, 11]. The purpose of this paper is to investigate the existence, multiplicity and infinite solvability of positive solutions for p-Laplacian dynamic equations on time scales

$$[\Phi_p(u^{\triangle}(t))]^{\nabla} + a(t)f(t,u(t)) = 0, \quad t \in [0,T]_{\mathbb{T}}, \tag{1.1}$$

satisfying the boundary conditions

$$u(0) - B_0(u^{\triangle}(\eta)) = 0, \quad u^{\triangle}(T) = 0, \tag{1.2}$$

or

$$u^{\triangle}(0) = 0, \quad u(T) + B_1(u^{\triangle}(\eta)) = 0, \tag{1.3}$$

2000 *Mathematics Subject Classification.* 34B10, 34B18, 39A10.
Key words and phrases. Time scales; p-Laplacian; boundary value problem; positive solution; existence; multiplicity; infinite solvability.

Submitted April 14, 2006. Published August 22, 2006.

where $\Phi_p(s)$ is p-Laplacian operator, i.e., $\Phi_p(s) = |s|^{p-2}s,\ p > 1, (\Phi_p)^{-1} = \Phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$, $\eta \in (0, \rho(T))_{\mathbb{T}}$. Here, by multiplicity we mean the existence of m solutions, where m is an arbitrary natural number.

In this paper we assume the following hypotheses:

(H1) $f : [0,T]_{\mathbb{T}} \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous ($\mathbb{R}^+$ denotes the nonnegative real numbers)

(H2) $a : \mathbb{T} \to \mathbb{R}^+$ is left dense continuous (i.e., $a \in C_{\rm ld}(\mathbb{T}, \mathbb{R}^+)$) and does not vanish identically on any closed subinterval of $[0,T]_{\mathbb{T}}$, where $C_{\rm ld}(\mathbb{T}, \mathbb{R}^+)$ denotes the set of all left dense continuous functions from $\mathbb{T}$ to $\mathbb{R}^+$.

(H3) $B_0(v)$ and $B_1(v)$ are both continuous odd functions defined on R and satisfy that there exist $C, D > 0$ such that

$$Dv \le B_j(v) \le Cv, \quad \text{for all } v \ge 0, \quad j = 0, 1.$$

We remark that by a solution u of (1.1), (1.2) (respectively (1.1),(1.3)), we mean $u : \mathbb{T} \to \mathbb{R}$ is delta differentiable, $u^{\triangle} : \mathbb{T}^\kappa \to \mathbb{R}$ is nabla differentiable on $\mathbb{T}^\kappa \cap \mathbb{T}_\kappa$ and $u^{\triangle\nabla} : \mathbb{T}^\kappa \cap \mathbb{T}_\kappa \to \mathbb{R}$ is continuous, and u satisfies boundary conditions (1.2) (respectively (1.3)). If $u^{\triangle\nabla}(t) \le 0$ on $[0,T]_{\mathbb{T}^\kappa \cap \mathbb{T}_\kappa}$, then we say u is concave on $[0,T]_{\mathbb{T}}$.

Anderson, Avery and Henderson [5] considered the problem

$$\begin{gathered} [\Phi_p(u^{\triangle}(t))]^{\nabla} + c(t)f(u(t)) = 0, \quad t \in (a,b)_{\mathbb{T}}, \\ u(a) - B_0(u^{\triangle}(v)) = 0, \quad u^{\triangle}(b) = 0, \end{gathered}$$

where $v \in (a,b)_{\mathbb{T}}$, $f \in C(\mathbb{R}^+, \mathbb{R}^+)$, $c \in C_{\rm ld}((a,b)_{\mathbb{T}}, \mathbb{R}^+)$ and $K_m x \le B_0(x) \le K_M x$ for some positive constants K_m, K_M. They established the existence result of at least one positive solution by a fixed point theorem of cone expansion and compression of functional type.

Very recently, in the case $f(t,l) = f(l)$, the existence of two positive solutions for the problem (1.1), (1.2) and (1.1), (1.3) has been established by He [13] by using a new double fixed-point theorem due to Avery, et al [7] in a cone.

In this paper, we shall apply the method arising in papers [17, 18] to problem (1.1), (1.2). The main ingredient is the Guo-Krasnosel'skii fixed point theorem in cone. By considering the properties of f on a bounded set of $[0,T]_{\mathbb{T}} \times \mathbb{R}^+$, we shall establish a basic existence criterion, that is theorem 3.1. Then, we shall prove the existence of m positive solutions (in section 4) and the existence of infinitely many positive solutions (in section 5).

In the remainder of this section we will provide without proof several foundational definitions and results from the calculus on time scales so that the paper is self-contained. For more details, one can see [1, 6, 8, 14, 16].

Definition 1.1. For $t < \sup \mathbb{T}$ and $r > \inf \mathbb{T}$, define the forward jump operator σ and the backward jump operator ρ, respectively,

$$\sigma(t) = \inf\{\tau \in \mathbb{T} | \tau > t\} \in \mathbb{T}, \quad \rho(r) = \sup\{\tau \in \mathbb{T} | \tau < r\} \in \mathbb{T}$$

for all t, $r \in \mathbb{T}$. If $\sigma(t) > t$, t is said to be right scattered, and if $\rho(r) < r$, r is said to be left scattered. If $\sigma(t) = t$, t is said to be right dense, and if $\rho(r) = r$, r is said to be left dense. If $\mathbb{T}$ has a right scattered minimum m, define $\mathbb{T}_\kappa = \mathbb{T} - \{m\}$; otherwise set $\mathbb{T}_\kappa = \mathbb{T}$. If $\mathbb{T}$ has a left scattered maximum M, define $\mathbb{T}^\kappa = \mathbb{T} - \{M\}$; otherwise set $\mathbb{T}^\kappa = \mathbb{T}$.

Definition 1.2. For $x:\mathbb{T}\to\mathbb{R}$ and $t\in\mathbb{T}^{\kappa}$, we define the delta derivative of $x(t)$, $x^{\triangle}(t)$, to be the number (when it exists), with the property that, for any $\varepsilon>0$, there is a neighborhood U of t such that

$$|[x(\sigma(t))-x(s)]-x^{\triangle}(t)[\sigma(t)-s]|<\varepsilon|\sigma(t)-s|,$$

for all $s\in U$. For $x:\mathbb{T}\to\mathbb{R}$ and $t\in\mathbb{T}_{\kappa}$, we define the nabla derivative of $x(t)$, $x^{\nabla}(t)$, to be the number (when it exists), with the property that, for any $\varepsilon>0$, there is a neighborhood V of t such that

$$|[x(\rho(t))-x(s)]-x^{\nabla}(t)[\rho(t)-s]|<\varepsilon|\rho(t)-s|,$$

for all $s\in V$.

If $\mathbb{T}=\mathbb{R}$, then $x^{\triangle}(t)=x^{\nabla}(t)=x'(t)$. If $\mathbb{T}=\mathbb{Z}$, then $x^{\triangle}(t)=x(t+1)-x(t)$ is the forward difference operator while $x^{\nabla}(t)=x(t)-x(t-1)$ is the backward difference operator.

Definition 1.3. If $F^{\triangle}(t)=f(t)$, then we define the delta integral by

$$\int_a^t f(s)\triangle s=F(t)-F(a).$$

If $\Phi^{\nabla}(t)=f(t)$, then we define the nabla integral by

$$\int_a^t f(s)\nabla s=\Phi(t)-\Phi(a).$$

Throughout this paper, we assume $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$ with $0\in\mathbb{T}_{\kappa}$, $T\in\mathbb{T}^{\kappa}$.

2. Preliminaries

Consider the Banach space $E=C_{\rm ld}([0,T]_{\mathbb{T}},\mathbb{R})$ with norm $\|u\|=\sup_{t\in[0,T]_{\mathbb{T}}}|u(t)|$. Then define the cone by

$$K=\big\{u\in E|\quad u \text{ is concave and nonnegative valued on } [0,T]_{\mathbb{T}},\text{ and } u^{\triangle}(T)=0\big\}.$$

From [13], we know that if $u\in K$, then

$$\inf_{t\in[\eta,T]_{\mathbb{T}}}u(t)\ge(\eta/T)\|u\|.$$

We define a operator $F:K\to E$ by

$$(Fu)(t)=B_0(\Phi_q(\int_\eta^T a(r)f(r,u(r))\nabla r))+\int_0^t\Phi_q(\int_s^T a(r)f(r,u(r))\nabla r)\triangle s,$$

and from [13], we also know $F:K\to K$ is completely continuous. We denote the constants

$$A=\Big[C\Phi_q(\int_\eta^T a(r)\nabla r)+\int_0^T\Phi_q(\int_s^T a(r)\nabla r)\triangle s\Big]^{-1},\quad B=\Big[D\Phi_q(\int_\eta^T a(r)\nabla r)\Big]^{-1}.$$

Clearly, $0 < A < B$. The following symbols are used in this paper:

$$\alpha(l) = \max\{f(t,c) : (t,c) \in [0,T]_{\mathbb{T}} \times [0,l]\},$$
$$\beta(l) = \min\{f(t,c) : (t,c) \in [\eta,T]_{\mathbb{T}} \times [(\eta/T)l, l]\},$$
$$\underline{\alpha}_0 = \liminf_{l\to 0} \alpha(l)/l^{p-1}, \quad \underline{\alpha}_\infty = \liminf_{l\to +\infty} \alpha(l)/l^{p-1},$$
$$\overline{\beta}_0 = \limsup_{l\to 0} \beta(l)/l^{p-1}, \quad \overline{\beta}_\infty = \limsup_{l\to +\infty} \beta(l)/l^{p-1};$$
$$\max \overline{f}_0 = \limsup_{l\to 0} \max_{t\in[0,T]_{\mathbb{T}}} f(t,l)/l^{p-1}, \quad \max \overline{f}_\infty = \limsup_{l\to +\infty} \max_{t\in[0,T]_{\mathbb{T}}} f(t,l)/l^{p-1},$$
$$\min \underline{f}_0 = \liminf_{l\to 0} \min_{t\in[\eta,T]_{\mathbb{T}}} f(t,l)/l^{p-1}, \quad \min \underline{f}_\infty = \liminf_{l\to +\infty} \min_{t\in[\eta,T]_{\mathbb{T}}} f(t,l)/l^{p-1},$$
$$\max f_0 = \lim_{l\to 0} \max_{t\in[0,T]_{\mathbb{T}}} f(t,l)/l^{p-1}, \quad \max f_\infty = \lim_{l\to +\infty} \max_{t\in[0,T]_{\mathbb{T}}} f(t,l)/l^{p-1},$$
$$\min f_0 = \lim_{l\to 0} \min_{t\in[\eta,T]_{\mathbb{T}}} f(t,l)/l^{p-1}, \quad \min f_\infty = \lim_{l\to +\infty} \min_{t\in[\eta,T]_{\mathbb{T}}} f(t,l)/l^{p-1}.$$

Lemma 2.1. (1) *If* $\max \overline{f}_0 < A^{p-1}$, *then* $\underline{\alpha}_0 < A^{p-1}$.
(2) *If* $\max \overline{f}_\infty < A^{p-1}$, *then* $\underline{\alpha}_\infty < A^{p-1}$.
(3) *If* $\min \underline{f}_0 > (TB^{p-1})/\eta$, *then* $\overline{\beta}_0 > B^{p-1}$.
(4) *If* $\min \underline{f}_\infty > (TB^{p-1})/\eta$, *then* $\overline{\beta}_\infty > B^{p-1}$.

Proof. It is easy to show that the following inequalities hold:

$$\limsup_{l\to 0} \max_{t\in[0,T]_{\mathbb{T}}} f(t,l)/l^{p-1} \ge \liminf_{l\to 0} \max\{f(t,c) : (t,c) \in [0,T]_{\mathbb{T}} \times [0,l]\}/l^{p-1},$$
$$\limsup_{l\to +\infty} \max_{t\in[0,T]_{\mathbb{T}}} f(t,l)/l^{p-1} \ge \liminf_{l\to +\infty} \max\{f(t,c) : (t,c) \in [0,T]_{\mathbb{T}} \times [0,l]\}/l^{p-1},$$
$$\begin{aligned}&\liminf_{l\to 0} \min_{t\in[\eta,T]_{\mathbb{T}}} f(t,l)/l^{p-1}\\ &\le \limsup_{l\to 0} \min\{f(t,c) : (t,c) \in [\eta,T]_{\mathbb{T}} \times [(\eta/T)l, l]\}/((l^{p-1}\eta)/T),\end{aligned}$$
$$\begin{aligned}&\liminf_{l\to +\infty} \min_{t\in[\eta,T]_{\mathbb{T}}} f(t,l)/l^{p-1}\\ &\le \limsup_{l\to +\infty} \min\{f(t,c) : (t,c) \in [\eta,T]_{\mathbb{T}} \times [(\eta/T)l, l]\}/((l^{p-1}\eta)/T).\end{aligned}$$

The statements of the lemma follow from these inequalities. □

The following Lemma is crucial in our argument, which is the well-known Guo-Krasnosel'skii fixed point theorem in cone.

Lemma 2.2 ([12, 15])**.** *Let X be a Banach space and $K \subset E$ be a cone in X. Assume Ω_1, Ω_2 are bounded open subsets of K with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$ and $T : K \to K$ is a completely continuous operator such that either:*
(1) $\|Tw\| \le \|w\|$, $w \in \partial\Omega_1$, *and* $\|Tw\| \ge \|w\|$, $w \in \partial\Omega_2$; *or*
(2) $\|Tw\| \ge \|w\|$, $w \in \partial\Omega_1$, *and* $\|Tw\| \le \|w\|$, $w \in \partial\Omega_2$.

Then T has a fixed point in $\overline{\Omega}_2 \backslash \Omega_1$.

3. Existence results

Theorem 3.1. *Assume that there exist two positive numbers a, b such that $\alpha(a) \le (aA)^{p-1}$, $\beta(b) \ge (bB)^{p-1}$. Then problem* (1.1), (1.2) *has at least one positive* *solution* $u^* \in K$ *satisfying*

$$\min\{a, b\} \le u^* \le \max\{a, b\}.$$

Proof. First of all, we claim $a \neq b$. If not, $a = b$. Noticing that $A < B$, then

$$\begin{aligned}
&\max\{f(t,l) : (t,l) \in [0,T]_{\mathbb{T}} \times [0,a]\} \\
&= \alpha(a) \le (aA)^{p-1} \\
&< (aB)^{p-1} \le \beta(a) \\
&= \min\{f(t,c) : (t,c) \in [\eta,T]_{\mathbb{T}} \times [(\eta/T)a, a]\}.
\end{aligned}$$

This is impossible.

Without loss of generality, we may assume $a < b$. We denote $\Omega_c = \{u \in K : \|u\| < c\}$, $\partial\Omega_c = \{u : \|u\| = c\}$. If $u \in \partial\Omega_a$, then $0 \le u \le a$, $t \in [0,T]_{\mathbb{T}}$. So,

$$f(t, u(t)) \le \alpha(a) \le (aA)^{p-1}, \quad t \in [0,T]_{\mathbb{T}}.$$

It follows that

$$\begin{aligned}
\|Fu\| &= B_0(\Phi_q(\int_\eta^T a(r)f(r,u(r))\nabla r)) + \int_0^T \Phi_q(\int_s^T a(r)f(r,u(r))\nabla r)\triangle s \\
&\le aAC\Phi_q(\int_\eta^T a(r)\nabla r) + aA\int_0^T \Phi_q(\int_s^T a(r)\nabla r)\triangle s \\
&= aA[C\Phi_q(\int_\eta^T a(r)\nabla r) + \int_0^T \Phi_q(\int_s^T a(r)\nabla r)\triangle s] \\
&= a = \|u\|.
\end{aligned}$$

If $u \in \partial\Omega_b$, then $(\eta/T)b = (\eta/T)\|u\| \le \min_{t\in[\eta,T]_{\mathbb{T}}} u(t) \le u(t) \le b$, $t \in [\eta,T]_{\mathbb{T}}$. So

$$f(t, u(t)) \ge \beta(b) \ge (bB)^{p-1}, \quad t \in [\eta,T]_{\mathbb{T}}.$$

It follows that

$$\begin{aligned}
\|Fu\| &= B_0(\Phi_q(\int_\eta^T a(r)f(r,u(r))\nabla r)) + \int_0^T \Phi_q(\int_s^T a(r)f(r,u(r))\nabla r)\triangle s \\
&\ge B_0(\Phi_q(\int_\eta^T a(r)f(r,u(r))\nabla r)) \\
&\ge DbB\Phi_q(\int_\eta^T a(r)\nabla r) \\
&= b = \|u\|.
\end{aligned}$$

By Lemma 2.2, F has a fixed point $u^* \in \overline{\Omega}_b \backslash \Omega_a$. □

Corollary 3.2. *Assume* $\inf_{l>0} \alpha(l)/l^{p-1} < A^{p-1}$ *and* $\sup_{l>0} \beta(l)/l^{p-1} > B^{p-1}$. *Then problem* (1.1), (1.2) *has at least one positive solution.*

By Theorem 3.1 and Lemma 2.1 we have the following results.

Corollary 3.3. *Assume that one of the following conditions holds:*

(1) $\underline{\alpha}_0 < A^{p-1}$ *and* $\overline{\beta}_\infty > B^{p-1}$ *(in particular,* $\underline{\alpha}_0 = 0$ *and* $\overline{\beta}_\infty = +\infty$*,*
(2) $\overline{\beta}_0 > B^{p-1}$ *and* $\underline{\alpha}_\infty < A^{p-1}$ *(in particular,* $\overline{\beta}_0 = +\infty$ *and* $\underline{\alpha}_\infty = 0$*).*

Then problem (1.1), (1.2) *has at least one positive solution.*

Corollary 3.4. *Assume that one of the following conditions holds:*

(1) $\max \overline{f}_0 < A^{p-1}$ *and* $\min \underline{f}_\infty > (TB^{p-1})/\eta$ *(in particular,* $\max f_0 = 0$ *and* $\min f_\infty = +\infty$*)*
(2) $\min \underline{f}_0 > (TB^{p-1})/\eta$ *and* $\max \overline{f}_\infty < A^{p-1}$ *(in particular,* $\min f_0 = +\infty$ *and* $\max f_\infty = 0$*).*

Then problem (1.1), (1.2) *has at least one positive solution.*

The special case of Corollary 3.4 is a useful result for superlinear and sublinear problems.

Corollary 3.5. *Assume that*
(1) $\inf_{l>0} \alpha(l)/l^{p-1} < A^{p-1}$ *(in particular, there exists* $a > 0$ *such that* $\alpha(a) < (aA)^{p-1}$*),*
(2) $\max\{\min \underline{f}_0, \min \underline{f}_\infty\} > (TB^{p-1})/\eta$ *(in particular when* $\min f_0 = +\infty$ *or* $\min f_\infty = +\infty$*).*

Then problem (1.1), (1.2) *has at least one positive solution.*

Corollary 3.6. *Assume that:*
(1) $\sup_{l>0} \beta(l)/l^{p-1} > B^{p-1}$ *(in particular, there exists* $b > 0$ *such that* $\beta(b) > (bB)^{p-1}$*),*
(2) $\min\{\max \overline{f}_0, \max \overline{f}_\infty\} < A^{p-1}$ *(in particular,* $\max f_0 = 0$ *or* $\max f_\infty = 0$*).*

Then problem (1.1), (1.2) *has at least one positive solution.*

4. Multiplicity

Let $[c]$ be the integer part of c.

Theorem 4.1. *Let* $0 < a_1 < a_2 < \dots < a_{m+1}$*. If one of the following conditions holds:*
(1) $\alpha(a_{2k-1}) < (a_{2k-1}A)^{p-1}$, $k = 1, \dots, [\frac{m+2}{2}]$, $\beta(a_{2k}) > (a_{2k}B)^{p-1}$, $k = 1, \dots, [\frac{m+1}{2}]$
(2) $\beta(a_{2k-1}) > (a_{2k-1}B)^{p-1}$, $k = 1, \dots, [\frac{m+2}{2}]$, $\alpha(a_{2k}) < (a_{2k}A)^{p-1}$, $k = 1, \dots, [\frac{m+1}{2}]$.

Then problem (1.1), (1.2) *has at least* m *positive solutions* $u_1^*, u_2^*, \dots, u_m^*$ *satisfying*
$$a_k < \|u_k^*\| < a_{k+1}, \quad k = 1, 2, \dots, m.$$

Proof. We prove only Case (2). The proof of Case (1) is similar. By the continuity of α and β, there exist
$$0 < b_1 < a_1 < c_1 < b_2 < a_2 < c_2 < \dots < c_m < b_{m+1} < a_{m+1} < +\infty$$
such that
$$\beta(b_{2k-1}) \geq (b_{2k-1}B)^{p-1}, \quad \beta(c_{2k-1}) \geq (c_{2k-1}B)^{p-1}, \quad k = 1, \dots, [\frac{m+2}{2}],$$
$$\alpha(b_{2k}) \leq (b_{2k}A)^{p-1}, \quad \alpha(c_{2k}) \leq (c_{2k}A)^{p-1}, \quad k = 1, \dots, [\frac{m+1}{2}].$$
Applying Theorem 3.1 for each pair of numbers $\{c_k, b_{k+1}\}$, $k = 1, 2, \dots, m$, we complete the proof. □

By Theorem 4.1 and Lemma 2.2 we have the following existence results of two or three positive solutions.

Corollary 4.2. *Assume that*

(1) $\inf_{l>0} \alpha(l)/l^{p-1} < A^{p-1}$ *(in particular, there exists* $a > 0$ *such that* $\alpha(a) < (aA)^{p-1}$*)*
(2) $\min\{\min \underline{f}_0,\ \min \underline{f}_\infty\} > (TB^{p-1})/\eta$ *(in particular,* $\min f_0 = \min f_\infty = +\infty$*).*

Then problem (1.1), (1.2) *has at least two positive solutions.*

Corollary 4.3. *Assume that:*
(1) $\sup_{l>0} \beta(l)/l^{p-1} > B^{p-1}$ *(in particular, there exists* $b > 0$ *such that* $\beta(b) > (bB)^{p-1}$*)*
(2) $\max\{\max \overline{f}_0, \max \overline{f}_\infty\} < A^{p-1}$ *(in particular,* $\max f_0 = \max f_\infty = 0$*).*

Then problem (1.1), (1.2) *has at least two positive solutions.*

Corollary 4.4. *Let* $0 < a_1 < a_2 < +\infty$*. If*
(1) $\min \underline{f}_0 > (TB^{p-1})/\eta$ *and* $\max \overline{f}_\infty < A^{p-1}$ *(in particular,* $\min f_0 = +\infty$ *and* $\max f_\infty = 0$*)*
(2) $\alpha(a_1) < (a_1 A)^{p-1}$ *and* $\beta(a_2) > (a_2 B)^{p-1}$.

Then problem (1.1), (1.2) *has at least three positive solutions.*

Corollary 4.5. *Let* $0 < a_1 < a_2 < +\infty$*. If*
(1) $\max \overline{f}_0 < A^{p-1}$ *and* $\min \underline{f}_\infty > (TB^{p-1})/\eta$ *(in particular,* $\max f_0 = 0$ *and* $\min f_\infty = +\infty$*)*
(2) $\beta(a_1) > (a_1 B)^{p-1}$ *and* $\alpha(a_2) < (a_2 A)^{p-1}$.

Then problem (1.1), (1.2) *has at least three positive solutions.*

Obviously, analogous results still hold for arbitrary number m. Also we have the following result.

Theorem 4.6. *Let* $0 < a_1 < a_2 < \cdots < a_{2m} < +\infty$*. If one of the following conditions holds:*
(1) $\alpha(a_{2k-1}) \le (a_{2k-1}A)^{p-1}$, $\beta(a_{2k}) \ge (a_{2k}B)^{p-1}$, $k = 1, \dots, m$
(2) $\beta(a_{2k-1}) \ge (a_{2k-1}B)^{p-1}$, $\alpha(a_{2k}) \le (a_{2k}A)^{p-1}$, $k = 1, \dots, m$;

then problem (1.1), (1.2) *has at least* m *positive solutions* $u_1^*, u_2^*, \dots, u_m^*$ *satisfying*

$$a_1 \le \|u_1^*\| < \|u_2^*\| < \cdots < \|u_m^*\| \le a_{2m}.$$

Proof. Applying Theorem 3.1 for each pair of numbers $\{a_{2k-1}, a_{2k}\}$, $k = 1, \dots, m$, the proof is completed. □

5. Infinite Solvability

Theorem 5.1. *Assume that* $\underline{\alpha}_0 < A^{p-1}$ *and* $\overline{\beta}_0 > B^{p-1}$ *(in particular,* $\underline{\alpha}_0 = 0$ *and* $\overline{\beta}_0 = +\infty$*). Then problem* (1.1), (1.2) *has a sequence of positive solutions* $\{u_k^*\}_{k=1}^\infty$ *satisfying* $\|u_k^*\| \to 0$ *as* $k \to \infty$.

Proof. Since $\liminf_{l\to 0} \alpha(l)/l^{p-1} < A^{p-1}$ and $\limsup_{l\to 0} \beta(l)/l^{p-1} > B^{p-1}$, there exist two sequences of positive numbers $a_k \to 0$ and $b_k \to 0$ such that

$$\alpha(a_k) \le (a_k A)^{p-1}, \quad \beta(b_k) \ge (b_k B)^{p-1}, \quad k = 1, \quad 2, \dots$$

Without loss of generality, we may assume

$$a_1 > b_1 > a_2 > b_2 > \cdots > a_k > b_k > \dots.$$

Now applying Theorem 3.1 to each pair of numbers $\{b_k, a_k\}$, $k = 1, 2, \dots$, problem (1.1), (1.2) has a sequence of positive solutions $\{u_k^* t\}_{k=1}^{\infty}$ satisfying $b_k \le \|u_k^*\| \le a_k$. The proof is complete. □

Corollary 5.2. *Assume that there exists an l_0 such that*

$$\inf_{0<l\le l_0} \min_{t\in[\eta,T]_{\mathbb{T}}} f(t,l)/\alpha(l) \ge k_1 > 0, \qquad \sup_{0<l\le l_0} \max_{t\in[0,T]_{\mathbb{T}}} f(t,(l\eta)/T)/\beta(l) \le k_2 < +\infty.$$

If $\max \overline{f}_0 > (k_2 T B^{p-1})/\eta$ and $\min \underline{f}_0 < k_1 A^{p-1}$ (in particular, $\max \overline{f}_0 = +\infty$ and $\min \underline{f}_0 = 0$), then problem (1.1), (1.2) has a sequence of positive solutions $\{u_k^*\}_{k=1}^{\infty}$ satisfying $\|u_k^*\| \to 0$ as $k \to \infty$.

Proof. Clearly, $\alpha(l) \le \min_{t\in[\eta,T]_{\mathbb{T}}} f(t,l)/k_1$, $\beta(l) \ge \max_{t\in[0,T]_{\mathbb{T}}} f(t,(l\eta)/T)/k_2$, for $l \in (0, l_0]$. Then

$$\begin{aligned}\underline{\alpha}_0 &= \liminf_{l\to 0} \alpha(l)/l^{p-1} \le (1/k_1) \liminf_{l\to 0} \min_{t\in[\eta,T]_{\mathbb{T}}} f(t,l)/l^{p-1} \\ &= (1/k_1) \min \underline{f}_0 < (1/k_1)(k_1 A^{p-1}) = A^{p-1},\end{aligned}$$

$$\begin{aligned}\overline{\beta}_0 &= \limsup_{l\to 0} \beta(l)/l^{p-1} \ge (1/k_2) \limsup_{l\to 0} \max_{t\in[0,T]_{\mathbb{T}}} f(t,(l\eta)/T)/l^{p-1} \\ &= \eta/(Tk_2) \limsup_{l\to 0} \max_{t\in[0,T]_{\mathbb{T}}} f(t,(l\eta)/T)/(l\eta/T) \\ &= \eta/(Tk_2) \max \overline{f}_0 > \eta/(Tk_2)(k_2 T B^{p-1}/\eta) = B^{p-1}.\end{aligned}$$

Now the conclusion follows from Theorem 5.1. □

Similarly, we have the following statement.

Theorem 5.3. *Assume that $\underline{\alpha}_\infty < A^{p-1}$ and $\overline{\beta}_\infty > B^{p-1}$ (in particular, $\underline{\alpha}_\infty = 0$ and $\overline{\beta}_\infty = +\infty$). Then problem (1.1), (1.2) has a sequence of positive solutions $\{u_k^*\}_{k=1}^{\infty}$ satisfying $\|u_k^*\| \to +\infty$ as $k \to \infty$.*

Corollary 5.4. *Assume that*

$$\inf_{0<l\le+\infty} \min_{t\in[\eta,T]_{\mathbb{T}}} f(t,l)/\alpha(l) \ge k_1 > 0,$$
$$\sup_{0<l\le+\infty} \max_{t\in[0,T]_{\mathbb{T}}} f(t,l\eta/T)/\beta(l) \le k_2 < +\infty.$$

If $\max \overline{f}_\infty > (k_2 T B^{p-1})/\eta$ and $\min \underline{f}_\infty < k_1 A^{p-1}$ (in particular, $\max \overline{f}_\infty = +\infty$ and $\min \underline{f}_\infty = 0$), then problem (1.1), (1.2) has a sequence of positive solutions $\{u_k^*\}_{k=1}^{\infty}$ satisfying $\|u_k^*\| \to +\infty$ as $k \to \infty$.

6. Examples

Example 6.1. Let $\mathbb{T} = \{1 - (\frac{1}{2})^{\mathbb{N}_0}\} \cup \{1\}$, where $\mathbb{N}_0$ denotes the set of all non-negative integers. Consider the p-Laplacian dynamic equation

$$[\Phi_p(u^{\triangle}(t))]^{\nabla} + f(u(t)) = 0, t \in [0,1]_{\mathbb{T}}, \tag{6.1}$$

satisfying the boundary conditions

$$u(0) - 2u^{\triangle}(1/2) = 0, \quad u^{\triangle}(1) = 0, \tag{6.2}$$

where $p = 3/2$, $q = 3$, $a(t) \equiv 1$, $C = D = 2$, $T = 1$ and

$$f(u) = \begin{cases} 1, & 0 \le u \le 1, \\ 4u - 3 & 1 \le u \le \frac{3}{2}, \\ 3 & \frac{3}{2} \le u \le 10, \\ 2u - 17 & 10 \le u \le 12, \\ 7 & 12 \le u \le 24. \end{cases}$$

Then problem (6.1), (6.2) has at least two positive solutions.

To proof the statement of the above example, choose $a_1 = 1$, $a_2 = 3$, $a_3 = 10$, $a_4 = 24$. It is easy to see that $A = 6/5$, $B = 2$, and

$$\begin{gathered} \alpha(a_1) = \max\{f(u) : u \in [0, 1]\} = 1 < \sqrt{\frac{6}{5}} = (a_1 A)^{p-1}, \\ \alpha(a_3) = \max\{f(u) : u \in [0, 10]\} = 3 < \sqrt{\frac{60}{5}} = (a_3 A)^{p-1}, \\ \beta(a_2) = \min\{f(u) : u \in [\frac{3}{2}, 3]\} = 3 > \sqrt{6} = (a_2 B)^{p-1}, \\ \beta(a_4) = \min\{f(u) : u \in [12, 24]\} = 7 > \sqrt{48} = (a_4 B)^{p-1}. \end{gathered}$$

Therefore, by Theorem 4.6, problem (6.1), (6.2) has at least two positive solutions u_1^*, u_2^* satisfying $1 \le \|u_1^*\| < \|u_2^*\| \le 24$.

The methods in this paper can also be used for studying problem (1.1), (1.3).

References

[1] R. P. Agarwal, M. Bohner; *Basic calculus on time scales and some of its applications*, Result. Math. **35**(1999), 3-22.

[2] R. P. Agarwal, M. Bohner and P. Wong; *Sturm-Liouville eigenvalue problem on time scales*, Appl. Math. Comput. **99** (1999), 153-166.

[3] R. P. Agarwal, D. O'Regan; *Nonlinear boundary value problems on time scales*, Nonlinear Anal. **44**(2001), 527-535.

[4] D. Anderson, *Solutions to second-order three-point problem on time scales*, J. Difference Equations Appl. **8**(2002), 673-688.

[5] D. Anderson, R. Avery and J. Henderson; *Existence of solutions for a one-dimensional p-Laplacian on time scales*, J. Difference Equations Appl. **10**(2004), 889-896.

[6] F. M. Atici, G. Sh. Guseinov; *On Green's functions and positive solutions for boundary value problems on time scales*, J. Comput. Appl. Math. **141**(2002), 75-99.

[7] R. I. Avery, C. J. Chyan and J. Henderson; *Twin solutions of boundary value problems for ordinary differential equations and finite difference equations*, Comput. Math. Appl. **42**(2001), 695-704.

[8] M. Bohner, A. Peterson, *Dynamic Equations on Time Scales, An Introduction with Applications*, Birkhäuser, Boston, 2001.

[9] C. J. Chyan, J. Henderson; *Eigenvalue problems for nonlinear differential equations on a measure chain*, J. Math. Anal. Appl. **245**(2000), 547-559.

[10] L. Erbe, A. Peterson; *Positive solutions for nonlinear differential equations on measure chain*, Math. Comput. Modelling. **32**(5-6)(2000), 571-585.

[11] L. Erbe, S. Hilger, Sturmian theory on measure chains, *Differential Equations Dyn. Systems*, **1**(1993), 223-246.

[12] D. Guo, V. Lakshmikantham; *Nonlinear Problems in Abstract Cones*, Academic press, Boston, 1988.

[13] Z. M. He; *Double positive solutions of three-point boundary value problems for p-Laplacian dynamic equations on time scales*, J. Comput. Appl. Math. **182**(2005), 304-315.

[14] S. Hilger; *Analysis on measure chains-a unified approach to continuous and discrete calculus*, Results Math., **18** (1990), 18-56.

[15] M. Q. Krasnosel'skii, *Positive solutions of operator equations*, P. Noordhoff Ltd., Groningen, The Netherlands, 1964.

[16] B. Kaymakcalan, V. Lakshmikantham and S. Sivasundaram; *Dynamic Systems on Measure Chains*, Kluwer Academic Publishers, Boston, 1996.

[17] Q. Yao; *Positive solutions for eigenvalue problems of fourth-order elastic beam equations*, Appl. Math. Letter. **17** (2004), 237-243.

[18] Q. Yao; *Existence, multiplicity and infinite solvability of positive solutions to a nonlinear fourth-order periodic boundary value problem*, Nonlinear Analysis, **63**(2005), 237-246.

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