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DIFFERENT TYPES OF SOLVABILITY CONDITIONS FOR DIFFERENTIAL OPERATORS

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Abstract. Solvability conditions for linear differential equations are usually formulated in terms of orthogonality of the right-hand side to solutions of the homogeneous adjoint equation. However, if the corresponding operator does not satisfy the Fredholm property such solvability conditions may be not applicable. For this case, we obtain another type of solvability conditions, for ordinary differential equations on the real axis, and for elliptic problems in unbounded cylinders.

1. Introduction

Many methods of linear and nonlinear analysis are based on Fredholm type solvability conditions. We recall that an operator L satisfies the Fredholm property if, by definition, the dimension of its kernel is finite, the image is closed, the codimension of the image is also finite. If it is the case then the nonhomogeneous equation $Lu = f$ is solvable if and only if $\phi(f) = 0$ for a finite number of linearly independent functionals ϕ from the dual space. These functionals are solutions of the homogeneous adjoint equation $L^*\phi = 0$.

General elliptic boundary-value problems in bounded domains satisfy the Fredholm property if they satisfy the conditions of ellipticity, proper ellipticity and the Lopatinskii condition (see [2], [3], [21] and the references therein). In the case of unbounded domains these conditions are not sufficient. Some additional conditions formulated in terms of limiting operator should be imposed (see [22] and the references therein). To illustrate these conditions consider the one-dimensional second order operator

$$Lu = a(x)u'' + b(x)u' + c(x)u, \quad x \in \mathbb{R}$$

where a, b, and c are bounded sufficiently smooth matrices. We can consider it as acting in Sobolev or in Hölder spaces. Let h_k be a sequence of numbers, $h_k \to +\infty$ or $h_k \to -\infty$. Consider the shifted coefficients $\tilde{a}_k(x) = a(x+h_k)$, $\tilde{b}_k(x) = b(x+h_k)$,

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$\tilde{c}_k(x) = c(x + h_k)$ and choose locally convergent subsequences of these sequences. Then the operator with the limiting coefficients

$$\hat{L}u = \hat{a}(x)u'' + \hat{b}(x)u' + \hat{c}(x)u, \quad x \in \mathbb{R}$$

is called limiting operator. There can exists many limiting operator for the same operator L. The operator L is Fredholm if in addition to the conditions mentioned above all limiting operators are invertible. This condition is necessary and sufficient.

It is known that if an elliptic operator in an unbounded domain satisfies the Fredholm property, then the bounded solutions of the homogeneous equation $Lu = 0$ decay exponentially at infinity. Suppose that, for the operator considered above, there exists a bounded solution $u_0(x)$ of this equation that does not converge to zero at infinity. Then there exists a sequence h_k and a subsequence of the shifted solutions $u_0(x+h_k)$ locally converging to some limiting function $\hat{u}(x)$ such that it is a bounded nonzero solution of one of the limiting problems $\hat{L}\hat{u} = 0$. Therefore the limiting operator is not invertible and the operator L does not satisfy the Fredholm property.

Thus, if the homogeneous equation has a bounded solution that does not decay at infinity, then the usual solvability conditions may be not applicable. In some cases it is possible to reduce an operator that does not satisfy the Fredholm property to an operator that satisfies it. It can be done by introduction of some special weighted spaces or replacing, for example, a differential operator by an integro-differential operator (see e.g. [8]). In this work we develop another approach to study non Fredholm operators. In the case where the Fredholm type solvability conditions are not applicable we obtain another type of solvability conditions. They are also formulated in terms of solutions of the homogeneous adjoint equation but they cannot be written in terms of linear functionals from the dual space.

First we obtain these solvability conditions for ordinary differential operators on the real axis. Then we apply these results to study elliptic problems in unbounded cylinders. Some spectral projections allow us to reduce them to a sequence of ordinary differential operators.

Consider the operators $\mathbf{L} : U \to X$,

$$\mathbf{L}u = u_{xx} + \Delta_y u + A_0(x,y)u_x + \sum_{k=1}^{m} A_k(x,y)u_{y_k} + B(x,y)u \tag{1.1}$$

in an unbounded cylinder $\Omega = \mathbb{R} \times \Omega'$ with the homogeneous Dirichlet boundary condition. Here Ω' is a bounded domain in $\mathbb{R}^m$ with $C^{2+\delta}$ boundary, $0 < \delta < 1$, the coefficients of the operator belong to $C^\delta(\bar{\Omega})$, x is a variable along the axis of the cylinder Ω, and $y = (y_1, \dots, y_m)$ is a vector variable in the section Ω'. The function spaces are

$$U = \{u \in C^{2+\delta}(\bar{\Omega}) : u|_{\partial\Omega} = 0\} \quad \text{and} \quad X = C^\delta(\bar{\Omega}).$$

Here $C^\delta(\bar{\Omega})$ is a Hölder space with the norm

$$\|u\| = \sup_{\Omega} |u(x)| + \sup_{x,y \in \Omega} \frac{|u(x) - u(y)|}{|x-y|^\delta},$$

$C^{2+\delta}(\bar{\Omega})$ is the space of functions whose second derivatives belong to $C^\delta(\bar{\Omega})$.

The Fredholm property of such operators is studied in [9]–[16]. The particular form of the operator $\mathbf{L}$,

$$\mathbf{L}u = u_{xx} + \Delta_y u + A(x)u_x + B(x)u, \tag{1.2}$$

where its coefficients are independent of the variable y is more convenient to study it by the Fourier decomposition (see below). In some cases more general operator (1.1) can be reduced to the form (1.2) by a continuous deformation in the class of Fredholm operators (see [9]) or be approximated by an operator (1.2).

We shall study the linear boundary problems

$$\mathbf{L}u = 0 \tag{1.3}$$

and the nonhomogeneous one

$$\mathbf{L}u = f, \quad f \in X. \tag{1.4}$$

Denote by ω_k eigenvalues of the Laplace operator Δ_y on the space

$$U' = \{v \in C^{2+\delta}(\bar{\Omega}') \,:\, v|_{\partial\Omega'} = 0\}$$

and by p_k their multiplicities. Note that all ω_k are negative, tend to $-\infty$ as $k \to \infty$ and their multiplicities p_k are finite [11]. The corresponding eigenfunctions φ_k^i ($k \in \mathbb{N}$, $i = 1, \dots, p_k$) form an orthogonal basis in the space $\mathbb{L}^2(\Omega')$, so the functions u and f can be presented as Fourier series

$$\begin{aligned} u(x,y) &= \sum_{k=1}^{\infty} \sum_{i=1}^{p_k} u_k^i(x)\varphi_k^i(y); \\ f(x,y) &= \sum_{k=1}^{\infty} \sum_{i=1}^{p_k} f_k^i(x)\varphi_k^i(y). \end{aligned} \tag{1.5}$$

Having denoted $\lambda_k = \sqrt{-\omega_k}$, $v_k^i = {u_k^i}'/\lambda_k$, $w_k^i = (u_k^i, v_k^i)^T$, we can reduce boundary problems (1.3) and (1.4) to infinite sequences of $2n$ - dimensional ordinary differential systems

$${w_k^i}' = P_k(x)w_k^i \tag{1.6}$$

and

$${w_k^i}' = P_k(x)w_k^i + F_k^i(x) \tag{1.7}$$

respectively. Here

$$P_k(x) = \begin{pmatrix} 0 & \lambda_k E_n \\ -\big(B(x)/\lambda_k\big) + \lambda_k E_n - A(x) & \end{pmatrix}; \quad F_k^i(x) = \begin{pmatrix} 0 \\ f_k^i(x)/\lambda_k \end{pmatrix},$$

where E_n is $n \times n$ unit matrix.

Definition 1.1 ([19])**.** Let I be closed convex subset of $\mathbb{R}$. Consider a $n \times n$ matrix $P(x)$, continuous and bounded on I. The system

$$u' = P(x)u$$

is *dichotomic* on I if there exist positive constants c and λ, and subspaces $U^s(x)$ and $U^u(x)$ of $\mathbb{R}^n$, defined for all $x \in I$ and such that

1. $\Phi(x,\xi)U^{s,u}(\xi) = U^{s,u}(x)$ for all $x, \xi \in I$;
2. $U^s(x) \oplus U^u(x) = \mathbb{R}^n$ for every $x \in I$;
3. $|\Phi(x,\xi)u_0| \le c\exp(-\lambda(x-\xi))|u_0|$ for all $x, \xi \in I$: $x \ge \xi$, $u_0 \in U^s(\xi)$;
4. $|\Phi(x,\xi)u_0| \le c\exp(\lambda(x-\xi))|u_0|$, if $x, \xi \in I$: $x \le \xi$, $u_0 \in U^u(\xi)$.

This property is also called hyperbolicity and the corresponding system is called hyperbolic. Nevertheless, we shall always call it dichotomic in order not to confuse this notion with hyperbolicity of partial differential equations. Note that Definition 1.1 coincides with the definition of exponential dichotomy given by Coppel [7, p. 10] with the additional assumption of the boundedness of the matrix P.

Here and below we denote by $|\cdot|$ the Euclidian vector norm and the corresponding matrix norm,while by $\|\cdot\|$ the norms in function spaces. We shall use the following hypotheses:

Condition 1.2. All systems (1.6) are dichotomic on $\mathbb{R}$.

Condition 1.3. All systems (1.6) are dichotomic both on $\mathbb{R}^+ = [0, +\infty)$ and on $\mathbb{R}^- = (-\infty, 0]$.

It is shown in [14] that there exists a number $N \in \mathbb{N}$ (which depends on the operator $\mathbf{L}$) such that every system (1.6) for $k > N$ is dichotomic on $\mathbb{R}$. Therefore it is sufficient to check conditions 1.2 and 1.3 for a finite set of systems (1.6).

The following results are established in [15].

Theorem 1.4. *The operator* $\mathbf{L}$ *of the form* (1.2) *is invertible if and only if it satisfies condtion 1.2.*

Theorem 1.5. *The operator* $\mathbf{L}$ *of the form* (1.2) *is Fredholm if and only if it satisfies 1.3. Its index, that is the difference between the dimension of the kernel and the codimension of the image is given by the expression*

$$\mathrm{ind}\mathbf{L} = \sum_{k=1}^{+\infty} p_k (d_k^+ - d_k^-),$$

where d_k^+ *and* d_k^- *are dimensions of spaces* $M_k^{s,+}(x)$ *and* $M_k^{s,-}(x)$*, stable for systems* (1.6) *for* $t \geq 0$ *and* $t \leq 0$ *respectively, and* p_k *is a multiplicity of the eigenvalue* ω_k*.*

These theorems show that the dichotomy condition for elliptic operators introduced by Palmer [17] (see also [5], [6]) can be reduced to a sequence of dichotomy conditions for systems (1.6).

If one of systems (1.6) has a bounded solution that does not converge to zero at infinity, then Conditions 1.2 and 1.3 are not satisfied, and the elliptic operator does not satisfy the Fredholm property. To study such operators we introduce almost dichotomic systems (Section 3, 4) and weakly hyperbolic systems (Section 5) and obtain for them solvability conditions. These results are applied in Section 6 to study elliptic operators. In the next section we present a simple example illustrating non Fredholm solvability conditions.

2. Example of non Fredholm solvability conditions

We present here a simple example that illustrates the classical Fredholm type solvability conditions and other type solvability conditions when the Fredholm property is not satisfied. Consider the scalar equation

$$\frac{du}{dt} = a(t)u + f(t), \quad t \in \mathbb{R}. \tag{2.1}$$

One of solutions of (2.1) is given by the equality

$$u(t) = u_0(t) \int_0^t v_0(\tau) f(\tau) d\tau, \tag{2.2}$$

where

$$u_0(t) = e^{\int_0^t a(\tau)d\tau}, \quad v_0(t) = e^{-\int_0^t a(\tau)d\tau} = \frac{1}{u_0(t)},$$

where $u_0(t)$ is a solution of the homogeneous equation, and $v_0(t)$ is a solution of the homogeneous adjoint equation

$$\frac{du_0}{dt} = a(t)u_0, \quad \frac{dv_0}{dt} = -a(t)v_0.$$

Let us introduce the functions

$$\Phi^+(t) = |u_0(t)| \int_0^t |v_0(\tau)|d\tau, \quad \Psi^+(t) = |u_0(t)| \int_t^\infty |v_0(\tau)|d\tau, \quad t > 0,$$
$$\Phi^-(t) = |u_0(t)| \int_t 0|v_0(\tau)|d\tau, \quad \Psi^-(t) = |u_0(t)| \int_{-\infty}^t |v_0(\tau)|d\tau, \quad t < 0.$$

Condition 2.1. There exists a positive constant M such that:
- either $\Phi^+(t) \le M$ for all $t \ge 0$ or the integral in the expressions for $\Psi^+(t)$ is defined and $\Psi^+(t) \le M$ for all $t \ge 0$,
- either $\Phi^-(t) \le M$ for all $t \le 0$ or the integral in the expressions for $\Psi^-(t)$ is defined and $\Psi^-(t) \le M$ for all $t \le 0$.

Proposition 2.2. *Let Condition 2.1 be satisfied. If at least one of the functions $\Phi^+(t)$ and $\Phi^-(t)$ is bounded then equation (2.1) has a bounded solution for any bounded function f. If both of them are not bounded, then a bounded solution exists if and only if*

$$\int_{-\infty}^\infty v_0(t)f(t)dt = 0. \tag{2.3}$$

Proof. Suppose that both functions $\Phi^+(t)$ and $\Phi^-(t)$ are bounded. Then the solution of equation (2.1) is given by expression (2.2), and it is obviously bounded.

Suppose next that $\Phi^+(t)$ is bounded and $\Phi^-(t)$ is not bounded. Then $\Psi^-(t)$ is defined. Put

$$u^-(t) = u_0(t) \int_{-\infty}^t v_0(\tau)f(\tau)d\tau, \tag{2.4}$$

It is easy to verify $u^-(t)$ is bounded on the whole axis for any bounded f. Moreover, since $u_0(t)$ is not bounded as $t \to -\infty$, this function $u^-(t)$ is the only solution of (2.1), bounded as $t \to -\infty$.

The case then $\Phi^-(t)$ is bounded and $\Phi^+(t)$ is not, is similar. The bounded solution is given by formula

$$u^+(t) = -u_0(t) \int_t^\infty v_0(\tau)f(\tau)d\tau. \tag{2.5}$$

This is the only solution, bounded as $t \to +\infty$.

If both functions $\Phi^+(t)$ and $\Phi^-(t)$ are not bounded but $\Psi^+(t)$ and $\Psi^-(t)$ are bounded, then $u_0(t)$ is not bounded as $t \to \pm\infty$. Therefore the functions u^- and u^+, defined by (2.4) and (2.5), are the only solutions, bounded as $t \to -\infty$ and $t \to +\infty$ respectively. The solution, bounded on the whole axis exists if and only if $u^+(0) = u^-(0)$. This gives us the necessity and sufficiency of condition (2.3) The proposition is proved. □

Example 2.3. Suppose that $a(t) = a^+$ for t sufficiently large, and $a(t) = a^-$ for $-t$ sufficiently large. If $a^{\pm} \neq 0$, then $u_0(t)$ and $v_0(t)$ behave exponentially at infinity. Then Condition 2.1 is satisfied.

Note that Proposition 2.2 shows that Condition 2.1 is sufficient for the Fredholm property. Condition (2.3) is a typical Fredholm type solvability condition. It may be not satisfied. Suppose for example that $v_0(t)$ is integrable. We can choose such t_0 that for the function

$$f(t) = \begin{cases} 1, & t \geq t_0 \\ -1, & t < t_0 \end{cases}$$

then Condition (2.3) is satisfied. From the integrability of $v_0(t)$ it follows that $u_0(t)$ is not bounded as $t \to \pm\infty$. Therefore, the functions $\Phi^+(t)$ and $\Phi^-(t)$ are not bounded neither. If Condition 2.1 is not satisfied, then at least one of the functions $\Psi^+(t)$ and $\Psi^-(t)$ is not bounded. Hence there is no bounded solution of equation (2.1) with such f. Thus, Condition (2.3) may be not sufficient for solvability of equation (2.1).

To illustrate another type of solvability conditions suppose that the function

$$b(t) = \int_0^t a(s)ds$$

is bounded uniformly. Then $v_0(t)$ is bounded and $|u_0(t)| \geq \varepsilon > 0$ for some ε. Therefore the solution given by (2.2) is bounded if and only if

$$\sup_t \Big| \int_0^t v_0(s)f(s)ds \Big| < \infty. \tag{2.6}$$

As above, the solvability condition is given in terms of bounded solutions of the homogeneous adjoint equation. However, the principal difference is that condition (2.6), contrary to Fredholm type solvability conditions, cannot be formulated in the form $\phi(f) = 0$, where ϕ is a functional from the dual space.

We will see below that solvability conditions of this type are also applicable for systems of equations.

3. Ordinary differential systems on the real line

In this section we study invertibility and Fredholm property for linear operators, corresponding to o.d.e. systems. Let $u \in \mathbb{R}^n$. Denote by $|\cdot|$ the Euclidian vector norm in $\mathbb{R}^n$ and the corresponding matrix norm and by $\langle \cdot, \cdot \rangle$ the scalar product in $\mathbb{R}^n$. Consider the linear system

$$u' = P(x)u \tag{3.1}$$

where the matrix $P(x)$ is defined, bounded and continuous on the interval $(a, b) \subset \mathbb{R}$. Here a is a real number or $-\infty$ and b is a real number or $+\infty$. Let $\Phi(x, t)$ be the Cauchy matrix of system (3.1).

Definition 3.1. *The system* (3.1) *is* almost dichotomic *on* (a, b) *with positive constants* c *and* λ *if for every* $x \in (a, b)$ *there exist three spaces* $M_S(x)$ *(stable space),* $M_U(x)$ *(unstable space) and* $M_B(x)$ *(zero space), satisfying following conditions:*

1. $M_S(x) \bigoplus M_U(x) \bigoplus M_B(x) = \mathbb{R}^n$ *for all* $x \in (a, b)$*;*
2. $\Phi(x, t)M_\sigma(t) = M_\sigma(x)$ *for all* $\sigma \in \{S, U, B\}$, $x, t \in (a, b)$*;*
3. $|\Phi(x, t)u_0| \leq c\exp(-\lambda(x - t))|u_0|$ *for all* $x \geq t$, $x, t \in (a, b)$, $u_0 \in M_S(t)$*;*

(4) $|\Phi(x,t)u_0| \le c\exp(\lambda(x-t))|u_0|$ *for all* $x \le t$, $x,t \in (a,b)$, $u_0 \in M_U(t)$;
(5) $|\Phi(x,t)u_0| \le c|u_0|$ *for all* $x,t \in (a,b)$, $u_0 \in M_B(t)$;

The following statement is evident.

Lemma 3.2. *Let matrix* $P(x)$ *be constant, i.e.* $P(x) \equiv P$. *The system* (3.1) *is almost dichotomic if and only if for every purely imaginary eigenvalue* λ *of the matrix* P *the number of linearly independent eigenvectors corresponding to* λ *is equal to the multiplicity of* λ.

Remark 3.3. In other words, the condition is the following: for every $\lambda \in i\mathbb{R}$ every block in the Jordan form of the matrix A corresponding to λ is simple.

Remark 3.4. The statement of the lemma holds true if the matrix P does not have purely imaginary eigenvalues at all. In this case the space M_B is trivial and system (3.1) is dichotomic.

Definition 3.5 ([1])**.** Consider the change of variables

$$u = L(x)v, \quad x \in \mathbb{R}. \tag{3.2}$$

It is called *Lyapunov* transform if the matrix $L(x)$ is C^1 - smooth invertible and all matrices $L(x)$, $L^{-1}(x)$ and $L'(x)$ are bounded.

Lemma 3.6. *Let system* (3.1) *be almost dichotomic and let the dimensions of the corresponding spaces* $M_S(x)$, $M_U(x)$ *and* $M_B(x)$ *be* n_S, n_U *and* n_B, *respectively. Then, for every* x *there exist continuous projectors* $\Pi_S(x)$, $\Pi_U(x)$ *and* $\Pi_B(x)$ *on the spaces* $M_S(x)$, $M_U(x)$ *and* $M_B(x)$ *respectively, such that* $\Pi_S(x)+\Pi_U(x)+\Pi_B(x) \equiv \mathrm{id}$. *These projectors are uniformly bounded.*

Also, there exists a Lyapunov transform (3.2)*, which reduces system* (3.1) *to the form*

$$v' = \widetilde{P}(x)v, \tag{3.3}$$

where $v = (v_S, v_U, v_B)$, $\widetilde{P}(x) = \mathrm{diag}(P_S(x), P_U(x), P_B(x))$, *and system* (3.3) *splits into three subsystems:*

$$v_S{}' = P_S(x)v_S, \tag{3.4}$$

$$v_U{}' = P_U(x)v_U, \tag{3.5}$$

$$v_B{}' = P_B(x)v_B. \tag{3.6}$$

Systems (3.4)–(3.6) *satisfy the following properties:*

(1) *The system* (3.4) *is steadily dichotomic, i.e. it is dichotomic and the corresponding stable space coincides with the space* $\mathbb{R}^{n_S}$ *for all* x.
(2) *The system* (3.5) *is unsteadily dichotomic, i.e. it is dichotomic and the corresponding unstable space coincides with the space* $\mathbb{R}^{n_U}$ *for all* x.
(3) *Every solution of the system* (3.6) *is bounded.*

Remark 3.7. The matrix $\widetilde{P}(x)$ can be found by the formula

$$\widetilde{P}(x) = L^{-1}(x)P(x)L(x) - L^{-1}(x)L'(x). \tag{3.7}$$

Since the matrix $P(x)$ is bounded, the matrix $\widetilde{P}(x)$ is also bounded. If for a certain $\delta \ge 0$, $P(x) \in C^\delta$ and $L(x) \in C^{1+\delta}$, then $\tilde{P}(x) \in C^\delta$.

The proof of Lemma 3.6 is the same as the proof for dichotomic (hyperbolic) ordinary differential systems [7, Lemma 3, p.41], [20, Theorem 0.1, p.14].

Lemma 3.8. *If the system* (3.1) *is steadily dichotomic, the dual system*

$$u' = -P^T(x)u, \tag{3.8}$$

is unsteadily dichotomic. If (3.1) *is an unsteadily dichotomic system, then the system* (3.8) *is steadily dichotomic. If the system* (3.1) *is almost dichotomic with all solutions bounded, the dual system also is.*

The lemma above follows from the fact that for every fundamental matrix $\Phi(x)$ of system (3.1), the matrix $(\Phi^{-1})^T(x)$ is fundamental for system (3.8).

The following statement is evident.

Lemma 3.9. *Any system* (3.1)*, which splits into almost dichotomic blocks, is almost dichotomic. The stable, unstable and bounded spaces are direct products of the corresponding spaces for blocks.*

Having fixed a number $\delta \geq 0$, define spaces $X = C^\delta(\mathbb{R} \to \mathbb{R}^n)$, $Y = C^{1+\delta}(\mathbb{R} \to \mathbb{R}^n)$ and consider a function $f \in X$.

Theorem 3.10. *Let system* (3.1) *be almost dichotomic on* $\mathbb{R}$*, and the matrix* $P(x)$ *be bounded in* $C^\delta(\mathbb{R} \to \mathbb{R}^{n^2})$*. Then for any* $f \in X$ *the system*

$$u' = P(x)u + f(x) \tag{3.9}$$

has a solution $v(x) \in Y$ *if and only if*

$$\sup_{x\in\mathbb{R}} \Big| \int_0^x \langle \varphi(s), f(s)\rangle \, ds \Big| < +\infty \tag{3.10}$$

for every bounded solution $\varphi(s)$ *of system* (3.8).

Proof. Transformation (3.2), which exists due to Lemma 3.6, reduces system (3.9) to the form

$$v' = \widetilde{P}(x)v + g(x) \tag{3.11}$$

where $\widetilde{P}(x)$ satisfies (3.7), and $g(x) = L^{-1}(x)f(x)$. If $f(x) \in X$, then $g(x) \in X$ and vice versa. System (3.11) splits into three subsystems

$$v_S{}' = P_S(x)v_S + g_S(x), \tag{3.12}$$

$$v_U{}' = P_U(x)v_U + g_U(x), \tag{3.13}$$

$$v_B{}' = P_B(x)v_B + g_B(x). \tag{3.14}$$

Here $g(x) = (g_S(x), g_U(x), g_B(x))$. Systems (3.9) and (3.11) have bounded solutions if and only if each system (3.12), (3.13), and (3.14) has a bounded solution.

Let $\Psi(x,t)$ be the Cauchy matrix of system (3.3). It can be written in the form

$$\Psi(x,t) = \operatorname{diag}\left(\Psi_S(x,t), \Psi_U(x,t), \Psi_B(x,t)\right)$$

where $\Psi_S(x,t)$, $\Psi_U(x,t)$ and $\Psi_B(x,t)$ are the Cauchy matrices for systems (3.4), (3.5) and (3.6), respectively. Since systems (3.4) and (3.5) are dichotomic, the nonhomogeneous systems (3.12) and (3.13) have for every g bounded solutions of the form

$$v_S(x) = \int_{-\infty}^x \Psi_S(x,t)g_S(t)\,dt; \quad v_U(x) = -\int_x^\infty \Psi_U(x,t)g_U(t)\,dt.$$

All solutions of the system (3.14) have the form

$$\Psi_B(x)C + \int_0^x \Psi_B(x,t)g_B(t)\,dt.$$

Here $\Psi_B(x) = \Psi_B(x,0)$. Every solution of system (3.6) is bounded. Therefore the matrix $\Psi_B(x)$ is also bounded. Hence, it is sufficient to verify that the solution

$$v_B(x) = \int_0^x \Psi_B(x,t)g_B(t)\,dt = \Psi_B(x)\int_0^x \Psi_B^{-1}(t)g_B(t)\,dt$$

is bounded. Let c be the constant from Definition 3.1 for system (3.1), and $K > 0$ be such that $\max(\|L(x)\|_{C^{1+\delta}}, \|L^{-1}(x)\|_{C^{1+\delta}}) < K$. Then every column of the matrices $\Psi_B(x)$ and $\Psi_B^{-1}(x)$ is bounded by cK. Hence $\max(\|\Psi_B(x)\|_{C^{1+\delta}}, \|\Psi_B^{-1}(x)\|_{C^{1+\delta}}) \le \sqrt{n}cK$.

Thus, $v_B(x)$ is bounded if and only if the integral

$$I(x) = \int_0^x \Psi_B^{-1}(t)g_B(t)\,dt$$

is bounded. Consider the matrix $\Xi(x)$ which is obtained from Ψ_B^{-1} by adding $n_U + n_S$ zero rows. It follows from Lemmas 3.8 and 3.9 that every bounded solution of the system

$$v' = -\widetilde{P}^T(x)v \tag{3.15}$$

is a linear combination of columns of $\Xi(x)$. Hence $I(x)$ is bounded if and only if the condition

$$\sup_{x\in\mathbb{R}} \Big|\int_0^x \langle \eta(t), g(t)\rangle\,dt\Big| < +\infty \tag{3.16}$$

is satisfied for every bounded solution $\eta(x)$ of (3.15).

On the other hand, $\Phi(x) = L(x)\Psi(x)$ is a fundamental matrix of system (3.1). Then $\Psi^{-1}(x) = \Phi^{-1}(x)L(x)$. Hence every bounded solution $\eta(x)$ of system (3.15) can be written in the form $\eta(x) = L^T(x)\varphi(x)$, where $\varphi(x)$ is a bounded solution of (3.8). It is easy to see that this correspondence is one to one. Consequently, we can rewrite the integral in (3.16) in the form

$$\int_0^x \langle L^T(t)\varphi(t), L^{-1}(t)f(t)\rangle\,dt = \int_0^x \langle \varphi(t), f(t)\rangle\,dt. \tag{3.17}$$

Thus, there exists a bounded solution of system (3.9) if and only if expression (3.17) is uniformly bounded. The theorem is proved. □

Remark 3.11. Condition (3.10) is not a Fredholm type solvability condition.

Bounded solutions of system (3.8) form a linear space H of the dimension

$$n_B = \dim M_B(x).$$

Therefore, it is sufficient to verify (3.10) for some basis in H, that is for solutions of (3.8) with initial data in a basis of $M_B(0)$.

For every function $f \in X$, satisfying (3.10), a bounded solution may be found by the formula

$$\begin{aligned}\mathcal{L}f(x) = \int_{-\infty}^x \Phi(x,s)\Pi_S(s)f(s)\,ds + \int_0^x \Phi(x,s)\Pi_B(s)f(s)\,ds\\ - \int_x^{+\infty} \Phi(t,s)\Pi_U(s)f(s)\,ds.\end{aligned} \tag{3.18}$$

If the integral

$$\int_0^x \Phi(x,s)\Pi_B(s)f(s)\,ds$$

is not bounded, it increases polynomially. On the other hand, any function $\Phi(x,0)C$ for any $C \in \mathbb{R}^n$ is bounded or increases exponentially (this follows from Definition 3.1). Hence, if the expression (3.18) is not bounded, then system (3.9) has no bounded solutions at all. If (3.18) is bounded, then all solutions of the form

$$u(x) = \mathcal{L}f(x) + \Phi(x,0)C, \tag{3.19}$$

where $C \in M_B(0)$, are also bounded.

Define the operator $\mathbf{T}_P : Y \to X$ by the formula $\mathbf{T}_P u = u' - P(x)u$. If the space $M_B(x)$ is not trivial, then the operator $\mathbf{T}_P$ is not Fredholm but it can satisfy the Fredholm property in other function spaces.

Assume that system (3.1) is almost dichotomic on all the line. Denote by $\mathcal{B}$ the set of all bounded solutions of this system and by $\mathcal{B}^*$ the set of bounded solutions of the adjoint system (3.8). Define the space

$$X_{P,\delta} = \big\{ f \in C^\delta(\mathbb{R} \to \mathbb{R}^n) : \big\| \int_0^x \langle f(s), \varphi(s)\rangle\, ds \big\|_{C^0} < +\infty \text{ for all } \varphi(x) \in \mathcal{B}^* \big\}.$$

It follows from [14, Theorem 3.10] that the codimension of the space $X_{P,\delta}$ in X is infinite if the space $M_B(x)$ is not trivial (otherwise $X_{P,\delta} = X$).

Let $\varphi_1(x), \dots, \varphi_{n_B}(x)$ be a basis in $\mathcal{B}^*$. The space $X_{P,\delta}$ with the norm

$$\|f\|_{P,\delta} = \|f\|_{C^\delta} + \sum_{k=1}^{n_B} \big\| \int_0^x \langle f(s), \varphi_k(s)\rangle\, ds \big\|_{C^0}$$

is a Banach space. We have $\mathbf{T}_P Y = X_{P,\delta}$ since every bounded solution of the system (3.9) is of the form (3.19). Taking into consideration the space $Y' = \mathcal{L}X_{P,\delta} \in Y$, we obtain $Y = \mathcal{B} \oplus Y'$. Thus, $\mathbf{T}_P$ considered as an operator from Y to $X_{P,\delta}$ is Fredholm, and $\operatorname{ind} \mathbf{T}_P = n_B$.

4. Systems on half-lines

Similarly to the previous section we can consider systems (3.1) almost dichotomic on half-axis $\mathbb{R}^-$ and $\mathbb{R}^+$. Let system (3.1) be almost dichotomic on $\mathbb{R}^+$. Denote the corresponding spaces by $M_S^+(x)$, $M_U^+(x)$ and $M_B^+(x)$ and their dimensions by n_S^+, n_U^+ and n_B^+, respectively.

System (3.1) has a bounded solution on the half-axis $\mathbb{R}^+$ if and only if

$$\sup_{x\ge 0} \big| \int_0^x \langle \varphi^+(s), f(s)\rangle\, ds \big| < +\infty \tag{4.1}$$

for any solution $\varphi^+(x)$ of the adjoint system (3.8) such that $\varphi^+(x)$ is bounded on $\mathbb{R}^+$. Note that if $\varphi^+(x)$ is exponentially decaying, then condition (4.1) is satisfied for any bounded f. If (4.1) is satisfied, the there exists a bounded on $\mathbb{R}^+$ solution of (3.9) given by the formula

$$\mathcal{L}^+ f(x) = \int_0^x \Phi(x,s)(\Pi_S^+(s) + \Pi_B^+(s)) f(s)\, ds - \int_x^{+\infty} \Phi(x,s)\Pi_U^+(s) f(s)\, ds.$$

Here Π_S^+, Π_U^+ and Π_B^+ are projectors on the corresponding spaces. All other solutions bounded for positive x have the form $u^+(x) = \mathcal{L}^+ f(x) + \Phi(x,0)C^+$, where C^+ is an arbitrary vector of the space $M^+ = M_S^+(0) \oplus M_B^+(0)$. Similarly, if the system (3.1) is almost dichotomic on $\mathbb{R}^-$, denote the corresponding spaces by $M_S^+(x)$, $M_U^+(x)$

and $M_B^+(x)$ and their dimensions by n_S^+, n_U^+ and n_B^+, respectively. Consider Π_S^-, Π_U^+ and Π_B^+ as projectors on M_S^-, M_U^+ and M_B^-. The solvability conditions are

$$\sup_{x\le 0}\Big|\int_0^x \langle \varphi^-(s), f(s)\rangle\, ds\Big| < +\infty \tag{4.2}$$

for any solution $\varphi^-(x)$ of (3.8), bounded for $x \le 0$. If this condition is satisfied, there is a bounded solution of the form

$$\mathcal{L}^- f(x) = \int_{-\infty}^x \Phi(x,s)\Pi_S^-(s) f(s)\, ds + \int_0^x \Phi(x,s)(\Pi_U^-(s) + \Pi_B^-(s)) f(s)\, ds.$$

All other bounded solutions are given by the expression

$$u^-(x) = \mathcal{L}^- f(x) + \Phi(x,0)C^-,$$

where C^- is an arbitrary vector of the space $M^- = M_U^-(0) \oplus M_B^-(0)$.

Assume that system (3.1) is almost dichotomic both for $x \ge 0$ and for $x \le 0$. If the function f satisfies conditions (4.1) and (4.2), then the existence of a solution $u(x) \in Y$ of system (3.9) is provided by the following condition

$$u^+(0) = u^-(0) \tag{4.3}$$

for certain values $C^+ \in M^+$ and $C^- \in M^-$. We can rewrite (4.3) in the form

$$\mathcal{L}^+ f(0) - \mathcal{L}^- f(0) \in M^+ + M^-.$$

This Fredholm condition provides the existence of an affine space of bounded solutions of the dimension $m_0 = \dim(M^+ \bigcap M^-)$.

Now we change the space X in order to make $\mathbf{T}_P$ Fredholm. Denote by $\varphi^+(x)$ an arbitrary solution of the system (3.8) bounded for $x \ge 0$. By $\varphi^-(x)$ we denote an arbitrary solution of the system (3.8) bounded for $x \le 0$. Consider the minimal linear space $\mathcal{A}$ containing all functions of the form

$$\varphi(x) = \begin{cases} 0 & \text{for } x < 0, \\ \varphi^+(x) & \text{for } x \ge 0 \end{cases}$$

and

$$\psi(x) = \begin{cases} \varphi^-(x) & \text{for } x \le 0, \\ 0 & \text{for } x > 0. \end{cases}$$

Denote by m^+ and m^- dimensions of spaces M^+ and M^- respectively. Then the dimension of $\mathcal{A}$ equals to $m^+ + m^-$. Define the space

$$X_{P,\delta} = \big\{ f \in C^\delta(\mathbb{R} \to \mathbb{R}^n) : \big\| \int_0^x \langle f(s), \varphi(s)\rangle\, ds \big\|_{C^0} < +\infty \text{ for all } \varphi(x) \in \mathcal{A} \big\}.$$

with the norm

$$\|f\|_{P,\delta} = \|f\|_{C^\delta} + \sum_{k=1}^{m^+ + m^-} \big\| \int_0^x \langle f(s), \varphi_k(s)\rangle\, ds \big\|_{C^0}.$$

Here $\varphi_1(x), \dots, \varphi_{m^+ + m^-}(x)$ is a basis in $\mathcal{A}$.

Since the system (3.9) is solvable in Y only if $f \in X_{P,\delta}$, one may consider $\mathbf{T}_P$ as an operator from Y to $X_{P,\delta}$. This operator is Fredholm. The dimension of the space $M^+ + M^-$ is $m^+ + m^- - m_0$, so

$$\operatorname{ind}\mathbf{T}_P = m_0 - (n - m^+ - m^- + m_0) = m^+ + m^- - n. \tag{4.4}$$

Taking into consideration the facts that $m^+ = n_S^+ + n_B^+$, that $m^- = n_U^- + n_B^-$ and that $n_S^\pm + n_U^\pm + n_B^\pm = n$, we obtain from (4.4) other formulae for index:

$$\operatorname{ind} \mathbf{T}_P = n_S^+ + n_B^+ - n_S^- = n_U^- + n_B^- - n_U^+. \tag{4.5}$$

5. Weakly hyperbolic systems

Suppose, that the linear system (3.1) is defined on the half-line $\mathbb{R}^+$.

Definition 5.1 ([12, 13])**.** Let $\lambda > 0$, and $\varepsilon \geq 0$. We call the system (3.1) *weakly hyperbolic* with constants λ and ε, if there exists such $K > 0$, that for every continuous vector function $g : [0, \infty) \to \mathbb{R}^n$, satisfying for $x \geq 0$ the estimate

$$|g(x)| \leq \exp(-\lambda(1+\varepsilon)x), \tag{5.1}$$

there is a solution $\varphi(x)$ of the nonhomogeneous system

$$u' = P(x)u + g(x), \tag{5.2}$$

such that

$$|\varphi(x)| \leq K \exp(-\lambda x) \quad \text{for } x \geq 0. \tag{5.3}$$

Assume that the matrix $P(x)$ in (3.1) is bounded. Denote the class, introduced by this definition by $\mathrm{WH}^+(\lambda, \varepsilon)$ (we shall write $P \in \mathrm{WH}^+(\lambda, \varepsilon)$). Here the superscript + underlines the fact that the solution $\varphi(x)$ exponentially decays on the right half-line.

Remark 5.2. If $\lambda_{1,2} > 0$ and $\varepsilon_{1,2} \geq 0$ are such that $\lambda_1(1+\varepsilon_1) \leq \lambda_2(1+\varepsilon_2)$ and $\lambda_1 \geq \lambda_2$, then $\mathrm{WH}^+(\lambda_1, \varepsilon_1) \subseteq \mathrm{WH}^+(\lambda_2, \varepsilon_2)$.

Lemma 5.3. Let $\lambda > 0, \varepsilon \geq 0$ and let $\Phi(x)$ be a fundamental matrix of the system (3.1). Suppose that there exist such continuous matrices $\Pi^s(x)$ and $\Pi^u(x)$, that

$$\Pi^s(x) + \Pi^u(x) \equiv E$$

is a $n \times n$ unit matrix and for a certain $K > 0$ the following inequality is satisfied

$$\begin{aligned} &\int_0^x |\Phi(x)\Phi^{-1}(t)\Pi^s(t)| \exp(-\lambda(1+\varepsilon)t)\, dt \\ &+ \int_x^\infty |\Phi(x)\Phi^{-1}(t)\Pi^u(t)| \exp(-\lambda(1+\varepsilon)t)\, dt \leq K \exp(-\lambda x). \end{aligned} \tag{5.4}$$

Then $P \in \mathrm{WH}^+(\lambda, \varepsilon)$.

Proof. Denote $\Phi(x,t) = \Phi(x)\Phi^{-1}(t)$,

$$\Phi^s(x,t) = \Phi(x)\Phi^{-1}(t)\Pi^s(t), \quad \Phi^u(x,t) = \Phi(x)\Phi^{-1}(t)\Pi^u(t).$$

Fix a vector function $g(x)$, satisfying (5.1), and define

$$\varphi(x) = \int_0^x \Phi^s(x,t)g(t)\, dt - \int_x^\infty \Phi^u(x,t)g(t)\, dt. \tag{5.5}$$

It follows from (5.4) that integrals in the right-hand side of (5.5) converge and the solution $\varphi(x)$ satisfies (5.3). The lemma is proved. □

Theorem 5.4. *If the system* (3.1) *is dichotomic on the real line, then there exists such a value* $\lambda_0 > 0$ *that* $P \in \mathrm{WH}^+(\lambda, 0)$ *for all* $0 < \lambda < \lambda_0$.

Proof. Consider constants c and λ from Definition 1.1 for the system (3.1), and take as $\Pi^s(x)$ and $\Pi^u(x)$ projectors on the stable and the unstable space of the system considered. It is well-known [20, Chapter 1] that $\max(|\Pi^s(x)|, |\Pi^u(x)|) \le M$ for a certain $M > 0$ and all $x \ge 0$. Fix a value $0 < \mu < \lambda$. Thus, we obtain

$$\begin{aligned}
&\int_0^x |\Phi^s(x,t)| \exp(-\mu t)\,dt + \int_x^\infty |\Phi^u(x,t)| \exp(-\mu t)\,dt \\
&\le \int_0^x Mc\exp(-\lambda(x-t))\exp(-\mu t)\,dt + \int_x^\infty Mc\exp(\lambda(x-t))\exp(-\mu t)\,dt \\
&= Mc\Big(\exp(-\lambda x)\int_0^x \exp((\lambda-\mu)t)\,dt + \exp(\lambda x)\int_x^\infty \exp(-(\lambda+\mu)t)\,dt\Big) \\
&\le K\exp(-\mu x).
\end{aligned}$$

The theorem is proved. □

Let $f(x)$ be a function (vector function, matrix function) defined on the interval $[0, +\infty)$.

Definition 5.5 ([1, 18])**.** The number (or the symbol $\pm\infty$), defined as

$$\chi^+[f] = \limsup_{x\to+\infty} \frac{1}{x} \ln|f(x)|$$

is called the *Lyapunov exponent* of the function $f(x)$.

For a function $f(x)$, defined on $\mathbb{R}^-$ one can define the Lyapunov exponent in negative direction

$$\chi^-[f] = \limsup_{x\to-\infty} \frac{1}{x} \ln|f(x)|.$$

Let $\Phi(x) = (\varphi_1(x), \dots, \varphi_n(x))$ be a fundamental matrix of system (3.1) and let $\chi^+[\varphi_j] = \lambda_j$ $(j = 1, \dots, n)$. Further, let $\Psi(x) = [\Phi^{-1}(x)]^* = (\psi_1(x), \dots, \psi_n(x))$ and $\chi^+[\psi_j] = \mu_j$ $(j = 1, \dots, n)$. Denote by $\gamma(\Phi) = \max(\lambda_i + \mu_i)$ the so-called *defect of reciprocal bases* $\{\varphi_j\}$ and $\{\psi_j\}$.

Definition 5.6 ([1, p. 67])**.** The system (3.1) is *regular* if there is such a fundamental matrix $\Phi(x)$ of this system that $\gamma(\Phi) = 0$.

It was shown by Grobmann [10], that this definition was equivalent to one, given by Lyapunov [18]. The class of regular systems is very wide. At least, it includes all systems with constant and periodic matrices of coefficients [1]. Note that regularity in positive direction does not imply the regularity in negative direction and vice versa.

Theorem 5.7. *If system* (3.1) *is regular, then for all* $\lambda, \varepsilon > 0$ *this system belongs to the class* $\mathrm{WH}^+(\lambda, \varepsilon)$.

Proof. Fix positive numbers λ and ε. Choose $\Phi(x)$, a fundamental matrix of system (3.1), which exists due to the Definition 5.6, and consider an $n \times n$ matrix $\Psi^s(x)$ which consists of those rows of the matrix $\Phi^{-1}(x)$, whose Lyapunov exponents are not less than $\lambda(1+\varepsilon)$, and zero strings. Without loss of generality, one may assume that first k rows of the matrix $\Psi^s(x)$ coincide with first k ones of the matrix $\Phi^{-1}(x)$, for a certain $0 \le k \le n$ and all other rows of the matrix $\Psi^s(x)$ are zero. Denote

$$\Pi^s(x) = \Phi(x)\Psi^s(x), \quad \Pi^u(x) = E - \Pi^s(x) = \Phi(x)\Psi^u(x),$$

where the matrix $\Psi^u(x)$ consists of k zero rows and $n-k$ last rows of the matrix $\Phi^{-1}(x)$.

Now we check inequality (5.4). Denote the elements of the matrix $\Phi^s(x,t)$ by $u^s_{ij}(x,t)$, and the elements of matrices $\Phi(x)$ and $\Phi^{-1}(x)$ by $u_{ij}(x)$ and $\eta_{ij}(x)$, respectively. Since $\Phi^{-1}(x)\Pi^s(x) = \Psi^s(x)$, we have

$$\begin{aligned} &\int_0^x |u^s_{ij}(x,t)| \exp(-\lambda(1+\varepsilon)t)\,dt \\ &= \int_0^x \Big|\sum_{r=1}^k u_{ir}(x)\eta_{rj}(t)\Big| \exp(-\lambda(1+\varepsilon)t)\,dt \\ &\le \sum_{r=1}^k |u_{ir}(x)| \int_0^x |\eta_{rj}(t)| \exp(-\lambda(1+\varepsilon)t)\,dt. \end{aligned} \tag{5.6}$$

Let $\eta_r(x)$ be the r-th row of the matrix $\Phi^{-1}(x)$. Due to the choice of k it is clear that $\chi^+(|\eta_r(x)|\exp(-\lambda(1+\varepsilon)x)) \ge 0$ for such r that $1 \le r \le k$. Thus,

$$\chi^+\Big(\int_0^x |\eta_{rj}(\tau)| \exp(-\lambda(1+\varepsilon)\tau)\,d\tau\Big) \le \chi^+(\eta_r(x)) - \lambda(1+\varepsilon).$$

Since system (3.1) is regular, for all $i, r = 1, \dots, n$ we have

$$\chi^+(u_{ir}(x)) + \chi^+(\eta_r(x)) - \lambda\varepsilon < 0.$$

Therefore, the Lyapunov exponent of the right-hand side of (5.6) is less than $-\lambda$ and this function could be estimated by $c_{ij}\exp(-\lambda t)$. Thus,

$$\begin{aligned} \int_0^x |\Phi^s(x,t)| \exp(-\lambda(1+\varepsilon)t)\,dt &= \int_0^x \max_i \sum_{j=1}^n |u^s_{ij}(x,t)| \exp(-\lambda(1+\varepsilon)t)\,dt \\ &\le \frac{K\exp(-\lambda x)}{2} \end{aligned} \tag{5.7}$$

for a certain $K > 0$. A similar estimate can be obtained for the second integral in (5.4). Together with (5.7) it gives (5.4). This proves the theorem. □

The following results allow us to obtain new weakly hyperbolic systems.

Theorem 5.8. *Let the matrix $P(x)$ be of the form*

$$P(x) = \begin{pmatrix} P_1(x) & 0 \\ 0 & P_2(x) \end{pmatrix},$$

and let the systems $u_1' = P_1(x)u_1$ and $u_2' = P_2(x)u_2$ of k and $n-k$ equations, respectively, belong to the class $\mathrm{WH}^+(\lambda,\varepsilon)$*. Then system* (3.1) *also belongs to the same class.*

The proof of the above theorem is evident; se we omit it.

Let us denote by $\exp(-\mu x)\mathbb{L}^\infty$ for any $\mu > 0$ the space of vector functions obtained as a product of $\exp(-\mu x)$ and a vector function, bounded for $x \ge 0$. The norm in this space is defined by the formula $\|h\|_\mu = \sup_{x\ge 0}(\exp(\mu x)|h(x)|)$.

Theorem 5.9. *Let system* (3.1) *belong to the class* $\mathrm{WH}^+(\lambda,\varepsilon)$*. Then there exists such a continuous linear mapping*

$$\mathcal{L}^+ : \exp(-\lambda(1+\varepsilon)x)\mathbb{L}^\infty \to \exp(-\lambda x)\mathbb{L}^\infty$$

that for any vector function $g \in \exp(-\lambda(1+\varepsilon)x)\mathbb{L}^\infty$ *the function* $\mathcal{L}^+ g(x)$ *is a solution of system* (5.2) *for the given* g.

Proof. Let k be the dimension of the space of all solutions of equation (3.1), which belongs to the space $\exp(-\lambda x)\mathbb{L}^\infty$. Denote by $\Phi(x)$ the fundamental matrix of system (3.1), whose first k columns belong to the space $\exp(-\lambda x)\mathbb{L}^\infty$ and no nontrivial combination of other columns does. We consider an arbitrary function $g \in \exp(-\lambda(1+\varepsilon)x)\mathbb{L}^\infty$. Provided $\|g\|_{\lambda(1+\varepsilon)} = K$, it follows from the conditions of the theorem that there exists the solution $\varphi(x)$ of system (5.2) satisfying the inequality

$$|\varphi(x)| \le cK \exp(-\lambda x) \quad \text{for } x \ge 0. \tag{5.8}$$

Obviously, there exists such a constant vector C_φ that

$$\varphi(x) = \Phi(x)\Big(C_\varphi + \int_0^x \Phi^{-1}(\tau) g(\tau)\, d\tau\Big).$$

One may split the vector C_φ into a sum $C_\varphi = C_\varphi^{(1)} + C_\varphi^{(2)}$ where the first k components of the vector $C_\varphi^{(1)}$ and the last $n-k$ ones of the vector $C_\varphi^{(2)}$ equal zero. We show that the vector $C_\varphi^{(1)}$ does not depend on φ for a fixed g. Then we can write $C_g^{(1)}$ instead of $C_\varphi^{(1)}$. Assume that for the same g there exist two solutions $\varphi_1(x)$ and $\varphi_2(x)$ of system (5.2) satisfying (5.8). So the solution $\varphi_1(x) - \varphi_2(x)$ of system (3.1) belongs to the space $\exp(-\lambda x)\mathbb{L}^\infty$. On the other hand,

$$\varphi_1(x) - \varphi_2(x) = \Phi(x)(C_{\varphi_1} - C_{\varphi_2}) = \Phi(x)(C_{\varphi_1}^{(1)} - C_{\varphi_2}^{(1)}) + \Phi(x)(C_{\varphi_1}^{(2)} - C_{\varphi_2}^{(2)}). \tag{5.9}$$

The second term in the right-hand side of equality (5.9) belongs to the space $\exp(-\lambda x)\mathbb{L}^\infty$. Therefore the whole sum does. So the equality $C_{\varphi_1}^{(1)} = C_{\varphi_2}^{(1)}$ follows from the choice of the matrix $\Phi(x)$. Let us define

$$\mathcal{L}^+ g(x) = \Phi(x)\Big(C_g^{(1)} + \int_0^x \Phi^{-1}(\tau) g(\tau)\, d\tau\Big).$$

We check now the properties of the mapping $\mathcal{L}^+$.
Linearity. Let $a, b \in \mathbb{R}$, $g_{1,2} \in \exp(-\lambda(1+\varepsilon)x)\mathbb{L}^\infty$. By virtue of the definition of the operator $\mathcal{L}^+$

$$\mathcal{L}^+(ag_1 + bg_2)(x) = \Phi(x) C_{ag_1+bg_2}^{(1)} + \int_0^x \Phi(t,\tau)(ag_1(\tau) + bg_2(\tau))\, d\tau. \tag{5.10}$$

The right-hand side of (5.10) belongs to the space $\exp(-\lambda(1+\varepsilon)x)\mathbb{L}^\infty$. It is a solution of system (5.2) with $g(x) = ag_1(x) + bg_2(x)$. Hence $C_{ag_1+bg_2}^{(1)} = aC_{g_1}^{(1)} + bC_{g_2}^{(1)}$ because of the uniqueness of $C_g^{(1)}$. This proves the linearity of the mapping $\mathcal{L}^+$.
Continuity. We will prove that there exists a constant $H > 0$ such that for every vector-function g,

$$\|g\|_{\lambda(1+\varepsilon)} = 1 \tag{5.11}$$

the inequality

$$\|\mathcal{L}^+ g\|_\lambda \le H \tag{5.12}$$

is true. We choose an arbitrary solution $\varphi(x)$ of system (5.2) such that

$$|\varphi(x)| \le c \exp(-\lambda x) \tag{5.13}$$

for every $x \geq 0$. According to the definition of the mapping $\mathcal{L}^+$,

$$\varphi(x) - \mathcal{L}^+ g(x) = \Phi(x) C_\varphi^{(2)} = \sum_{i=1}^{k} c_i X_i(x),$$

where $X_i(x)$ are columns of the matrix $\Phi(x)$ and $C_\varphi^{(2)} = (c_1, \dots, c_k, 0, \dots, 0)^T$. Assuming that the numbers M and l are such that $\max(|X_1(x)|, \dots, |X_k(x)|) < M \exp(-\lambda x)$ for any $x \geq 0$ and $|c_1| + \dots + |c_k| < l |C_\varphi|$, we obtain

$$|\varphi(x) - \mathcal{L}^+ g(x)| \leq \sum_{i=1}^{k} |c_i| \max_{i \leq k} |X_i(x)| \leq l M |C_\varphi| \exp(-\lambda x). \tag{5.14}$$

On the other hand, $\varphi(0) = \Phi(0) C_\varphi$ and

$$|C_\varphi| \leq |\Phi^{-1}(0)| |\varphi(0)| \leq c |\Phi^{-1}(0)|.$$

Substituting this estimate into (5.14), we obtain

$$\|\varphi - \mathcal{L}^+ g\|_\lambda \leq L M c |\Phi^{-1}(0)|. \tag{5.15}$$

Suppose $H = c(1 + LM|\Phi^{-1}(0)|)$. The inequality (5.12) follows from (5.13) and (5.15). The theorem is proved. □

Theorem 5.10. *Let system* (3.1) *belong to the class* $\mathrm{WH}^+(\lambda, \varepsilon)$ *and let the invertible matrix* $L(x)$ *be such that*

$$\begin{gathered} L(x) \in C^1([0, \infty)), \\ \chi^+(|L(x)| + |L^{-1}(x)|) = 0. \end{gathered} \tag{5.16}$$

Then for any λ_1 *and* ε_1 *such that* $\lambda_1 < \lambda$, $\lambda_1(1 + \varepsilon_1) > \lambda(1 + \varepsilon)$ *the system*

$$v' = \widetilde{P}(x) v, \tag{5.17}$$

with the matrix $\widetilde{P}(x) = L^{-1}(x) P(x) L(x) - L^{-1}(x) \dot{L}(x)$ *obtained from* (3.1) *by the transformation*

$$u = L(x) v, \tag{5.18}$$

belongs to the class $\mathrm{WH}^+(\lambda_1, \varepsilon_1)$.

Proof. Let us choose a constant $c_1 > 0$ such that

$$|L(x)| \leq c_1 \exp((\lambda_1(1 + \varepsilon_1) - \lambda(1 + \varepsilon))x), \quad |L^{-1}(x)| \leq c_1 \exp((\lambda - \lambda_1)x)$$

for all $x \geq 0$. Consider a vector function

$$g(x) \in \exp(-\lambda_1(1 + \varepsilon_1)x) \mathbb{L}^\infty$$

and the system

$$v' = \widetilde{P}(x) v + g(x). \tag{5.19}$$

The transformation inverse to (5.18) reduces this system to the form

$$u' = P(x) u + L(x) g(x). \tag{5.20}$$

Since $-\lambda_1(1 + \varepsilon_1) < -\lambda(1 + \varepsilon)$, the vector function $L(x) g(x)$ belongs to the space $\exp(-\lambda(1 + \varepsilon)x) \mathbb{L}^\infty$. Hence system (5.19) has a solution $\varphi(x) \in \exp(-\lambda x) \mathbb{L}^\infty$, and system (5.20) has a solution

$$\psi(x) = L^{-1}(x) \varphi(x) \in \exp(-\lambda_1 x) \mathbb{L}^\infty.$$

Let $c = \|\mathcal{L}^+\|$, where $\mathcal{L}^+$ is the operator which corresponds to the weakly hyperbolic system (3.1). Clearly,

$$\|\psi\|_{\lambda_1} \le cc2_1\|g\|_{\lambda_1(1+\varepsilon_1)}.$$

Therefore, system (5.17) is weakly hyperbolic with constants λ_1 and ε_1. The theorem is proved. □

Remark 5.11. Linear transformations (5.18) satisfying (5.16) are called *generalized Lyapunov* transformations. It is proved in [4], see also [1], that system (3.1) is regular if and only if it can be reduced to a system with a constant matrix by a generalized Lyapunov transformation.

One can also consider weakly hyperbolic systems in the negative direction, that is on a half-axis $\mathbb{R}^-$. All results similar to theorems of this section may be proved. Denote the corresponding classes by $\mathrm{WH}^-(\lambda,\varepsilon)$ and corresponding operators by $\mathcal{L}^-$.

Consider the class $\mathrm{WH}^0(\lambda,\varepsilon)$ which consists of systems (3.1), defined on $\mathbb{R}$ which are weakly hyperbolic both on the left and the right half-axis with constants λ and ε. Let $\Phi(t)$ be such a fundamental matrix of (3.1) that $\Phi(0) = E$. Consider the following two spaces

$$M^+ = \{u_0 \in \mathbb{R}^n : |\Phi(t)u_0| \le c\exp(-\lambda t) \quad \text{for all } t \ge 0\},$$
$$M^- = \{u_0 \in \mathbb{R}^n : |\Phi(t)u_0| \le c\exp(\lambda t) \quad \text{for all } t \le 0\}.$$

Let $\dim M^+ = m^+$, $\dim M^- = m^-$, $M^0 = M^+ \bigcap M^-$, $\widetilde{M} = M^+ + M^-$.

Fix nonnegative parameters δ and μ and take into consideration two sets of functional spaces

$$U_{\delta,\mu} = \{u(x) : \mathbb{R} \to \mathbb{R}^n : \exp(\mu\sqrt{1+x^2})u(x) \in C^{1+\delta}(\mathbb{R} \to \mathbb{R}^n)\};$$
$$X_{\delta,\mu} = \{f(x) : \mathbb{R} \to \mathbb{R}^n : \exp(\mu\sqrt{1+x^2})f(x) \in C^{\delta}(\mathbb{R} \to \mathbb{R}^n)\}.$$

One can define norms in the space $X_{\delta,\mu}$ by the formula

$$\|f\|_{\delta,\mu} = \|\exp(\mu\sqrt{1+x^2})f(x)\|_{C^\delta}.$$

The norm in $U_{\delta,\mu}$ can be defined similarly.

Theorem 5.12. *If system* (3.1) *belongs to the class* $\mathrm{WH}^0(\lambda,0)$*, then the operator*

$$\mathbf{T}_P : U_{\delta,\lambda} \to X_{\delta,\lambda},$$

defined by the formula $\mathbf{T}_P u = u' - P(x)u$ *is Fredholm and* $\operatorname{ind}\mathbf{T}_P = m^+ + m^- - 2n$. *If* $M^0 = \{0\}$ *and* $\widetilde{M} = \mathbb{R}^n$*, the operator* $\mathbf{T}_P$ *is invertible.*

The proof of this statement is similar to the reasonings presented in Section 4. The following statement is a corollary of the theory of Fredholm operators [11, 3, §19.1].

Theorem 5.13. *If system* (3.1) *belongs to the class* $\mathrm{WH}^0(\lambda,0)$ *and* $\widetilde{M} = \mathbb{R}^n$*, then there is an operator*

$$\mathcal{L}_P \in C(X_{\delta,\lambda} \to U_{\delta,\lambda}),$$

which transforms the function $f \in X_{\delta,\mu}$ *to a solution* $\mathcal{L}_P f$ *of system* (3.9)*, that is* $\mathbf{T}_P\mathcal{L}_P f = f$ *for any* $f \in X_{\delta,\mu}$.

These results can be used in the following theorem. To simplify its formulation we will assume that there exist bounded solutions of systems (5.16) and will not present the existence conditions.

Theorem 5.14. *Let $\lambda_0 > 0$ be a number such that the system* (3.1) *belongs to all classes* $\mathrm{WH}^0(\lambda, \varepsilon)$ *for any* $\lambda \in (0, \lambda_0)$ *and* $\varepsilon > 0$. *Consider a function* $f \in C^\delta(\mathbb{R} \to \mathbb{R}^n)$, *where* $\delta \in (0,1)$. *Suppose that* $P(x) \in C^\delta$ *and that there are two sequences* λ_k *and* ε_k *of positive numbers and a sequence of functions* f_k *satisfying the following conditions:*

1. $\lambda_k \to 0$ *as* $k \to \infty$,
2. $\lambda_k \varepsilon_k \to 0$ *as* $k \to \infty$,
3. *the norms* $\|f_k\|_{C^\delta}$ *are uniformly bounded and for every compact set* $K \subset \mathbb{R}$ *the sequence* f_k *converges to* f *in* $C(K \to \mathbb{R}^n)$,
4. *There is a sequence* φ_k *of solutions of systems*
$$u' = P(x)u + f_k(x) \tag{5.21}$$
such that $\sup_k \|\varphi_k\|_{C^0} < +\infty$.

Then system (3.9) *is solvable in* $C^{1+\delta}(\mathbb{R} \to \mathbb{R}^n)$.

Proof. Since the functions φ_k are uniformly bounded in $C(\mathbb{R} \to \mathbb{R}^n)$, then by virtue of the conditions on the matrix $P(x)$ and on the functions f_k they are also uniformly bounded in $C^{1+\delta}(\mathbb{R} \to \mathbb{R}^n)$. Therefore we can choose a subsequence φ_{k_l} that converges to $\varphi_0 \in C^{1+\delta}(\mathbb{R} \to \mathbb{R}^n)$ uniformly on every compact set $K \subset \mathbb{R}$. The function φ_0 satisfies equation (3.9). The theorem is proved. □

6. Applications to elliptic partial differential operators

The results of the previous sections will be applied to obtain solvability conditions for elliptic operators in unbounded cylinders considered in Section 1. Let $\mathbf{L}$ be the operator defined by (1.2). The following lemma is essential for what follows.

Lemma 6.1. *There exists a number* $N \in \mathbb{N}$ *such that for* $k > N$ *every system* (1.6) *is dichotomic on* $\mathbb{R}$ *with constants* $c = 2$ *and* $\lambda = 1/2$. *Furthermore, the norms of the projectors* $\Pi^{s,u}$ *do not exceed 2.*

Proof. Consider the change of the independent variable $t = \lambda_k x$. It reduces system (1.6) to
$$\dot{w}_k^i = Q_k(t) w_k^i. \tag{6.1}$$
Here
$$Q_k(t) = \begin{pmatrix} 0 & E_m \\ (\frac{B(t/\lambda_k)}{\omega_k} + E_m) & -\frac{A(t/\lambda_k)}{\lambda_k} \end{pmatrix}.$$
Evidently, the system
$$\dot{w} = \begin{pmatrix} 0 & E_m \\ E_m & 0 \end{pmatrix} w \tag{6.2}$$
is dichotomic with constants $c = 1$, $\lambda = 1$. Moreover, the stable and the unstable subspace are always orthogonal. Therefore, the norms of the projectors on these subspaces equal 1. Due to the Perron theorem [7, Proposition 1, p.34] there is such a value $\varepsilon > 0$ that if
$$\|B(x)\|/|\omega_k| < \varepsilon \quad \text{and} \quad \|A(x)\|/\lambda_k < \varepsilon, \tag{6.3}$$

then system (6.1) is dichotomic with constants $c = 2$ and $\lambda = 1/2$. We can take this ε so small that the angle between stable spaces of systems (6.1) and (6.2) for every x is less then $\pi/100$. Then the norms of the corresponding projectors are less than 2. Hence system (1.6) is dichotomic with constants $c = 2$ and $\lambda = \lambda_k/2$. The norms of the projectors rest the same because they do not depend on the scaling of the independent variable.

Thus, we can choose the number N big enough in order to obtain the estimate $|\lambda_N| > \max(1, M/\varepsilon)$. The lemma is proved. □

Remark 6.2. The dichotomicity constants for systems (1.6) can be chosen independently of k.

Assume that the operator $\mathbf{L}$ and the function f satisfy the condition.

Condition 6.3. Every system (1.7) is solvable in $C^0(\mathbb{R} \to \mathbb{R}^n)$.

This condition implies that system (1.7) is solvable in the space $C^{1+\delta}(\mathbb{R} \to \mathbb{R}^n)$ because the coefficients of this system belong to the space $C^\delta(\mathbb{R} \to \mathbb{R}^n)$.

Note that if system (3.1) is dichotomic then for every bounded g the corresponding system (5.2) has a bounded solution, which can be found by the following formula [7, p.22]:

$$\varphi(x) = \int_{-\infty}^{x} \Phi(x,t)\Pi_S(t)g(t)\,dt - \int_{x}^{\infty} \Phi(x,t)\Pi_U(t)g(t)\,dt.$$

This solution depends linearly on the right-hand side g and satisfies the inequality

$$|\varphi(x)| \le 2cH/\lambda.$$

Here c and λ are the constants of dichotomicity for system (3.1) and H is a constant, which bounds norms of projectors on the stable and unstable subspaces. Thus, due to Lemma 6.1 it is sufficient to verify Condition 6.3 for systems (1.7) with $k = 1, \dots, N$. To check the solvability of these systems one can either use the results on almost dichotomic systems (Sections 3 and 4) or use the the theorems on weakly hyperbolic systems (Section 5). The last approach is applicable if the right-hand sides F_k^i decay exponentially or satisfy conditions of Theorem 5.14.

Theorem 6.4. *Let the operator* $\mathbf{L}$ *defined by* (1.1) *and the function* f *satisfy Condition 6.3. Then problem* (1.4) *is solvable in* U.

Proof. We will prove convergence of the series (1.5). We take a number N, which exists due to Lemma 6.1 and consider the spectral decomposition of the operator $\mathbf{L}$ developed in [9]. Consider first the projector P'_N acting in the space $C^\delta(\bar{\Omega}')$ and corresponding to the first N eigenvalues of the Laplace operator Δ' in the section of the cylinder,

$$P'_N v = \frac{1}{2i\pi}\int_\Gamma (\Delta' - \lambda)^{-1} v\, d\lambda.$$

Here Γ is the contour in the complex plane containing the first N eigenvalues. Consider the operator Q'_N acting in the same space and defined by the equality

$$Q'_N u = u - P'_N u.$$

Denote

$$E'_N = P'_N(C^\delta(\bar{\Omega}')), \quad \tilde{E}'_N = Q'_N(C^\delta(\bar{\Omega}')).$$

Then

$$C^\delta(\bar{\Omega}') = E'_N \oplus \tilde{E}'_N.$$

Let us set

$$E_N = \{u \in C^\delta(\bar\Omega) : \forall x \in \mathbb{R}, u(x,\cdot) \in E'_N\},$$
$$\tilde E_N = \{u \in C^\delta(\bar\Omega) : \forall x \in \mathbb{R}, u(x,\cdot) \in \tilde E'_N\}.$$

We define now the operators

$$(P_N u)(x,\cdot) = P'_N(u(x,\cdot)), \quad (Q_N u)(x,\cdot) = Q'_N(u(x,\cdot)).$$

It is shown in [9] that P_N and Q_N are bounded projectors in $C^\delta(\bar\Omega)$ that commute with the operator $\mathbf{L}$. The subspace E_N is invariant with respect to P_N, and $\tilde E_N$ is invariant with respect to Q_N.

The operator $\mathbf{L}$ can be considered as an unbounded operator acting in $C^\delta(\bar\Omega')$ with the domain

$$D(\mathbf{L}) = \{u \in C^{2+\delta}(\bar\Omega),\ u|_{\partial\Omega} = 0\}.$$

Denote by $\mathbf{L}_{\rm I}$ and $\mathbf{L}_{\rm II}$ the restrictions of the operator $\mathbf{L}$ to the subspaces E_N and $\tilde E'_N$, respectively. The domains of these operators are the intersections of the corresponding subspaces with the domain of the operator $\mathbf{L}$.

It is proved in [9] that for N sufficiently large $\mathbf{L}_{\rm I}$ is a Fredholm operator with the zero index. Note that this result remains valid without the assumption that the coefficients have limits at infinity.

Since its kernel is empty, then it is invertible. We can represent a function $f \in C^\delta(\bar\Omega)$ as a sum, $f = f_{\rm I} + f_{\rm II}$, where $f_{\rm I} \in E_N$ and $f_{\rm II} \in \tilde E_N$. Then the equation $\mathbf{L}_{\rm I} u = f_{\rm I}$ is solvable in E_N. Denote its solution by $u_{\rm I}$. Then $\mathbf{L} u_{\rm I} = \mathbf{L}_{\rm I} u_{\rm I} = f_{\rm I}$. On the other hand, if we look for the solution of the equation $\mathbf{L}u = f_{\rm I}$ in the form of the Fourier series with respect to the eigenfunctions of the Laplace operator in the section of the cylinder,

$$u_{\rm I}(x,y) = \sum_{k=N+1}^{\infty} \sum_{j=1}^{p_k} u_k^j(x)\varphi_k^j(y), \quad f_{\rm I}(x,y) = \sum_{k=N+1}^{\infty} \sum_{j=1}^{p_k} f_k^j(x)\varphi_k^j(y),$$

then by virtue of the condition of the theorem we find unique solutions u_k^j of the corresponding ordinary differential systems of equations. Hence for $k > N$

$$u_k^j(x) = \int_{\Omega'} u_{\rm I}(x,y)\varphi_k^j(y)dy,$$

and the Fourier series converges to $u_{\rm I}(x)$. It remains to note that (1.5) differs from the Fourier representation for $u_{\rm I}$ by a finite number of terms and, consequently, converges. The theorem is proved. □

In the remaining part of this section we will consider almost dichotomic systems on half-lines.

Condition 6.5. Every system (1.6) is almost dichotomic both on the left- and on the right-half axis.

Let $M^{\pm}_{S,k}(x)$, $M^{\pm}_{U,k}(x)$ and $M^{\pm}_{B,k}(x)$ be respectively stable, unstable and bounded subspaces of systems (1.6) in positive and negative direction. Let $n^{\pm}_{S,k}$, $n^{\pm}_{U,k}$ and $n^{\pm}_{B,k}$ be corresponding dimensions. Denote by $\overline{M}_{B,k}$ the space of solutions of system (1.6) bounded on *all* the axis. Let $d_{B,k} = \dim \overline{M}_{B,k}(x)$,

$$N_B = \sum_{k=1}^{\infty} p_k d_{B,k}.$$

Since there is only finite number of nonzero values $d_{B,k}$ (see Lemma 6.1), the number N_B is finite. Let us select a basis $\eta_k 1(x), \dots, \eta_k^{d_{B,k}}(x)$ in every space $M_{B,k}(x)$ such that every $\eta_k^j(x)$ is a solution of the corresponding system (1.6). Consider the problem

$$\mathbf{L}^* u = 0, \tag{6.4}$$

adjoint to (1.3). It is described by operator $\mathbf{L}^* u = u_{xx} + \Delta_y u - A(x)u_x + B(x)u$.

Then the space $\mathbf{B}$ of bounded solutions of the problem (6.4) has a finite basis

$$\{\eta_k^l(x)\varphi_k^j(y) : \quad k = 1, \dots \mathbb{N}, \quad j = 1, \dots p_k, \quad l = 1, \dots d_{B,k}\}$$

and $\dim \mathbf{B} = N_B$. For every $F \in X \times X$ and $\eta \in \mathbf{B}$ we consider a continuous function $R[F, \eta] : \mathbb{R} \to \mathbb{R}^{2n}$ defined by the formula

$$R[F, \eta](x) = \int_0^x dt \int_{\Omega'} \langle F(t, y), \eta(t, y) \rangle dy.$$

This function depends linearly both on F and on η.

Consider the following condition

$$\sup_{x \in \mathbb{R}} |R[F, \eta](x)| < +\infty \quad \forall \eta \in \mathbf{B}. \tag{6.5}$$

It can be also written in the form

$$\sup_{x \in \mathbb{R}} |R[F, \eta_k^l \varphi_k^j](x)| < +\infty \quad \forall k = 1, \dots \mathbb{N}, \quad j = 1, \dots p_k, \quad l = 1, \dots, d_{B,k}.$$

For every $f \in X$ we take the corresponding $F(f) = (0, f) \in X \times X$. We shall say that f satisfies (6.5) if it is true for $F(f)$.

Denote by $\widetilde{X} \subset X$ the subspace of functions f satisfying (6.5). It becomes a Banach space with the norm

$$\|f\|_{\widetilde{X}} = \|f\|_X + \max_{j,k,l} \|R[f, \eta_k^l \varphi_k^j](x)\|_{C^0}.$$

If every system (1.6) is dichotomic both on the left- and the right-half axis, we may consider the corresponding operators $\mathcal{L}_k^\pm$ and spaces $M_k^\pm$ introduced in Section 4.

Theorem 6.6. *If conditions* (6.5) *and*

$$\mathcal{L}_k^+ f_k^i(0) - \mathcal{L}_k^- f_k^i(0) \in M_k^+ + M_k^-$$

are satisfied for all $k \in \mathbb{N}$, then problem (1.4) *is solvable. Moreover, the operator $\mathbf{L} : U \to \widetilde{X}$ is Fredholm with the index*

$$\operatorname{ind} \mathbf{L} = \sum_{k=1}^{\infty} (n_{S,k}^+ + n_{B,k}^+ - n_{S,k}^-) p_k = \sum_{k=1}^{\infty} (n_{U,k}^- + n_{B,k}^- - n_{U,k}^+) p_k. \tag{6.6}$$

Both sums in (6.6) *are finite.*

Proof. Split the space $\widetilde{X}$ into the direct sum $\widetilde{X} = \widetilde{X}_1 \oplus \widetilde{X}_2$, where

$$\widetilde{X}_1 = \{f \in \widetilde{X} : f(x, y) = \sum_{k=1}^{N} \sum_{i=1}^{p_k} f_k^i(x) \varphi_k^i(y)\},$$

$$\widetilde{X}_2 = \{f \in \widetilde{X} : f_k^i(x) = 0, \ k = 1, \dots, N, \ i = 1, \dots, p_k\},$$

the number $N \in \mathbb{N}$ is the same as in Lemma 6.1. As it was mentioned above, one may consider the splitting $U = U_1 \oplus U_2$ and the restrictions $\mathbf{L}_i : U_i \to \widetilde{X}_i$ of the corresponding operators to these subspaces. It is shown in the proof of Theorem 6.4 that the operator $\mathbf{L}_2$ is invertible.

The space $\widetilde{X}_1$ splits into the direct sum

$$\widetilde{X}_1 = p_1 X^{(1)} \oplus p_2 X^{(2)} \oplus \dots \oplus p_N X^{(N)}.$$

Every term in this sum corresponds to an eigenfunction φ_k^i. Similarly, we may present the space U. The operator $\mathbf{L}_1$ is the sum of operators $\mathbf{L}^{(k)} : U^{(k)} \to X^{(k)}$,

$$\mathbf{L}^{(k)} = \frac{d^2}{dx^2} + A(x)\frac{d}{dx} + B(x) + \omega_k E_n.$$

and the operators

$$\mathbf{T}_k = \frac{d}{dx} - P_k(x) : Y^{(k)} \to X^{(k)} \times X^{(k)},$$

where the matrices P_k are defined in Section 1 and $Y^{(k)}$ is the subspace in $C^{1+\delta}(\mathbb{R} \to \mathbb{R}^{2n})$, containing functions $F = (f_1, f_2)$, which satisfy the condition

$$\sup_{x\geq 0} \Big| \int_0^x \langle F(t), \eta(t)\rangle \, dt \Big| < +\infty$$

for every $\eta(x) \in M^+_{B,k}(x)$ and

$$\sup_{x\leq 0} \Big| \int_0^x \langle F(t), \eta(t)\rangle \, dt \Big| < +\infty$$

for every $\eta(x) \in M^-_{B,k}(x)$. It has been shown that all operators T_k are Fredholm and their indices satisfy (4.5). On the other hand, it is proved in [9] that

$$\operatorname{ind} \mathbf{L} = \sum_{k=1}^{N} p_k \operatorname{ind} \mathbf{L}^{(k)}.$$

Therefore it remains to prove the following lemma. □

Lemma 6.7. *If the operator $\mathbf{T}_k$ is Fredholm, then the operator $\mathbf{L}^{(k)}$ is also Fredholm and their indices are equal to each other.*

Proof. The substitution $v(x) = u'(x)/\lambda_k$ defines an isomorphism of the spaces $\ker \mathbf{L}^{(k)}$ and $\ker \mathbf{T}_k$. We shall show that numbers of solvability conditions are also equal. Consider the system

$$\begin{gathered} u' = \lambda_k v + f_1(x), \\ v' = \big(-\frac{B(x)}{\lambda_k} + \lambda_k E_m \big) u - A(x) v + f_2(x), \end{gathered} \tag{6.7}$$

where $f_1(x) \in C^{1+\delta}(\mathbb{R} \to \mathbb{R}^m)$ and $f_2(x) \in C^{\delta}(\mathbb{R} \to \mathbb{R}^m)$.

The transformation $q = v + f_1(x)/\lambda_k$ reduces system (6.7) to

$$\begin{gathered} u' = \lambda_k q, \\ q' = \left(-\frac{B(x)}{\lambda_k} + \lambda_k E_m \right) u - A(x) q + g(x), \end{gathered} \tag{6.8}$$

where

$$g(x) = \frac{f_1'(x) + A(t) f_1(x)}{\lambda_k} + f_2(x) =: \pi(f_1, f_2)(x) \in C^{\delta}(\mathbb{R} \to \mathbb{R}^m).$$

System (6.8) has a bounded solution if and only if system (6.7) has one. If $\alpha_k = \dim\ker \mathbf{T}_k$, then there exist $\beta_k = \alpha_k + 2n - m_k^+ - m_k^-$ linearly independent functions $f^{(j)} = (f_1^{(j)}, f_2^{(j)})^T$, $j = 1, \dots, N$ such that for every

$$0 \neq f = (f_1, f_2)^T \in \operatorname{Lin}\{f^{(1)}, \dots, f^{(\beta_k)}\}$$

system (6.7) has no bounded solutions. Without loss of generality we can assume that all components $f_1^{(j)}$, $(j = 1, \dots, \beta_k)$ belong to $C^{1+\delta}(\mathbb{R})$.

Denote $g^{(j)} = \pi(f_1^{(j)}, f_2^{(j)})$. Then for every nontrivial linear combination

$$g(x) = c_1 g^{(1)} + \dots + c_{\beta_k} g^{(\beta_k)}$$

the corresponding system (6.8) has no bounded solutions. In particular this means that all functions $g^{(k)}$ are linearly independent. This system of linearly independent functions is complete, otherwise the number of solvability conditions for the operator $\mathbf{T}_k$ would exceed β_k.

On the other hand, system (6.8) has a solution bounded in $C^{2+\delta}(\mathbb{R} \to \mathbb{R}^m)$ if and only if it is true for the system

$$u'' + A(x)u' + (B(x) + \omega_k E_m)u = g(x).$$

Therefore, the numbers of solvability conditions for the corresponding operators are equal to each other. Due to Lemma 6.1 the operator $\mathbf{L}_2$ is invertible, so that $\operatorname{ind}\mathbf{L}_2 = 0$. Hence,

$$\operatorname{ind}\mathbf{L} = \operatorname{ind}\mathbf{L}_1 = \sum_{k=1}^{N}(n_{S,k}^+ + n_{B,k}^+ - n_{S,k}^-)p_k = \sum_{k=1}^{N}(n_{U,k}^- + n_{B,k}^- - n_{U,k}^+)p_k.$$

This completes the proof of Lemma 6.7 and of Theorem 6.6. □

Corollary 6.8. *If* $M_k^+ + M_k^- = \mathbb{R}^n$ *and* $M_k^+ \bigcap M_k^- = 0$ *for* $k = 1, \dots, N$, *then the operator* $\mathbf{L} : U \to \widetilde{X}$ *is continuously invertible.*

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