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A MINIMAX INEQUALITY FOR A CLASS OF FUNCTIONALS AND APPLICATIONS TO THE EXISTENCE OF SOLUTIONS FOR TWO-POINT BOUNDARY-VALUE PROBLEMS

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Abstract. In this paper, we establish an equivalent statement to minimax inequality for a special class of functionals. As an application, we prove the existence of three solutions to the Dirichlet problem

$$\begin{gathered}-u''(x)+m(x)u(x)=\lambda f(x,u(x)), \quad x\in(a,b),\\ u(a)=u(b)=0,\end{gathered}$$

where $\lambda>0$, $f:[a,b]\times\mathbb{R}\to\mathbb{R}$ is a continuous function which changes sign on $[a,b]\times\mathbb{R}$ and $m(x)\in C([a,b])$ is a positive function.

1. Introduction

Given two Gâteaux differentiable functionals Φ and T on a real Banach space X, the minimax inequality

$$\sup_{\lambda\geq 0}\inf_{u\in X}(\Phi(u)+\lambda(\rho-T(u)))<\inf_{u\in X}\sup_{\lambda\geq 0}(\Phi(u)+\lambda(\rho-T(u))), \quad \rho\in\mathbb{R}, \tag{1.1}$$

plays a fundamental role for establishing the existence of at least three critical points for the functional $\Phi(u)-\lambda T(u)$.

In this work some conditions that imply the minimax inequality (1.1) are pointed out and equivalent formulations are proved.

In this paper, our approach is based on a three critical-point theorem proved in [8] (Theorem 2.1) which stated below for the reader's convenience. Also we state a technical lemma that enables us to apply the theorem.

Lemma 2.2 below establishes an equivalent statement of minimax inequality (1.1) for a special class of functionals, while its consequences (Lemmas 2.5 and 2.7) guarantee some conditions so that minimax inequality holds.

Finally, we apply Theorem 2.1 to elliptic equations, by using an immediate consequence of Lemma 2.2: We consider the boundary-value problem

$$\begin{gathered}-u''(x)+m(x)u(x)=\lambda f(x,u(x)), \quad x\in(a,b),\\ u(a)=u(b)=0,\end{gathered}\tag{1.2}$$

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where $\lambda > 0$, $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function which changes sign on $[a, b] \times \mathbb{R}$, m is a continuous, positive function and we establish some conditions on f so that problem (1.2) admits at least three weak solutions.

We say that u is a weak solution to (1.2) if $u \in W_0^{1,2}([a, b])$ and

$$\int_a^b u'(x)v'(x)dx + \int_a^b m(x)u(x)v(x)dx - \lambda \int_a^b f(x, u(x))v(x)dx = 0$$

for every $v \in W_0^{1,2}([a, b])$.

By arguments similar to those in problem (1.2), we will have the existence of at least three weak solutions for the problem

$$\begin{gathered} -u''(x) + m(x)u(x) = \lambda h_1(x)h_2(u(x)), \quad x \in (a, b) \\ u(a) = u(b) = 0, \end{gathered} \tag{1.3}$$

where $h_1 \in C([a, b])$ is a function which changes sign on $[a, b]$ and $h_2 \in C(\mathbb{R})$ is a positive function. The existence of at least three weak solutions is also proved for the problem

$$\begin{gathered} -u''(x) + m(x)u(x) = \lambda f(u(x)), \quad x \in (a, b) \\ u(a) = u(b) = 0, \end{gathered} \tag{1.4}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function which changes sign on $\mathbb{R}$.

Conditions that guarantee the existence of multiple solutions to differential equations are of interest because physical processes described by differential equations can exhibit more that one solution. For example, certain chemical reactions in tubular reactors can be mathematically described by a nonlinear, two-point boundary value problem with the interest in seeing if multiple steady-states to the problem exist. For a recent treatment of chemical reactor theory and multiple solutions see [2, section 7] and references therein.

In recent years, many authors have studied multiple solutions from several points of view and with different approaches and we refer to [1, 3, 4, 7] and the references therein for more details, for instance, in their interesting paper [3], the authors studied problem

$$\begin{gathered} u'' + \lambda f(u) = 0, \\ u(0) = u(1) = 0, \end{gathered} \tag{1.5}$$

(in the case independent of λ) by using a multiple fixed-point theorem to obtain three symmetric positive solutions under growth conditions on f.

Also, in [4], the author proves multiplicity results for the problem (1.5) which for each $\lambda \in [0, +\infty[$, admits at least three solutions in $W_0^{1,2}([0, 1])$ when f is a continuous function.

In particular, in [1] we obtained the existence of an interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q such that, such that for each $\lambda \in \Lambda$ problem

$$\begin{gathered} \Delta_p u + \lambda f(x, u) = a(x)|u|^{p-2}u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.6}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $\Omega \subset \mathbb{R}^N (N \geq 2)$ is non-empty bounded open set with smooth boundary $\partial\Omega$, $p > N$, $\lambda > 0$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function and positive weight function $a(x) \in C(\overline{\Omega})$, admits at least three weak solutions whose norms in $W_0^{1,p}(\Omega)$ are less than q.

For additional approaches to the existence of multiple solutions to boundary-value problems, see [2, 5, 6] and references therein.

2. Main results

First, we recall the three critical point theorem by Ricceri [8] when choosing $h(\lambda) = \lambda\rho$.

Theorem 2.1. *Let X be a separable and reflexive real Banach space; $\Phi : X \to \mathbb{R}$ a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*; $\Psi : X \to \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that*

$$\lim_{\|u\|\to+\infty}(\Phi(u) + \lambda\Psi(u)) = +\infty$$

for all $\lambda \in [0, +\infty[$, and that there exists $\rho \in \mathbb{R}$ such that

$$\sup_{\lambda\geq 0}\inf_{u\in X}(\Phi(u) + \lambda\Psi(u) + \lambda\rho) < \inf_{u\in X}\sup_{\lambda\geq 0}(\Phi(u) + \lambda\Psi(u) + \lambda\rho).$$

Then, there exists an open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, the equation

$$\Phi'(u) + \lambda\Psi'(u) = 0$$

has at least three solutions in X whose norms are less than q.

Here and in the sequel, X will denote the Sobolev space $W_0^{1,2}([a, b])$ with the norm

$$\|u\| := \Big(\int_a^b |u'(x)|^2 dx\Big)^{1/2},$$

$f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $g : [a, b] \times \mathbb{R} \to \mathbb{R}$ is defined by

$$g(x, t) = \int_0^t f(x, \xi) d\xi$$

for each $(x, t) \in [a, b] \times \mathbb{R}$. Now, we define

$$\|u\|_* := \Big(\int_a^b (|u'(x)|^2 + m(x)|u(x)|^2) dx\Big)^{1/2}.$$

So the Poincaré's inequality and the positivity of the function $m(x) \in C([a, b])$, there exist positive suitable constants c_1 and c_2 such that

$$c_1\|u\| \leq \|u\|_* \leq c_2\|u\| \tag{2.1}$$

(i.e., the above norms are equivalent). We now introduce two positive special functionals on the Sobolev space X: For $u \in X$, let

$$\Phi(u) := \frac{\|u\|_*^2}{2},$$

$$T(u) := \int_a^b g(x, u(x)) dx$$

Let $\rho, r \in \mathbb{R}$, $w \in X$ be such that $0 < \rho < T(w)$ and $0 < r < \Phi(w)$. We put

$$\beta_1(\rho, w) := \rho \frac{\Phi(w)}{T(w)}, \tag{2.2}$$

$$\beta_2(r, w) := r \frac{T(w)}{\Phi(w)}, \tag{2.3}$$

$$\beta_3(\rho, w) := \frac{1}{c_1}\big(\frac{b-a}{2}\beta_1(\rho, w)\big)^{1/2}, \tag{2.4}$$

Clearly, $\beta_1(\rho, w)$, $\beta_2(r, w)$ and $\beta_3(\rho, w)$ are positive. Now, we put

$$\delta_1 := \inf\{\frac{(b-a)^{1/2}}{2c_1}\|u\|_* \in \mathbb{R}^+; T(u) \geq \rho\},$$

$$\begin{aligned} \delta_2 := \inf\big\{ & \frac{(b-a)^{1/2}}{2c_1}\|u\|_* \in \mathbb{R}^+, \text{ such that} \\ & (b-a) \max_{(x,t)\in[a,b]\times[-\frac{(b-a)^{1/2}}{2c_1}\|u\|_*,\ \frac{(b-a)^{1/2}}{2c_1}\|u\|_*]} g(x,t) \geq \rho\big\} \end{aligned}$$

and

$$\delta_\rho := \delta_1 - \delta_2. \tag{2.5}$$

Clearly, $\delta_1 \geq \delta_2$. Taking into account that for every $u \in X$,

$$\max_{x\in[a,b]} |u(x)| \leq \frac{(b-a)^{1/2}}{2}\|u\|$$

and (2.1), we have

$$\max_{x\in[a,b]} |u(x)| \leq \frac{(b-a)^{1/2}}{2c_1}\|u\|_*$$

for each $u \in X$. So that

$$T(u) = \int_a^b g(x, u(x))dx \leq (b-a)\max g(x,t)$$

where $(x,t) \in [a,b] \times [-\frac{(b-a)^{1/2}}{2c_1}\|u\|_*\ ,\ \frac{(b-a)^{1/2}}{2c_1}\|u\|_*]$. Namely

$$T(u) \leq (b-a)\max g(x,t),$$

where $(x,t) \in [a,b] \times [-\frac{(b-a)^{1/2}}{2c_1}\|u\|_*\ ,\ \frac{(b-a)^{1/2}}{2c_1}\|u\|_*\ ]$; therefore, $\{\frac{(b-a)^{1/2}}{2c_1}\|u\|_* \in \mathbb{R}^+; T(u) \geq \rho\}$ is a subset of

$$\begin{aligned} \big\{ & \frac{(b-a)^{1/2}}{2c_1}\|u\|_* \in \mathbb{R}^+ \text{ such that} \\ & (b-a) \max_{(x,t)\in[a,b]\times[-\frac{(b-a)^{1/2}}{2c_1}\|u\|_*\ ,\ \frac{(b-a)^{1/2}}{2c_1}\|u\|_*\ ]} g(x,t) \geq \rho\big\}. \end{aligned}$$

So, we have $\delta_1 \geq \delta_2$ and $\delta_\rho \geq 0$.

Our main results depend on the following lemma:

Lemma 2.2. *Assume that there exist $\rho \in \mathbb{R}$, $w \in X$ such that*

(i) $0 < \rho < T(w)$,
(ii) $(b-a)\max_{(x,t)\in[a,b]\times[-\beta_3(\rho,w)+\delta_\rho,\ \beta_3(\rho,w)-\delta_\rho]} g(x,t) < \rho$, *where $\beta_3(\rho, w)$ is given by (2.4) and δ_ρ by (2.5).*

Then, there exists $\rho \in \mathbb{R}$ such that

$$\sup_{\lambda \geq 0} \inf_{u \in X} (\Phi(u) + \lambda(\rho - T(u))) < \inf_{u \in X} \sup_{\lambda \geq 0} (\Phi(u) + \lambda(\rho - T(u))).$$

Proof. From (ii), we obtain

$$\beta_3(\rho, w) - \delta_\rho \notin \{l \in \mathbb{R}^+ : (b-a) \max_{(x,t) \in [a,b] \times [-l,l]} g(x,t) \geq \rho\}.$$

Moreover

$$\inf\{l \in \mathbb{R}^+ ; (b-a) \max_{(x,t) \in [a,b] \times [-l,l]} g(x,t) \geq \rho\} \geq \beta_3(\rho, w) - \delta_\rho;$$

in fact, arguing by contradiction, we assume that there is $l_1 \in \mathbb{R}^+$ such that

$$(b-a) \max_{(x,t) \in [a,b] \times [-l_1,l_1]} g(x,t) \geq \rho$$

and

$$l_1 < \beta_3(\rho, w) - \delta_\rho,$$

so

$$(b-a) \max_{(x,t) \in [a,b] \times [-\beta_3(\rho,w)+\delta_\rho\ ,\ \beta_3(\rho,w)-\delta_\rho]} g(x,t) \geq (b-a) \max_{(x,t) \in [a,b] \times [-l_1,l_1]} g(x,t) \geq \rho\,.$$

This is a contradiction. So

$$\inf\{l \in \mathbb{R}^+ ; (b-a) \max_{(x,t) \in [a,b] \times [-l,l]} g(x,t) \geq \rho\} > \beta_3(\rho, w) - \delta_\rho.$$

Therefore,

$$\begin{aligned}
&\inf\{\frac{(b-a)^{1/2}}{2c_1} \|u\|_* \in \mathbb{R}^+ : \\
&(b-a) \max_{(x,t) \in [a,b] \times [-\frac{(b-a)^{1/2}}{2c_1}\|u\|_*\ ,\ \frac{(b-a)^{1/2}}{2c_1}\|u\|_*]} g(x,t) \geq \rho\} \\
&> \beta_3(\rho, w) - \delta_\rho;
\end{aligned}$$

namely $\beta_3(\rho, w) < \delta_1$. So, we have

$$\inf\{\frac{\|u\|_*^2}{2} \in \mathbb{R}^+ ; T(u) \geq \rho\} > \beta_1(\rho, w),$$

or equivalently

$$\inf\{\Phi(u);\ u \in T^{-1}([\rho, +\infty[)\} > \rho \frac{\Phi(w)}{T(w)},$$

and, taking in to account that (i) holds, one has

$$\frac{\inf\{\Phi(u);\ u \in T^{-1}([\rho, +\infty[)\}}{\rho} > \frac{\Phi(w) - \inf\{\Phi(u);\ u \in T^{-1}([\rho, +\infty[)\}}{T(w) - \rho}.$$

Now, there exists $\lambda \in \mathbb{R}$ such that

$$\lambda > \frac{\Phi(w) - \inf\{\Phi(u);\ u \in T^{-1}([\rho, +\infty[)\}}{T(w) - \rho}$$

and

$$\lambda < \frac{\inf\{\Phi(u);\ u \in T^{-1}([\rho, +\infty[)\}}{\rho}.$$

or equivalently

$$\inf\{\Phi(u);\ u \in T^{-1}([\rho, +\infty[)\} > \Phi(w) + \lambda(\rho - T(w))$$

and

$$\lambda\rho < \inf\{\Phi(u);\ u \in T^{-1}([\rho, +\infty[)\}.$$

Therefore, thanks to the $0 < \rho < T(w)$, we obtain

$$\inf_{u\in X}(\Phi(u) + \lambda(\rho - T(u))) < \inf\{\Phi(u);\ u \in T^{-1}([\rho, +\infty[)\}. \tag{2.6}$$

On other hand,

$$\inf_{u\in X}(\Phi(u) + \lambda(\rho - T(u))) \le (\Phi(0) + \lambda(\rho - T(0))) = \lambda\rho. \tag{2.7}$$

So, with (2.6) and (2.7), one has

$$\sup_{\lambda\ge 0}\inf_{u\in X}(\Phi(u) + \lambda(\rho - T(u))) < \inf\{\Phi(u);\ u \in T^{-1}([\rho, +\infty[)\}.$$

Therefore, thanks to the

$$\inf_{u\in X}\sup_{\lambda\ge 0}(\Phi(u) + \lambda(\rho - T(u))) = \inf\{\Phi(u);\ u \in T^{-1}([\rho, +\infty[)\},$$

we have

$$\sup_{\lambda\ge 0}\inf_{u\in X}(\Phi(u) + \lambda(\rho - T(u))) < \inf_{u\in X}\sup_{\lambda\ge 0}(\Phi(u) + \lambda(\rho - T(u))).$$

□

Remark 2.3. Note that $\sup_{\lambda\ge 0}\inf_{u\in X}(\Phi(u)+\lambda(\rho-T(u)))$ is well defined, because $\lambda \to \inf_{u\in X}(\Phi(u) + \lambda(\rho - T(u)))$ is upper semicontinuous in $[0, +\infty[$ and tends to $-\infty$ as $\lambda \to +\infty$.

Remark 2.4. If $\beta_3(\rho, w) - \delta_\rho \le 0$ in Lemma 2.2,; then then the lemma still holds. Because, $\beta_3(\rho, w) \le \delta_1 - \delta_2 \le \delta_1$, and by arguing as in the proof of Lemma 2.2, the results holds.

If instead of condition (ii) in Lemma 2.2, we put

$$(b-a)\max_{(x,t)\in[a,b]\times[-\beta_3(\rho,w)\ ,\ \beta_3(\rho,w)]} g(x,t) < \rho,$$

then the result holds, because

$$\begin{aligned}(b-a)&\max_{(x,t)\in[a,b]\times[-\beta_3(\rho,w)+\delta_\rho\ ,\ \beta_3(\rho,w)-\delta_\rho]} g(x,t)\\ &\le (b-a)\max_{(x,t)\in[a,b]\times[-\beta_3(\rho,w)\ ,\ \beta_3(\rho,w)]} g(x,t) < \rho.\end{aligned}$$

So, we have the following result.

Lemma 2.5. *Assume that there exist $\rho \in \mathbb{R},\ w \in X$ such that*

(i) $0 < \rho < T(w)$,
(ii) $(b-a)\max_{(x,t)\in[a,b]\times[-\beta_3(\rho,w)\ ,\ \beta_3(\rho,w)]} g(x,t) < \rho$, *where $\beta_3(\rho, w)$ is given by* (2.4)

Then, there exists $\rho \in \mathbb{R}$ such that

$$\sup_{\lambda\ge 0}\inf_{u\in X}(\Phi(u) + \lambda(\rho - T(u))) < \inf_{u\in X}\sup_{\lambda\ge 0}(\Phi(u) + \lambda(\rho - T(u))).$$

Proposition 2.6. *The following assertions are equivalent:*

(a) *There are $\rho \in \mathbb{R}$, $w \in X$ such that*
(i) $0 < \rho < T(w)$,
(ii) $(b-a)\max_{(x,t)\in[a,b]\times[-\beta_3(\rho,w)\ ,\ \beta_3(\rho,w)]} g(x,t) < \rho$, *where $\beta_3(\rho, w)$ is given by* (2.4).

(b) *There are* $r \in \mathbb{R}$, $w \in X$ *such that*
(iii) $0 < r < \Phi(w)$,
(iv) $(b-a)\max_{(x,t)\in[a,b]\times[-\frac{1}{c_1}\sqrt{\frac{b-a}{2}r}\ ,\ \frac{1}{c_1}\sqrt{\frac{b-a}{2}r}]} g(x,t) < \beta_2(r,w)$, *where* $\beta_2(r,w)$ *is given by* (2.3).

Proof. (a) $\Rightarrow$ (b). First we note that $0 < \Phi(w)$, because if $\Phi(w) = 0$, one has $\frac{(b-a)^{1/2}}{2c_1}\|w\|_* = 0$. Hence, taking into account (ii), one has

$$T(w) \le (b-a) \max_{(x,t)\in[a,b]\times[-\frac{(b-a)^{1/2}}{2c_1}\|w\|_*\ ,\ \frac{(b-a)^{1/2}}{2c_1}\|w\|_*\ ]} g(x,t) = 0,$$

and that is in contradiction to (i). We now put $\beta_1(\rho,w) = r$. We obtain $\rho = \beta_2(r,w)$ and $\beta_3(\rho,w) = \frac{1}{c_1}\sqrt{\frac{b-a}{2}r}$. Therefore, from (i) and (ii), one has $0 < r < \Phi(w)$ and

$$(b-a) \max_{(x,t)\in[a,b]\times[-\frac{1}{c_1}\sqrt{\frac{b-a}{2}r}\ ,\ \frac{1}{c_1}\sqrt{\frac{b-a}{2}r}]} g(x,t) < \beta_2(r,w).$$

(b) $\Rightarrow$ (a) First we note that $0 < T(w)$, because if $0 \ge T(w)$, from (iii) one has $r\frac{T(w)}{\Phi(w)} \le 0$; namely, $\beta_2(r,w) \le 0$. Hence, from (iv) one has

$$0 = T(0) \le (b-a) \max_{(x,t)\in[a,b]\times[-\frac{1}{c_1}\sqrt{\frac{b-a}{2}r}\ ,\ \frac{1}{c_1}\sqrt{\frac{b-a}{2}r}]} g(x,t) < 0,$$

and this is a contradiction. We now put $\beta_2(r,w) = \rho$. We obtain $r = \beta_1(\rho,w)$ and $\frac{1}{c_1}\sqrt{\frac{b-a}{2}r} = \beta_3(\rho,w)$. Therefore, from (iii) and (iv), we have the conclusion. $\square$

The following lemma is another consequence of Lemma 2.2.

Lemma 2.7. *Assume that there exist* $r \in \mathbb{R}$, $w \in X$ *such that*
(i) $0 < r < \Phi(w)$,
(ii) $(b-a)\max_{(x,t)\in[a,b]\times[-\frac{1}{c_1}\sqrt{\frac{b-a}{2}r}\ ,\ \frac{1}{c_1}\sqrt{\frac{b-a}{2}r}]} g(x,t) < \beta_2(r,w)$, *where* $\beta_2(r,w)$ *is given by* (2.3).

Then, there exists $\rho \in \mathbb{R}$ *such that*

$$\sup_{\lambda\ge0}\inf_{u\in X}(\Phi(u) + \lambda(\rho - T(u))) < \inf_{u\in X}\sup_{\lambda\ge0}(\Phi(u) + \lambda(\rho - T(u))).$$

The above lemma follows from Lemma 2.5 and Proposition 2.6.

Finally, we are interested in ensuring the existence of at least three weak solutions for the Dirichlet problem (1.2). Now, we have the following result.

Theorem 2.8. *Assume that there exist* $\rho \in \mathbb{R}$, $a_1 \in L^1([a,b])$, $w \in X$ *and a positive constant* γ *with* $\gamma < 2$ *such that*
(i) $0 < \rho < \int_a^b g(x,w(x))dx$,
(ii) $(b-a)\max_{(x,t)\in[a,b]\times[-\beta_3(\rho,w)\ ,\ \beta_3(\rho,w)]} g(x,t) < \rho$
(iii) $g(x,t) \le a_1(x)(1+|t|^\gamma)$ *almost everywhere in* $[a,b]$ *and for each* $t \in \mathbb{R}$, *where* $\beta_3(\rho,w)$ *is given by* (2.4).

Then, there exists an open interval $\Lambda \subseteq [0,+\infty[$ *and a positive real number* q *such that, for each* $\lambda \in \Lambda$, *problem* (1.2) *admits at least three solutions in* X *whose norms are less than* q.

Proof. For each $u \in$X, we put

$$\Phi(u) = \frac{\|u\|_*^2}{2},$$
$$\Psi(u) = -\int_a^b g(x, u(x))dx.$$
$$J(u) = \Phi(u) + \lambda\Psi(u).$$

In particular, for each $u, v \in X$ one has

$$\Phi'(u)(v) = \int_a^b (u'(x)v'(x) + m(x)u(x)v(x))dx, \Psi'(u)(v) = -\int_a^b f(x, u(x))v(x)dx.$$

It is well known that the critical points of J are the weak solutions of (1.2), our goal is to prove that Φ and Ψ satisfy the assumptions of Theorem 2.1. Clearly, Φ is a continuously Gâteaux differentiable and sequentially weakly lower semi continuous functional whose Gâteaux derivative admits a continuous inverse on X^* and Ψ is a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact.

Thanks to (iii), for each $\lambda > 0$ one has

$$\lim_{\|u\|\to+\infty}(\Phi(u) + \lambda\Psi(u)) = +\infty\,.$$

Furthermore, thanks to Lemma 2.5, from (i) and (ii), we have

$$\sup_{\lambda\geq 0}\inf_{u\in X}(\Phi(u) + \lambda\Psi(u) + \lambda\rho) < \inf_{u\in X}\sup_{\lambda\geq 0}(\Phi(u) + \lambda\Psi(u) + \lambda\rho).$$

Therefore, we can apply Theorem 2.1. It follows that there exists an open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, problem (1.2) admits at least three solutions in X whose norms are less than q. □

We also have the following existence result.

Theorem 2.9. *Assume that there exist* $r \in \mathbb{R}$, $a_2 \in L^1([a, b])$, $w \in X$ *and a positive constant* γ *with* $\gamma < 2$ *such that*

(i) $0 < r < \frac{\|w\|_*^2}{2}$;
(ii) $(b - a)\max_{(x,t)\in[a,b]\times[-\frac{1}{c_1}\sqrt{\frac{b-a}{2}r}\ ,\ \frac{1}{c_1}\sqrt{\frac{b-a}{2}r}]} g(x, t) < \beta_2(r, w)$;
(iii) $g(x, t) \leq a_2(x)(1 + |t|^\gamma)$ *almost everywhere in* $[a, b]$ *and for each* $t \in \mathbb{R}$, *where* $\beta_2(r, w)$ *is given by* (2.3).

Then, there exists an open interval $\Lambda \subseteq [0, +\infty[$ *and a positive real number* q *such that, for each* $\lambda \in \Lambda$, *problem* (1.2) *admits at least three solutions in* X *whose norms are less than* q.

The above theorem follows from Lemma 2.7 and Theorem 2.8.

Let $h_1 \in C([a, b])$ be a function which changes sign on $[a, b]$ and $h_2 \in C(\mathbb{R})$ be a positive function. For for $(x, t) \in [a, b] \times \mathbb{R}$, put

$$f(x, t) = h_1(x)h_2(t).$$

For for $t \in \mathbb{R}$, put

$$\alpha(t) = \int_0^t h_2(\xi)d\xi\,.$$

For almost every $x \in [a, b]$, put

$$a_3(x) = \frac{a_1(x)}{h_1(x)}$$

Then, using Theorem 2.8, we have the following result.

Theorem 2.10. *Assume that there exist $\rho \in \mathbb{R}$, $a_3 \in L^1([a, b])$, $w \in X$ and a positive constant γ with $\gamma < 2$ such that*

(i) $0 < \rho < \int_a^b (h_1(x)\alpha(w(x)))dx$;
(ii) $(b - a) \max_{x \in [a,b]} h_1(x) < \frac{\rho}{\alpha(\beta_3(\rho,w))}$;
(iii) $\alpha(t) \leq a_3(x)(1 + |t|^\gamma)$ *almost everywhere in* $[a, b]$ *and for each* $t \in \mathbb{R}$, *where* $\beta_3(\rho, w)$ *is given by* (2.4).

Then, there exists an open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, problem (1.3) *admits at least three solutions in X whose norms are less than q.*

Put

$$a_4(x) = \frac{a_2(x)}{h_1(x)}$$

for almost every $x \in [a, b]$. Then, by Theorem 2.9, we have the following existence result.

Theorem 2.11. *Assume that there exist $r \in \mathbb{R}$, $a_4 \in L^1([a, b])$, $w \in X$ and a positive constant γ with $\gamma < 2$ such that*

(i) $0 < r < \frac{\|w\|_*^2}{2}$;
(ii) $(b - a) \max_{x \in [a,b]} h_1(x) < \frac{\beta_2(r,w)}{\alpha(\frac{1}{c_1}\sqrt{\frac{b-a}{2}r})}$;
(iii) $\alpha(t) \leq a_4(x)(1 + |t|^\gamma)$ *almost everywhere in* $[a, b]$ *and for each* $t \in \mathbb{R}$, *where* $\beta_2(r, w)$ *is given by* (2.3).

Then, there exists an open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, problem (1.3) *admits at least three solutions in X whose norms are less than q.*

We now want to point out two simple consequences of Theorems 2.8 and 2.9. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function which changes sign on $\mathbb{R}$. For $t \in \mathbb{R}$, put $g(t) = \int_0^t f(\xi)d\xi$. So we have the following results.

Theorem 2.12. *Assume that there exist $\rho \in \mathbb{R}$, $w \in X$ and two positive constants γ and η with $\gamma < 2$ such that*

(i) $0 < \rho < \int_a^b g(w(x))dx$;
(ii) $(b - a) \max_{t \in [-\beta_3(\rho,w)\ ,\ \beta_3(\rho,w)]} g(t) < \rho$;
(iii) $g(t) \leq \eta(1 + |t|^\gamma)$ *for each* $t \in \mathbb{R}$, *where* $\beta_3(\rho, w)$ *is given by* (2.4).

Then, there exists an open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, problem (1.4) *admits at least three solutions in X whose norms are less than q.*

Theorem 2.13. *Assume that there exist $r \in \mathbb{R}$, $w \in X$ and two positive constants γ and μ with $\gamma < 2$ such that*

(i) $0 < r < \frac{\|w\|_*^2}{2}$;
((ii) $(b - a) \max_{t \in [-\frac{1}{c_1}\sqrt{\frac{b-a}{2}r},\ \frac{1}{c_1}\sqrt{\frac{b-a}{2}r}]} g(t) < \beta_2(r, w)$;

(iii) $g(t) \leq \mu(1+|t|^{\gamma})$ *for each* $t \in \mathbb{R}$, *where* $\beta_2(r,w)$ *is given by* (2.3).

Then, there exists an open interval $\Lambda \subseteq [0,+\infty[$ *and a positive real number* q *such that, for each* $\lambda \in \Lambda$, *problem* (1.4) *admits at least three solutions in* X *whose norms are less than* q.

Example 2.14. Let $\Omega = (0,1)$ and consider the problem

$$\begin{gathered} -u'' + e^x u = \lambda(e^u u^2(3+u)), \quad x \in (0,1) \\ u(0) = u(1) = 0. \end{gathered} \tag{2.8}$$

Then, there exists an open interval $\Lambda \subseteq [0,+\infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, problem (2.8) admits at least three solutions in $W_0^{1,2}([0,1])$ whose norms are less than q. In fact, by choosing $\rho = \frac{1}{4}$ and

$$w(x) = \begin{cases} x, & x \in (0,1) \\ 0, & \text{otherwise} \end{cases}$$

so that $\beta_3(\rho,w) = \frac{1}{c_1}(\frac{e-1}{96-32e})^{1/2}$, all assumptions of Theorem 2.12, are satisfied with $\gamma = 1$, c_1 is positive constant such that the inequality (2.1) hold for $m(x) = e^x$ and η sufficiently large, also with choose $r = \frac{1}{2}$ so that $\beta_2(r,w) = \frac{6-2e}{e-1}$, all assumptions of Theorem 2.13, are satisfied with μ sufficiently large.

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