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SPECTRAL PROPERTIES OF NON-LOCAL UNIFORMLY-ELLIPTIC OPERATORS

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Abstract. In this paper we consider the spectral properties of a class of non-local uniformly elliptic operators, which arise from the study of non-local uniformly elliptic partial differential equations. Such equations arise naturally in the study of a variety of physical and biological systems with examples ranging from Ohmic heating to population dynamics. The operators studied here are bounded perturbations of linear (local) differential operators, and the non-local perturbation is in the form of an integral term. We study the eigenvalues, the multiplicities of these eigenvalues, and the existence of corresponding positive eigenfunctions. It is shown here that the spectral properties of these non-local operators can differ considerably from those of their local counterpart. However, we show that under suitable hypotheses, there still exists a principal eigenvalue of these operators.

1. Introduction

This paper studies the spectral properties of a class of linear integro-differential operators

$$[L_\epsilon u](x) = \sum_{i,j=1}^{n} (a_{ij}(x)u_{x_i}(x))_{x_j} + b(x)u(x) + \epsilon c(x)\int_U d(x)u(x)dx, \quad x \in U, \tag{1.1}$$

where U is a bounded connected subset of $\mathbb{R}^n$ with a suitably smooth boundary, ∂U, and $-\sum_{i,j=1}^{n}(a_{ij}(x)u_{x_i}(x))_{x_j}$ is uniformly elliptic on U.

The operator L_ϵ is defined on a domain that incorporates homogeneous Dirichlet boundary conditions. By varying the real parameter ϵ, the non-local operator can be viewed as a continuous, bounded perturbation of the (local) differential operator,

$$[Au](x) = \sum_{i,j=1}^{n} (a_{ij}(x)u_{x_i}(x))_{x_j} + b(x)u(x). \tag{1.2}$$

In this paper this structure will be exploited to study the spectral properties of L_ϵ. Results will not be restricted to small ϵ; rather, ϵ should be viewed as a homotopy parameter from the local operator A to the general form L_ϵ.

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Operators of the type given in (1.1) arise from the study of non-local nonlinear parabolic problems of the form

$$u_t(x,t) = \sum_{i,j=1}^{n} (a_{ij}(x)u_{x_i}(x,t))_{x_j} + f(x,u(x,t),\bar{u}(t)), \quad \bar{u}(t) = \epsilon \int_U g(x,u(x,t))dx, \tag{1.3}$$

for sufficiently differentiable functions f and g. Here ϵ is a real parameter which can be viewed as a measure of strength of the non-local interactions. System (1.3) requires to be augmented with appropriate boundary conditions, for example homogeneous Dirichlet boundary conditions, as studied here, as well as an initial condition. Non-local boundary value problems of this type appear in a wide variety of applications, including Ohmic heating [13, 21], the formation of shear bands in materials [4], heat transfer in thermistors [11], combustion theory [22], the electric ballast resistor [8], microwave heating of ceramic materials [7, 20], and population dynamics [17]. An extensive survey of results, techniques and applications of non-local reaction-diffusion equations of this form is given in [15].

It is often desirable to identify steady states of (1.3) and determine their stability, as stable steady states represent possible asymptotic states of the system under consideration, and are thus of most physical relevance. Let us assume that a steady state, u^*, of (1.3) exists. Then linearizing (1.3) around u^*, leads to

$$u_t(x,t) = \sum_{i,j=1}^{n} (a_{ij}(x)u_{x_i}(x,t))_{x_j} + b(x)u(x,t) + \epsilon c(x)\int_U d(x)u(x,t)dx, \tag{1.4}$$

where $b(x) = f_u(x,u^*(x),\bar{u^*})$, $c(x) = f_{\bar{u}}(x,u^*(x),\bar{u^*})$ and $d(x) = g_u(x,u^*(x))$. We will assume that f and g, and consequently b, c and d are real valued functions. Formally at least, it is straightforward to see that the values of λ for which

$$\begin{gathered}\sum_{i,j=1}^{n} (a_{ij}(x)u_{x_i}(x))_{x_j} + b(x)u(x) + \epsilon c(x)\int_U d(x)u(x)dx = \lambda u(x), \quad x \in U; \\ u(x) = 0 \quad \text{on } \partial U,\end{gathered} \tag{1.5}$$

has a solution, determine the growth properties of the solutions of (1.4) and hence the (local asymptotic) stability of the steady state u^* of (1.3). This connection can be made rigorous as detailed in [18]. Indeed, an operator equation $L_\epsilon u = \lambda u$ can be defined on a suitable domain, which is equivalent to (1.5) and hence, the spectral properties of L_ϵ determine the stability of steady states of (1.3).

The spectral properties of (1.2) are well-known, and in [12], certain corresponding properties for the non-local operator L_ϵ are derived using the perturbation theory of linear operators (see e.g. [19]), for the special case $n = 1$. Results in [12] deal with the structure of the set of eigenvalues of L_ϵ, $\sigma(L_\epsilon)$, when considered as functions of the parameter ϵ, and show that the Fourier coefficients of the functions c and d in (1.1) with respect to the eigenfunctions of (1.2), are fundamental to determining the qualitative structure of $\sigma(L_\epsilon)$. Further results concerning the spectrum of L_ϵ, multiplicities of the eigenvalues of L_ϵ, and the nodal properties of the associated eigenfunctions for the case $n = 1$, are presented in [9]. In [16], some of the work done in [12] is extended to $n \geq 1$ in the case where A is of the form $Au = \Delta u + a(x)u$, where $\Delta\cdot$ denotes the Laplacian operator, as is standard. Other papers that include related results on spectral properties of non-local operators are [1, 5, 6, 10, 14, 23].

In this paper we extend results from [9] to the general $n \geq 1$ case and present some further spectral results, which are new to all cases. As is shown in the above references, the presence of the non-local term in L_ϵ gives a much wider variety of possible behaviour of the spectrum, than that of the corresponding local operator.

The following section contains relevant definitions, along with some basic results. In Section 3 we consider how the eigenvalues of L_ϵ change with ϵ, and results regarding multiplicities of eigenvalues of L_ϵ are given in Section 4. The existence of a principal eigenvalue, i.e. an eigenvalue to which there corresponds a positive eigenfunction, is important to a number of results including those related to determining stability and the existence of positive solutions of associated nonlinear problems (via bifurcation theory). [9, Theorem 5.5], [12, Lemma 3.16] and [14, Proposition 6.1] deal with the existence of positive eigenfunctions of L_ϵ. We present further, more general results dealing with not only the existence, but also the uniqueness of a principal eigenvalue of L_ϵ in Section 5.

2. Preliminaries

Let U be an open bounded connected subset of $\mathbb{R}^n$ where $n \geq 1$, and assume that the boundary of U, ∂U is C^{k+1} where $k := \lfloor \frac{n+4}{2} \rfloor$. Let $A, B, L_\epsilon : H^2(U) \cap H_0^1(U) \subset L^2(U) \to L^2(U)$ be defined by

$$\begin{gathered} [Au](x) := \sum_{i,j=1}^{n} (a_{ij}(x) u_{x_i}(x))_{x_j} + b(x)u(x), \\ [Bu](x) := c(x) \int_U d(x)u(x)dx, \\ L_\epsilon := A + \epsilon B, \end{gathered} \tag{2.1}$$

where $a_{ij}, b \in C^k(\bar{U})$; $c \in H^{k-1}(U)$; $d \in L^2(U)$; $c, d \not\equiv 0$; and $\epsilon \in \mathbb{R}$. Note that following standard regularity arguments, the condition on c is necessary for the eigenfunctions of L_ϵ to lie in $C^2(U)$. This in turn is necessary to ensure the equivalence of the operator equation $L_\epsilon u = \lambda u$ and (1.5). Also, assume that $-A$ is uniformly elliptic.

Then A is a densely defined, closed, self-adjoint operator with compact resolvent. Its spectrum is real, bounded above and consists entirely of isolated eigenvalues of finite multiplicity. Denote these eigenvalues by γ_i, $i = 1, 2, 3, \dots$ Then it is well-known that $\gamma_1 > \gamma_2 \geq \dots \geq \gamma_i \geq \gamma_{i+1} \geq \dots$ and $\gamma_i \to -\infty$ as $i \to \infty$. The largest eigenvalue of A, γ_1, is simple and the eigenfunction v_1 corresponding to γ_1 can be chosen to be strictly positive on U. Furthermore this is the only eigenvalue of A to which there corresponds a positive eigenfunction, and the set of eigenfunctions of A, $\{v_i\}_{i=1}^{\infty}$ can be chosen to form an orthonormal basis for $L^2(U)$.

Clearly B is a bounded linear operator, and therefore it can be shown that for each fixed ϵ, L_ϵ is a densely defined, closed operator with compact resolvent. Hence, for each fixed ϵ, the spectrum, $\sigma(L_\epsilon)$, consists entirely of isolated eigenvalues of finite multiplicity. Denote these eigenvalues by $\lambda_i(\epsilon)$ and for consistency let $\lambda_i(0) = \gamma_i$ for each $i \in \mathbb{N}$. Denote the eigenfunctions of L_ϵ by $u_i(\epsilon)$, where $u_i(\epsilon)$ corresponds to the eigenvalue $\lambda_i(\epsilon)$. Then in this way, we generate a set of functions, $\Sigma := \{\lambda_i(\epsilon)\}_{i=1}^{\infty}$, which we shall also refer to as eigenvalues of L_ϵ. Similarly, the functions of ϵ, $u_i(\epsilon)$ will be referred to as eigenfunctions. Then we may apply results contained in [19, Sections II-1, III-6.4, IV-3.5 and VII-1.3] to our problem to give:

Lemma 2.1. (a) *For each i, $\lambda_i(\epsilon)$ is a continuous function of ϵ, $\forall \epsilon \in \mathbb{R}$.*

(b) *Fix j. If $\lambda_j(\epsilon) \neq \lambda_i(\epsilon)$ for all $i \neq j$ and $\forall \epsilon \in (\epsilon_1, \epsilon_2)$, then $\lambda_j(\epsilon)$ is an analytic function of ϵ $\forall \epsilon \in (\epsilon_1, \epsilon_2)$, and the eigenprojection corresponding to $\lambda_j(\epsilon)$ is an analytic function of ϵ $\forall \epsilon \in (\epsilon_1, \epsilon_2)$.*

(c) *Let $S \subset \Sigma$ contain only a finite number of elements. If $\lambda_i(\epsilon) \neq \lambda_j(\epsilon)$ for any $\lambda_i(\epsilon) \in S$ and $\lambda_j(\epsilon) \in \Sigma \backslash S$, $\forall \epsilon \in (\epsilon_1, \epsilon_2)$, then the sum of the eigenvalues in S is an analytic function of ϵ for all $\epsilon \in (\epsilon_1, \epsilon_2)$. Furthermore, the total eigenprojection corresponding to all the eigenvalues in S is an analytic function of ϵ $\forall \epsilon \in (\epsilon_1, \epsilon_2)$.*

Also, by standard regularity theory, given the assumptions on the coefficient functions a_{ij}, b, c, and the boundary ∂U, it can be shown that any eigenfunction of L_ϵ is actually in $C^2(\bar{U})$, i.e. the spectral properties of $L_\epsilon : H^2(U) \cap H_0^1(U) \subset L^2(U) \to L^2(U)$ are identical to the spectral properties of the corresponding non-local differential equation.

It is useful to distinguish between those eigenvalues of L_ϵ which change with ϵ, and those which do not.

Definition 2.2. We call $\lambda_i(\epsilon)$ a *fixed eigenvalue* iff $\lambda_i(\epsilon) \equiv \gamma_i$. If $\lambda_i(\epsilon)$ is not fixed, then it is referred to as a *moving eigenvalue*.

Note that an eigenfunction $u_i(\epsilon)$ corresponding to a fixed eigenvalue $\lambda_i(\epsilon)$, may or may not vary with ϵ. If the latter holds, i.e. $u_i(\epsilon) \equiv v_i$, then we refer to such an eigenfunction as being *fixed*.

Let γ_i be a fixed eigenvalue of L_ϵ, and let $X = \bigcap_{\epsilon \in \mathbb{R}} N(L_\epsilon - \gamma_i I)$. We call X the *fixed eigenspace* of L_ϵ corresponding to γ_i.

Finally, the adjoint of L_ϵ, denoted L_ϵ^* is defined by

$$L_\epsilon^* u = Au + \epsilon B^* u, \quad \epsilon \in \mathbb{R}, \quad u \in H^2(U) \cap H_0^1(U)$$

where

$$[B^* u](x) = d(x) \int_U c(x) u(x) dx.$$

As already noted A is self-adjoint, i.e. L_0 is self-adjoint. Moreover, L_ϵ is self-adjoint if and only if $c \equiv d$, and clearly if L_{ϵ^*} is self-adjoint for some $\epsilon^* \neq 0$, then L_ϵ is self-adjoint for all $\epsilon \in \mathbb{R}$.

Whilst [16] considered non-local perturbations of $Au = \Delta u + a(x)u(x)$, it was noted there that it is straightforward to extend all of the spectral theory results in [16] to a wider class of non-local operators, including the operators given by (2.1). Hence, where we refer to relevant results from [16], we do so in terms of (2.1).

Understanding the key differences in spectral structure of L_ϵ between the case $n = 1$ studied in [9, 12] and the general case studied here is central to extending many results from [9] and [12]. Hence we now proceed to highlight the important differences between the structure of Σ in these two cases.

One of the main differences between the case $n = 1$ previously studied in [9, 12], and the general $n \geq 1$ case, is that for $n > 1$, eigenvalues of A can have arbitrarily large (but finite) multiplicities. Therefore, although from Lemma 2.1 the eigenvalues and eigenprojections are continuous, it may be possible for an eigenvalue to "split" at $\epsilon = 0$ to form multiple distinct eigenvalues of L_ϵ. However in [16], it was shown that:

Lemma 2.3 ([16]). *If an eigenvalue of A, γ_i, has geometric multiplicity $m > 1$, then γ_i is an eigenvalue of L_ϵ $\forall \epsilon \in \mathbb{R}$, and the fixed eigenspace, X, of L_ϵ corresponding to γ_i is such that* $\dim X \geq (m-1)$.

Note that by this result it follows that all moving eigenvalues of L_ϵ are of geometric multiplicity 1 until they "collide" with another eigenvalue of L_ϵ, (see Section 4 for a detailed discussion of multiplicities). The effect of A having eigenvalues of higher multiplicity is simply to add more (possibly high multiplicity) fixed eigenvalues to Σ, or to increase the multiplicities of the fixed eigenvalues already in Σ. Therefore, for differential operators of the form given above, the set of moving eigenvalues has the same properties as the set of moving eigenvalues of the operators studied in [9, 12], i.e. in the case $n = 1$. As shown in [16], these properties include that $\lambda_j(\epsilon_1) = \lambda_j(\epsilon_2)$ for any $j \in \mathbb{N}$ and any $\epsilon_1 \neq \epsilon_2$ only if $\lambda_j(\epsilon_1) = \lambda_j(\epsilon_2) = \gamma_i$. for some $i \in \mathbb{N}$. Hence if a real moving eigenvalue starts moving in one direction along the real line, then it can not turn back on itself as long as it remains real. Also, $\lambda_j(\epsilon_1) = \lambda_k(\epsilon_2)$ for any $j \neq k \in \mathbb{N}$ and any $\epsilon_1 \neq \epsilon_2$ only if $\lambda_j(\epsilon_1) = \lambda_k(\epsilon_2) = \gamma_i$ for some $i \in \mathbb{N}$. Hence if λ_j and λ_k are 2 real moving eigenvalues which are real $\forall \epsilon \in \mathbb{R}$, and if $\gamma_j < \gamma_k$, then $\lambda_j(\epsilon_1) \leq \lambda_k(\epsilon_2)$ for any $\epsilon_1, \epsilon_2 \in \mathbb{R}$.

3. The Spectrum of L_ϵ

As noted above, in general the spectrum of L_ϵ will vary with the parameter ϵ. In this section we present results that deal with precisely how the eigenvalues of L_ϵ change.

Ideally we would like to determine explicit formulae for the functions $\{\lambda_i(\epsilon)\}_{i=1}^\infty$. We have been unable to do this, but we have obtained the following implicit (but nevertheless useful) expression.

Lemma 3.1. *Take any real number $\lambda \neq \gamma_i$ for any $i \in \mathbb{N}$. Take an orthonormal basis of eigenfunctions of A, $\{v_i\}_{i=1}^\infty$, and let*

$$c(x) = \sum_{i=1}^\infty c_i v_i(x), \quad d(x) = \sum_{i=1}^\infty d_i v_i(x),$$

i.e. $\{c_i\}_{i=1}^\infty$ and $\{d_i\}_{i=1}^\infty$ are the Fourier coefficients of c and d respectively. Then the solution ϵ^* of the equation $\lambda = \lambda_{k^*}(\epsilon^*)$ is unique if it exists, and is given by

$$\epsilon^* = \Big(\sum_{i=1}^\infty \frac{c_i d_i}{(\lambda - \gamma_i)}\Big)^{-1}. \tag{3.1}$$

Proof. The uniqueness of the value ϵ^* follows from the arguments at the end of the preceding section. Let

$$u(x) = \sum_{i=1}^\infty \beta_i v_i(x). \tag{3.2}$$

Then, substituting the above expressions for c, d and u into the equation $L_\epsilon u = \lambda u$ and comparing the coefficients of v_i for each $i \in \mathbb{N}$ gives

$$\beta_i(\gamma_i - \lambda) + \epsilon c_i \int_U d(x)u(x)dx = 0.$$

Hence, either $c_i = \beta_i = 0$ or

$$\epsilon = \frac{\beta_i(\lambda - \gamma_i)}{c_i \int_U d(x)u(x)dx}.$$

But this will hold for all i such that $c_i \neq 0$, and so in this case

$$\frac{\beta_i(\lambda - \gamma_i)}{c_i} = K, \tag{3.3}$$

for some constant K independent of i, and without loss of generality we take $K = 1$. Hence,

$$u(x) = \sum_{i=1}^{\infty} \frac{c_i}{(\lambda - \gamma_i)} v_i(x), \tag{3.4}$$

and it follows directly from (3.3) and (3.4) that

$$\epsilon = \frac{1}{\int_U d(x)u(x)dx}.$$

By Parseval's formula we can interchange the order of summation and integration in $\int_U d(x) \sum_{i=1}^{\infty} \frac{c_i}{(\lambda-\gamma_i)} v_i(x)dx$, as $\{v_i\}_{i=1}^{\infty}$ is an orthonormal basis for $L^2(U)$, to give

$$\epsilon = \Big(\sum_{i=1}^{\infty} \frac{c_i d_i}{(\lambda - \gamma_i)} \Big)^{-1}.$$

□

Note that as observed in [16],

$$\begin{aligned} \lambda_i(\epsilon) \equiv \gamma_i \;&\Rightarrow\; \lambda_i'(0) = 0 \\ &\Leftrightarrow\; \int_U c(x)v_i(x)dx \int_U d(x)v_i(x)dx = 0 \qquad (3.5) \\ &\Leftrightarrow\; \text{either } Bv_i \equiv 0 \text{ or } B^* v_i \equiv 0, \end{aligned}$$

also

$$B[u_i(\epsilon^*)] \equiv 0 \text{ for some } \epsilon^* \in \mathbb{R} \;\Rightarrow\; \lambda_i(\epsilon) \equiv \gamma_i \text{ and } u_i(\epsilon) \equiv v_i = u_i(\epsilon^*).$$

Furthermore, as a result of Lemma 3.1

Lemma 3.2. *No moving eigenvalue of L_ϵ emanates from γ_j if and only if*

$$\int_U c(x)v(x)dx \int_U d(x)v(x)dx = 0 \tag{3.6}$$

for all $v \in N(A - \gamma_j I)$.

Proof. If no moving eigenvalue of L_ϵ emanates from γ_j, then (3.6) follows by the results listed immediately above. Now, suppose that (3.6) holds for all $v \in N(A - \gamma_j I)$, and assume that there exists a moving eigenvalue $\lambda_j(\epsilon)$ emanating from γ_j at $\epsilon = 0$, with corresponding eigenfunction $u_j(\epsilon)$. Then by Lemma 2.3, there exists an orthogonal basis $\{v_1, ..., v_n\}$ of $N(A - \gamma_j I)$, such that

$$\text{Span}\{v_1, ..., v_{n-1}\} = N(L_\epsilon - \gamma_j I), \quad \forall \epsilon \neq 0. \tag{3.7}$$

Moreover, by Lemma 2.1(c),

$$\lim_{\epsilon \to 0} u_j(\epsilon) = v_n.$$

Therefore putting $u = u_j(\epsilon)$ in (3.2) gives $\beta_n \neq 0$ for ϵ sufficiently small. Hence, from (3.4), $c_n \neq 0$, and noting that (3.6) is equivalent to $c_i d_i = 0$ for $i = 1, ..., n$, we have $d_n = 0$. However, again from above $d_n = 0 \Rightarrow v_n \in N(L_\epsilon - \gamma_j I)\ \forall \epsilon \in \mathbb{R}$, but this contradicts (3.7). □

In [12] an expression for $\lambda_j'(0)$ was obtained for the case $n = 1$. In [9] an expression was obtained for $\lambda_j'(\epsilon)$, $\forall\epsilon \in \mathbb{R}$, for the case $n = 1$ when L_ϵ is self-adjoint. The following theorem extends these two results in the case $n = 1$, and furthermore holds for general n.

Theorem 3.3. *Let $\lambda_j(\epsilon)$ be a moving eigenvalue of L_ϵ of algebraic multiplicity 1, and let $u_j(\epsilon)$ and $u_j^*(\epsilon)$ be eigenfunctions of L_ϵ and L_ϵ^* respectively, corresponding to $\lambda_j(\epsilon)$. Then*

$$\lambda_j'(\epsilon) = \frac{\int_U d(x)[u_j(\epsilon)](x)dx \int_U c(x)[u_j^*(\epsilon)](x)dx}{\int_U [u_j(\epsilon)](x)[u_j^*(\epsilon)](x)dx}. \tag{3.8}$$

Proof. First suppose that $\lambda_j(\epsilon) \neq \gamma_i \ \forall\, i \in \mathbb{N}$. Then by Lemma 3.1 $[u_j(\epsilon)](x) := \sum_{i=1}^\infty \frac{c_i}{\lambda_j(\epsilon)-\gamma_i} v_i(x)$ and $[u_j^*(\epsilon)](x) := \sum_{i=1}^\infty \frac{d_i}{\lambda_j(\epsilon)-\gamma_i} v_i(x)$ are eigenfunctions of L_ϵ and L_ϵ^* respectively corresponding to the eigenvalue $\lambda_j(\epsilon)$. Also by Lemma 3.1,

$$\epsilon(\lambda_j) = \Big(\sum_{i=1}^\infty \frac{c_i d_i}{\lambda_j(\epsilon) - \gamma_i}\Big)^{-1}. \tag{3.9}$$

Now as $\lambda_j(\epsilon) \neq \gamma_i$ for any $i \in \mathbb{N}$, it is straightforward to show that $\sum_{i=1}^\infty \frac{c_i d_i}{\lambda_j(\epsilon)-\gamma_i}$ is uniformly convergent in a suitable neighbourhood of ϵ. Hence, we can differentiate the series in (3.9) term by term and use the chain rule to obtain

$$\frac{d\epsilon}{d\lambda} = \Big(\sum_{i=1}^\infty \frac{c_i d_i}{\lambda_j(\epsilon) - \gamma_i}\Big)^{-2} \Big(\sum_{i=1}^\infty \frac{c_i d_i}{(\lambda_j(\epsilon) - \gamma_i)^2}\Big),$$

and so

$$\frac{d\lambda}{d\epsilon} = \Big(\sum_{i=1}^\infty \frac{c_i d_i}{\lambda_j(\epsilon) - \gamma_i}\Big)^{2} \Big(\sum_{i=1}^\infty \frac{c_i d_i}{(\lambda_j(\epsilon) - \gamma_i)^2}\Big)^{-1}. \tag{3.10}$$

If we substitute the series expansions for c, d, $u(\epsilon)$ and $u^*(\epsilon)$ into (3.8), and interchange the order of summation and integration, then the equivalence of (3.8) and (3.10) is proven.

For $\lambda_j(\epsilon) = \gamma_j$ (i.e. when $\epsilon = 0$), the result follows by the analyticity of the eigenvalues and eigenfunctions, as discussed in Lemma 2.1. □

Hence, as in the case $n = 1$, we have

Corollary 3.4. *If L_ϵ is self-adjoint, and if $\lambda_j(\epsilon)$ is an eigenvalue of L_ϵ of geometric multiplicity 1, then*

$$\lambda_j'(\epsilon) = \frac{\Big(\int_U c(x)[u_j(\epsilon)](x)dx\Big)^2}{\int_U [u_j(\epsilon)](x)^2 dx}.$$

When deriving an expression for $\lambda_j'(0)$, we must deal with the case where an eigenvalue of A "splits" to form a moving (simple) eigenvalue and a fixed eigenvalue of L_ϵ.

Theorem 3.5. *Suppose that there exists an eigenvalue of L_ϵ, $\lambda_j(\epsilon)$ for which $\lambda_j(0) = \gamma_j$, but $\lambda_j(\epsilon) \neq \gamma_j$ for $\epsilon \neq 0$. If $\tilde{v} := \lim_{\epsilon\to 0} u_j(\epsilon)$, then*

$$\lambda_j'(0) = \frac{\int_U c(x)\tilde{v}(x)dx \int_U d(x)\tilde{v}(x)dx}{\int_U \tilde{v}(x)^2 dx}.$$

If in addition, L_ϵ is self-adjoint, then

$$\lambda_j'(0) = \max_{u \in N(A-\gamma_j I)} \frac{\left(\int_U c(x)u(x)dx\right)^2}{\int_U u(x)^2 dx}.$$

Proof. Even in the case where the moving eigenvalue $\lambda_j(\epsilon)$ intersects a fixed eigenvalue at γ_j, by Lemma 2.1 $\lambda_j(\epsilon)$ is analytic at $\epsilon = 0$, and hence as a consequence of Theorem 3.3 and the analyticity of the eigenfunctions, we have

$$\lambda_j'(0) = \frac{\int_U c(x)\tilde{v}(x)dx \int_U d(x)\tilde{v}(x)dx}{\int_U \tilde{v}(x)^2 dx}.$$

If L_ϵ is self-adjoint, $\tilde{v}$ will be perpendicular to the fixed eigenspace X, corresponding to γ_j, whilst for any $v \in X$, $\int_U c(x)v(x)dx = 0$. Therefore it follows that

$$\max_{u \in N(A-\gamma_j I)} \frac{\left(\int_U c(x)u(x)dx\right)^2}{\int_U u(x)^2 dx} = \frac{\left(\int_U c(x)\tilde{v}(x)dx\right)^2}{\int_U \tilde{v}(x)^2 dx} = \lambda_j'(0).$$

□

Note that the above result is consistent with Corollary 3.2 proved earlier.

Whilst it is possible in general for all of the eigenvalues of L_ϵ to be fixed, such behaviour is not possible if L_ϵ is self-adjoint as illustrated by the following result.

Theorem 3.6. *If L_ϵ is self-adjoint, then at least one of the eigenvalues of L_ϵ is not fixed.*

Proof. The set of eigenfunctions of A, $\{v_i\}_{i=1}^\infty$ forms a basis for $L^2(U)$, and hence $\exists\, v_j \in \{v_i\}_{i=1}^\infty$ such that $\int_U c(x)v_j(x)dx = \int_U d(x)v_j(x)dx \neq 0$. Then, by Corollary 3.2, there exists a moving eigenvalue emanating from γ_j. □

In the case where $\sigma(L_\epsilon) = \sigma(A)$ for all $\epsilon \in \mathbb{R}$, varying ϵ affects the corresponding eigenfunctions as is now shown.

Theorem 3.7. *If $\sigma(L_\epsilon) = \sigma(A)$ for all $\epsilon \in \mathbb{R}$, then at least one of the eigenfunctions of L_ϵ is not fixed.*

Proof. Suppose that $\lambda_i(\epsilon) \equiv \gamma_i$, $\forall i \in \mathbb{N}$. Then, similar to the proof of Theorem 3.6, there is at least one eigenfunction of A, v_j say, such that $Bv_j \not\equiv 0$. But by assumption $\lambda_j(\epsilon) \equiv \gamma_j$ and so it follows from the equation

$$Au_j(\epsilon) + \epsilon B u_j(\epsilon) = \lambda_j(\epsilon)u_j(\epsilon),$$

that $u_j(\epsilon) \not\equiv v_j$. □

As noted in Lemma 2.1, the eigenvalues of L_ϵ are continuous functions of ϵ. We shall prove that the eigenvalues of L_ϵ are also *equicontinuous* functions of ϵ. Equicontinuity is defined as follows.

Definition 3.8. Let X be a normed vector space, and let (a, b) be a (possibly unbounded) interval of the real line. A sequence of functions, $f_n : (a, b) \to X$ $n = 1, 2, \dots$ is *uniformly equicontinuous* if for each $\epsilon > 0$, there exists $\delta > 0$, such that

$$|x - y| < \delta \;\Rightarrow\; \|f_n(x) - f_n(y)\| < \epsilon,\ \forall x, y \in (a, b),\ \forall n \in \mathbb{N}.$$

We shall prove the equicontinuity of the eigenfunctions with the aid of the following two lemmas.

Lemma 3.9. *If L_ϵ is self-adjoint, then for each $\epsilon \in \mathbb{R}$, the eigenfunctions of L_ϵ can be chosen to form an orthonormal basis for $L^2(U)$.*

Proof. Fix ϵ and assume without loss of generality that 0 is not an eigenvalue of L_ϵ. (If 0 is an eigenvalue then simply consider the operator $L_\epsilon + KI$ for some constant K suitably chosen.) The only spectral values of L_ϵ are eigenvalues, therefore $L_\epsilon^{-1} : L^2(U) \to L^2(U)$ is bounded. Furthermore, $L_\epsilon = L_\epsilon^*$ implies $L_\epsilon^{-1} = (L_\epsilon^{-1})^*$, and as noted above, L_ϵ^{-1} is compact. Also, 0 is not an eigenvalue of L_ϵ^{-1}. Hence

$$\ker L_\epsilon^{-1} = \{0\}.$$

Applying Corollary 6.35 in [24] and using the equivalence of the eigenfunctions of L_ϵ and L_ϵ^{-1} concludes the proof. □

Now, define $\rho(L_\epsilon)$ to be the resolvant set of L_ϵ, i.e. $\rho(L_\epsilon) := \mathbb{C} \backslash \sigma(L_\epsilon)$. Then we have:

Lemma 3.10. *Fix $\epsilon^* \in \mathbb{R}$. If L_ϵ is self-adjoint, and if $(\lambda, \epsilon) \in \mathbb{R}^2$ satisfies*

$$|\epsilon - \epsilon^*| \|B\| < \min_{i \in \mathbb{N}} |\lambda_i(\epsilon^*) - \lambda|, \tag{3.11}$$

then $\lambda \in \rho(L_\epsilon)$.

Proof. As noted in Section II-5.1 of [19], for any fixed ϵ^*, if $\lambda \in \rho(L_{\epsilon^*})$, and if

$$|\epsilon - \epsilon^*| \|B\| < \|(L_{\epsilon^*} - \lambda I)^{-1}\|^{-1},$$

then $\lambda \in \rho(L_\epsilon)$. As L_ϵ is self-adjoint, by Lemma 3.9 the eigenfunctions of L_ϵ form an orthonormal basis for $L^2(U)$. Hence, it is straightforward to show that

$$\|(L_{\epsilon^*} - \lambda I)^{-1}\| = \frac{1}{\min_{i \in \mathbb{N}} |\lambda_i(\epsilon^*) - \lambda|}.$$

Therefore if

$$|\epsilon - \epsilon^*| \|B\| < \min_{i \in \mathbb{N}} |\lambda_i(\epsilon^*) - \lambda|,$$

then $\lambda \in \rho(L_\epsilon)$. □

Theorem 3.11. *If L_ϵ is self-adjoint, then the set of eigenvalues, $\Sigma = \{\lambda_i(\epsilon)\}_{i=1}^\infty$, is uniformly equicontinuous for $\epsilon \in \mathbb{R}$.*

Proof. We first make some observations about the eigenvalues of L_ϵ in the self-adjoint case. As a consequence of (3.10), the moving eigenvalues of L_ϵ all increase with respect to ϵ. Then as noted in the final sentence of Section 2, as the eigenvalues of a self-adjoint operator are real, 2 moving eigenvalues of L_ϵ never intersect. Fixed eigenvalues are clearly analytic $\forall \epsilon \in \mathbb{R}$. In the case where a moving eigenvalue, $\lambda_i(\epsilon)$ intersects a fixed eigenvalue, $\lambda_j(\epsilon) \equiv \gamma_j$, of L_ϵ, $\lambda_i(\epsilon) + \gamma_j$ is analytic at the point of intersection by Lemma 2.1(c), and hence by Lemma 2.1(b), a moving eigenvalue $\lambda_i(\epsilon)$ is also analytic $\forall \epsilon \in \mathbb{R}$.

Now, consider an eigenvalue $\lambda_j(\epsilon)$, and take ϵ_1, $\epsilon_2 \in \mathbb{R}$. As $\lambda_j(\epsilon_2) \notin \rho(L_{\epsilon_2})$, it follows from Lemma 3.10 that

$$|\epsilon_1 - \epsilon_2| \|B\| \geq \min_{i \in \mathbb{N}} |\lambda_i(\epsilon_1) - \lambda_j(\epsilon_2)|. \tag{3.12}$$

First suppose that $\lambda_j(\epsilon_2)$ is not a point of intersection of a moving eigenvalue and a fixed eigenvalue of L_ϵ. Then for $|\epsilon_1 - \epsilon_2|$ sufficiently small,

$$\min_{i \in \mathbb{N}} |\lambda_i(\epsilon_1) - \lambda_j(\epsilon_2)| = |\lambda_j(\epsilon_1) - \lambda_j(\epsilon_2)|.$$

Hence from (3.12),

$$\frac{|\lambda_j(\epsilon_1) - \lambda_j(\epsilon_2)|}{|\epsilon_1 - \epsilon_2|} \le \|B\|.$$

This holds for any $j \in \mathbb{N}$, $\forall \epsilon_1, \epsilon_2 \in \mathbb{R}$, such that $|\epsilon_1 - \epsilon_2|$ is sufficiently small, whenever $\lambda_j(\epsilon_2) \neq \gamma_i$ $\forall i \in \mathbb{N}$. Therefore

$$|\lambda_j'(\epsilon)| \le \|B\| \quad \forall j \in \mathbb{N},\ \forall \epsilon \in \mathbb{R},$$

whenever $\lambda_j(\epsilon) \neq \gamma_i$ $\forall i \in \mathbb{N}$. But then since $\lambda_j(\epsilon)$ is analytic for all $\epsilon \in \mathbb{R}$, it is certainly continuously differentiable, and therefore it follows that the previous inequality also holds for $\lambda_j(\epsilon) = \gamma_i$ for some $i \in \mathbb{N}$. i.e.

$$|\lambda_j'(\epsilon)| \le \|B\| \quad \forall j \in \mathbb{N},\ \forall \epsilon \in \mathbb{R},$$

and the result follows. □

4. Algebraic and Geometric Multiplicity

The geometric and algebraic multiplicities of the eigenvalues are of importance in establishing conditions for results on nodal properties of eigenfunctions, and for bifurcation in associated non-linear problems, respectively. Hence, we now consider whether the multiplicities of the eigenvalues $\lambda_i(\epsilon)$ change as the parameter ϵ is varied. Geometric multiplicity of an eigenvalue λ of L_ϵ, can be defined in the usual way, i.e. $\dim(N(L_\epsilon - \lambda I))$. Since L_ϵ is a closed linear operator with compact resolvent, the algebraic multiplicity of an eigenvalue, λ of L_ϵ can be defined to be the algebraic multiplicity of the eigenvalue, $1/\lambda$ of L_ϵ^{-1}. (Here we are assuming without loss of generality that L_ϵ^{-1} does exist; if L_ϵ is not invertible, then consider $L_\epsilon + KI$ for an appropriate constant, K.) A *simple eigenvalue* is defined to be an eigenvalue of algebraic multiplicity 1, (see e.g. [3]). Note that since A is self-adjoint its eigenvalues will have equal algebraic and geometric multiplicities.

4.1. **Algebraic Multiplicity.**

As was noted earlier, the algebraic multiplicity of an eigenvalue of A, although finite, can be arbitrarily large. However, the following theorem can be deduced from Section IV-3.5 of [19].

Theorem 4.1. *Let $S \subset \Sigma$ contain only a finite number of elements. If $\lambda_i(\epsilon) \neq \lambda_j(\epsilon)$ for any $\lambda_i(\epsilon) \in S$ and $\lambda_j(\epsilon) \in \Sigma \backslash S$ $\forall \epsilon \in (\epsilon_1, \epsilon_2)$, then the sum of the algebraic multiplicities of the eigenvalues in S is constant with respect to ϵ, $\forall \epsilon \in (\epsilon_1, \epsilon_2)$.*

Remark 4.2. As a consequence of the above theorem and Lemma 2.3, a moving eigenvalue $\lambda_i(\epsilon)$ of L_ϵ is simple for $0 < |\epsilon| < \hat{\epsilon}$, where $\hat{\epsilon} = \min_{\epsilon \neq 0}\{\epsilon | \lambda_i(\epsilon) = \lambda_j(\epsilon) \text{ for } i \neq j\}$.

4.2. **Geometric Multiplicity.**

In the self-adjoint case, since it is well-known that eigenvalues of self-adjoint operators have equal geometric and algebraic multiplicities, a direct consequence of Theorem 4.1 is

Theorem 4.3. *Suppose that L_ϵ is self-adjoint. If $\lambda_i(\epsilon)$ is a moving eigenvalue of L_ϵ, γ_j is a fixed eigenvalue of L_ϵ, and if $\lambda_i(\epsilon^*) = \gamma_j$ then* $\dim(N(L_{\epsilon^*} - \gamma_j I)) \ge 2$.

However it is possible for geometric multiplicity to not be preserved in the general case, even where no eigenvalues of L_ϵ intersect, as shown by the following simple example.

Example 4.4. *Let A and γ be such that γ is an eigenvalue of A of multiplicity 2, and let v_1 and v_2 be orthogonal eigenfunctions corresponding to γ. Let $c = v_1$ and $d = v_2$. Then, using Corollary 3.2, it can be shown that all of the eigenvalues of L_ϵ are fixed and $N(L_0 - \gamma I) = N(A - \gamma I) = Span\{v_1, v_2\}$, whilst for any $\epsilon \neq 0$, $N(L_\epsilon - \gamma I) = Span\{v_1\}$.*

In the remainder of this section we present results that restrict the behaviour of the geometric multiplicities in the general case.

Theorem 4.5. *An eigenvalue $\lambda_j(\epsilon)$ has geometric multiplicity 1 provided $\lambda_j(\epsilon) \neq \gamma_i$ for any $i \in \mathbb{N}$.*

Proof. Suppose that for some ϵ^* and some j, $N(L_{\epsilon^*} - \lambda_j(\epsilon^*)) = \mathrm{span}\{u, v\}$ with u and v linearly independent where $\lambda_j(\epsilon^*) \neq \gamma_i$ for any $i \in \mathbb{N}$. Then, there exist constants a and b with $|a| + |b| \neq 0$ such that $B(au + bv) \equiv 0$. Hence, $au + bv = v_i$ and $\lambda_j(\epsilon^*) = \gamma_i$ for some $i \in \mathbb{N}$, which is a contradiction and so the result is proved. □

Then since $\sigma(A) \subset \mathbb{R}$, we have

Corollary 4.6. *Any complex eigenvalue of L_ϵ has geometric multiplicity 1.*

The following theorem is an extension of [9, Theorem 4.8]. The proof of [9, Theorem 4.8] uses results concerning a corresponding initial value problem. Such results are not available here, so we require an alternative method, which is similar to that used in the proof of a different result (Lemma 3.9) in [12].

Theorem 4.7. *Suppose that γ_j is an eigenvalue of A of geometric multiplicity m. Then, there exists at most one value, ϵ_j, such that $\dim(N(L_{\epsilon_j} - \gamma_j I)) = m + 1$.*

Note 4.8. Let $S : \hat{X} \to X$ be a closed operator, where $\hat{X}$ is a dense subset of a Banach space, X, and consider $T : \hat{X} \to X$. Then T is said to be *relatively degenerate with respect to S* if and only if (i) $\exists\, k_1, k_2 \geq 0$, such that

$$\|Tu\| \leq k_1\|u\| + k_2\|Su\| \quad \forall u \in \hat{X},$$

and (ii) $R(T)$ is finite dimensional.

Proof of Theorem 4.7. This proof will use the Weinstein-Aronszajn (W-A) determinant and the W-A formula for relatively degenerate perturbations, details of which can be found in [19, Section IV-6.1].

Note that ϵB is a relatively degenerate perturbation with respect to $(A - \lambda I)$, since it is bounded and its range is finite dimensional. In the same way as shown in [12], the W-A determinant, ω, associated with ϵB and $(A - \lambda I)$ has the form

$$\omega(\lambda, \epsilon) = 1 + \epsilon G(\lambda), \tag{4.1}$$

for some function G, not dependent upon ϵ. The W-A formula for degenerate perturbations gives

$$M(\lambda, L_\epsilon) = M(\lambda, A) + N(\lambda, \omega),$$

where for $T \in \{A, L_\epsilon\}$

$$M(\lambda, T) = \begin{cases} 0 & \text{if } \lambda \in \rho(T) \\ \text{the algebraic multiplicity of } \lambda & \text{if } \lambda \in \sigma(T), \end{cases}$$

and

$$N(\lambda,\omega)=\begin{cases} k & \text{if } \lambda \text{ is a zero of } \omega \text{ of order } k\\ -k & \text{if } \lambda \text{ is a pole of } \omega \text{ of order } k\\ 0 & \text{otherwise.}\end{cases}$$

Hence, it follows that for an eigenvalue λ of A, of multiplicity m, to be an eigenvalue of multiplicity $m+1$ of L_ϵ requires $N(\lambda,\omega)>0$ and this is true only when $\omega(\lambda,\epsilon)$ has a zero. However, from the form of (4.1), it can be seen that for any fixed λ, $\omega(\lambda,\epsilon)$ equals 0 for at most one value of ϵ. □

The case where a moving eigenvalue and a fixed eigenvalue emanate from the same point γ_i, can be thought of as a moving eigenvalue intersecting γ_i at $\epsilon=0$. In this case, by Lemma 2.3 there exists an orthonormal basis, $\{w_1,\dots,w_m\}$, for $N(A-\gamma_i I)$ such that $\operatorname{Span}\{w_1,\dots,w_{m-1}\}\subset N(L_\epsilon-\gamma_i I)$ $\forall\epsilon\in\mathbb{R}$, and $w_m=\lim_{\epsilon\to 0}u_m(\epsilon)$, where $u_m(\epsilon)$ is the eigenfunction corresponding to the moving eigenvalue. Hence $\dim N(A-\gamma_i I)=1+\dim N(L_\epsilon-\gamma_i I)$, for any $\epsilon\neq 0$.

We now consider when a moving eigenvalue, $\lambda_j(\epsilon)$, intersects a fixed eigenvalue, $\lambda_i(\epsilon)\equiv\gamma_i$, of L_ϵ at $\epsilon\neq 0$.

Theorem 4.9. *Let $\lambda_i(\epsilon)\equiv\gamma_i$ be a fixed eigenvalue of L_ϵ, such that no moving eigenvalue of L_ϵ emanates from γ_i at $\epsilon=0$.*

(a) *If either $Bv\not\equiv 0$ or $B^*v\not\equiv 0$, for any $v\in N(A-\gamma_i I)$, then* $\dim(N(L_\epsilon-\gamma_i I))\le\dim(N(A-\gamma_i I))$ $\forall\epsilon\in\mathbb{R}$.

(b) *If $Bv=B^*v\equiv 0$, $\forall v\in N(A-\gamma_i I)$, and if a moving eigenvalue $\lambda_j(\epsilon_i)=\gamma_i$ for some $\epsilon_i\in\mathbb{R}$, $\epsilon_i\neq 0$, then* $\dim(N(L_{\epsilon_i}-\gamma_i I))=\dim(N(A-\gamma_i I))+1$.

Remark 4.10. Whilst part (a) of the above result gives that $\dim(N(L_\epsilon-\gamma_i I))\not>\dim(N(A-\gamma_i I))$ $\forall\epsilon\in\mathbb{R}$, we may have $\dim(N(L_\epsilon-\gamma_i I))<\dim(N(A-\gamma_i I))$ for some $\epsilon\neq 0$, as in Example 4.4.

Proof. **(a)** If $\exists\, v\in N(A-\gamma_i I)$ such that $Bv\not\equiv 0$, then by Lemma 2.3, there exists an orthonormal basis, $\{v_1,v_2,\dots v_m\}$ for $N(A-\gamma_i I)$, such that $B(v_1)=B(v_2)=\dots=B(v_{m-1})\equiv 0$, whilst $B(v_m)\not\equiv 0$. Now suppose that $\dim(N(L_{\epsilon_i}-\gamma_i I))=m+1$ for some $\epsilon_i\neq 0$, i.e. suppose that there exists an orthonormal basis of $N(L_{\epsilon_i}-\gamma_i I)$ of the form $\{v_1,v_2,\dots v_{m-1},z_1,z_2\}$. Define the functions

$$w_n:=\Big(\int_U d(x)v_m(x)dx\Big)\,z_n+\Big(\int_U d(x)z_n(x)dx\Big)\,v_m,\quad n=1,2.$$

Then $\{v_1,v_2,\dots v_{m-1},w_1,w_2\}$ are linearly independent and it can be shown that $w_1,\,w_2\in N(L_{\epsilon_i/2}-\gamma_i I)$, i.e. $\dim(N(L_{\epsilon_i/2}-\gamma_i I))=m+1$. But this contradicts Theorem 4.7.

If $B^*v\not\equiv 0$, for some $v\in N(A-\gamma_i I)$, then by considering the adjoint problem, $(L_\epsilon^*-\lambda I)u=0$, and using the equivalence of the spectrums and multiplicities of L_ϵ and L_ϵ^*, a similar contradiction can be obtained and result (a) is proven.

(b) Since by assumption $Bv\equiv 0$, $N(A-\gamma_i I)\subseteq N(L_\epsilon-\gamma_i I)$ $\forall\epsilon\in\mathbb{R}$. As in the proof of Theorem 4.10 in [9] it can be shown that

$$u(x):=\sum_{j\neq i}\frac{c_i}{(\gamma_i-\gamma_j)}v_j(x)$$

is in $N(L_{\epsilon_i}-\gamma_i I)$, where in the above summation, the eigenvalues are repeated according to their geometric multiplicity. The function u is orthogonal to $N(A-\gamma_i I)$, and hence the result is proven. □

5. Existence and Uniqueness of Principal Eigenvalues

As mentioned earlier, for differential operators of the type given in (2.1), it is known that the eigenfunction corresponding to the first eigenvalue of A is unique and is of one sign on U, but in general less is known about the nodal properties of the eigenfunctions corresponding to the other eigenvalues. By the continuity of the (1-dimensional) eigenprojection corresponding to $\lambda_1(\epsilon)$, it follows that for ϵ sufficiently small, the eigenfunction $u_1(\epsilon)$, corresponding to $\lambda_1(\epsilon)$ is of one sign on U, i.e. L_ϵ has a principal eigenvalue for sufficiently small ϵ. In this section we are concerned with proving the existence and in some cases the uniqueness of a principal eigenvalue of L_ϵ, without restriction on $|\epsilon|$.

Theorem 5.1. *Suppose that either $c \geq 0$ on U or $c \leq 0$ on U. Then, L_ϵ has a principal eigenvalue $\lambda_p(\epsilon) \geq \gamma_1$ (i.e. an eigenvalue to which there corresponds a positive eigenfunction) either $\forall \epsilon \geq 0$ or $\forall \epsilon \leq 0$.*

Proof. Following similar methods to those used in the proof of Theorem 5.5 in [9], it can be shown that L_ϵ has an eigenvalue greater than or equal to γ_1, either $\forall \epsilon \geq 0$ or $\forall \epsilon \leq 0$.

We now show that any eigenvalue, λ of L_ϵ, satisfying $\lambda \geq \gamma_1$ has a corresponding eigenfunction which is positive on U. If $\lambda > \gamma_1$, then $(A - \lambda I)$ satisfies the strong maximum principle (see for example Theorem 2.4 in [2]). Let $u(x)$ be an eigenfunction corresponding to $\lambda > \gamma_1$. Then we note that $\int_U d(x)u(x)dx \neq 0$, as there exist no fixed eigenvalues greater than γ_1. Hence, supposing without loss of generality that $-\epsilon c(x) \int_U d(x)u(x)dx$ is a non-negative function which is not identical to zero, applying the strong maximum principle yields

$$u(x) = (A - \lambda I)^{-1}\Big[-\epsilon c(x) \int_U d(x)u(x)dx\Big] > 0 \quad \forall x \in U.$$

We are left to consider the case $\lambda_i(\epsilon) = \gamma_1$ for some $i \in \mathbb{N}$. If $\epsilon = 0$ is the only solution to $\lambda_i(\epsilon) = \gamma_1$, then as the corresponding eigenfunction, v_1 has no interior zeros, and the result follows directly. If $\lambda_i(\epsilon) = \gamma_1$ for some $i \in \mathbb{N}$ and some $\epsilon \neq 0$, then by the arguments at the end of Section 2, $\lambda_1(\epsilon) \equiv \gamma_1$. A corresponding eigenfunction does not change sign as the following argument shows. The function c is either non-negative or non-positive, therefore $B^*(v_1) \not\equiv 0$, and hence as $\lambda_1(\epsilon) \equiv \gamma_1$, $B(v_1) \equiv 0$. Therefore v_1 is an eigenfunction corresponding to γ_1, $\forall \epsilon \in \mathbb{R}$. and the result is proved. □

Now, having established the existence of a principal eigenvalue of L_ϵ under certain hypotheses, we finish this section by considering the uniqueness of the principal eigenvalue.

Theorem 5.2. *If L_ϵ is self-adjoint, if c is strictly of one sign on U and if $\epsilon > 0$, then the eigenfunction corresponding to $\lambda_1(\epsilon)$ is the only eigenfunction of L_ϵ with no interior zeros.*

Proof. The case $\epsilon = 0$ corresponds to the well studied local problem. Now consider $\epsilon > 0$ and suppose that either $u_1(\epsilon)$ has an interior zero, or $u_k(\epsilon)$ has no interior zeros for $k \neq 1$. As a consequence of Lemma 2.1, the eigenfunctions of L_ϵ can be chosen to be continuous functions of ϵ. Hence $\exists \hat{\epsilon} > 0$, and $u_k \in C^2(\bar{U})$ such that $[u_k(\hat{\epsilon})](x) \geq 0$ $\forall x \in \bar{U}$, whilst $\exists \hat{x} \in \bar{U}$ such that $[u_k(\hat{\epsilon})](\hat{x}) = [u_k(\hat{\epsilon})]_{x_1}(\hat{x}) =$

$[u_k(\hat{\epsilon})]_{x_2}(\hat{x}) = \cdots = [u_k(\hat{\epsilon})]_{x_n}(\hat{x}) = 0$. Then, it follows from standard properties of uniformly elliptic operators that

$$\sum_{i,j=1}^{n} [a_{ij}(\hat{x}) u_{k\ x_i}(\hat{x})]_{x_j} \geq 0,$$

and also $(b(\hat{x}) - \lambda)u_k(\hat{x}) = 0$, for any $\lambda \in \mathbb{R}$. Note that by an appropriate Sobolev embedding, $c \in C(\bar{U})$, and then, $a_{ij}, u_k \in C^2(\bar{U})$ and $b, c \in C(\bar{U})$ implies that the differential equation

$$\sum_{i,j=1}^{n} [a_{ij}(\hat{x}) u_{k\ x_i}(\hat{x})]_{x_j} + (b(x) - \lambda)u_k(x) + \epsilon c(x) \int_U c(x) u_k(x) dx = 0,$$

holds not only for $x \in U$, but also for $x \in \bar{U}$. Hence,

$$\epsilon c(\hat{x}) \int_U c(x) u_k(x) dx \leq 0,$$

which contradicts our assumptions on the sign of ϵ. Hence the result is proven. □

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