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GLOBAL WELL-POSEDNESS OF NLS-KDV SYSTEMS FOR PERIODIC FUNCTIONS

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Abstract. We prove that the Cauchy problem of the Schrödinger-Korteweg-deVries (NLS-KdV) system for periodic functions is globally well-posed for initial data in the energy space $H^1 \times H^1$. More precisely, we show that the non-resonant NLS-KdV system is globally well-posed for initial data in $H^s(\mathbb{T}) \times H^s(\mathbb{T})$ with $s > 11/13$ and the resonant NLS-KdV system is globally well-posed with $s > 8/9$. The strategy is to apply the I-method used by Colliander, Keel, Staffilani, Takaoka and Tao. By doing this, we improve the results by Arbieto, Corcho and Matheus concerning the global well-posedness of NLS-KdV systems.

1. Introduction

We consider the Cauchy problem of the Schrödinger-Korteweg-deVries (NLS-KdV) system

$$\begin{gathered} i\partial_t u + \partial_x^2 u = \alpha uv + \beta |u|^2 u, \\ \partial_t v + \partial_x^3 v + \tfrac{1}{2}\partial_x(v^2) = \gamma \partial_x(|u|^2), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad t \in \mathbb{R}. \end{gathered} \tag{1.1}$$

This system appears naturally in fluid mechanics and plasma physics as a model of interaction between a short-wave $u = u(x,t)$ and a long-wave $v = v(x,t)$.

In this paper we are interested in global solutions of the NLS-KdV system for rough initial data. Before stating our main results, let us recall some of the recent theorems of local and global well-posedness theory of the Cauchy problem (1.1).

For continuous spatial variable (i.e., $x \in \mathbb{R}$), Corcho and Linares [5] recently proved that the NLS-KdV system is locally well-posed for initial data $(u_0, v_0) \in H^k(\mathbb{R}) \times H^s(\mathbb{R})$ with $k \ge 0$, $s > -3/4$ and

$$\begin{gathered} k - 1 \le s \le 2k - 1/2 \quad \text{if } k \le 1/2, \\ k - 1 \le s < k + 1/2 \quad \text{if } k > 1/2. \end{gathered}$$

Furthermore, they prove the global well-posedness of the NLS-KdV system in the energy $H^1 \times H^1$ using three conserved quantities discovered by Tsutsumi [7], whenever $\alpha\gamma > 0$.

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Also, Pecher [6] improved this global well-posedness result by an application of the I-method of Colliander, Keel, Stafillani, Takaoka and Tao (see for instance [3]) combined with some refined bilinear estimates. In particular, Pecher proved that, if $\alpha\gamma > 0$, the NLS-KdV system is globally well-posed for initial data $(u_0, v_0) \in H^s \times H^s$ with $s > 3/5$ in the resonant case $\beta = 0$ and $s > 2/3$ in the non-resonant case $\beta \neq 0$.

On the other hand, in the periodic setting (i.e., x isn the space of periodic functions $\mathbb{T}$), Arbieto, Corcho and Matheus [1] proved the local well-posedness of the NLS-KdV system for initial data $(u_0, v_0) \in H^k \times H^s$ with $0 \leq s \leq 4k-1$ and $-1/2 \leq k-s \leq 3/2$. Also, using the same three conserved quantities discovered by Tsutsumi, one obtains the global well-posedness of NLS-KdV on $\mathbb{T}$ in the energy space $H^1 \times H^1$ whenever $\alpha\gamma > 0$.

Motivated by this scenario, we combine the new bilinear estimates of Arbieto, Corcho and Matheus [1] with the I-method of Tao and his collaborators to prove the following result.

Theorem 1.1. *The NLS-KdV system* (1.1) *on* $\mathbb{T}$ *is globally well-posed for initial data* $(u_0, v_0) \in H^s(\mathbb{T}) \times H^s(\mathbb{T})$ *with* $s > 11/13$ *in the non-resonant case* $\beta \neq 0$ *and* $s > 8/9$ *in the resonant case* $\beta = 0$*, whenever* $\alpha\gamma > 0$.

The paper is organized as follows. In the section 2, we discuss the preliminaries for the proof of the theorem 1.1: Bourgain spaces and its properties, linear estimates, standard estimates for the non-linear terms $|u|^2 u$ and $\partial_x(v^2)$, the bilinear estimates of Arbieto, Corcho and Matheus [1] for the coupling terms uv and $\partial_x(|u|^2)$, the I-operator and its properties. In the section 3, we apply the results of the section 2 to get a variant of the local well-posedness result of [1]. In the section 4, we recall some conserved quantities of (1.1) and its modification by the introduction of the I-operator; moreover, we prove that two of these modified energies are almost conserved. Finally, in the section 5, we combine the almost conservation results in section 4 with the local well-posedness result in section 3 to conclude the proof of the theorem 1.1.

2. Preliminaries

A successful procedure to solve some dispersive equations (such as the nonlinear Schrödinger and KdV equations) is to use the Picard's fixed point method in the following spaces:

$$\|f\|_{X^{k,b}} := \Big(\int \sum_{n\in\mathbb{Z}} \langle n\rangle^{2k} \langle \tau + n^2\rangle^{2b} |\widehat{f}(n,\tau)| d\tau \Big)^{1/2} = \|U(-t)f\|_{H^b_t(\mathbb{R}, H^k_x)},$$

$$\|g\|_{Y^{s,b}} := \Big(\int \sum_{n\in\mathbb{Z}} \langle n\rangle^{2s} \langle \tau - n^3\rangle^{2b} |\widehat{g}(n,\tau)| d\tau \Big)^{1/2} = \|V(-t)f\|_{H^b_t(\mathbb{R}, H^s_x)},$$

where $\langle\cdot\rangle := 1 + |\cdot|$, $U(t) = e^{it\partial_x^2}$ and $V(t) = e^{-t\partial_x^3}$. These spaces are called Bourgain spaces. Also, we introduce the restriction in time norms

$$\|f\|_{X^{k,b}(I)} := \inf_{\widetilde{f}|_I = f} \|\widetilde{f}\|_{X^{k,b}} \quad \text{and} \quad \|g\|_{Y^{s,b}(I)} := \inf_{\widetilde{g}|_I = g} \|\widetilde{g}\|_{Y^{s,b}}$$

where I is a time interval.

The interaction of the Picard method has been based around the spaces $Y^{s,1/2}$. Because we are interested in the continuity of the flow associated to (1.1) and the

$Y^{s,1/2}$ norm do not control the $L^\infty_t H^s_x$ norm, we modify the Bourgain spaces as follows:

$$\|u\|_{X^k} := \|u\|_{X^{k,1/2}} + \|\langle n\rangle^k \widehat{u}(n,\tau)\|_{L^2_n L^1_\tau},$$
$$\|v\|_{Y^s} := \|v\|_{Y^{s,1/2}} + \|\langle n\rangle^s \widehat{v}(n,\tau)\|_{L^2_n L^1_\tau}$$

and, given a time interval I, we consider the restriction in time of the X^k and Y^s norms

$$\|u\|_{X^k(I)} := \inf_{\widetilde{u}|_I = u} \|\widetilde{u}\|_{X^k} \quad \text{and} \quad \|v\|_{Y^s(I)} := \inf_{\widetilde{v}|_I = v} \|\widetilde{v}\|_{Y^s}$$

Furthermore, the mapping properties of $U(t)$ and $V(t)$ naturally leads one to consider the companion spaces

$$\|u\|_{Z^k} := \|u\|_{X^{k,-1/2}} + \Big\| \frac{\langle n\rangle^k \widehat{u}(n,\tau)}{\langle \tau + n^2\rangle} \Big\|_{L^2_n L^1_\tau},$$
$$\|v\|_{W^s} := \|v\|_{Y^{s,-1/2}} + \Big\| \frac{\langle n\rangle^s \widehat{v}(n,\tau)}{\langle \tau - n^3\rangle} \Big\|_{L^2_n L^1_\tau}$$

In the sequel, ψ denotes a non-negative smooth bump function supported on $[-2,2]$ with $\psi = 1$ on $[-1,1]$ and $\psi_\delta(t) := \psi(t/\delta)$ for any $\delta > 0$.

Notation. Fix (k,s) a pair of indices such that the local well-posedness of the periodic NLS-KdV system holds. Given two non-negative real numbers A and B, we write $A \lesssim B$ whenever $A \le C\cdot B$, where $C = C(k,s)$ is a constant which may depend only on (k,s). Also, we write $A \gtrsim B$ if $A \ge c \cdot B$, where $c = c(k,s)$ is sufficiently small (depending only on (k,s)), and $A \sim B$ if $A \lesssim B \lesssim A$. Furthermore, we use $A \ll B$ to mean $A \le c\dot{B}$ where $c = c(k,s)$ is a small constant (depending only on (k,s)), and $A \gg B$ to denote $A \ge C \cdot B$ with $C = C(k,s)$ a large constant. Finally, given, for instance, a function ψ and a number b, we put also $A \lesssim_{\psi,b} B$ to mean $A \le C \cdot B$ where $C = C(k,s,\psi,b)$ is a constant depending also on the specified function ψ and number b (besides (k,s)).

Next, we recall some properties of the Bourgain spaces:

Lemma 2.1. *$X^{0,3/8}([0,1]), Y^{0,1/3}([0,1]) \subset L^4(\mathbb{T}\times[0,1])$. More precisely,*

$$\|\psi(t) f\|_{L^4_{xt}} \lesssim \|f\|_{X^{0,3/8}} \quad \textit{and} \quad \|\psi(t) g\|_{L^4_{xt}} \lesssim \|g\|_{Y^{0,1/3}}.$$

For the proof of the above lemma see [2]. Another basic property of these spaces are their stability under time localization:

Lemma 2.2. *Let $X^{s,b}_{\tau=h(\xi)} := \{f : \langle \tau - h(\xi)\rangle^b \langle \xi\rangle^s |\widehat{f}(\tau,\xi)| \in L^2\}$. Then*

$$\|\psi(t) f\|_{X^{s,b}_{\tau=h(\xi)}} \lesssim_{\psi,b} \|f\|_{X^{s,b}_{\tau=h(\xi)}}$$

for any $s,b \in \mathbb{R}$. Moreover, if $-1/2 < b' \le b < 1/2$, then for any $0 < T < 1$, we have

$$\|\psi_T(t) f\|_{X^{s,b'}_{\tau=h(\xi)}} \lesssim_{\psi,b',b} T^{b-b'} \|f\|_{X^{s,b}_{\tau=h(\xi)}}.$$

Proof. First of all, note that $\langle \tau - \tau_0 - h(\xi)\rangle^b \lesssim_b \langle \tau_0\rangle^{|b|} \langle \tau - h(\xi)\rangle^b$, from which we obtain

$$\|e^{it\tau_0} f\|_{X^{s,b}_{\tau=h(\xi)}} \lesssim_b \langle \tau_0\rangle^{|b|} \|f\|_{X^{s,b}_{\tau=h(\xi)}}.$$

Using that $\psi(t) = \int \widehat{\psi}(\tau_0) e^{it\tau_0} d\tau_0$, we conclude

$$\|\psi(t)f\|_{X^{s,b}_{\tau=h(\xi)}} \lesssim_b \Big(\int |\widehat{\psi}(\tau_0)| \langle \tau_0 \rangle^{|b|} \Big) \|f\|_{X^{s,b}_{\tau=h(\xi)}}.$$

Since ψ is smooth with compact support, the first estimate follows.

Next we prove the second estimate. By conjugation we may assume $s = 0$ and, by composition it suffices to treat the cases $0 \le b' \le b$ or $\le b' \le b \le 0$. By duality, we may take $0 \le b' \le b$. Finally, by interpolation with the trivial case $b' = b$, we may consider $b' = 0$. This reduces matters to show that

$$\|\psi_T(t) f\|_{L^2} \lesssim_{\psi,b} T^b \|f\|_{X^{0,b}_{\tau=h(\xi)}}$$

for $0 < b < 1/2$. Partitioning the frequency spaces into the cases $\langle \tau - h(\xi) \rangle \ge 1/T$ and $\langle \tau - h(\xi) \le 1/T$, we see that in the former case we'll have

$$\|f\|_{X^{0,0}_{\tau=h(\xi)}} \le T^b \|f\|_{X^{0,b}_{\tau=h(\xi)}}$$

and the desired estimate follows because the multiplication by ψ is a bounded operation in Bourgain's spaces. In the latter case, by Plancherel and Cauchy-Schwarz

$$\begin{aligned}
\|f(t)\|_{L^2_x} &\lesssim \|\widehat{f(t)}(\xi)\|_{L^2_\xi} \\
&\lesssim \Big\| \int_{\langle \tau - h(\xi) \rangle \le 1/T} |\widehat{f}(\tau,\xi)| d\tau) \Big\|_{L^2_\xi} \\
&\lesssim_b T^{b-1/2} \Big\| \int \langle \tau - h(\xi) \rangle^{2b} |\widehat{f}(\tau,\xi)|^2 d\tau)^{1/2} \Big\|_{L^2_\xi} \\
&= T^{b-1/2} \|f\|_{X^{s,b}_{\tau=h(\xi)}}.
\end{aligned}$$

Integrating this against ψ_T concludes the proof of the lemma. $\square$

Also, we have the following duality relationship between X^k (resp., Y^s) and Z^k (resp., W^s):

Lemma 2.3. *We have*

$$\begin{gathered}
\Big| \int \chi_{[0,1]}(t) f(x,t) g(x,t)\, dx\, dt \Big| \lesssim \|f\|_{X^s} \|g\|_{Z^{-s}}, \\
\Big| \int \chi_{[0,1]}(t) f(x,t) g(x,t)\, dx\, dt \Big| \lesssim \|f\|_{Y^s} \|g\|_{W^{-s}}
\end{gathered}$$

for any s and any f, g on $\mathbb{T} \times \mathbb{R}$.

Proof. See [4, p. 182–183] (note that, although this result is stated only for the spaces Y^s and W^s, the same proof adapts for the spaces X^k and Z^k). $\square$

Now, we recall some linear estimates related to the semigroups $U(t)$ and $V(t)$:

Lemma 2.4 (Linear estimates)**.** *It holds*

$$\begin{aligned}\|\psi(t)U(t)u_0\|_{Z^k} &\lesssim \|u_0\|_{H^k},\\ \|\psi(t)V(t)v_0\|_{W^s} &\lesssim \|v_0\|_{H^s};\\ \|\psi_T(t)\int_0^t U(t-t')F(t')dt'\|_{X^k} &\lesssim \|F\|_{Z^k},\\ \|\psi_T(t)\int_0^t V(t-t')G(t')dt'\|_{Y^s} &\lesssim \|G\|_{W^s}.\end{aligned}$$

For a proof of the above lemma, see [3], [4] or [1]. Furthermore, we have the following well-known multiinear estimates for the cubic term $|u|^2u$ of the nonlinear Schrödinger equation and the nonlinear term $\partial_x(v^2)$ of the KdV equation:

Lemma 2.5. $\|uv\overline{w}\|_{Z^k} \lesssim \|u\|_{X^{k,\frac{3}{8}}}\|v\|_{X^{k,\frac{3}{8}}}\|w\|_{X^{k,\frac{3}{8}}}$ *for any* $k \geq 0$.

For the proof of the above lemma, see See [2] and [1].

Lemma 2.6. $\|\partial_x(v_1v_2)\|_{W^s} \lesssim \|v_1\|_{Y^{s,\frac{1}{3}}}\|v_2\|_{Y^{s,\frac{1}{2}}} + \|v_1\|_{Y^{s,\frac{1}{2}}}\|v_2\|_{Y^{s,\frac{1}{3}}}$ *for any* $s \geq -1/2$*, if* $v_1 = v_1(x,t)$ *and* $v_2 = v_2(x,t)$ *are* x*-periodic functions having zero* x*-mean for all* t.

The proof of the above lemma can be found in [2], [3] and [1]. Next, we revisit the bilinear estimates of mixed Schrödinger-Airy type of Arbieto, Corcho and Matheus [1] for the coupling terms uv and $\partial_x(|u|^2)$ of the NLS-KdV system.

Lemma 2.7. $\|uv\|_{Z^k} \lesssim \|u\|_{X^{k,\frac{3}{8}}}\|v\|_{Y^{s,\frac{1}{2}}} + \|u\|_{X^{k,\frac{1}{2}}}\|v\|_{Y^{s,\frac{1}{3}}}$ *whenever* $s \geq 0$ *and* $k - s \leq 3/2$.

Lemma 2.8. $\|\partial_x(u_1\overline{u_2})\|_{W^s} \lesssim \|u_1\|_{X^{k,3/8}}\|u_2\|_{X^{k,1/2}} + \|u_1\|_{X^{k,1/2}}\|u_2\|_{X^{k,3/8}}$ *whenever* $1 + s \leq 4k$ *and* $k - s \geq -1/2$.

Remark 2.9. Although the lemmas 2.7 and 2.8 are not stated as above in [1], it is not hard to obtain them from the calculations of Arbieto, Corcho and Matheus.

Finally, we introduce the I-operator: let $m(\xi)$ be a smooth non-negative symbol on $\mathbb{R}$ which equals 1 for $|\xi| \leq 1$ and equals $|\xi|^{-1}$ for $|\xi| \geq 2$. For any $N \geq 1$ and $\alpha \in \mathbb{R}$, denote by I_N^α the spatial Fourier multiplier

$$\widehat{I_N^\alpha f}(\xi) = m\big(\frac{\xi}{N}\big)^\alpha \widehat{f}(\xi).$$

For latter use, we recall the following general interpolation lemma.

Lemma 2.10 ([4, Lemma 12.1])**.** *Let* $\alpha_0 > 0$ *and* $n \geq 1$. *Suppose* $Z, X_1, \dots, X_n$ *are translation-invariant Banach spaces and* T *is a translation invariant* n*-linear operator such that*

$$\|I_1^\alpha T(u_1,\dots,u_n)\|_Z \lesssim \prod_{j=1}^n \|I_1^\alpha u_j\|_{X_j},$$

for all $u_1,\dots,u_n$ *and* $0 \leq \alpha \leq \alpha_0$. *Then*

$$\|I_N^\alpha T(u_1,\dots,u_n)\|_Z \lesssim \prod_{j=1}^n \|I_N^\alpha u_j\|_{X_j}$$

for all $u_1,\dots,u_n$, $0 \leq \alpha \leq \alpha_0$ *and* $N \geq 1$. *Here the implicit constant is independent of* N.

After these preliminaries, we can proceed to the next section where a variant of the local well-posedness of Arbieto, Corcho and Matheus is obtained. In the sequel we take $N \gg 1$ a large integer and denote by I the operator $I = I_N^{1-s}$ for a given $s \in \mathbb{R}$.

3. A variant local well-posedness result

This section is devoted to the proof of the following proposition.

Proposition 3.1. *For any $(u_0, v_0) \in H^s(\mathbb{T}) \times H^s(\mathbb{T})$ with $\int_{\mathbb{T}} v_0 = 0$ and $s \geq 1/3$, the periodic NLS-KdV system* (1.1) *has a unique local-in-time solution on the time interval $[0, \delta]$ for some $\delta \leq 1$ and*

$$\delta \sim \begin{cases} (\|Iu_0\|_{X^1} + \|Iv_0\|_{Y^1})^{-\frac{16}{3}-}, & \text{if } \beta \neq 0, \\ (\|Iu_0\|_{X^1} + \|Iv_0\|_{Y^1})^{-8-}, & \text{if } \beta = 0. \end{cases} \tag{3.1}$$

Moreover, we have $\|Iu\|_{X^1} + \|Iv\|_{Y^1} \lesssim \|Iu_0\|_{X^1} + \|Iv_0\|_{Y^1}$.

Proof. We apply the I-operator to the NLS-KdV system (1.1) so that

$$\begin{gathered} iIu_t + Iu_{xx} = \alpha I(uv) + \beta I(|u|^2 u), \\ Iv_t + Iv_{xxx} + I(vv_x) = \gamma I(|u|^2)_x, \\ Iu(0) = Iu_0, \quad Iv(0) = Iv_0. \end{gathered}$$

To solve this equation, we seek for some fixed point of the integral maps

$$\begin{gathered} \Phi_1(Iu, Iv) := U(t)Iu_0 - i\int_0^t U(t-t')\{\alpha I(u(t')v(t')) + \beta I(|u(t')|^2 u(t'))\}dt', \\ \Phi_2(Iu, Iv) := V(t)Iv_0 - \int_0^t V(t-t')\{I(v(t')v_x(t')) - \gamma I(|u(t')|^2)_x\}dt'. \end{gathered}$$

The interpolation lemma 2.10 applied to the linear and multilinear estimates in the lemmas 2.4, 2.5, 2.6, 2.7 and 2.8 yields, in view of the lemma 2.2,

$$\begin{gathered} \|\Phi_1(Iu, Iv)\|_{X^1} \lesssim \|Iu_0\|_{H^1} + \alpha\delta^{\frac{1}{8}-}\|Iu\|_{X^1}\|Iv\|_{Y^1} + \beta\delta^{\frac{3}{8}-}\|Iu\|_{X^1}^3, \\ \|\Phi_2(Iu, Iv)\|_{Y^1} \lesssim \|Iv_0\|_{H^1} + \delta^{\frac{1}{6}-}\|Iv\|_{Y^1}^2 + \gamma\delta^{\frac{1}{8}-}\|Iu\|_{X^1}^2. \end{gathered}$$

In particular, these integrals maps are contractions provided that $\beta\delta^{\frac{3}{8}-}(\|Iu_0\|_{H^1} + \|Iv_0\|_{H^1})^2 \ll 1$ and $\delta^{\frac{1}{8}-}(\|Iu_0\|_{H^1} + \|Iv_0\|_{H^1}) \ll 1$. This completes the proof. □

4. Modified energies

We define the following three quantities:

$$M(u) := \|u\|_{L^2}, \tag{4.1}$$

$$L(u, v) := \alpha\|v\|_{L^2}^2 + 2\gamma\int \Im(u\overline{u_x})dx, \tag{4.2}$$

$$E(u, v) := \alpha\gamma\int v|u|^2 dx + \gamma\|u_x\|_{L^2}^2 + \frac{\alpha}{2}\|v_x\|_{L^2}^2 - \frac{\alpha}{6}\int v^3 dx + \frac{\beta\gamma}{2}\int |u|^4 dx. \tag{4.3}$$

In the sequel, we suppose $\alpha\gamma > 0$. Note that

$$|L(u, v)| \lesssim \|v\|_{L^2}^2 + M\|u_x\|_{L^2}, \tag{4.4}$$

$$\|v\|_{L^2}^2 \lesssim |L| + M\|u_x\|_{L^2}. \tag{4.5}$$

Also, the Gagliardo-Nirenberg and Young inequalities implies

$$\|u_x\|_{L^2}^2 + \|v_x\|_{L^2}^2 \lesssim |E| + |L|^{\frac{5}{3}} + M^8 + 1, \tag{4.6}$$

$$|E| \lesssim \|u_x\|_{L^2}^2 + \|v_x\|_{L^2}^2 + |L|^{\frac{5}{3}} + M^8 + 1 \tag{4.7}$$

In particular, combining the bounds (4.4) and (4.7),

$$|E| \lesssim \|u_x\|_{L^2}^2 + \|v_x\|_{L^2}^2 + \|v\|_{L^2}^{\frac{10}{3}} + M^{10} + 1. \tag{4.8}$$

Moreover, from the bounds (4.5) and (4.6),

$$\|v\|_{L^2}^2 \lesssim |L| + M|E|^{1/2} + M^6 + 1 \tag{4.9}$$

and hence

$$\|u\|_{H^1}^2 + \|v\|_{H^1}^2 \lesssim |E| + |L|^{5/3} + M^8 + 1 \tag{4.10}$$

$$\begin{aligned}
&\frac{d}{dt} L(Iu, Iv) \\
&= 2\alpha \int Iv(IvIv_x - I(vv_x))dx + 2\alpha\gamma \int Iv(I(|u|^2) - |Iu|^2)_x dx \\
&\quad + 4\alpha\gamma \Re \int I\overline{u}_x (IuIv - I(uv))dx + 4\beta\gamma \Re \int ((Iu)^2 I\overline{u} - I(u^2\overline{u}))I\overline{u}_x dx \\
&=: \sum_{j=1}^{4} L_j.
\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}
&\frac{d}{dt} E(Iu, Iv) \\
&= \alpha \int (I(vv_x) - IvIv_x)Iv_{xx} dx + \frac{\alpha}{2} \int (Iv)^2 (I(vv_x) - IvIv_x)dx + \\
&\quad + 2\beta\gamma \Im \int (I(|u|^2 u)_x - ((Iu)^2 I\overline{u})_x) I\overline{u}_x dx \\
&\quad + \alpha\gamma \int |Iu|^2 (IvIv_x - I(vv_x))dx + \alpha\gamma \int (|Iu|^2 - I(|u|^2))IvIv_x dx \\
&\quad + \alpha\gamma \int Iv_{xx} (|Iu|^2 - I(|u|^2))_x dx - 2\alpha\gamma \Im \int Iu_x (I(\overline{u}v) - I\overline{u}Iv)_x dx \\
&\quad + \alpha\gamma^2 \int (I(|u|^2) - |Iu|^2)_x |Iu|^2 dx + 2\alpha^2\gamma \Im \int IvIu(I(\overline{u}v - I\overline{u}Iv))dx \\
&\quad + 2\beta^2\gamma \Im \int Iu(I\overline{u})^2 (I(|u|^2 u) - (Iu)^2 I\overline{u})dx \\
&\quad - 2\alpha\beta\gamma \Im \int IvIu(I(|u|^2\overline{u}) - Iu(I\overline{u}))^2 dx - 2\alpha\beta\gamma \Im \int (Iu)^2 I\overline{u}(I(\overline{u}v) - I\overline{u}Iv)dx \\
&=: \sum_{j=1}^{12} E_j
\end{aligned} \tag{4.12}$$

4.1. **Estimates for the modified L-functional.**

Proposition 4.1. *Let* (u, v) *be a solution of* (1.1) *on the time interval* $[0, \delta]$*. Then, for any* $N \geq 1$ *and* $s > 1/2$*,*

$$\begin{aligned} &|L(Iu(\delta), Iv(\delta)) - L(Iu(0), Iv(0))| \\ &\lesssim N^{-1+}\delta^{\frac{19}{24}-}(\|Iu\|_{X^{1,1/2}} + \|Iv\|_{Y^{1,1/2}})^3 + N^{-2+}\delta^{\frac{1}{2}-}\|Iu\|^4_{X^{1,1/2}}. \end{aligned} \tag{4.13}$$

Proof. Integrating (4.11) with respect to $t \in [0, \delta]$, it follows that we have to bound the (integral over $[0, \delta]$ of the) four terms on the right hand side. To simplify the computations, we assume that the Fourier transform of the functions are non-negative and we ignore the appearance of complex conjugates (since they are irrelevant in our subsequent arguments). Also, we make a dyadic decomposition of the frequencies $|n_i| \sim N_j$ in many places. In particular, it will be important to get extra factors N_j^{0-} everywhere in order to sum the dyadic blocks.

We begin with the estimate of $\int_0^\delta L_1$. It is sufficient to show that

$$\begin{aligned} &\int_0^\delta \sum_{n_1+n_2+n_3=0} \Big|\frac{m(n_1+n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)}\Big| \widehat{v_1}(n_1, t)|n_2|\widehat{v_2}(n_2, t)\widehat{v_3}(n_3, t) \\ &\lesssim N^{-1}\delta^{\frac{5}{6}-}\prod_{j=1}^3 \|v_j\|_{Y^{1,1/2}} \end{aligned} \tag{4.14}$$

- $|n_1| \ll |n_2| \sim |n_3|$, $|n_2| \gtrsim N$. In this case, note that

$$\Big|\frac{m(n_1+n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)}\Big| \lesssim \Big|\frac{\nabla m(n_2) \cdot n_1}{m(n_2)}\Big| \lesssim \frac{N_1}{N_2}, \text{if } |n_1| \leq N,$$
$$\Big|\frac{m(n_1+n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)}\Big| \lesssim \Big(\frac{N_1}{N}\Big)^{1/2}, \text{if } |n_1| \geq N.$$

Hence, using the lemmas 2.1 and 2.2, we obtain

$$\Big|\int_0^\delta L_1\Big| \lesssim \frac{N_1}{N_2}\|v_1\|_{L^4}\|(v_2)_x\|_{L^4}\|v_3\|_{L^2} \lesssim N^{-2+}\delta^{\frac{5}{6}-}N_{\max}^{0-}\prod_{j=1}^3 \|v_i\|_{Y^{1,1/2}}$$

if $|n_1| \leq N$, and

$$\Big|\int_0^\delta L_1\Big| \lesssim \Big(\frac{N_1}{N}\Big)^{1/2}\frac{1}{N_1N_3}\delta^{\frac{5}{6}-}\prod_{j=1}^3 \|v_i\|_{Y^{1,1/2}} \lesssim N^{-2+}\delta^{\frac{5}{6}-}N_{\max}^{0-}\prod_{j=1}^3 \|v_i\|_{Y^{1,1/2}}.$$

- $|n_2| \ll |n_1| \sim |n_3|$, $|n_1| \gtrsim N$. This case is similar to the previous one.
- $|n_1| \sim |n_2| \gtrsim N$. The multiplier is bounded by

$$\Big|\frac{m(n_1+n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)}\Big| \lesssim \Big(\frac{N_1}{N}\Big)^{1-}.$$

In particular, using the lemmas 2.1 and 2.2,

$$\Big|\int_0^\delta L_1\Big| \lesssim \Big(\frac{N_1}{N}\Big)^{1-}\|v_1\|_{L^2}\|(v_2)_x\|_{L^4}\|v_3\|_{L^4} \lesssim N^{-1+}\delta^{\frac{5}{6}-}N_{\max}^{0-}\prod_{j=1}^3 \|v_i\|_{Y^{1,1/2}}.$$

Now, we estimate $\int_0^\delta L_2$. Our task is to prove that

$$\begin{aligned}&\int_0^\delta \sum_{n_1+n_2+n_3=0} \Big|\frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}\Big| |n_1+n_2| \widehat{u_1}(n_1,t)\widehat{u_2}(n_2,t)\widehat{v_3}(n_3,t)\\ &\lesssim N^{-1+}\delta^{\frac{19}{24}-}\|u_1\|_{X^{1,1/2}}\|u_2\|_{X^{1,1/2}}\|v_3\|_{Y^{1,1/2}}\end{aligned} \tag{4.15}$$

- $|n_2| \ll |n_1| \sim |n_3| \gtrsim N$. We estimate the multiplier by

$$\Big|\frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}\Big| \lesssim \langle (\frac{N_2}{N})^{1/2}\rangle.$$

Thus, using $L^2_{xt}L^4_{xt}L^4_{xt}$ Hölder inequality and the lemmas 2.1 and 2.2

$$\begin{aligned}\int_0^\delta L_2 &\lesssim \langle\big(\frac{N_2}{N}\big)^{1/2}\rangle \frac{1}{\langle N_2\rangle N_3}\delta^{\frac{19}{24}-}\|u_1\|_{X^{1,1/2}}\|u_2\|_{X^{1,1/2}}\|v_3\|_{Y^{1,1/2}}\\ &\lesssim N^{-1+}\delta^{\frac{19}{24}-}N^{0-}_{\max}\|u_1\|_{X^{1,1/2}}\|u_2\|_{X^{1,1/2}}\|v_3\|_{Y^{1,1/2}}.\end{aligned}$$

- $|n_1| \ll |n_2| \sim |n_3|$. This case is similar to the previous one.
- $|n_1| \sim |n_2| \gtrsim N$. Estimating the multiplier by

$$\Big|\frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}\Big| \lesssim \big(\frac{N_2}{N}\big)^{1-}$$

we conclude

$$\begin{aligned}\int_0^\delta L_2 &\lesssim \big(\frac{N_2}{N}\big)^{1-}\frac{1}{N_1N_2}\delta^{\frac{19}{24}-}\|u_1\|_{X^{1,1/2}}\|u_2\|_{X^{1,1/2}}\|v_3\|_{Y^{1,1/2}}\\ &\lesssim N^{-2+}\delta^{\frac{19}{24}-}N^{0-}_{\max}\|u_1\|_{X^{1,1/2}}\|u_2\|_{X^{1,1/2}}\|v_3\|_{Y^{1,1/2}}.\end{aligned}$$

Next, let us compute $\int_0^\delta L_3$. We claim that

$$\begin{aligned}&\int_0^\delta \sum_{n_1+n_2+n_3=0} \Big|\frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}\Big| \widehat{u_1}(n_1,t)\widehat{v_2}(n_2,t)|n_3|\widehat{u_3}(n_3,t)\\ &\lesssim N^{-2+}\delta^{\frac{19}{24}-}\|u_1\|_{X^{1,1/2}}\|v_2\|_{Y^{1,1/2}}\|u_3\|_{X^{1,1/2}}\end{aligned} \tag{4.16}$$

- $|n_2| \ll |n_1| \sim |n_3|$, $|n_1| \gtrsim N$. The multiplier is bounded by

$$\Big|\frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}\Big| \lesssim \begin{cases} \big|\frac{\nabla m(n_1)\cdot n_2}{m(n_1)}\big| \lesssim \frac{N_2}{N_1}, & \text{if } |n_2|\le N,\\ \big(\frac{N_2}{N}\big)^{1/2}, & \text{if } |n_2|\ge N.\end{cases}$$

So, it is not hard to see that

$$\int_0^\delta L_3 \lesssim N^{-2+}\delta^{\frac{19}{24}-}N^{0-}_{\max}\|u_1\|_{X^{1,1/2}}\|v_2\|_{Y^{1,1/2}}\|u_3\|_{X^{1,1/2}}$$

- $|n_1| \ll |n_2| \sim |n_3|$, $|n_2| \gtrsim N$. This case is completely similar to the previous one.
- $|n_1| \sim |n_2| \gtrsim N$. Since the multiplier is bounded by N_2/N, we get

$$\int_0^\delta L_3 \lesssim N^{-2+}\delta^{\frac{19}{24}-}N^{0-}_{\max}\|u_1\|_{X^{1,1/2}}\|v_2\|_{Y^{1,1/2}}\|u_3\|_{X^{1,1/2}}.$$

Finally, it remains to estimate the contribution of $\int_0^\delta L_4$. It suffices to see that

$$\begin{aligned} &\int_0^\delta \sum_{n_1+n_2+n_3+n_4=0} \Big| \frac{m(n_1+n_2+n_3)-m(n_1)m(n_2)m(n_3)}{m(n_1)m(n_2)m(n_3)} \Big| |n_4| \prod_{j=1}^4 \widehat{u_j}(n_j,t) \\ &\lesssim N^{-2+}\delta^{\frac{1}{2}-} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}} \end{aligned} \tag{4.17}$$

• $N_1, N_2, N_3 \gtrsim N$. Since the multiplier verifies

$$\Big| \frac{m(n_1+n_2+n_3)-m(n_1)m(n_2)m(n_3)}{m(n_1)m(n_2)m(n_3)} \Big| \lesssim \Big(\frac{N_1}{N}\frac{N_2}{N}\frac{N_3}{N} \Big)^{1/2},$$

appliying $L^4_{xt}L^4_{xt}L^4_{xt}L^4_{xt}$ Hölder inequality and the lemmas 2.1, 2.2, we have

$$\begin{aligned} \int_0^\delta L_4 &\lesssim \Big(\frac{N_1}{N}\frac{N_2}{N}\frac{N_3}{N} \Big)^{1/2} \frac{\delta^{\frac{1}{2}-}}{N_1N_2N_3} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}} \\ &\lesssim N^{-3+}\delta^{\frac{1}{2}-} N_{\max}^{0-} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}}. \end{aligned}$$

• $N_1 \sim N_2 \gtrsim N$ and $N_3, N_4 \ll N_1, N_2$. Here the multiplier is bounded by $\big(\frac{N_1}{N}\frac{N_2}{N}\big)^{1/2} \langle (\frac{N_3}{N})^{1/2} \rangle$. Hence,

$$\begin{aligned} \int_0^\delta L_4 &\lesssim \Big(\frac{N_1}{N}\frac{N_2}{N} \Big)^{1/2} \langle \big(\frac{N_3}{N}\big)^{1/2} \rangle \frac{\delta^{\frac{1}{2}-}}{N_1N_2\langle N_3\rangle} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}} \\ &\lesssim N^{-2+}\delta^{\frac{1}{2}-} N_{\max}^{0-} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}}. \end{aligned}$$

• $N_1 \sim N_4 \gtrsim N$ and $N_2, N_3 \ll N_1, N_4$. In this case we have the following estimates for the multiplier

$$\begin{aligned} &\Big| \frac{m(n_1+n_2+n_3)-m(n_1)m(n_2)m(n_3)}{m(n_1)m(n_2)m(n_3)} \Big| \\ &\lesssim \begin{cases} \big| \frac{\nabla m(n_1)(n_2+n_3)}{m(n_1)} \big| \lesssim \frac{N_2+N_3}{N_1}, & \text{if } N_2, N_3 \le N \\ \big(\frac{N_1}{N}\frac{N_2}{N}\big)^{1/2} \langle (\frac{N_3}{N})^{1/2} \rangle, & \text{if } N_2 \ge N, \\ \big(\frac{N_1}{N}\frac{N_3}{N}\big)^{1/2} \langle (\frac{N_2}{N})^{1/2} \rangle, & \text{if } N_3 \ge N. \end{cases} \end{aligned}$$

Therefore, it is not hard to see that, in any of the situations $N_2, N_3 \le N$, $N_2 \ge N$ or $N_3 \ge N$, we have

$$\int_0^\delta L_4 \lesssim N^{-2+}\delta^{\frac{1}{2}-} N_{\max}^{0-} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}}.$$

• $N_1 \sim N_2 \sim N_4 \gtrsim N$ and $N_3 \ll N_1, N_2, N_4$. Here we have the following bound

$$\int_0^\delta L_4 \lesssim \Big(\frac{N_1}{N}\frac{N_2}{N} \Big)^{1/2} \langle \big(\frac{N_3}{N}\big)^{1/2} \rangle \frac{\delta^{\frac{1}{2}-}}{N_1N_2N_3} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}}.$$

At this point, clearly the bounds (4.14), (4.15), (4.16) and (4.17) concludes the proof of the proposition 4.1. □

4.2. **Estimates for the modified E-functional.**

Proposition 4.2. *Let (u, v) be a solution of (1.1) on the time interval $[0, \delta]$ such that $\int_{\mathbb{T}} v = 0$. Then, for any $N \geq 1$, $s > 1/2$,*

$$
\begin{aligned}
&|E(Iu(\delta), Iv(\delta)) - E(Iu(0), Iv(0))| \\
&\lesssim \Big(N^{-1+}\delta^{\frac{1}{6}-} + N^{-\frac{2}{3}+}\delta^{\frac{3}{8}-} + N^{-\frac{3}{2}+}\delta^{\frac{1}{8}-}\Big)(\|Iu\|_{X^1} + \|Iv\|_{Y^1})^3 \\
&\quad + N^{-1+}\delta^{\frac{1}{2}-}(\|Iu\|_{X^1} + \|Iv\|_{Y^1})^4 + N^{-2+}\delta^{\frac{1}{2}-}\|Iu\|^4_{X^1}(\|Iu\|^2_{X^1} + \|Iv\|_{Y^1}).
\end{aligned} \tag{4.18}
$$

Proof. Again we integrate (4.12) with respect to $t \in [0, \delta]$, decompose the frequencies into dyadic blocks, etc., so that our objective is to bound the (integral over $[0, \delta]$ of the) E_j for each $j = 1, \dots, 12$.

For the expression $\int_0^\delta E_1$, apply the lemma 2.3. We obtain

$$
|\int_0^\delta E_1| \lesssim \|Iv_{xx}\|_{Y^{-1}} \|IvIv_x - I(vv_x)\|_{W^1} \lesssim \|Iv\|_{Y^1} \|IvIv_x - I(vv_x)\|_{W^1}
$$

Writing the definition of the norm W^1, it suffices to prove the bound

$$
\begin{aligned}
&\Big\| \frac{\langle n_3\rangle}{\langle \tau_3 - n_3^3\rangle^{1/2}} \int \sum \frac{m(n_1+n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \widehat{v_1}(n_1,\tau_1)\ n_2\ \widehat{v_2}(n_2,\tau_2)\Big\|_{L^2_{n_3,\tau_3}} \\
&+ \Big\| \frac{\langle n_3\rangle}{\langle \tau_3 - n_3^3\rangle} \int \sum \frac{m(n_1+n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \widehat{v_1}(n_1,\tau_1)\ n_2\ \widehat{v_2}(n_2,\tau_2)\Big\|_{L^2_{n_3}L^1_{\tau_3}} \\
&\lesssim N^{-1+}\delta^{\frac{1}{6}-}\|v_1\|_{Y^{1,1/2}}\|v_2\|_{Y^{1,1/2}}.
\end{aligned} \tag{4.19}
$$

Recall that the dispersion relation $\sum_{j=1}^3 \tau_j - n_j^3 = -3n_1n_2n_3$ implies that, since $n_1n_2n_3 \neq 0$, if we put $L_j := |\tau_j - n_j^3|$ and $L_{\max} = \max\{L_j; j = 1, 2, 3\}$, then $L_{\max} \gtrsim \langle n_1\rangle\langle n_2\rangle\langle n_3\rangle$.

- $|n_2| \sim |n_3| \gtrsim N$, $|n_1| \ll |n_2|$. The multiplier is bounded by

$$
\Big|\frac{m(n_1+n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)}\Big| \lesssim \begin{cases} \frac{N_1}{N_2}, & \text{if } |n_1| \leq N, \\ \big(\frac{N_1}{N}\big)^{1/2}, & \text{if } |n_1| \geq N. \end{cases}
$$

Thus, if $|\tau_3 - n_3^3| = L_{\max}$, we have

$$
\begin{aligned}
&\Big\| \frac{\langle n_3\rangle}{\langle \tau_3 - n_3^3\rangle^{1/2}} \int \sum \frac{m(n_1+n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \widehat{v_1}(n_1,\tau_1)\ n_2\ \widehat{v_2}(n_2,\tau_2)\Big\|_{L^2_{n_3,\tau_3}} \\
&\lesssim \begin{cases} \frac{N_1}{N_2}\frac{N_3}{(N_1N_2N_3)^{1/2}}\|v_1\|_{L^4_{xt}}\|(v_2)_x\|_{L^4_{xt}} \\ \lesssim N^{-1+}\delta^{\frac{1}{3}-}N_{\max}^{0-}\|v_1\|_{Y^{1,1/2}}\|v_2\|_{Y^{1,1/2}}, & \text{if } |n_1| \leq N, \\ \big(\frac{N_1}{N_2}\big)^{1/2}\frac{N_3}{N_1}\frac{1}{(N_1N_2N_3)^{1/2}}\|v_1\|_{L^4_{xt}}\|(v_2)_x\|_{L^4_{xt}} \\ \lesssim N^{-\frac{3}{2}+}\delta^{\frac{1}{3}-}N_{\max}^{0-}\|v_1\|_{Y^{1,\frac{1}{2}}}\|v_2\|_{Y^{1,\frac{1}{2}}}, & \text{if } |n_1| \geq N. \end{cases}
\end{aligned}
$$

and

$$\Big\|\frac{\langle n_3\rangle}{\langle \tau_3-n_3^3\rangle}\int\sum\frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}\widehat{v_1}(n_1,\tau_1)\ n_2\ \widehat{v_2}(n_2,\tau_2)\Big\|_{L^2_{n_3}L^1_{\tau_3}}$$
$$\lesssim\begin{cases}\frac{N_1}{N_2}\frac{N_3}{(N_1N_2N_3)^{\frac12-}}\|v_1\|_{L^4_{xt}}\|(v_2)_x\|_{L^4_{xt}} \\ \lesssim N^{-1+}\delta^{\frac13-}N_{\max}^{0-}\|v_1\|_{Y^{1,1/2}}\|v_2\|_{Y^{1,1/2}}, & \text{if } |n_1|\le N,\\ \big(\frac{N_1}{N_2}\big)^{1/2}\frac{N_3}{N_1}\frac{\delta^{\frac13-}}{(N_1N_2N_3)^{\frac12-}}\|v_1\|_{Y^{1,\frac12}}\|v_2\|_{Y^{1,\frac12}} \\ \lesssim N^{-\frac32+}\delta^{\frac13-}N_{\max}^{0-}\|v_1\|_{Y^{1,\frac12}}\|v_2\|_{Y^{1,\frac12}}, & \text{if } |n_1|\ge N.\end{cases}$$

If either $|\tau_1-n_1^3|=L_{\max}$ or $|\tau_2-n_2^3|=L_{\max}$, we have

$$\Big\|\frac{\langle n_3\rangle}{\langle \tau_3-n_3^3\rangle^{1/2}}\int\sum\frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}\widehat{v_1}(n_1,\tau_1)\ n_2\ \widehat{v_2}(n_2,\tau_2)\Big\|_{L^2_{n_3,\tau_3}}$$
$$\lesssim\begin{cases}\frac{N_1}{N_2}\frac{N_3}{(N_1N_2N_3)^{1/2}}\frac{\delta^{\frac16-}}{N_1}\|v_1\|_{Y^{1,\frac12}}\|v_2\|_{Y^{1,\frac12}} \\ \lesssim N^{-1+}\delta^{\frac16-}N_{\max}^{0-}\|v_1\|_{Y^{1,1/2}}\|v_2\|_{Y^{1,1/2}}, & \text{if } |n_1|\le N,\\ \big(\frac{N_1}{N_2}\big)^{1/2}\frac{N_3}{N_1}\frac{1}{(N_1N_2N_3)^{1/2}}\delta^{\frac16-}\|v_1\|_{Y^{1,\frac12}}\|v_2\|_{Y^{1,\frac12}} \\ \lesssim N^{-\frac32+}\delta^{\frac13-}N_{\max}^{0-}\|v_1\|_{Y^{1,\frac12}}\|v_2\|_{Y^{1,\frac12}}, & \text{if } |n_1|\ge N.\end{cases}$$

and

$$\Big\|\frac{\langle n_3\rangle}{\langle \tau_3-n_3^3\rangle}\int\sum\frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}\widehat{v_1}(n_1,\tau_1)\ n_2\ \widehat{v_2}(n_2,\tau_2)\Big\|_{L^2_{n_3}L^1_{\tau_3}}$$
$$\lesssim\begin{cases}\frac{N_1}{N_2}\frac{N_3}{(N_1N_2N_3)^{\frac12-}}\frac{\delta^{\frac16-}}{N_1}\|v_1\|_{Y^{1,\frac12}}\|v_2\|_{Y^{1,\frac12}} \\ \lesssim N^{-1+}\delta^{\frac16-}N_{\max}^{0-}\|v_1\|_{Y^{1,1/2}}\|v_2\|_{Y^{1,1/2}}, & \text{if } |n_1|\le N,\\ \big(\frac{N_1}{N_2}\big)^{1/2}\frac{N_3}{N_1}\frac{\delta^{\frac16-}}{(N_1N_2N_3)^{\frac12-}}\|v_1\|_{Y^{1,\frac12}}\|v_2\|_{Y^{1,\frac12}} \\ \lesssim N^{-\frac32+}\delta^{\frac16-}N_{\max}^{0-}\|v_1\|_{Y^{1,\frac12}}\|v_2\|_{Y^{1,\frac12}}, & \text{if } |n_1|\ge N.\end{cases}$$

- $|n_1|\sim|n_2|\gtrsim N$. Estimating the multiplier by

$$\Big|\frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}\Big|\lesssim\big(\frac{N_1}{N}\big)^{1-},$$

we have that, if $|\tau_3-n_3^3|=L_{\max}$,

$$\begin{aligned}&\Big\|\frac{\langle n_3\rangle}{\langle \tau_3-n_3^3\rangle^{1/2}}\int\sum\frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}\widehat{v_1}(n_1,\tau_1)\ n_2\ \widehat{v_2}(n_2,\tau_2)\Big\|_{L^2_{n_3,\tau_3}}\\ &+\Big\|\frac{\langle n_3\rangle}{\langle \tau_3-n_3^3\rangle}\int\sum\frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}\widehat{v_1}(n_1,\tau_1)\ n_2\ \widehat{v_2}(n_2,\tau_2)\Big\|_{L^2_{n_3}L^1_{\tau_3}}\\ &\lesssim\Big\{\big(\frac{N_1}{N}\big)^{1-}\frac{N_3}{(N_1N_2N_3)^{1/2}}\frac{\delta^{\frac13-}}{N_1}+\big(\frac{N_1}{N}\big)^{1-}\frac{N_3}{(N_1N_2N_3)^{\frac12-}}\frac{\delta^{\frac13-}}{N_1}\Big\}\|v_1\|_{Y^{1,\frac12}}\|v_2\|_{Y^{1,\frac12}}\\ &\lesssim N^{-\frac32+}\delta^{\frac13-}N_{\max}^{0-}\|v_1\|_{Y^{1,\frac12}}\|v_2\|_{Y^{1,\frac12}}\end{aligned}$$

and, if either $|\tau_1 - n_1^3| = L_{\max}$ or $|\tau_2 - n_2^3| = L_{\max}$,

$$\begin{aligned}
&\Big\| \frac{\langle n_3\rangle}{\langle \tau_3 - n_3^3\rangle^{1/2}} \int \sum \frac{m(n_1+n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \widehat{v_1}(n_1,\tau_1)\ n_2\ \widehat{v_2}(n_2,\tau_2)\Big\|_{L^2_{n_3,\tau_3}} \\
&+ \Big\| \frac{\langle n_3\rangle}{\langle \tau_3 - n_3^3\rangle} \int \sum \frac{m(n_1+n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \widehat{v_1}(n_1,\tau_1)\ n_2\ \widehat{v_2}(n_2,\tau_2)\Big\|_{L^2_{n_3} L^1_{\tau_3}} \\
&\lesssim \big\{ \big(\frac{N_1}{N}\big)^{1-} \frac{N_3}{(N_1N_2N_3)^{1/2}} \frac{\delta^{\frac16 -}}{N_1} + \big(\frac{N_1}{N}\big)^{1-} \frac{N_3}{(N_1N_2N_3)^{\frac12 -}} \frac{\delta^{\frac16 -}}{N_1} \big\} \|v_1\|_{Y^{1,\frac12}} \|v_2\|_{Y^{1,\frac12}} \\
&\lesssim N^{-\frac32 +} \delta^{\frac16 -} N_{\max}^{0-} \|v_1\|_{Y^{1,\frac12}} \|v_2\|_{Y^{1,\frac12}}.
\end{aligned}$$

For the expression $\int_0^\delta E_2$, it suffices to prove that

$$\begin{aligned}
&\Big| \int_0^\delta \sum \frac{m(n_3+n_4) - m(n_3)m(n_4)}{m(n_3)m(n_4)} \widehat{v_1}(n_1,t)\widehat{v_2}(n_2,t)\widehat{v_3}(n_3,t)\ n_4\ \widehat{v_4}(n_4,t)\Big| \\
&\lesssim N^{-2+}\delta^{\frac23 -} \prod_{j=1}^4 \|v_j\|_{Y^{1,1/2}}.
\end{aligned} \tag{4.20}$$

Since at least two of the N_i are bigger than $N/3$, we can assume that $N_1 \geq N_2 \geq N_3$ and $N_1 \gtrsim N$. Hence,

$$\int_0^\delta E_2 \lesssim \begin{cases}
\big(\frac{N_1}{N}\big)^{1-} \frac{\delta^{\frac23 -}}{N_1N_2N_3} \prod_{j=1}^4 \|v_j\|_{Y^{1,1/2}} \lesssim N^{-2+}\delta^{\frac23 -} N_{\max}^{0-} \prod_{j=1}^4 \|v_j\|_{Y^{1,1/2}}, \\
\quad \text{if } |n_3| \sim |n_4| \gtrsim N, \\
\frac{N_3}{N_4} \frac{\delta^{\frac23 -}}{N_1N_2N_3} \prod_{j=1}^4 \|v_j\|_{Y^{1,1/2}} \lesssim N^{-2+}\delta^{\frac23 -} N_{\max}^{0-} \prod_{j=1}^4 \|v_j\|_{Y^{1,1/2}}, \\
\quad \text{if } |n_3| \ll |n_4|, |n_3| \leq N |n_4| \gtrsim N, \\
\big(\frac{N_3}{N}\big)^{1/2} \frac{\delta^{\frac23 -}}{N_1N_2N_3} \prod_{j=1}^4 \|v_j\|_{Y^{1,\frac12}} \lesssim N^{-2+}\delta^{\frac23 -} N_{\max}^{0-} \prod_{j=1}^4 \|v_j\|_{Y^{1,\frac12}}, \\
\quad \text{if } |n_3| \ll |n_4|, |n_3| \geq N, |n_4| \gtrsim N.
\end{cases}$$

Next, we estimate the contribution of $\int_0^\delta E_3$. We claim that

$$\begin{aligned}
&\int_0^\delta \sum \frac{m(n_1n_2n_3) - m(n_1)m(n_2)m(n_3)}{m(n_1)m(n_2)m(n_3)} \widehat{u_1}(n_1,t)\widehat{u_2}(n_2,t)\widehat{u_3}(n_3,t)\ |n_4|^2\ \widehat{u_4}(n_4,t) \\
&\lesssim N^{-1+}\delta^{\frac12 -} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}}.
\end{aligned} \tag{4.21}$$

- $|n_1| \sim |n_2| \sim |n_3| \sim |n_4| \gtrsim N$. Since the multiplier satisfies

$$\frac{m(n_1n_2n_3) - m(n_1)m(n_2)m(n_3)}{m(n_1)m(n_2)m(n_3)} \lesssim \big(\frac{N_1}{N}\big)^{\frac32}$$

we obtain

$$\int_0^\delta E_3 \lesssim \big(\frac{N_1}{N}\big)^{\frac32} \frac{N_4}{N_1N_2N_3} \delta^{\frac12 -} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}} \lesssim N^{-2+}\delta^{\frac12 -} N_{\max}^{0-} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}}.$$

• Exactly two frequencies are bigger than $N/3$. We consider the most difficult case $|n_4| \gtrsim N$, $|n_1| \sim |n_4|$ and $|n_2|, |n_3| \ll |n_1|, |n_4|$. The multiplier is estimated by

$$\frac{m(n_1 n_2 n_3) - m(n_1)m(n_2)m(n_3)}{m(n_1)m(n_2)m(n_3)} \lesssim \begin{cases} \langle \big(\frac{N_3}{N}\big)^{1/2} \rangle \big(\frac{N_2}{N}\big)^{1/2}, & \text{if } |n_2| \geq N, \\ \langle \big(\frac{N_2}{N}\big)^{1/2} \rangle \big(\frac{N_3}{N}\big)^{1/2}, & \text{if } |n_3| \geq N, \\ \frac{N_2+N_3}{N_1}, & \text{if } |n_2|, |n_3| \leq N. \end{cases}$$

Thus,

$$\int_0^\delta E_3 \lesssim N^{-1+} \delta^{\frac{1}{2}-} N_{\max}^{0-} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}}.$$

• Exactly three frequencies are bigger than $N/3$. The most difficult case is $|n_1| \sim |n_2| \sim |n_4| \gtrsim N$ and $|n_3| \ll |n_1|, |n_2|, |n_4|$. Here the multiplier is bounded by

$$\frac{m(n_1 n_2 n_3) - m(n_1)m(n_2)m(n_3)}{m(n_1)m(n_2)m(n_3)} \lesssim \Big(\frac{N_1}{N}\frac{N_2}{N}\Big)^{1/2} \langle \big(\frac{N_3}{N}\big)^{1/2} \rangle.$$

Hence,

$$\begin{aligned} \int_0^\delta E_3 &\lesssim \Big(\frac{N_1}{N}\frac{N_2}{N}\Big)^{1/2} \langle \big(\frac{N_3}{N}\big)^{1/2} \rangle \frac{N_4}{N_1 N_2 N_3} \delta^{\frac{1}{2}-} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}} \\ &\lesssim N^{-1+} \delta^{\frac{1}{2}-} N_{\max}^{0-} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}}. \end{aligned}$$

The contribution of $\int_0^\delta E_4$ is controlled if we are able to show that

$$\begin{aligned} &\int_0^\delta \sum \frac{m(n_1+n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \widehat{v_1}(n_1,t)\ |n_2|\ \widehat{v_2}(n_2,t)\widehat{u_3}(n_3,t)\widehat{u_4}(n_4,t) \lesssim \\ &N^{-1+}\delta^{\frac{7}{12}-} \prod_{j=1}^2 \|u_j\|_{X^{1,1/2}} \|v_j\|_{Y^{1,1/2}}. \end{aligned} \tag{4.22}$$

We crudely bound the multiplier by

$$\Big|\frac{m(n_1+n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)}\Big| \lesssim \big(\frac{N_{\max}}{N}\big)^{1-}.$$

The most difficult case is $|n_2| \geq N$. We have two possibilities:

• Exactly two frequencies are bigger than $N/3$. We can assume $N_3 \ll N_2$. In particular,

$$\begin{aligned} \int_0^\delta E_4 &\lesssim \big(\frac{N_{\max}}{N}\big)^{1-} \frac{\delta^{\frac{7}{12}}}{N_1 N_3 N_4} \prod_{j=1}^2 \|u_j\|_{X^{1,\frac{1}{2}}} \|v_j\|_{Y^{1,\frac{1}{2}}} \\ &\lesssim N^{-1+}\delta^{\frac{7}{12}-} N_{\max}^{0-} \prod_{j=1}^2 \|u_j\|_{X^{1,\frac{1}{2}}} \|v_j\|_{Y^{1,\frac{1}{2}}}. \end{aligned}$$

• At least three frequencies are bigger than $N/3$. In this case,

$$\int_0^\delta E_4 \lesssim N^{-2+}\delta^{\frac{7}{12}-} N_{\max}^{0-} \prod_{j=1}^2 \|u_j\|_{X^{1,\frac{1}{2}}} \|v_j\|_{Y^{1,\frac{1}{2}}}.$$

The expression $\int_0^\delta E_5$ is controlled if we are able to prove

$$\begin{aligned}&\int_0^\delta \sum \frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}\widehat{u_1}(n_1,t)\widehat{u_2}(n_2,t)\widehat{v_3}(n_3,t)\ |n_4|\ \widehat{v_4}(n_4,t)\\ &\lesssim N^{-1+}\delta^{\frac{7}{12}-}\prod_{j=1}^{2}\|u_j\|_{X^{1,1/2}}\|v_j\|_{Y^{1,1/2}}.\end{aligned}\tag{4.23}$$

This follows directly from the previous analysis for (4.22).

For the term $\int_0^\delta E_6$, we apply the lemma 2.3 to obtain

$$\int_0^\delta E_6 \lesssim \|(Iv)_{xx}\|_{Y^{-1}}\|(|Iu|^2-I(|u|^2))_x\|_{W^1} \lesssim \|Iv\|_{Y^1}\|(|Iu|^2-I(|u|^2))_x\|_{W^1}.$$

So, the definition of the W^1 norm means that we have to prove

$$\begin{aligned}&\Big\|\frac{\langle n_3\rangle}{\langle \tau_3-n_3^3\rangle^{1/2}}|n_3|\int\sum\frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}\widehat{u_1}(n_1,\tau_1)\widehat{u_2}(n_2,\tau_2)\Big\|_{L^2_{n_3,\tau_3}}\\ &+\Big\|\frac{\langle n_3\rangle}{\langle \tau_3-n_3^3\rangle^{1/2}}|n_3|\int\sum\frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}\widehat{u_1}(n_1,\tau_1)\widehat{u_2}(n_2,\tau_2)\Big\|_{L^2_{n_3}L^1_{\tau_3}}\\ &\lesssim \Big\{N^{-\frac{3}{2}+}\delta^{\frac{1}{8}-}+N^{-\frac{2}{3}}\delta^{\frac{3}{8}-}\Big\}\|u_1\|_{X^{1,1/2}}\|u_2\|_{X^{1,1/2}}.\end{aligned}\tag{4.24}$$

Note that $\sum\tau_j=0$ and $\sum n_j=0$. In particular, we obtain the dispersion relation

$$\tau_3-n_3^3+\tau_2+n_2^2+\tau_1+n_1^2=-n_3^3+n_1^2+n_2^2.$$

• $|n_1|\gtrsim N$, $|n_2|\ll|n_1|$. Denoting by $L_1:=|\tau_1+n_1^2|$, $L_2:=|\tau_2+n_2^2|$ and $L_3:=|\tau_3-n_3^3|$, the dispersion relation says that in the present situation $L_{\max}:=\max\{L_j\}\gtrsim N_3^3$. Since the multiplier is bounded by

$$\Big|\frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}\Big|\lesssim\begin{cases}\frac{\nabla m(n_1)n_2}{m(n_1)}\lesssim\frac{N_2}{N_1}, & \text{if } |n_2|\le N,\\ \big(\frac{N_2}{N}\big)^{1/2}, & \text{if } |n_2|\ge N,\end{cases}$$

we deduce that

$$\begin{aligned}&\Big\|\frac{\langle n_3\rangle}{\langle \tau_3-n_3^3\rangle^{1/2}}|n_3|\int\sum\frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}\widehat{u_1}(n_1,\tau_1)\widehat{u_2}(n_2,\tau_2)\Big\|_{L^2_{n_3,\tau_3}}\\ &+\Big\|\frac{\langle n_3\rangle}{\langle \tau_3-n_3^3\rangle^{1/2}}|n_3|\int\sum\frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}\widehat{u_1}(n_1,\tau_1)\widehat{u_2}(n_2,\tau_2)\Big\|_{L^2_{n_3}L^1_{\tau_3}}\\ &\lesssim\frac{N_3^2}{N_3^{\frac{3}{2}-}}\frac{\delta^{\frac{1}{8}-}}{NN_1}\|u_1\|_{X^{1,1/2}}\|u_2\|_{X^{1,1/2}}\\ &\lesssim N^{-\frac{3}{2}+}\delta^{\frac{1}{8}-}N_{\max}^{0-}\|u_1\|_{X^{1,1/2}}\|u_2\|_{X^{1,1/2}}.\end{aligned}$$

• $|n_1| \sim |n_2| \gtrsim N$, $|n_3|^3 \gg |n_2|^2$. In the present case the multiplier is bounded by $\big(\frac{N_1}{N}\big)^{1-}$ and the dispersion relation says that $L_{\max} \gtrsim N_3^3$. Thus,

$$\begin{aligned}
&\Big\|\frac{\langle n_3\rangle}{\langle \tau_3 - n_3^3\rangle^{1/2}}|n_3| \int\sum \frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}\widehat{u_1}(n_1,\tau_1)\widehat{u_2}(n_2,\tau_2)\Big\|_{L^2_{n_3,\tau_3}}\\
&+\Big\|\frac{\langle n_3\rangle}{\langle \tau_3 - n_3^3\rangle^{1/2}}|n_3| \int\sum \frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}\widehat{u_1}(n_1,\tau_1)\widehat{u_2}(n_2,\tau_2)\Big\|_{L^2_{n_3}L^1_{\tau_3}}\\
&\lesssim \frac{N_3^2}{N_3^{\frac{3}{2}-}}\big(\frac{N_1}{N}\big)^{1-}\frac{\delta^{\frac18-}}{N_1N_2}\|u_1\|_{X^{1,1/2}}\|u_2\|_{X^{1,1/2}}\\
&\lesssim N^{-\frac32+}\delta^{\frac18-}N_{\max}^{0-}\|u_1\|_{X^{1,1/2}}\|u_2\|_{X^{1,1/2}}.
\end{aligned}$$

• $|n_1| \sim |n_2| \gtrsim N$ and $|n_3|^3 \lesssim |n_2|^2$. Here the dispersion relation does not give useful information about $L_{\max}$. Since the multiplier is estimated by $\big(\frac{N_2}{N}\big)^{1/2}$, we obtain the crude bound

$$\begin{aligned}
&\Big\|\frac{\langle n_3\rangle}{\langle \tau_3 - n_3^3\rangle^{1/2}}|n_3| \int\sum \frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}\widehat{u_1}(n_1,\tau_1)\widehat{u_2}(n_2,\tau_2)\Big\|_{L^2_{n_3,\tau_3}}\\
&+\Big\|\frac{\langle n_3\rangle}{\langle \tau_3 - n_3^3\rangle^{1/2}}|n_3| \int\sum \frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}\widehat{u_1}(n_1,\tau_1)\widehat{u_2}(n_2,\tau_2)\Big\|_{L^2_{n_3}L^1_{\tau_3}}\\
&\lesssim N_3^2\big(\frac{N_2}{N}\big)^{1/2}\frac{\delta^{\frac38-}}{N_1N_2}\|u_1\|_{X^{1,1/2}}\|u_2\|_{X^{1,1/2}}\\
&\lesssim N^{-\frac23+}\delta^{\frac38-}N_{\max}^{0-}\|u_1\|_{X^{1,1/2}}\|u_2\|_{X^{1,1/2}}.
\end{aligned}$$

Next, the desired bound related to $\int_0^\delta E_7$ follows from

$$\begin{aligned}
&\int_0^\delta \sum\Big|\frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}\Big||n_1+n_2|\widehat{u_1}(n_1,t)\widehat{v_2}(n_2,t)|n_3|\widehat{u_3}(n_3,t)\\
&\lesssim N^{-1+}\delta^{\frac{19}{24}-}\|u_1\|_{X^{1,1/2}}\|v_2\|_{Y^{1,1/2}}\|u_3\|_{X^{1,1/2}}
\end{aligned} \tag{4.25}$$

• $|n_1| \ll |n_2| \gtrsim N$. The multiplier is $\lesssim (|n_2|/N)^{1/2}$ so that

$$\begin{aligned}
\int_0^\delta E_7 &\lesssim \frac{1}{N^{1/2}}\int_0^\delta \sum |n_1+n_2|\widehat{u_1}(n_1,t)|n_2|^{1/2}\widehat{v_2}(n_2,t)|n_3|\widehat{u_3}(n_3,t)\\
&\lesssim N^{-1}\delta^{\frac{19}{24}-}\|u_1\|_{X^{1,1/2}}\|v_2\|_{Y^{1,1/2}}\|u_3\|_{X^{1,1/2}}.
\end{aligned}$$

• $|n_1| \sim |n_2| \gtrsim N$. The multiplier is $\lesssim |n_2|/N$. Hence,

$$\int_0^\delta E_7 \lesssim N^{-1}\delta^{\frac{19}{24}-}\|u_1\|_{X^{1,1/2}}\|v_2\|_{Y^{1,1/2}}\|u_3\|_{X^{1,1/2}}.$$

$|n_1| \gtrsim N$, $|n_2| \le N$. The multiplier is again $\lesssim N_2/N$, so that it can be estimated as above.

Now we turn to the term $\int_0^\delta E_8$. The objective is to show that

$$\int_0^\delta \Big|\frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)}\Big||n_1+n_2|\prod_{j=1}^4 \widehat{u_j}(n_j,t) \lesssim N^{-1+}\delta^{\frac12-}\prod_{j=1}^4\|u_j\|_{X^{1,1/2}} \tag{4.26}$$

• At least three frequencies are bigger than $N/3$. We can assume $|n_1| \geq |n_2|$. The multiplier is bounded by $N_{\max}/N$ so that

$$\int_0^\delta E_8 \lesssim \frac{N_{\max}}{N} \frac{\delta^{\frac{1}{2}-}}{N_2 N_3 N_4} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}} \lesssim N^{-2+}\delta^{\frac{1}{2}-} N_{\max}^{0-} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}}.$$

• Exactly two frequencies are bigger than $N/3$. Without loss of generality, we suppose $|n_1| \sim |n_2| \gtrsim N$ and $|n_3|, |n_4| \ll N$. Since the multiplier satisfies

$$\Big|\frac{m(n_1+n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)}\Big| \lesssim \big(\frac{N_{\max}}{N}\big)^{1-},$$

we get the bound

$$\int_0^\delta E_8 \lesssim \big(\frac{N_{\max}}{N}\big)^{1-} \frac{\delta^{\frac{1}{2}-}}{N_2 N_3 N_4} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}} \lesssim N^{-1+}\delta^{\frac{1}{2}-} N_{\max}^{0-} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}}.$$

The contribution of $\int_0^\delta E_9$ is estimated if we prove that

$$\begin{aligned} &\int_0^\delta \Big|\frac{m(n_1+n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)}\Big| \widehat{u_1}(n_1,t)\widehat{v_2}(n_2,t)\widehat{u_3}(n_3,t)\widehat{v_4}(n_4,t) \\ &\lesssim N^{-2+}\delta^{\frac{7}{12}-} \|u_1\|_{X^{1,1/2}} \|v_2\|_{Y^{1,1/2}} \|u_3\|_{X^{1,1/2}} \|v_4\|_{Y^{1,1/2}}. \end{aligned} \tag{4.27}$$

This follows since at least two frequencies are bigger than $N/3$ and the multiplier is always bounded by $(N_{\max}/N)^{1-}$, so that

$$\begin{aligned} \int_0^\delta E_9 &\lesssim \big(\frac{N_{\max}}{N}\big)^{1-} \|u_1\|_{L^4}\|v_2\|_{L^4}\|u_3\|_{L^4}\|v_4\|_{L^4} \\ &\lesssim \big(\frac{N_{\max}}{N}\big)^{1-} \frac{\delta^{\frac{1}{4}+\frac{1}{3}-}}{N_1N_2N_3N_4} \|u_1\|_{X^{1,1/2}} \|v_2\|_{Y^{1,1/2}} \|u_3\|_{X^{1,1/2}} \|v_4\|_{Y^{1,1/2}} \\ &\lesssim N^{-2+}\delta^{\frac{7}{12}-} \|u_1\|_{X^{1,1/2}} \|v_2\|_{Y^{1,1/2}} \|u_3\|_{X^{1,1/2}} \|v_4\|_{Y^{1,1/2}}. \end{aligned}$$

Now, we treat the term $\int_0^\delta E_{10}$. It is sufficient to prove

$$\begin{aligned} &\int_0^\delta \sum \Big|\frac{m(n_4+n_5+n_6) - m(n_4)m(n_5)m(n_6)}{m(n_4)m(n_5)m(n_6)}\Big| \prod_{j=1}^6 \widehat{u_j}(n_j,t) \\ &\lesssim N^{-2+}\delta^{\frac{1}{2}-} \prod_{j=1}^6 \|u_j\|_{X^1}. \end{aligned} \tag{4.28}$$

This follows easily from the facts that the multiplier is bounded by $(N_{\max}/N)^{3/2}$, at least two frequencies are bigger than $N/3$, say $|n_{i_1}| \geq |n_{i_2}| \gtrsim N$, the Strichartz bound $X^{0,3/8} \subset L^4$ and the inclusion[1] $X^{\frac{1}{2}+} \subset L^\infty_{xt}$. Indeed, if we combine these informations, it is not hard to get

$$\begin{aligned} \int_0^\delta E_{10} &\lesssim \big(\frac{N_{\max}}{N}\big)^{\frac{3}{2}} \frac{1}{N_{i_1}N_{i_2}N_{i_3}N_{i_4}} \delta^{\frac{1}{2}-} \frac{1}{(N_{i_5}N_{i_6})^{1/2-}} \prod_{j=1}^6 \|u_j\|_{X^1} \\ &\lesssim N^{-2+}\delta^{\frac{1}{2}-} N_{\max}^{0-} \prod_{j=1}^6 \|u_j\|_{X^1} \end{aligned}$$

[1] This inclusion is an easy consequence of Sobolev embedding.

For the expression $\int_0^\delta E_{11}$, we use again that the multiplier is bounded by $(N_{\max}/N)^{3/2}$, at least two frequencies are bigger than $N/3$ (say $|n_{i_1}| \geq |n_{i_2}| \gtrsim N$), the Strichartz bounds in lemma 2.1 and the inclusions $X^{\frac{1}{2}+}, Y^{\frac{1}{2}+} \subset L^\infty_{xt}$ to obtain

$$
\begin{aligned}
&\int_0^\delta \sum \Big| \frac{m(n_1+n_2+n_3) - m(n_1)m(n_2)m(n_3)}{m(n_1)m(n_2)m(n_3)} \Big| \prod_{j=1}^4 \widehat{u_j}(n_j,t)\widehat{v_5}(n_5,t) \\
&\lesssim \big(\frac{N_{\max}}{N}\big)^{\frac{3}{2}} \frac{1}{N_{i_1}N_{i_2}N_{i_3}N_{i_4}} \frac{\delta^{\frac{1}{2}-}}{N_{i_5}^{1/2-}} \prod_{j=1}^4 \|u_j\|_{X^1}\|v_5\|_{Y^1} \\
&\lesssim N^{-2+}\delta^{\frac{1}{2}-} \prod_{j=1}^4 \|u_j\|_{X^1}\|v_5\|_{Y^1}.
\end{aligned}
\tag{4.29}
$$

The analysis of $\int_0^\delta E_{12}$ is similar to the $\int_0^\delta E_{11}$. This completes the proof. □

5. Global well-posedness below the energy space

In this section we combine the variant local well-posedness result in proposition 3.1 with the two almost conservation results in the propositions 4.1 and 4.2 to prove the theorem 1.1.

Remark 5.1. Note that the spatial mean $\int_{\mathbb{T}} v(t,x)dx$ is preserved during the evolution (1.1). Thus, we can assume that the initial data v_0 has zero-mean, since otherwise we make the change $w = v - \int_{\mathbb{T}} v_0 dx$ at the expense of two harmless linear terms (namely, $u\int_{\mathbb{T}} v_0 dx$ and $\partial_x v \int_{\mathbb{T}} v_0$).

The definition of the I-operator implies that the initial data satisfies $\|Iu_0\|^2_{H^1} + \|Iv_0\|^2_{H^1} \lesssim N^{2(1-s)}$ and $\|Iu_0\|^2_{L^2} + \|Iv_0\|^2_{L^2} \lesssim 1$. By the estimates (4.4) and (4.8), we get that $|L(Iu_0, Iv_0)| \lesssim N^{1-s}$ and $|E(Iu_0,Iv_0)| \lesssim N^{2(1-s)}$.

Also, any bound for $L(Iu,Iv)$ and $E(Iu,Iv)$ of the form $|L(Iu,Iv)| \lesssim N^{1-s}$ and $|E(Iu,Iv)| \lesssim N^{2(1-s)}$ implies that $\|Iu\|^2_{L^2} \lesssim M$, $\|Iv\|^2_{L^2} \lesssim N^{1-s}$ and $\|Iu\|^2_{H^1} + \|Iv\|^2_{H^1} \lesssim N^{2(1-s)}$.

Given a time T, if we can uniformly bound the H^1-norms of the solution at times $t=\delta$, $t = 2\delta$, etc., the local existence result in proposition 3.1 says that the solution can be extended up to any time interval where such a uniform bound holds. On the other hand, given a time T, if we can interact $T\delta^{-1}$ times the local existence result, the solution exists in the time interval $[0,T]$. So, in view of the propositions 4.1 and 4.2, it suffices to show

$$
(N^{-1+}\delta^{\frac{19}{24}-}N^{3(1-s)} + N^{-2+}\delta^{\frac{1}{2}-}N^{4(1-s)})T\delta^{-1} \lesssim N^{1-s} \tag{5.1}
$$

and

$$
\begin{aligned}
&\big\{(N^{-1+}\delta^{\frac{1}{6}-} + N^{-\frac{2}{3}+}\delta^{\frac{3}{8}-} + N^{-\frac{3}{2}+}\delta^{\frac{1}{8}-})N^{3(1-s)} \\
&+ N^{-1+}\delta^{\frac{1}{2}-}N^{4(1-s)} + N^{-2+}\delta^{\frac{1}{2}-}N^{6(1-s)}\big\}\frac{T}{\delta} \lesssim N^{2(1-s)}
\end{aligned}
\tag{5.2}
$$

At this point, we recall that the proposition 3.1 says that $\delta \sim N^{-\frac{16}{3}(1-s)-}$ if $\beta \neq 0$ and $\delta \sim N^{-8(1-s)-}$ if $\beta = 0$. Hence,

- $\beta \neq 0$. The condition (5.1) holds for

$$
-1 + \frac{5}{24}\frac{16}{3}(1-s) + 3(1-s) < (1-s), \quad \text{i.e. }, s > 19/28
$$

and
$$-2+\frac{1}{2}\frac{16}{3}(1-s)+4(1-s)<(1-s), \quad \text{i.e. } , s>11/17;$$
Similarly, condition (5.2) is satisfied if
$$\begin{gathered}
-1+\frac{5}{6}\frac{16}{3}(1-s)+3(1-s)<2(1-s), \quad \text{i.e. } , s>40/49; \\
-\frac{2}{3}+\frac{5}{6}\frac{16}{3}(1-s)+3(1-s)<2(1-s), \quad \text{i.e. } , s>11/13; \\
-\frac{3}{2}+\frac{7}{8}\frac{16}{3}(1-s)+3(1-s)<2(1-s), \quad \text{i.e. } , s>25/34; \\
-1+\frac{1}{2}\frac{16}{3}(1-s)+4(1-s)<2(1-s), \quad \text{i.e. } , s>11/14; \\
-2+\frac{1}{2}\frac{16}{3}(1-s)+6(1-s)<2(1-s), \quad \text{i.e. } , s>7/10.
\end{gathered}$$
Thus, we conclude that the non-resonant NLS-KdV system is globally well-posed for any $s>11/13$.

• $\beta=0$. Condition (5.1) is fulfilled when
$$-1+\frac{5}{24}8(1-s)+3(1-s)<(1-s), \quad \text{i.e. } , s>8/11$$
and
$$-2+\frac{1}{2}8(1-s)+4(1-s)<(1-s), \quad \text{i.e. } , s>5/7;$$
Similarly, the condition (5.2) is verified for
$$\begin{gathered}
-1+\frac{5}{6}8(1-s)+3(1-s)<2(1-s), \quad \text{i.e. } , s>20/23; \\
-\frac{2}{3}+\frac{5}{6}8(1-s)+3(1-s)<2(1-s), \quad \text{i.e. } , s>8/9; \\
-\frac{3}{2}+\frac{7}{8}8(1-s)+3(1-s)<2(1-s), \quad \text{i.e. } , s>13/16; \\
-1+\frac{1}{2}8(1-s)+4(1-s)<2(1-s), \quad \text{i.e. } , s>5/6; \\
-2+\frac{1}{2}8(1-s)+6(1-s)<2(1-s), \quad \text{i.e. } , s>3/4.
\end{gathered}$$
Hence, we obtain that the resonant NLS-KdV system is globally well-posed for any $s>8/9$.

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