Electronic Journal of Differential Equations, Vol. 2007(2007), No. 14, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

BOUNDARY-VALUE PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATIONS WITH MATRIX COEFFICIENTS CONTAINING A SPECTRAL PARAMETER

MOHAMED DENCHE, AMARA GUERFI

Abstract. In the present work, we study a multi-point boundary-value problem for an ordinary differential equation with matrix coefficients containing a spectral parameter in the boundary conditions. Assuming some regularity conditions, we show that the characteristic determinant has an infinite number of zeros, and specify their asymptotic behavior. Using the asymptotic behavior of Green matrix on contours expending at infinity, we establish the series expansion formula of sufficiently smooth functions in terms of residuals solutions to the given problem. This formula actually gives the completeness of root functions as well as the possibility of calculating the coefficients of the series.

1. Introduction

We study a multi-point boundary-value problem

$$y' - \lambda A(x,\lambda) y = f(x), \quad -\infty < a \le x \le b < \infty, \tag{1.1}$$

$$L(y) = \sum_{k=0}^{P} \lambda^k \big(\alpha^{(k)} y(a,\lambda) + \beta^{(k)} y(b,\lambda)\big) = 0, \tag{1.2}$$

with

$$A(x,\lambda) = \sum_{j=0}^{\infty} \lambda^{-j} A_j(x)\,,$$

where $\lambda \gg 1$, $A_j(x)$, $(j = 0, 1, \dots)$, $\alpha^{(k)}, \beta^{(k)}$ are matrices of order $n \times n$, $f(x)$ is a vector function of order n, which is continuous (or integrable bounded) in $[a, b]$.

The study of the boundary-value problem (1.1)–(1.2) in the case of ordinary differential equations originates in the papers by Birkhoff [2], [3]. Later, Tamarkin [12] considered the same problem, under more general hypothesis, and introduced the classes of regular and strongly regular problems.

We note that boundary-value problems with a parameter in the boundary conditions have interesting applications, since many concrete problems of mathematical physics (e.g., [13]) lead to problems of this form. This happens whenever one applies

2000 *Mathematics Subject Classification.* 34L10, 34E05, 47E05.
Key words and phrases. Characteristic determinant; expansion formula; Green matrix; regularity conditions.

Submitted August 15, 2006. Published Juanary 8, 2007.

the method of separation of variables to solve the corresponding partial differential equation with boundary conditions, which contain a directional derivative.

In general, the spectral properties of (1.1)–(1.2) are mainly determined not only by the boundary conditions, but also by the highest coefficients of all the polynomials in λ, $A(x,\lambda)$. Hence, for the same boundary conditions, but different matrix functions $A(x,\lambda)$, the problems can be both regular and non regular.

Various questions connected with the theory of ordinary differential operators have been studied intensively; see for example [4, 6, 7, 8, 9, 10, 11, 15, 16, 17, 18, 19]. Problems of the form (1.1)–(1.2), for the case of first order systems, have been studied in [8], where the results by Tamarkin [12] were generalized. In this paper, we consider more general problems of the form (1.1)–(1.2), where regularity conditions are taken to be more general than those in [6, 8], and coincide with the one in [9], when the coefficients of the equation are independent of λ. Using the asymptotic behavior of the system of fundamental solutions of equation (1.1) given in [14], we formulate the regularity notion of the problem (1.1)–(1.2). For the introduced regular problems, we show that the characteristic determinant $\Delta(\lambda)$ has an infinite number of zeros. We establish that in the exterior of δ-neighboring of those zeros the elements of the Green matrix have the uniform estimate $G_{pq}(x,\xi,\lambda)=0(1)$. Using this estimate on the contours which expand at infinity, we obtain the series expansion formula of sufficiently smooth functions in terms of solutions residuals to the given problem. In fact, this formula gives the completeness of root functions as well as the possibility of calculating the coefficients of the series.

2. Preliminaries

Suppose that:

(1) $A_j(x)$ belongs to $C[a,b]$ for $j=0,1,\dots$.
(2) For $x\in[a,b]$, the roots $\varphi_1(x),\dots,\varphi_n(x)$ of the characteristic equation in the sense of Birkhoff [8]
$$\det(A_0(x)-\varphi E)=0, \tag{2.1}$$
are distinct, not identically zero, their arguments and the arguments of their differences are independent of x.

Let $M(x)$ be a matrix which transforms $A_0(x)$ to the diagonal matrix $D(x)$ i.e.
$$M^{-1}(x)A_0(x)M(x)=D(x)=\operatorname{diag}(\varphi_1(x),\dots,\varphi_n(x)).$$
We require that at least one of the matrix $M'(x)$, $A_1(x)$ belong to the Holder space H_α

(3) For $|\lambda|$ sufficiently large, the following matrix has rank $n\times 2np$:
$$\begin{pmatrix} \alpha_{11}^{(1)}\dots\alpha_{1n}^{(1)} & \dots & \alpha_{11}^{(P)}\dots\alpha_{1n}^{(P)} & \beta_{11}^{(1)}\dots\beta_{1n}^{(1)} & \dots & \beta_{11}^{(P)}\dots\alpha_{1n}^{(P)}\\ \vdots & & & & & \vdots\\ \alpha_{n1}^{(1)}\dots\alpha_{nn}^{(1)} & \dots & \alpha_{n1}^{(P)}\dots\alpha_{nn}^{(P)} & \beta_{n1}^{(1)}\dots\beta_{nn}^{(1)} & \dots & \beta_{n1}^{(P)}\dots\beta_{nn}^{(P)}\end{pmatrix}$$

We first start by giving the notion of sectors that we need later on. For this purpose, we consider the set of values λ that satisfy
$$\operatorname{Re}\lambda\varphi_k(x)=\operatorname{Re}\lambda\varphi_s(x)\quad k\neq s\ x\in[a,b]. \tag{2.2}$$

This equality determines a finite number of sectors (Σ_j) for which by a convenable numeration of zeros of (2.1), we have the inequalities

$$\operatorname{Re}\lambda\varphi_1(x) \le \operatorname{Re}\lambda\varphi_2(x) \le \dots \le \operatorname{Re}\lambda\varphi_n(x).$$

Consider now the set of values λ satisfying

$$\operatorname{Re}\lambda\omega_s = 0\ , \quad s = \overline{1,n}\ , \tag{2.3}$$

where $\omega_s = \int_a^b \varphi_s(t)dt$. By condition 2 the equalities (2.3) define a certain number of straight lines coming through the origin of λ-plane, and each is applied by the origin into two straight-half lines through this origin. We denote them by $d_1, d_2, \dots, d_{2\mu}$, and the argument of d_j by $-\alpha_j + \frac{\pi}{2}$, where α_j are numerated as follows:

$$0 \le \alpha_1 < \alpha_2 < \dots < \alpha_{2\mu} < 2\pi.$$

Consider a second set of straight half lines d'_j $(j = \overline{1, 2\mu})$ distributed as

$$d'_1, d_1, d'_2, d_2, \dots, d'_{2\mu}, d_{2\mu}, d'_1.$$

The rays d'_j divided the λ-plane into 2μ sectors $T_1, T_2, \dots, T_{2\mu}$. Let us consider an arbitrary T_j. Let $\omega_{1j}, \dots, \omega_{\upsilon_j j}$ be taken from the numbers $\omega_1, \dots, \omega_p$, which are situated on a straight line issued from the origin and making an angle α_j with the real axis:

$$\omega_{sj} = \mu_{sj} e^{\alpha_j \sqrt{-1}}, \quad s = \overline{1, \nu_j}.$$

In addition, we can always choose a numeration of the numbers ω_{sj} such that we have the following inequalities hold:

$$\mu_{1j} < \mu_{2j} < \dots < \mu_{s_j j} < 0 < \mu_{s_j+1 j} < \dots < \mu_{\nu_j j}.$$

If all μ_{sj} are strictly positive, then we put $s_j = 0$. Otherwise, if μ_{sj} are strictly negative, then $s_j = \nu_j$.

After excluding ω_{sj} $(\overline{1, \nu_j})$ from the set $\{\omega_1, \dots, \omega_n\}$, the remaining ω_s can be divided into two groups: $(\omega_s^{(1)}), (\omega_s^{(2)})$. The first group is formed by those one for which $\operatorname{Re}\lambda\omega_s \to -\infty$, whereas the second group those for $\operatorname{Re}\lambda\omega_s \to +\infty$. Hence, in each (T_j) the roots of equation (2.1) are numerated as

$$\omega_1^{(1)}, \dots, \omega_{\kappa_j}^{(1)}, \omega_{1j}, \dots, \omega_{\nu_j j}, \omega_{\kappa_j+\nu_j+1}^{(2)}, \dots, \omega_n^{(2)}.$$

The boundaries of the sectors (Σ_j) and (T_j) divide the whole λ-plane into a finite number of sectors (R_j), where each of those is simultaneously situated in one of the sectors (T_j) and in one of the sectors (Σ_j). So, in (R_j) we have

$$\begin{aligned} \operatorname{Re}\lambda\varphi_1(x) \le \operatorname{Re}\lambda\varphi_2(x) \le \dots \le \operatorname{Re}\lambda\varphi_{\tau_j}(x) \le 0 \\ \le \operatorname{Re}\lambda\varphi_{\tau_{j+1}}(x) \le \dots \le \operatorname{Re}\lambda\varphi_n(x), \end{aligned}$$

where $\tau_j = \kappa_j + s_j$.

Definition 2.1. A sequence of curves Γ_ν in the λ-plane is called an expanding sequence, if there is a constant K such that, for $\lambda \in \Gamma_\nu$ and all positive integer ν, the inequalities $\operatorname{meas}\Gamma_\nu \le K r_\nu$, and $|d\lambda| \le r'_\nu d\theta$ hold, where r_ν is the distance from the origin of λ-plane to the nearest point of the Γ_ν, r'_ν is the largest distance between points of curve Γ_ν, and $d\theta$ is the angle subtended by the chord $d\lambda$ at the origin.

Lemma 2.2 ([15])**.** *Let $\xi(\lambda,z,x)$ be a continuous function defined in the half-plane $\operatorname{Re} c\lambda \leq 0$, with c constant not equal to zero, $x\in[a,b]$, $z\in(0,Z)$. Suppose that*

$$|\xi(\lambda,z,x)| \leq c/|\lambda|^{\alpha}\ ,\quad \alpha>\frac{1}{2},\quad |\lambda|\gg 1.$$

Let $\psi(z)$ be a bounded function. Then

$$J(\psi)=\int_0^Z \psi(z)dz\int_{\Gamma_\nu}\xi(\lambda,z,x)e^{c\lambda Z}d\lambda$$

tends to zero uniformly with respect to $x\in[a,b]$, as ν approaches infinity on the contour Γ_ν (where Γ_ν is an expanding sequence situated in the half-plane $\operatorname{Re} c\lambda\leq 0$).

3. Main Results

Construction of the Green Matrix. The Green matrix of problem (1.1)–(1.2) is

$$G(x,\xi,\lambda)=g(x,\xi,\lambda)-y^0(x,\lambda)U^{-1}(\lambda)L(g(x,\xi,\lambda))\,,$$

where

$$G(x,\xi,\lambda)=\big(G_{pq}(x,\xi,\lambda)\big)_{p,q=1}^n,\quad U(\lambda)=L(y^0(x,\lambda)=\big(U_{pq}(\lambda)\big)_{p,q=1}^n,$$
$$L(g(x,\xi,\lambda))=\big(L_{pq}(g(x,\xi,\lambda))\big)_{p,q=1}^n,$$

$y^0(x,\lambda)$ is the solution of the homogeneous equation (1.1), and

$$G_{pq}(x,\xi,\lambda)=\frac{\Delta_{pq}(x,\xi,\lambda)}{\Delta(\lambda)},$$

where

$$\Delta_{pq}(x,\xi,\lambda)=\det\begin{pmatrix} g_{pq}(x,\xi,\lambda) & y^0_{p1}(x,\lambda) & \dots & y^0_{pn}(x,\lambda)\\ L_{1q}(g) & U_{11}(\lambda) & \dots & U_{1n}(\lambda)\\ \vdots & \vdots & & \vdots\\ L_{nq}(g) & U_{n1}(\lambda) & \dots & U_{nn}(\lambda)\end{pmatrix},$$

$$g_{pq}(x,\xi,\lambda)=\begin{cases}\frac{1}{2}\sum_{s=1}^n y^0_{pq}(x,\lambda)Z_{sq}(\xi,\lambda) & \text{if } a\leq\xi\leq x\leq b\\ -\frac{1}{2}\sum_{s=1}^n y^0_{pq}(x,\lambda)Z_{sq}(\xi,\lambda) & \text{if } a\leq x\leq\xi\leq b,\end{cases}$$

$Z(x,\lambda)=T(x,\lambda)/W(x,\lambda)$, where $T(x,\lambda)$ is the matrix of order $n\times n$ when we take the transposed of the matrix made up using the co-factors of the elements of the matrix $y^0(x,\lambda)$, and $W(x,\lambda)=\det y^0(x,\lambda)$,

$$L_{pq}(g(x,\xi,\lambda))=\sum_{s=1}^n\sum_{k=0}^P\lambda^k\big(\alpha_{ps}^{(k)}g_{sq}(a,\xi,\lambda)+\beta_{ps}^{(k)}g_{sq}(b,\xi,\lambda)\big),$$
$$U_{pq}(\lambda)=\sum_{s=1}^n\sum_{k=0}^P\lambda^k\big(\alpha_{ps}^{(k)}y^0_{sq}(a,\lambda)+\beta_{ps}^{(k)}y^0_{sq}(b,\lambda)\big),$$

where

$$\Delta(\lambda)=\det U(\lambda) \tag{3.1}$$

is the characteristic determinant of problem (1.1)–(1.2). Thus, the general solution of problem (1.1)–(1.2) is

$$y(x,\lambda,f)=\int_a^b G(x,\xi,\lambda)f(\xi)d\xi,$$

for $x \in [a,b]$.

Asymptotic Representation of the Zeros of the Characteristic Determinant. According to the Vagabov theorem [14], the fundamental system of solutions for the homogeneous equation corresponding to (1.1), have in each sector (Σ_j) the asymptotic behavior

$$y^0(x,\lambda) = \Big(M(x) + 0(\frac{1}{|\lambda|^\alpha})\Big)\exp\Big(\lambda\int_a^x D(\xi)d\xi\Big), \tag{3.2}$$

where $0 < \alpha \le 1$, $x \in [a,b]$, and $M(x) = (M_{pq}(x))_{p,q=1}^n$ is one of the matrix indicated in condition 2. Using the notation

$$\widehat{\Phi}(x) = \Phi(x) + 0(\frac{1}{|\lambda|^\alpha}),$$

and substituting (3.2) from the boundary conditions (1.2), we obtain

$$U_{pq}(\lambda) = A_{pq}(\lambda) + B_{pq}(\lambda)e^{\lambda\omega_q}\,, \quad p,q = \overline{1,n}, \tag{3.3}$$

where

$$A_{pq}(\lambda) = \sum_{s=1}^n\sum_{k=0}^P \lambda^k\alpha_{ps}^{(k)}\widehat{M}_{sq}(a)\,, \tag{3.4}$$

and

$$B_{pq}(\lambda) = \sum_{s=1}^n\sum_{k=0}^P \lambda^k\beta_{ps}^{(k)}\widehat{M}_{sq}(b). \tag{3.5}$$

On the other hand, if we denote

$$A^{(q)} = \begin{pmatrix} A_{1q}\\ \vdots \\ A_{nq}\end{pmatrix}, \quad B^{(q)} = \begin{pmatrix} B_{1q}\\ \vdots \\ B_{nq}\end{pmatrix},$$

then $\Delta(\lambda)$ can be written in the form

$$\Delta(\lambda) = \det\big(A^{(1)} + B^{(1)}e^{\lambda\omega_1} \quad \dots \quad A^{(n)} + B^{(n)}e^{\lambda\omega_n}\big)\,. \tag{3.6}$$

Using (3.1), (3.3), (3.4), and (3.5) we conclude from (3.6) that the following asymptotic relations hold:

$$\Delta(\lambda)e^{-\lambda\sum_{s=\kappa_j+\nu_j+1}^n \omega_s^{(2)}} = \widehat{M}_{1j}(\lambda)e^{m_{1j}Z} + \dots + \widehat{M}_{\sigma_j j}(\lambda)e^{m_{\sigma_j j}Z}\,, \tag{3.7}$$

where $m_{1j} < m_{2j} < \dots < m_{\sigma_j j}$, $Z = \lambda e^{\exp(\alpha_j\sqrt{-1})}$, and

$$m_{1j} = \begin{cases}\sum_{s=1}^{s_j}\mu_{sj} & \text{for } s_j > 0\\ 0 & \text{for } s_j = 0,\end{cases} \quad m_{\sigma_j j} = \begin{cases}\sum_{s=s_j+1}^{\nu_j}\mu_{sj} & \text{for } s_j < \nu_j\\ 0 & \text{for } s_j = \nu_j,\end{cases}$$

$$M_{1j}(\lambda) = \det\Big(A^{(1)}\dots A^{(\kappa_j)}B^{(\kappa_j+1)}\dots B^{(\kappa_j+s_j)}A^{(\kappa_j+s_j+1)}\dots \\ A^{(\kappa_j+\nu_j)}B^{(\kappa_j+\nu_j+1)}\dots B^{(n)}\Big),$$

$$M_{\sigma_j j} = \det\big(A^{(1)} \quad \dots \quad A^{(\kappa_j+s_j)}B^{(\kappa_j+s_j+1)} \quad \dots \quad B^{(n)}\big)\,.$$

Definition 3.1. A function $f(\lambda)$ is called an asymptotic power function of degree κ, if there exist $a \in \mathbb{C}\backslash\{0\}$, $0 < \alpha \le 1$ and $\kappa \in \mathbb{Z}$ such that

$$f(\lambda) = \lambda^\kappa\Big(a + 0(\frac{1}{|\lambda|^\alpha})\Big), \quad |\lambda| \to \infty.$$

A similar definition is given in [1] and [5] for $\alpha = 1$.

Definition 3.2 (Regularity). The boundary-value problem (1.1)–(1.2) is said to be regular if in all sectors R_j, the functions $M_{1j}(\lambda)$ are asymptotic power functions of degree κ where κ is a positive integer, and all the other determinants built by different columns of the matrix $(A^{(1)}\dots A^{(n)}B^{(1)}\dots B^{(n)})$ are asymptotic power functions of degree $\le \kappa$.

Theorem 3.3. *Suppose that the boundary-value problem* (1.1)–(1.2) *is regular, and the conditions 1, 2, 3, of section 2 are satisfied, then in each sector* (T_j) *we have*

(1) $\Delta(\lambda)$ *admits an infinite number of zeros which can be divided into* 2μ *groups. The values of* $j^{th}-$ *group are contained in the strip* (D_j) *of finite width and parallel to rays* d_j *which is inside* (D_j)*.*
(2) *If the interiors of circles of sufficiently small radius* δ *with centers at zeros of* $\Delta(\lambda)$ *are removed, then in the remained plane, we get*

$$\big|\lambda^{-\kappa}\Delta(\lambda)\exp\big(-\lambda\sum_{s=\kappa_j+\nu_j+1}^{n}\omega_s^{(2)}\big)\big| \ge k_\delta\,,$$

where k_δ *is a positive number depending only on* δ*.*
(3) *The number of zeros of* $\Delta(\lambda)$ *which are near to the origin is finite. The zeros* $\lambda_N^{(j)}$ *of* j^{th}*-group have the asymptotic representation*

$$|\lambda_N^{(j)}| = \frac{2N\pi}{m_{\sigma_j j} - m_{1j}}\big(1 + 0\big(\frac{1}{N}\big)\big).$$

(4) *Each zero of* $\Delta(\lambda)$ *is a pole of the solution of problem* (1.1)*)*–(1.2)*.*

The proof of this theorem can be done as in [9, Theorem 4, page 205].

Asymptotic Representation of a Solution of Boundary Value Problem (1.1)–(1.2). According to condition 2 of section 2, the root arguments of the characteristic equation (2.1) are independent of x. So, we can write

$$\varphi_s(x) = \pi_s q_s(x), \quad x \in [a,b],\ s = \overline{1,n}\,,$$

where $\pi_s$is in general a complex constant, $q_s(x) > 0$, hence from (3.2) it results

$$\operatorname{Re}\lambda\pi_1 \le \operatorname{Re}\lambda\pi_2 \le \dots \le \operatorname{Re}\lambda\pi_{\tau_j} \le 0 \le \operatorname{Re}\lambda\pi_{\tau_{j+1}} \le \dots \le \operatorname{Re}\lambda\pi_n\,. \tag{3.8}$$

Let us set

$$x_s = \int_a^x q_s(t)dt, \quad \xi_s = \int_a^\xi q_s(t)dt, \quad x_{0s} = \int_a^b q_s(x)dx.$$

By appropriate transformations, the matrix $G(x,\xi,\lambda)$ can be written, in each sector $R_j(\delta)$ (where $R_j(\delta)$ denotes the remaining part of sector R_j after removing the interior of the circle of sufficiently small rays δ centered in the zeros of $\Delta(\lambda)$), in the following form

$$
\begin{aligned}
G_{pq}(x,\xi,\lambda) = g^0_{pq}(x,\xi,\lambda) + \Big(& \sum_{l=1}^{\tau_j} \sum_{s=\tau_j+1}^{n} P_{ls}(\lambda)\widehat{M}_{pl}(x)\widehat{V}_{sq}(\xi) e^{\lambda\pi_l x_l - \lambda\pi_s\xi_s} \\
& + \sum_{l=\tau_j+1}^{n} \sum_{s=\tau_j+1}^{n} P_{ls}(\lambda)\widehat{M}_{pl}(x)\widehat{V}_{sq}(\xi) e^{\lambda\pi_l (x_l - x_{0l}) - \lambda\pi_s\xi_s} \\
& + \sum_{l=1}^{\tau_j} \sum_{s=1}^{\tau_j} Q_{ls}(\lambda)\widehat{M}_{pl}(x)\widehat{V}_{sq}(\xi) e^{\lambda\pi_l x_l - \lambda\pi_s(\xi_s - x_{0s})} \\
& + \sum_{l=\tau_j+1}^{n} \sum_{s=1}^{\tau_j} Q_{ls}(\lambda)\widehat{M}_{pl}(x)\widehat{V}_{sq}(\xi) e^{\lambda\pi_l (x_l - x_{0l}) - \lambda\pi_s(\xi_s - x_{0s})} \Big),
\end{aligned}
\tag{3.9}
$$

where

$$
P_{ls}(\lambda) = \begin{cases} \dfrac{\lambda^{-\kappa} e^{-\lambda W} \sum_{m=1}^{n} A_{ms}(\lambda)\Delta_{ml}(\lambda)}{\lambda^{-\kappa} e^{-\lambda W}\Delta(\lambda)} & \text{if } l \le \tau_j \\ \dfrac{\lambda^{-\kappa} e^{-\lambda W + \lambda\pi_l x_{0l}} \sum_{m=1}^{n} A_{ms}(\lambda)\Delta_{ml}(\lambda)}{\lambda^{-\kappa} e^{-\lambda W}\Delta(\lambda)} & \text{if } l \ge \tau_j + 1 \end{cases}
\tag{3.10}
$$

$$
Q_{ls}(\lambda) = \begin{cases} \dfrac{\lambda^{-\kappa} e^{-\lambda W} \sum_{m=1}^{n} B_{ms}(\lambda)\Delta_{ml}(\lambda)}{\lambda^{-\kappa} e^{-\lambda W}\Delta(\lambda)} & \text{if } l \le \tau_j \\ \dfrac{\lambda^{-\kappa} e^{-\lambda W + \lambda\pi_l x_{0l}} \sum_{m=1}^{n} B_{ms}(\lambda)\Delta_{ml}(\lambda)}{\lambda^{-\kappa} e^{-\lambda W}\Delta(\lambda)} & \text{if } l \ge \tau_j + 1, \end{cases}
\tag{3.11}
$$

where

$$
g^0_{pq}(x,\xi,\lambda) = \begin{cases} \sum_{s=1}^{\tau_j} \widehat{M}_{ps}(x)\widehat{V}_{sq}(\xi) e^{\lambda\pi_s(x_s - \xi_s)} & \text{if } a \le \xi \le x \le b \\ -\sum_{s=\tau_j+1}^{n} \widehat{M}_{ps}(x)\widehat{V}_{sq}(\xi) e^{\lambda\pi_s(x_s - \xi_s)} & \text{if } a \le x \le \xi \le b, \end{cases}
\tag{3.12}
$$

$W = \sum_{s=\kappa_j+\nu_j+1}^{n} \omega_s^{(2)}$, the $V_{sq}(\xi)$ is the element of the matrix $V(x)$ which verifies $M(x)V(x) = I$, $\Delta_{ms}(\lambda)$ is the complement algebraic of the element (m,s) in $\Delta(\lambda)$.

Theorem 3.4. *Suppose that the boundary-value problem* (1.1)–(1.2) *is regular, and the conditions 1, 2, 3, of section 2 are satisfied. Then, in each sector $R_j(\delta)$ the elements $G_{pq}(x,\xi,\lambda)$ of the Green matrix admits the estimate*

$$
G_{pq}(x,\xi,\lambda) = 0(1)\,. \tag{3.13}
$$

Proof. Numerators in (3.10), (3.11) are bounded in $R_j(\delta)$ for large λ. It follows from Theorem 3.3 that the denominators are bounded below by a positive number in $R_j(\delta)$. In other words, the functions $P_{ls}(\lambda)$ and $Q_{ls}(\lambda)$ are uniformly bounded outside δ-neighborhoods of the zeros. Then (3.13) follows directly from (3.9)–(3.12). □

An Expansion Formula.

Theorem 3.5. *If the boundary-value problem* (1.1)–(1.2) *is regular, the Holder power satisfies $\frac{1}{2} < \alpha \le 1$, and the conditions 1, 2, 3, of section 2, are satisfied, then for all $f(x) \in L_2[a,b]$, the following expansion formula holds in the sense of $L_2[a,b]$:*

$$
\frac{-1}{2\pi\sqrt{-1}} \sum_{\nu} \int_{\Gamma_\nu} y(x,\lambda,f)d\lambda = \sum_{\nu} \operatorname{Res} y(x,\lambda,f) = D^{-1}(x)f(x)\,, \tag{3.14}
$$

where Γ_ν is a simple closed contour containing only one pole λ_ν of the integrand, and the sum over ν is extended to all poles of this function. Here, $\operatorname{Res}_{z_\nu} F(z)$ *denotes the residual of $F(z)$ at z_ν.*

Proof. Theorem 3.3 implies that the distance between the zeros of $\Delta(\lambda)$ is larger than some sufficiently small positive number 2δ. Then, we can choose a sequence of closed expanding contours Γ_ν, which does not intersect circles of radius δ centered at the zeros of $\Delta(\lambda)$. Since each Γ_ν is the union of its parts in the sectors R_j, we can conclude from (3.9), that

$$\begin{aligned}
&\int_{\Gamma_\nu} d\lambda \sum_{q=1}^{n} \int_a^b G_{pq}(x,\xi,\lambda) f_q(\xi) d\xi \\
&= \sum_j \int_{\Gamma_\nu \cap R_j} d\lambda \Big(\sum_{q=1}^{n} \int_a^b g^0_{pq}(x,\xi,\lambda) f_q(\xi) d\xi \\
&\quad + \sum_{q=1}^{n} \int_a^b \Big(\sum_{l=1}^{\tau_j} \sum_{s=\tau_j+1}^{n} P_{ls}(\lambda) \widehat{M}_{pl}(x) \widehat{V}_{sq}(\xi) e^{\lambda\pi_l x_l - \lambda\pi_s\xi_s)} \\
&\quad + \sum_{l=\tau_j+1}^{n} \sum_{s=\tau_j+1}^{n} P_{ls}(\lambda) \widehat{M}_{pl}(x) \widehat{V}_{sq}(\xi) e^{\lambda\pi_l (x_l - x_{0l}) - \lambda\pi_s\xi_s)} \\
&\quad + \sum_{l=1}^{\tau_j} \sum_{s=1}^{\tau_j} Q_{ls}(\lambda) \widehat{M}_{pl}(x) \widehat{V}_{sq}(\xi) e^{\lambda\pi_l x_l - \lambda\pi_s(\xi_s - x_{0s})} \\
&\quad + \sum_{l=\tau_j+1}^{n} \sum_{s=1}^{\tau_j} Q_{ls}(\lambda) \widehat{M}_{pl}(x) \widehat{V}_{sq}(\xi) e^{\lambda\pi_l (x_l - x_{0l}) - \lambda\pi_s(\xi_s - x_{0s})} \Big) \Big),
\end{aligned} \tag{3.15}$$

here, $\sum_j$ denotes the sum over all R_j. From (3.10)-(3.11), the regularity of problem (1.1)–(1.2) and the choice of Γ_ν, it follows that the $P_{ls}(\lambda)$, $Q_{ls}(\lambda)$ are uniformly bounded on all Γ_ν. Inequalities (3.8) imply that the real parts of all exponents in the right side of (3.15) are non-positive. Using [9, Lemma 1], [9, Lemma 3] and Lemma 2.2, it follows that

$$\begin{aligned}
&\lim_{\nu\to+\infty} \int_{\Gamma_\nu} d\lambda \sum_{q=1}^{n} \int_a^b G_{pq}(x,\xi,\lambda) f_q(\xi) d\xi \\
&= \lim_{\nu\to+\infty} \sum_j \int_{\Gamma_\nu \cap R_j} d\lambda \sum_{q=1}^{n} \int_a^b g^0_{pq}(x,\xi,\lambda) f_q(\xi) d\xi.
\end{aligned} \tag{3.16}$$

By substituting the expression (3.12) into (3.16), and using Lemma 2.2, appropriate transformations yield

$$\sum_\nu \int_{\Gamma_\nu} y(x,\lambda,f) d\lambda = \sum_\nu \operatorname{Res} \int_a^b G(x,\xi,\lambda) f(\xi) d\xi = -2\pi\sqrt{-1} D^{-1}(x) f(x).$$

□

References

[1] H. E. Benzinger,*Green's function for ordinary differential operators,* J. Diff. Equations, **7** (1970), 3, 478-496 .

[2] G. D. Birkhoff,*On the asymptotic character of the solution of certain linear differential equations containing a parameter,* Trans. Am. Math. Soc. **9** (1908), 219-231.
[3] G. D. Birkhoff, *Boundary value and expansion problems of ordinary linear differential equations*, Trans. Am. Math. Soc. **9** (1908), 373-395.
[4] N. Dunford and J.T. Schwartz, *Linear Operators. PART III*, Wiley 1971.
[5] W. Eberhard, *G. Freling, Ston-regular eigenwert problem,* Math. Z, **160** (1978), 2, 139-161.
[6] M. A. Naimark, *Linear differential operators*. UNGAR, New-York 1967.
[7] A. P. Khromov, *Finite-Dimensional Perturbations of Volterra operators*, Journal of Mathematical Sciences, **138** (2006), 5, 5894-6066.
[8] M. L. Rassulov, *Methods of Contour Integration*. Amesterdam, North-Holland Pubshing Company 1967.
[9] M. L. Rassulov, *Application of Residuals Method for the Solution of Problems for Differential Equations* (in Russian). Elm AZSSR 1989.
[10] A. A. Shkalikov, *Boundary value problems for ordinary differential equation with a parameter in the boundary conditions*, J. Soviet. Math. **33** (1986), 1311-1342.
[11] A. A. Shkalikov, *Boundary value problems with the spectral parameter in the boundary conditions*, Z. Angew. Mech. **76**, Suppl. 2 (1996), 233-235.
[12] Ya. D. Tamarkin, *General Problems of the Theory of Ordinary Differential Equations and Expansions of Arbitrary Functions in Series* (in Russian), Petrograd 1917.
[13] A. N. Tikhonov and A. A. Samarskii, *Equations of Mathematical Physics* (in Russian). Gostekhizdat, Moscow 1953.
[14] A. I. Vagabov, *Asymptotics of solutions of differential equations with respect to a parameter and applications*, (in Russian), Acad. sci. Dokl. Math. **46** (1993), 2, 240-244.
[15] A. I. Vagabov, *Introduction to Spectral Theory of Differential Operators* (in Russian). Rostov na Danou 1994.
[16] A. I. Vagabov, V. S. Galyaev, *On the completeness of root functions of pencils of linear ordinary differential operators with general boundary conditions* (in Russian), Differ. Uravn. **40** (2004), 1, 5–14, 141; translation in Differ. Equ. **40** (2004), 1, 3–13.
[17] A. I. Vagabov, *Regularity conditions for a pencil of ordinary differential operators of general type* (in Russian), Differ. Uravn. 40 (2004), no. 3, 299–309, 429; translation in Differ. Equ. **40** (2004), 3, 318–328.
[18] A. I. Vagabov, V. R. Ragimkhanov, *(C,1)-Summability of Fourier Series in Root Functions of Ordinary Linear Differential Operators in a space of Vector Functions* (in Russian), Differ. Uravn. **41** (2005), 8, 1037-1045; translation in Differ. Equ. **41** (2005), 8, 1087-1096.
[19] S. Ya. Yakubov, *Completeness of Root Functions of Regular Differential Operators*, Longman, 1994.

Mohamed Denche
Laboratoire Equations Differentielles, Departement de Mathematiques, Faculté des Sciences, Université Mentouri, Constantine, 25000 Constantine, Algeria
E-mail address: `denech@wissal.dz`

Amara Guerfi
Department of Mathematics and Computer engineering, Faculty of Science and Engineering, University of Ouargla, 30000 Ouargla, Algeria
E-mail address: `amaraguerfi@yahoo.fr`