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EXISTENCE AND UNIQUENESS OF GLOBAL SOLUTIONS TO A MODEL FOR THE FLOW OF AN INCOMPRESSIBLE, BAROTROPIC FLUID WITH CAPILLARY EFFECTS

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Abstract. We study the initial-value problem for a system of nonlinear equations that models the flow of an inviscid, incompressible, barotropic fluid with capillary stress effects. We prove the global-in-time existence of a unique, classical solution to this system of equations, with a small initial velocity gradient. The key to the proof lies in using an L^2 estimate for the density ρ, and using the smallness of the initial velocity gradient, to obtain uniqueness for the density.

1. Introduction

In this paper, we consider equations which arise from a model of the multi-dimensional flow of an incompressible, barotropic fluid with capillary stresses. When viscosity is neglected, these equations reduce to the following system, written in terms of the density ρ, the pressure p, and velocity $\mathbf{v}$:

$$\frac{D\mathbf{v}}{Dt} + \rho^{-1}\nabla p = c\nabla\Delta\rho, \tag{1.1}$$

$$\nabla\cdot\mathbf{v} = 0. \tag{1.2}$$

Here c is a coefficient of capillarity which is a small, positive constant, and the material derivative $D/Dt = \partial/\partial t + \mathbf{v}\cdot\nabla$. The term $c\nabla\Delta\rho$ arises from capillary stresses, as described in the theory of Korteweg-type materials developed by Dunn and Serrin [4]. The fluid's thermodynamic state is determined by the density ρ. The pressure p is determined from the density by an equation of state $p = \hat{p}(\rho)$. In related work, the existence of a solution to a similar system of equations for the case of viscous fluid flow, which also includes a hyperbolic equation for density and a parabolic equation for temperature, has been proven by Hattori and Li [7, 8], and by Bresch, Desjardins, and Lin [2]. Anderson, McFadden and Wheeler [1] have given a review of related theories and applications to diffuse-interface modelling.

In this model, it will be shown that $\partial\rho/\partial t$, $\nabla\rho$, and $\nabla\mathbf{v}$ are small, for suitable initial data. Although the conservation of mass equation is only approximately satisfied, the model equations (1.1), (1.2) might be useful as an approximation in

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the case of almost constant density, and nearly incompressible fluid flow with a small velocity gradient. We write equation (1.1) equivalently as

$$\frac{D\mathbf{v}}{Dt} + \rho^{-1}\hat{p}'(\rho)\nabla\rho = c\nabla\Delta\rho \tag{1.3}$$

The purpose of this paper is to prove the existence of a unique, global-in-time, classical solution $\mathbf{v}$, ρ to equations (1.2), (1.3) with suitable initial velocity data $\mathbf{v}_0 \in H^s$, under periodic boundary conditions. That is, we choose for our domain the N-dimensional torus $\mathbb{T}^N$, where $N = 2$ or $N = 3$. The proof of the existence theorem is based on the method of successive approximations, in which an iteration scheme, based on solving a linearized version of the equations, is designed and convergence of the sequence of approximating solutions to a unique solution satisfying the nonlinear equations is sought. The framework of the proof follows one used, for example, by A. Majda for proving the existence of a solution to a system of conservation laws [10]. Embid [5] also uses the same general framework to prove the existence of a solution to equations for zero Mach number combustion. Under this framework, the convergence proof is presented in two steps. In the first step, we prove uniform boundedness of the approximating sequence of solutions in a high Sobolev norm. The second step is to prove contraction of the sequence in a low Sobolev norm. Standard compactness arguments will be used to finish the proof. The key to the proof lies in using an L^2 estimate for the density ρ, and using the smallness of the initial velocity gradient, to obtain uniqueness for the density.

2. A priori estimates

The main tools utilized in the existence proof are a priori estimates. We will work with the Sobolev space $H^s(\Omega)$ (where $s \geq 0$ is an integer) of real-valued functions in $L^2(\Omega)$ whose distribution derivatives up to order s are in $L^2(\Omega)$, with norm given by $\|f\|_s^2 = \sum_{|\alpha|\leq s}\int_\Omega |D^\alpha f|^2 dx$ and inner product $(f,g)_s = \sum_{|\alpha|\leq s}\int_\Omega (D^\alpha f)\cdot(D^\alpha g)dx$. Here, we adopt the standard multi-index notation. For convenience, we will denote derivatives by $f_\alpha = D^\alpha f$. We will let Df denote the gradient of f. Also, we will denote the L^2 inner product by $(f,g) = \int_\Omega f \cdot g\ dx$. We will use standard function spaces. $L^\infty([0,T], H^s)$ is the space of bounded measurable functions from $[0,T]$ into $H^s(\Omega)$, with the norm $\|f\|_{s,T}^2 = \operatorname{ess\,sup}_{0\leq t\leq T}\|f(t)\|_s^2$. $C([0,T], H^s)$ is the space of continuous functions from $[0,T]$ into $H^s(\Omega)$. The following technical lemmas will be needed for the proof of the existence of a classical solution to the initial-value problem for the system (1.2), (1.3).

Lemma 2.1 (Standard Calculus Inequalities)**.**

(a) *If $f \in H^{s_1}(\Omega)$, $g \in H^{s_2}(\Omega)$ and $s_3 = \min\{s_1, s_2, s_1 + s_2 - s_0\} \geq 0$, where $s_0 = [\frac{N}{2}]+1$, then $fg \in H^{s_3}(\Omega)$, and $\|fg\|_{s_3} \leq C\|f\|_{s_1}\|g\|_{s_2}$. We note that $s_0 = 2$ for $N = 2$ or $N = 3$.*

(b) *If $f \in H^s(\Omega)$, $g \in H^{s-1}(\Omega) \cap L^\infty(\Omega)$, $Df \in L^\infty(\Omega)$, and $|\alpha| \leq s$, then $\|D^\alpha(fg) - fD^\alpha g\|_0 \leq C(|Df|_{L^\infty}\|g\|_{s-1} + |g|_{L^\infty}\|Df\|_{s-1})$.*

In (a) the constant C depends on s_1, s_2, and Ω, while in (b) the constant C depends on s and Ω. These inequalities are well known. Proofs may be found, for example, in [9, 11].

Lemma 2.2 (Low-Norm Commutator Estimate)**.** *If $Df \in H^{r_1}(\Omega)$, $g \in H^{r-1}(\Omega)$, where $r_1 = \max\{r-1, s_0\}$, $s_0 = [\frac{N}{2}]+1$, then for any $r \geq 1$, f, g satisfy the estimate*

$\|D^\alpha(fg) - fD^\alpha g\|_0 \le C\|Df\|_{r_1}\|g\|_{r-1}$, *where* $r = |\alpha|$, *and the constant* C *depends on* r, Ω.

Proof. The proof is based on the Sobolev calculus inequalities from Lemma 2.1. We consider separately the cases $r-1 < s_0$ and $r-1 \ge s_0$, where $r \ge 1$. If $r-1 < s_0$, we expand the term $D^\alpha(fg)$ using the Leibniz rule and then apply inequality (a) from Lemma 2.1 to obtain the desired estimate. If $r-1 \ge s_0$, we apply the inequality (b) from Lemma 2.1 and the Sobolev inequality $|h|_{L^\infty} \le C\|h\|_{s_0}$ for $s_0 = [\frac{N}{2}] + 1$, to obtain the estimate for this case. Combining these two results then completes the proof. □

Lemma 2.3. *If* $\mathbf{u}$, $\mathbf{v}$, a, $\mathbf{f}$, *and* ρ *are sufficiently smooth in*

$$D\mathbf{u}/Dt = -a\nabla\rho + c\nabla\Delta\rho + \mathbf{f},$$
$$\nabla\cdot\mathbf{u} = 0,$$

where $\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x})$, $\nabla\cdot\mathbf{u}_0 = 0$, $\Omega = \mathbb{T}^N$, *where* $N = 2,3$, *and where* c *is a positive constant,* $D/Dt = \partial/\partial t + \mathbf{v}\cdot\nabla$, *and* $\nabla\cdot\mathbf{v} = 0$, *then for any* $r \ge 1$, $D\mathbf{u}$ *and* $\mathbf{u}$ *satisfy the estimates*

$$\|D\mathbf{u}\|_{r-1}^2 \le Ce^{2t}(1+te^{2t}e^{\beta(t)}\|D\mathbf{v}\|_{r_1,T}^2)(\|D\mathbf{u}_0\|_{r-1}^2+\int_0^t(\|\mathbf{f}\|_r^2+\|Da\|_{r_1}^2\|\nabla\rho\|_{r-1}^2)d\tau)$$

and

$$\begin{aligned}\|\mathbf{u}\|_r^2 \le e^{2t}\|\mathbf{u}_0\|_0^2 + Ce^{2t}(1+te^{2t}e^{\beta(t)}\|D\mathbf{v}\|_{r_1,T}^2)(\|D\mathbf{u}_0\|_{r-1}^2 \\ + \int_0^t(\|\mathbf{f}\|_r^2 + \|Da\|_{r_1}^2\|\nabla\rho\|_{r-1}^2)d\tau)\end{aligned}$$

where $\beta(t) = te^{2t}\|D\mathbf{v}\|_{r_1,T}^2$. *Here* $r_1 = \max\{r-1, s_0\}$, *and* $s_0 = [\frac{N}{2}] + 1 = 2$ *for* $N = 2,3$. *The constant* C *depends on* r, Ω.

Proof. First, we obtain an L^2 estimate for $\mathbf{u}$. Let $\bar\rho = \rho - \frac{1}{|\Omega|}\int_\Omega \rho d\mathbf{x}$. We then compute

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|_0^2 &= (\mathbf{u}_t,\mathbf{u})\\ &= -(\mathbf{v}\cdot\nabla\mathbf{u},\mathbf{u}) - (a\nabla\rho,\mathbf{u}) + c(\nabla\Delta\rho,\mathbf{u}) + (\mathbf{f},\mathbf{u})\\ &= \frac{1}{2}(\mathbf{u}\nabla\cdot\mathbf{v},\mathbf{u}) + (a\bar\rho,\nabla\cdot\mathbf{u}) + (\bar\rho\nabla a,\mathbf{u}) - c(\Delta\rho,\nabla\cdot\mathbf{u}) + (\mathbf{f},\mathbf{u}) \quad (2.1)\\ &\le |Da|_{L^\infty}\|\mathbf{u}\|_0\|\bar\rho\|_0 + \|\mathbf{f}\|_0\|\mathbf{u}\|_0\\ &\le \|\mathbf{u}\|_0^2 + \frac{1}{2}|Da|_{L^\infty}^2\|\bar\rho\|_0^2 + \frac{1}{2}\|\mathbf{f}\|_0^2\end{aligned}$$

where we used the facts that $\nabla\cdot\mathbf{u} = 0$, and $\nabla\cdot\mathbf{v} = 0$. And we used Cauchy's inequality. After applying Gronwall's inequality, we obtain the estimate

$$\|\mathbf{u}\|_0^2 \le e^{2t}\|\mathbf{u}_0\|_0^2 + e^{2t}\int_0^t C(|Da|_{L^\infty}^2\|\bar\rho\|_0^2 + \|\mathbf{f}\|_0^2)d\tau, \quad (2.2)$$

Next, we obtain an estimate for $\|D\mathbf{u}\|_{r-1}^2$. After applying the operator $D^{\gamma+\alpha}$ to the equation for $\mathbf{u}$, where $0 \le |\alpha| \le r-1$ and $|\gamma| = 1$, we obtain

$$\frac{D\mathbf{u}_{\gamma+\alpha}}{Dt} = -a\nabla\rho_{\gamma+\alpha} + c\nabla\Delta\rho_{\gamma+\alpha} + \mathbf{F}_{\gamma+\alpha} \quad (2.3)$$

where $\mathbf{F}_{\gamma+\alpha} = \mathbf{f}_{\gamma+\alpha} - [(\mathbf{v}\cdot\nabla\mathbf{u})_{\gamma+\alpha} - \mathbf{v}\cdot\nabla\mathbf{u}_{\gamma+\alpha}] - [(a\nabla\rho)_{\gamma+\alpha} - a\nabla\rho_{\gamma+\alpha}]$. For (2.3), estimate (2.1) becomes

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_{\gamma+\alpha}\|_0^2 \le \|\mathbf{u}_{\gamma+\alpha}\|_0^2 + \frac{1}{2}|Da|_{L^\infty}^2\|\bar{\rho}_{\gamma+\alpha}\|_0^2 + \frac{1}{2}\|\mathbf{F}_{\gamma+\alpha}\|_0^2 \tag{2.4}$$

Next, we estimate $\|\mathbf{F}_{\gamma+\alpha}\|_0^2$. Using the commutator estimate from Lemma 2.2, we obtain

$$\begin{aligned}\|\mathbf{F}_{\gamma+\alpha}\|_0^2 &\le C\|\mathbf{f}_{\gamma+\alpha}\|_0^2 + C\|(\mathbf{v}\cdot\nabla\mathbf{u})_{\gamma+\alpha} - \mathbf{v}\cdot\nabla\mathbf{u}_{\gamma+\alpha}\|_0^2\\ &\quad + C\|(a\nabla\rho)_{\gamma+\alpha} - a\nabla\rho_{\gamma+\alpha}\|_0^2\\ &\le C\|\mathbf{f}\|_k^2 + C\|D\mathbf{v}\|_{k_1}^2\|D\mathbf{u}\|_{k-1}^2 + C\|Da\|_{k_1}^2\|\nabla\rho\|_{k-1}^2\end{aligned} \tag{2.5}$$

where $k = |\gamma+\alpha|$, $k_1 = \max\{k-1, s_0\}$, and $s_0 = [\frac{N}{2}] + 1$. Here, we used the triangle inequality and Cauchy's inequality. Substituting estimate (2.5) into (2.4) yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_{\gamma+\alpha}\|_0^2 &\le \|\mathbf{u}_{\gamma+\alpha}\|_0^2 + \frac{1}{2}|Da|_{L^\infty}^2\|\bar{\rho}_{\gamma+\alpha}\|_0^2 + C\|\mathbf{f}\|_k^2\\ &\quad + C\|D\mathbf{v}\|_{k_1}^2\|D\mathbf{u}\|_{k-1}^2 + C\|Da\|_{k_1}^2\|\nabla\rho\|_{k-1}^2\end{aligned}$$

Applying Gronwall's inequality yields the following:

$$\begin{aligned}\|\mathbf{u}_{\gamma+\alpha}\|_0^2 &\le e^{2t}\|(\mathbf{u}_0)_{\gamma+\alpha}\|_0^2 + Ce^{2t}\int_0^t \Big(|Da|_{L^\infty}^2\|\bar{\rho}_{\gamma+\alpha}\|_0^2 + \|\mathbf{f}\|_k^2\\ &\quad + \|D\mathbf{v}\|_{k_1}^2\|D\mathbf{u}\|_{k-1}^2 + \|Da\|_{k_1}^2\|\nabla\rho\|_{k-1}^2\Big)d\tau\\ &\le Ce^{2t}\|D\mathbf{u}_0\|_{k-1}^2 + Ce^{2t}\int_0^t \Big(|Da|_{L^\infty}^2\|\bar{\rho}\|_k^2 + \|\mathbf{f}\|_k^2\\ &\quad + \|D\mathbf{v}\|_{k_1}^2\|D\mathbf{u}\|_{k-1}^2 + \|Da\|_{k_1}^2\|\nabla\rho\|_{k-1}^2\Big)d\tau,\end{aligned} \tag{2.6}$$

where $|\gamma| = 1$, and $|\gamma+\alpha| = k \ge 1$, and $k_1 = \max\{k-1, s_0\}$, with $s_0 = [\frac{N}{2}] + 1$.

After adding the above inequality (2.6) over all γ, where $|\gamma| = 1$, and then adding over all α, where $0 \le |\alpha| \le r-1$, we obtain the estimate

$$\begin{aligned}\|D\mathbf{u}\|_{r-1}^2 &\le Ce^{2t}\|D\mathbf{u}_0\|_{r-1}^2 + Ce^{2t}\int_0^t \Big(|Da|_{L^\infty}^2\|\bar{\rho}\|_r^2 + \|\mathbf{f}\|_r^2\\ &\quad + \|D\mathbf{v}\|_{r_1,T}^2\|D\mathbf{u}\|_{r-1}^2 + \|Da\|_{r_1}^2\|\nabla\rho\|_{r-1}^2\Big)d\tau,\end{aligned} \tag{2.7}$$

where $r_1 = \max\{r-1, s_0\}$, and where $s_0 = [\frac{N}{2}] + 1 = 2$ for $N = 2, 3$. Here, we used the fact that $\sum_{0\le|\alpha|\le r-1}\sum_{|\gamma|=1}\|\mathbf{u}_{\gamma+\alpha}\|_0^2 = \sum_{0\le|\alpha|\le r-1}\sum_{i=1}^N\int_\Omega|\frac{\partial\mathbf{u}_\alpha}{\partial x_i}|^2 d\mathbf{x} = \sum_{0\le|\alpha|\le r-1}\|D\mathbf{u}_\alpha\|_0^2 = \|D\mathbf{u}\|_{r-1}^2$. After applying Gronwall's inequality to (2.7), we get

$$\begin{aligned}\|D\mathbf{u}\|_{r-1}^2 &\le Ce^{2t}(1 + te^{2t}e^{\beta(t)}\|D\mathbf{v}\|_{r_1,T}^2)(\|D\mathbf{u}_0\|_{r-1}^2\\ &\quad + \int_0^t (|Da|_{L^\infty}^2\|\bar{\rho}\|_r^2 + \|\mathbf{f}\|_r^2 + \|Da\|_{r_1}^2\|\nabla\rho\|_{r-1}^2)d\tau),\end{aligned} \tag{2.8}$$

with $\beta(t) = te^{2t}\|D\mathbf{v}\|^2_{r_1,T}$. Adding the estimates (2.2), (2.8), we obtain

$$\begin{aligned}
\|\mathbf{u}\|_r^2 &\leq (\|\mathbf{u}\|_0^2 + C\|D\mathbf{u}\|_{r-1}^2) \\
&\leq e^{2t}\|\mathbf{u}_0\|_0^2 + Ce^{2t}(1 + te^{2t}e^{\beta(t)}\|D\mathbf{v}\|^2_{r_1,T})(\|D\mathbf{u}_0\|^2_{r-1} \\
&\quad + \int_0^t (|Da|^2_{L^\infty}\|\bar{\rho}\|_r^2 + \|\mathbf{f}\|_r^2 + \|Da\|^2_{r_1}\|\nabla\rho\|^2_{r-1})d\tau) \\
&\leq e^{2t}\|\mathbf{u}_0\|_0^2 + Ce^{2t}(1 + te^{2t}e^{\beta(t)}\|D\mathbf{v}\|^2_{r_1,T})(\|D\mathbf{u}_0\|^2_{r-1} \\
&\quad + \int_0^t (\|\mathbf{f}\|_r^2 + \|Da\|^2_{r_1}\|\nabla\rho\|^2_{r-1})d\tau)
\end{aligned}$$

where $r_1 = \max\{r-1, s_0\}$, and where $s_0 = [\frac{N}{2}] + 1 = 2$ for $N = 2, 3$. Here, we used Poincaré's inequality to estimate $\|\bar{\rho}\|_0^2 \leq C\|\nabla\rho\|_0^2$, and $\|\bar{\rho}\|_r^2 \leq C(\|\nabla\rho\|^2_{r-1} + \|\bar{\rho}\|_0^2) \leq C\|\nabla\rho\|^2_{r-1}$. We also used Sobolev's lemma $|h|_{L^\infty} \leq C\|h\|_{s_0}$ where $s_0 = [\frac{N}{2}] + 1$. Using the estimate $|Da|^2_{L^\infty}\|\bar{\rho}\|_r^2 \leq C\|Da\|^2_{r_1}\|\nabla\rho\|^2_{r-1}$ in the right-hand side of (2.8) completes the proof. □

Lemma 2.4. *Let u, w be C^1 functions on a bounded, open, connected, convex domain Ω. And let $u(\mathbf{x}_0) = w(\mathbf{x}_0)$ at a single, fixed point $\mathbf{x}_0 \in \Omega$. Then $u - w$ and u satisfy the estimates*

$$\begin{gathered}
\|u - w\|_0^2 \leq C\|\nabla(u - w)\|_2^2, \\
\|u\|_0^2 \leq C_0\|w\|_0^2 + C_0\|\nabla w\|_2^2 + C_0\|\nabla u\|_2^2
\end{gathered}$$

Here C, C_0 are constants which depend only on Ω.

Proof. First, we obtain an estimate for $\|u - w\|_0^2$. From the mean value theorem, and since $u - w$ is sufficiently smooth on the convex domain Ω, we have

$$(u - w)(\mathbf{x}) = (u - w)(\mathbf{x}_0) + \nabla(u - w)(\mathbf{x}_*) \cdot (\mathbf{x} - \mathbf{x}_0)$$

where $\mathbf{x}_*$ is a point on the line segment joining the points $\mathbf{x}_0 \in \Omega$ and $\mathbf{x} \in \Omega$.

Since we are given that $u(\mathbf{x}_0) = w(\mathbf{x}_0)$, at a single fixed point $\mathbf{x}_0 \in \Omega$, it follows that

$$(u - w)(\mathbf{x}) = \nabla(u - w)(\mathbf{x}_*) \cdot (\mathbf{x} - \mathbf{x}_0)$$

Taking the absolute value of both sides yields

$$\begin{aligned}
|(u - w)(\mathbf{x})| &= |\nabla(u - w)(\mathbf{x}_*) \cdot (\mathbf{x} - \mathbf{x}_0)| \\
&\leq |\nabla(u - w)|_{L^\infty}|\mathbf{x} - \mathbf{x}_0| \\
&\leq C|\nabla(u - w)|_{L^\infty}
\end{aligned}$$

Here C depends only on Ω. Squaring $|(u-w)(\mathbf{x})|$ and integrating over Ω, and using the above inequality, yields

$$\int_\Omega |(u - w)(\mathbf{x})|^2 d\mathbf{x} \leq C\int_\Omega |\nabla(u - w)|^2_{L^\infty} d\mathbf{x} \leq C\|\nabla(u - w)\|_2^2$$

where we used Sobolev's inequality $|h|_{L^\infty} \leq C\|h\|_{s_0}$, where $s_0 = [\frac{N}{2}] + 1 = 2$ for $N = 2, 3$ and C depends on Ω.

Next, we obtain an estimate for $\|u\|_0^2$. From using the triangle inequality and Cauchy's inequality, and from using the previous estimate for $\|u-w\|_0^2$, we obtain

$$\begin{aligned}\|u\|_0^2 &\le C\|w\|_0^2 + C\|u-w\|_0^2\\ &\le C\|w\|_0^2 + C\|\nabla(u-w)\|_2^2\\ &\le C_0\|w\|_0^2 + C_0\|\nabla w\|_2^2 + C_0\|\nabla u\|_2^2\end{aligned}$$

Here C_0 is a constant which depends only on Ω. □

Lemma 2.5. *If g is a sufficiently smooth function on the domain $\Omega = \mathbb{T}^N$, then g satisfies the estimate $\|\nabla g\|_r^2 \le C\|\Delta g\|_{r-1}^2$ where $r \ge 1$ and C depends on r.*

Proof. First, we integrate $-\Delta g$ by parts with the function $\bar{g} = g - \frac{1}{|\Omega|}\int_\Omega g\,d\mathbf{x}$ over $\Omega = \mathbb{T}^N$, to obtain

$$\begin{aligned}(\nabla g, \nabla g) &= -(\Delta g, \bar{g})\\ &\le C(\epsilon)\|\Delta g\|_0^2 + \epsilon\|\bar{g}\|_0^2\\ &\le C(\epsilon)\|\Delta g\|_0^2 + \epsilon C\|\nabla g\|_0^2\end{aligned}$$

where we used Cauchy's inequality with ϵ. We also used Poincaré's inequality to estimate $\|\bar{g}\|_0^2 \le C\|\nabla g\|_0^2$. Then D^α is applied, yielding $-\Delta g_\alpha$, which is then integrated by parts with the function g_α over $\Omega = \mathbb{T}^N$, to obtain

$$\begin{aligned}(\nabla g_\alpha, \nabla g_\alpha) &= -(\Delta g_\alpha, g_\alpha)\\ &= (\Delta g_{\alpha-\gamma}, g_{\alpha+\gamma})\\ &\le C(\epsilon)\|\Delta g_{\alpha-\gamma}\|_0^2 + \epsilon\|g_{\alpha+\gamma}\|_0^2\\ &\le C(\epsilon)\|\Delta g\|_{k-1}^2 + \epsilon C\|\nabla g\|_k^2\end{aligned}$$

where $|\gamma| = 1$, and $|\alpha| = k \ge 1$. Here, we used Cauchy's inequality with ϵ. Adding these two inequalities, for $|\alpha| = k \le r$, and moving the terms containing ϵ to the left-hand side, completes the proof. □

Lemma 2.6. *If $\mathbf{u}$, $\mathbf{v}$, a, $\mathbf{f}$, and ρ are sufficiently smooth in the equation*

$$\Delta^2\rho = \frac{1}{c}\nabla\cdot(a\nabla\rho) + \frac{1}{c}\nabla\cdot(\mathbf{v}\cdot\nabla\mathbf{u}) - \frac{1}{c}\nabla\cdot\mathbf{f} \tag{2.9}$$

where c is a positive constant, $\nabla\cdot\mathbf{u} = 0$, $\nabla\cdot\mathbf{v} = 0$, $a(\mathbf{x},t) \ge c_1$, with $c_1 > 0$, and where $\Omega = \mathbb{T}^N$, $N = 2,3$, and $r \ge 1$, we obtain the following estimates for $\|\Delta\rho\|_0^2 + \|\nabla\rho\|_0^2$ and for $\|\nabla\rho\|_{r+1}^2$

$$\begin{aligned}\|\Delta\rho\|_0^2 + \|\nabla\rho\|_0^2 &\le C|D\mathbf{v}|_{L^\infty}^2\|D\mathbf{u}\|_0^2 + C\|\mathbf{f}\|_0^2,\\ \|\nabla\rho\|_{r+1}^2 &\le C(\|\Delta\rho\|_r^2 + \|\nabla\rho\|_r^2)\\ &\le C\|Da\|_{r_1}^2\|\nabla\rho\|_{r-1}^2 + C\|\mathbf{f}\|_{r-1}^2 + C\|D\mathbf{v}\|_{r_2+1}^2\|D\mathbf{u}\|_{r-1}^2\end{aligned}$$

where $r_1 = \max\{r-1, s_0\}$, $r_2 = \max\{r-2, s_0\}$, with $s_0 = [\frac{N}{2}] + 1$, and $N = 2,3$.

Proof. First, we obtain an L^2 estimate. Integrating equation (2.9) by parts with $\bar\rho$, where $\bar\rho = \rho - \frac{1}{|\Omega|}\int_\Omega \rho d\mathbf{x}$, yields

$$\begin{aligned}
(\Delta\rho, \Delta\rho) = (\Delta^2\rho, \bar\rho) &= \frac{1}{c}(\nabla\cdot(a\nabla\rho), \bar\rho) + \frac{1}{c}(\nabla\cdot(\mathbf{v}\cdot\nabla\mathbf{u}), \bar\rho) - \frac{1}{c}(\nabla\cdot\mathbf{f}, \bar\rho) \\
&= -\frac{1}{c}(a\nabla\rho, \nabla\rho) + \frac{1}{c}(\nabla\mathbf{v}^T : \nabla\mathbf{u}, \bar\rho) + \frac{1}{c}(\mathbf{f}, \nabla\rho) \\
&\le -\frac{c_1}{c}\|\nabla\rho\|_0^2 + C(\epsilon)|D\mathbf{v}|_{L^\infty}^2\|D\mathbf{u}\|_0^2 + \epsilon\|\bar\rho\|_0^2 + C(\epsilon)\|\mathbf{f}\|_0^2 + \epsilon\|\nabla\rho\|_0^2 \\
&\le -\frac{c_1}{c}\|\nabla\rho\|_0^2 + C(\epsilon)|D\mathbf{v}|_{L^\infty}^2\|D\mathbf{u}\|_0^2 + \epsilon\|\nabla\rho\|_0^2 + C(\epsilon)\|\mathbf{f}\|_0^2 + \epsilon\|\nabla\rho\|_0^2
\end{aligned}$$

where we used Cauchy's inequality with ϵ, and where we used the fact that $a(\mathbf{x}, t) \ge c_1$ where $c_1 > 0$. We also used Poincaré's inequality to estimate $\|\bar\rho\|_0^2 \le C\|\nabla\rho\|_0^2$. And we used the fact that $\nabla\cdot(\mathbf{v}\cdot\nabla\mathbf{u}) = \nabla\mathbf{v}^T : \nabla\mathbf{u}$, because $\nabla\cdot\mathbf{u} = 0$. Moving the $\|\nabla\rho\|_0^2$ terms to the left-hand side yields the estimate

$$\|\Delta\rho\|_0^2 + \|\nabla\rho\|_0^2 \le C|D\mathbf{v}|_{L^\infty}^2\|D\mathbf{u}\|_0^2 + C\|\mathbf{f}\|_0^2 \tag{2.10}$$

Next, after applying D^α to the equation (2.9), we obtain the equation:

$$\Delta^2\rho_\alpha = \frac{1}{c}\nabla\cdot(a\nabla\rho_\alpha) + \frac{1}{c}\nabla\cdot(\mathbf{v}\cdot\nabla\mathbf{u})_\alpha + F_\alpha \tag{2.11}$$

where $F_\alpha = -\frac{1}{c}\nabla\cdot\mathbf{f}_\alpha + \frac{1}{c}[\nabla\cdot(a\nabla\rho)_\alpha - \nabla\cdot(a\nabla\rho_\alpha)]$. Integrating equation (2.11) by parts with ρ_α, when $|\alpha| \ge 1$, yields

$$\begin{aligned}
&(\Delta\rho_\alpha, \Delta\rho_\alpha) = (\Delta^2\rho_\alpha, \rho_\alpha) \\
&= \frac{1}{c}(\nabla\cdot(a\nabla\rho_\alpha), \rho_\alpha) + \frac{1}{c}(\nabla\cdot(\mathbf{v}\cdot\nabla\mathbf{u})_\alpha, \rho_\alpha) + (F_\alpha, \rho_\alpha) \\
&= -\frac{1}{c}(a\nabla\rho_\alpha, \nabla\rho_\alpha) + \frac{1}{c}((\nabla\mathbf{v}^T : \nabla\mathbf{u})_\alpha, \rho_\alpha) + (F_\alpha, \rho_\alpha) \\
&= -\frac{1}{c}(a\nabla\rho_\alpha, \nabla\rho_\alpha) - \frac{1}{c}((\nabla\mathbf{v}^T : \nabla\mathbf{u})_{\alpha-\gamma}, \rho_{\alpha+\gamma}) + (F_\alpha, \rho_\alpha) \\
&\le -\frac{c_1}{c}(\nabla\rho_\alpha, \nabla\rho_\alpha) + C\|(\nabla\mathbf{v}^T : \nabla\mathbf{u})_{\alpha-\gamma}\|_0\|\rho_{\alpha+\gamma}\|_0 + |(F_\alpha, \rho_\alpha)| \\
&\le -\frac{c_1}{c}(\nabla\rho_\alpha, \nabla\rho_\alpha) + C(\epsilon)\|(\nabla\mathbf{v}^T : \cdot\nabla\mathbf{u})_{\alpha-\gamma}\|_0^2 + \epsilon\|\rho_{\alpha+\gamma}\|_0^2 + |(F_\alpha, \rho_\alpha)|
\end{aligned} \tag{2.12}$$

where $|\gamma| = 1$. Here, we used Cauchy's inequality with ϵ, and we used the fact that $a(\mathbf{x}, t) \ge c_1$ where $c_1 > 0$. Next, we estimate the terms on the right-hand side of the above inequality. First, we estimate $\|(\nabla\mathbf{v}^T : \nabla\mathbf{u})_{\alpha-\gamma}\|_0^2$. When $|\alpha| = 1$, we choose $\gamma = \alpha$ and obtain

$$\|(\nabla\mathbf{v}^T : \nabla\mathbf{u})_{\alpha-\gamma}\|_0^2 = \|(\nabla\mathbf{v}^T : \nabla\mathbf{u})\|_0^2 \le C|D\mathbf{v}|_{L^\infty}^2\|D\mathbf{u}\|_0^2 \tag{2.13}$$

Next, we use the triangle inequality and the commutator estimate from Lemma 2.2 to estimate $\|(\nabla\mathbf{v}^T : \nabla\mathbf{u})_{\alpha-\gamma}\|_0^2$, when $|\alpha| > |\gamma| = 1$, obtaining

$$\begin{aligned}
\|(\nabla\mathbf{v}^T : \nabla\mathbf{u})_{\alpha-\gamma}\|_0^2 &\le C(\|\nabla\mathbf{v}^T : \nabla\mathbf{u}_{\alpha-\gamma}\|_0^2 + \|(\nabla\mathbf{v}^T : \nabla\mathbf{u})_{\alpha-\gamma} - \nabla\mathbf{v}^T : \nabla\mathbf{u}_{\alpha-\gamma}\|_0^2) \\
&\le C|D\mathbf{v}|_{L^\infty}^2\|D\mathbf{u}_{\alpha-\gamma}\|_0^2 + C\|D^2\mathbf{v}\|_{k_2}^2\|D\mathbf{u}\|_{k-2}^2 \\
&\le C\|D\mathbf{v}\|_{k_2+1}^2\|D\mathbf{u}\|_{k-1}^2,
\end{aligned} \tag{2.14}$$

Here $|\gamma| = 1$, $k = |\alpha|$, and $k_2 = \max\{k-2, s_0\}$, with $s_0 = [\frac{N}{2}] + 1$. Here, we also used the Sobolev inequality $|h|_{L^\infty} \le C\|h\|_{s_0}$.

Next, we estimate the following term from the right-hand side of (2.12), obtaining

$$\epsilon\|\rho_{\alpha+\gamma}\|_0^2 \le \epsilon C\|\nabla\rho\|_k^2 \tag{2.15}$$

where $|\gamma| = 1$ and $|\alpha| = k$.

Next, we use the commutator estimate from Lemma 2.2 to estimate $|(F_\alpha, \rho_\alpha)|$ from the right-hand side of (2.12), obtaining

$$\begin{aligned} |(F_\alpha,\rho_\alpha)| &\le |\frac{1}{c}(\nabla\cdot\mathbf{f}_\alpha,\rho_\alpha)| + |\frac{1}{c}([\nabla\cdot(a\nabla\rho)_\alpha - \nabla\cdot(a\nabla\rho_\alpha)],\rho_\alpha)| \\ &= |\frac{1}{c}(\mathbf{f}_{\alpha-\gamma},\nabla\rho_{\alpha+\gamma})| + |\frac{1}{c}([(a\nabla\rho)_\alpha - a\nabla\rho_\alpha],\nabla\rho_\alpha)| \\ &\le C\|\mathbf{f}\|_{k-1}\|\nabla\rho\|_{k+1} + C\|Da\|_{k_1}\|\nabla\rho\|_{k-1}\|\nabla\rho\|_k \\ &\le C(\epsilon)\|\mathbf{f}\|_{k-1}^2 + \epsilon\|\nabla\rho\|_{k+1}^2 + C(\epsilon)\|Da\|_{k_1}^2\|\nabla\rho\|_{k-1}^2 + \epsilon\|\nabla\rho\|_k^2 \end{aligned} \tag{2.16}$$

where $|\gamma| = 1$, $k = |\alpha|$, and $k_1 = \max\{k-1, s_0\}$, with $s_0 = [\frac{N}{2}] + 1$. Again, we used Cauchy's inequality with ϵ. Substituting (2.13)-(2.16) into (2.12), and adding (2.12) over $|\alpha| = k \le r$, including the L^2 estimate (2.10), we obtain for $r \ge 1$ the estimate

$$\begin{aligned} \|\Delta\rho\|_r^2 + \|\nabla\rho\|_r^2 &\le C\|Da\|_{r_1}^2\|\nabla\rho\|_{r-1}^2 + C\|\mathbf{f}\|_{r-1}^2 + C\|D\mathbf{v}\|_{r_2+1}^2\|D\mathbf{u}\|_{r-1}^2 \\ &\quad + \epsilon C\|\nabla\rho\|_{r+1}^2 + \epsilon C\|\nabla\rho\|_r^2, \end{aligned} \tag{2.17}$$

where $r_1 = \max\{r-1, s_0\}$, $r_2 = \max\{r-2, s_0\}$, with $s_0 = [\frac{N}{2}] + 1$. Here, we also used the Sobolev inequality $|h|_{L^\infty} \le C\|h\|_{s_0}$.

From Lemma 2.5, we have $\epsilon\|\nabla\rho\|_{r+1}^2 \le \epsilon C\|\Delta\rho\|_r^2$. After substituting this estimate into the right-hand side of (2.17), we obtain the estimate

$$\|\Delta\rho\|_r^2 + \|\nabla\rho\|_r^2 \le C\|Da\|_{r_1}^2\|\nabla\rho\|_{r-1}^2 + C\|\mathbf{f}\|_{r-1}^2 + C\|D\mathbf{v}\|_{r_2+1}^2\|D\mathbf{u}\|_{r-1}^2, \tag{2.18}$$

where we have moved the terms $\epsilon C\|\Delta\rho\|_r^2$ and $\epsilon C\|\nabla\rho\|_r^2$ to the left-hand side. Finally, using the estimate for $\|\nabla\rho\|_{r+1}^2$ from Lemma 2.5, we obtain from (2.18) the estimate

$$\begin{aligned} \|\nabla\rho\|_{r+1}^2 &\le C(\|\Delta\rho\|_r^2 + \|\nabla\rho\|_r^2) \\ &\le C\|Da\|_{r_1}^2\|\nabla\rho\|_{r-1}^2 + C\|\mathbf{f}\|_{r-1}^2 + C\|D\mathbf{v}\|_{r_2+1}^2\|D\mathbf{u}\|_{r-1}^2 \end{aligned}$$

□

Lemma 2.7. *If* $\mathbf{u}$, $\mathbf{v}$, a, $\mathbf{f}$, *and* ρ *are sufficiently smooth in*

$$\Delta^2\rho = \frac{1}{c}\nabla\cdot(a\nabla\rho) + \frac{1}{c}\nabla\cdot(\mathbf{v}\cdot\nabla\mathbf{u}) - \frac{1}{c}\nabla\cdot\mathbf{f} \tag{2.19}$$

where c *is a positive constant,* $\nabla\cdot\mathbf{u} = 0$, $\nabla\cdot\mathbf{v} = 0$, $a(\mathbf{x},t) \ge c_1$, *with* $c_1 > 0$, *and* $\Omega = \mathbb{T}^N$, $N = 2, 3$, *then for* $r \ge 1$, ρ *satisfies the following estimate*

$$\begin{aligned} \|\nabla\rho\|_{r+1}^2 &\le C\left(\|\Delta\rho\|_r^2 + \|\nabla\rho\|_r^2\right) \\ &\le C\Big[1 + \sum_{j=1}^r \|Da\|_{r_1}^{2j}\Big](\|\mathbf{f}\|_{r-1}^2 + \|D\mathbf{v}\|_{r_2+1}^2\|D\mathbf{u}\|_{r-1}^2) \end{aligned}$$

where $r_1 = \max\{r-1, s_0\}$, $r_2 = \max\{r-2, s_0\}$, *with* $s_0 = [\frac{N}{2}] + 1$, *and* C *depends on* r.

Proof. From Lemma 2.6 applied to equation (2.19), we have the estimate

$$\|\Delta\rho\|_s^2 + \|\nabla\rho\|_s^2 \le C\|Da\|_{s_1}^2\|\nabla\rho\|_{s-1}^2 + C\|\mathbf{f}\|_{s-1}^2 + C\|D\mathbf{v}\|_{s_2+1}^2\|D\mathbf{u}\|_{s-1}^2 \tag{2.20}$$

where $s \ge 1$, and where $s_1 = \max\{s-1, s_0\}$, $s_2 = \max\{s-2, s_0\}$, with $s_0 = [\frac{N}{2}]+1$. Letting $s = r$ in the estimate (2.20) yields

$$\|\Delta\rho\|_r^2 + \|\nabla\rho\|_r^2 \le C\|Da\|_{r_1}^2\|\nabla\rho\|_{r-1}^2 + C\|\mathbf{f}\|_{r-1}^2 + C\|D\mathbf{v}\|_{r_2+1}^2\|D\mathbf{u}\|_{r-1}^2 \tag{2.21}$$

Applying the estimate (2.20), letting $s = r-1$, to the term $C\|Da\|_{r_1}^2\|\nabla\rho\|_{r-1}^2$ which appears on the right-hand side of (2.21) yields

$$\begin{aligned} \|\Delta\rho\|_r^2 + \|\nabla\rho\|_r^2 &\le C\|Da\|_{r_1}^2\Big[\|Da\|_{r_2}^2\|\nabla\rho\|_{r-2}^2 + \|\mathbf{f}\|_{r-2}^2 + \|D\mathbf{v}\|_{r_3+1}^2\|D\mathbf{u}\|_{r-2}^2\Big] \\ &\quad + C\|\mathbf{f}\|_{r-1}^2 + C\|D\mathbf{v}\|_{r_2+1}^2\|D\mathbf{u}\|_{r-1}^2 \\ &\le C\|Da\|_{r_1}^4\|\nabla\rho\|_{r-2}^2 + C\|Da\|_{r_1}^2\Big(\|\mathbf{f}\|_{r-2}^2 + \|D\mathbf{v}\|_{r_2+1}^2\|D\mathbf{u}\|_{r-2}^2\Big) \\ &\quad + C\|\mathbf{f}\|_{r-1}^2 + C\|D\mathbf{v}\|_{r_2+1}^2\|D\mathbf{u}\|_{r-1}^2 \end{aligned} \tag{2.22}$$

where $r_1 = \max\{r-1, s_0\}$, $r_2 = \max\{r-2, s_0\}$, and $r_3 = \max\{r-3, s_0\}$, $r_3 \le r_2 \le r_1$, with $s_0 = [\frac{N}{2}] + 1 = 2$ for $N = 2, 3$. Similarly, by repeatedly applying the estimate (2.20), letting $s = r - j$, to the term $C\|Da\|_{r_1}^{2j}\|\nabla\rho\|_{r-j}^2$, for $j = 2, 3, \dots, r-1$, which will appear on the right-hand side of (2.22) yields

$$\begin{aligned} \|\Delta\rho\|_r^2 + \|\nabla\rho\|_r^2 &\le C\sum_{j=1}^{r-1}\|Da\|_{r_1}^{2j}\Big(\|\mathbf{f}\|_{r-1-j}^2 + \|D\mathbf{v}\|_{r_2+1}^2\|D\mathbf{u}\|_{r-1-j}^2\Big) \\ &\quad + C\|Da\|_{r_1}^{2r}\|\nabla\rho\|_0^2 + C\|\mathbf{f}\|_{r-1}^2 + C\|D\mathbf{v}\|_{r_2+1}^2\|D\mathbf{u}\|_{r-1}^2 \\ &\le C\Big[1 + \sum_{j=1}^{r-1}\|Da\|_{r_1}^{2j}\Big]\Big(\|\mathbf{f}\|_{r-1}^2 + \|D\mathbf{v}\|_{r_2+1}^2\|D\mathbf{u}\|_{r-1}^2\Big) \\ &\quad + C\|Da\|_{r_1}^{2r}\|\nabla\rho\|_0^2 \end{aligned} \tag{2.23}$$

Substituting the estimate for $\|\nabla\rho\|_0^2$ from Lemma 2.6 into the right-hand side of (2.23) yields

$$\begin{aligned} \|\Delta\rho\|_r^2 + \|\nabla\rho\|_r^2 &\le C\big[1 + \sum_{j=1}^{r-1}\|Da\|_{r_1}^{2j}\Big]\Big(\|\mathbf{f}\|_{r-1}^2 + \|D\mathbf{v}\|_{r_2+1}^2\|D\mathbf{u}\|_{r-1}^2\Big) \\ &\quad + C\|Da\|_{r_1}^{2r}\Big(\|\mathbf{f}\|_0^2 + |D\mathbf{v}|_{L^\infty}^2\|D\mathbf{u}\|_0^2\Big) \\ &\le C\Big[1 + \sum_{j=1}^{r}\|Da\|_{r_1}^{2j}\Big]\Big(\|\mathbf{f}\|_{r-1}^2 + \|D\mathbf{v}\|_{r_2+1}^2\|D\mathbf{u}\|_{r-1}^2\Big) \end{aligned}$$

This completes the proof. □

3. Existence theorem

In this section, we prove the existence of a unique classical solution to the initial-value problem for equations (1.2), (1.3), on any given time interval $0 \le t \le T$, with periodic boundary conditions, for sufficiently small initial velocity gradient.

Theorem 3.1. *Suppose $s > \frac{N}{2} + 3$ and $\Omega = \mathbb{T}^N$, $N = 2, 3$. For any given time interval $0 \le t \le T$, equations (1.2), (1.3) have a unique classical solution ρ, $\mathbf{v}$, for initial data $\mathbf{v}_0(\mathbf{x}) \in H^s(\Omega)$, $\nabla \cdot \mathbf{v}_0 = 0$, and for given data $\rho^0(\mathbf{x}, t)$ and $\mathbf{x}_0$, provided $D\mathbf{v}_0$ is sufficiently small. Here $\rho^0(\mathbf{x}, t)$ is a given positive function with sufficiently small gradient $\nabla\rho^0$, and $\mathbf{x}_0$ is a point in the domain Ω. The regularity of the solution is*

$$\begin{aligned} \rho &\in C([0,T], C^5) \cap L^\infty([0,T], H^{s+2}), \\ \mathbf{v} &\in C([0,T], C^3) \cap L^\infty([0,T], H^s), \\ \frac{\partial \mathbf{v}}{\partial t} &\in C([0,T], C^2) \cap L^\infty([0,T], H^{s-1}). \end{aligned}$$

and $\rho(\mathbf{x}, t) \ge c_1$, and $\rho^{-1}\hat{p}'(\rho)(\mathbf{x}, t) \ge c_1$, for some positive constant c_1, for $\mathbf{x} \in \Omega$, and $0 \le t \le T$.

Proof. We will construct the solution of the problem for (1.2), (1.3), with $\Omega = \mathbb{T}^N$, through an iteration scheme. To define the iteration scheme, we will let the sequence of approximate solutions be $\mathbf{v}^k$ and ρ^k. Set $\mathbf{v}^0(\mathbf{x}, t) = \mathbf{v}_0(\mathbf{x})$, the initial velocity data, and set $\rho^0(\mathbf{x}, t) \in C([0,T], C^5) \cap L^\infty([0,T], H^{s+2})$ to be a positive function satisfying $\|\nabla\rho^0\|_{s+1,T} \le \|D\mathbf{v}_0\|_{s-1}$. For $k = 0, 1, 2, \dots$, construct $\mathbf{v}^{k+1}$, ρ^{k+1} from the previous iterates $\mathbf{v}^k$, ρ^k by solving the linear system of equations

$$\frac{D^k \mathbf{v}^{k+1}}{Dt} + (\rho^k)^{-1}\hat{p}'(\rho^k)\nabla\rho^{k+1} = c\nabla\Delta\rho^{k+1}, \tag{3.1}$$

$$\nabla \cdot \mathbf{v}^{k+1} = 0, \tag{3.2}$$

where $D^k/Dt = \partial/\partial t + \mathbf{v}^k \cdot \nabla$, and with initial data $\mathbf{v}^{k+1}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x})$. Since the solution ρ^{k+1} is unique up to an arbitrary function of t, we specify that the solution ρ^{k+1} satisfy $\rho^{k+1}(\mathbf{x}_0, t) = \rho^k(\mathbf{x}_0, t)$ at a single, fixed point $\mathbf{x}_0 \in \Omega$, for all $k \ge 0$. Hence $\rho^{k+1}(\mathbf{x}_0, t) = \rho^0(\mathbf{x}_0, t)$.

Existence of a sufficiently smooth solution to equations (3.1), (3.2) for fixed k follows from the results stated in Section 4. We proceed now to prove convergence of the iterates as $k \to \infty$ to a unique classical solution of (1.2), (1.3). We assume that p is a sufficiently smooth function of the thermodynamic state variable ρ in an open interval $G \subset \mathbb{R}$. We fix connected, bounded open sets G_0 and G_1 such that $\bar{G}_0 \subset G_1$ and $\bar{G}_1 \subset G$, and we require that the initial iterate ρ^0 satisfies $\rho^0 \in G_0$. We fix $\delta = \hat{\delta}(G_0, G_1)$ so that $0 < \delta < \operatorname{dist}(\bar{G}_0, \partial G_1)$; therefore if $|\rho^k - \rho^0|_{L^\infty} \le \delta$, then $\rho^k(\mathbf{x}, t) \in G_1$ for all $\mathbf{x} \in \mathbb{T}^N$, and for $t \in [0,T]$. The values of $\rho^k \in G_1$ are assumed to be strictly positive, bounded, and bounded away from zero. And the values of $(\rho^k)^{-1}\hat{p}'(\rho^k)$ for $\rho^k \in G_1$ are also assumed to be strictly positive, bounded, and bounded away from zero. Using a proof by induction on k, we assume that $\rho^k \in G_1$, and then later we will show that $\rho^{k+1} \in G_1$. First, we proceed with the proof of uniform boundedness of the approximating sequence in a high Sobolev norm.

Proposition 3.2. *Assume that the hypotheses of Theorem 3.1 hold. Let ρ^0, $\mathbf{v}_0$ satisfy $C_0\|\rho^0\|_0^2 + C_0\|\nabla\rho^0\|_2^2 \le L_0^2$ and $e^{2T}\|\mathbf{v}_0\|_0^2 \le L_0^2$, where L_0 is a constant and where C_0 is the constant from Lemma 2.4. There are constants ϵ_0, L_1, L_2, L_3, where $0 < \epsilon_0 < 1$, such that the following estimates hold for $k = 1, 2, 3 \dots$, provided $\|D\mathbf{v}_0\|_{s-1}$ is sufficiently small:*

(a) $\|\nabla\rho^k\|_{s+1,T}^2 \le \epsilon_0$, $\|\rho^k\|_{s+2,T} \le L_1$,

(b) $\|D\mathbf{v}^k\|_{s-1,T}^2 \le \epsilon_0$, $\|\mathbf{v}^k\|_{s,T} \le L_2$,
(c) $\|\partial\mathbf{v}^k/\partial t\|_{s-1,T}^2 \le \epsilon_0 L_3$.

Proof. The proof is by induction on k. We show only the inductive step. We will derive estimates for ρ^{k+1} and $\mathbf{v}^{k+1}$, and then use these estimates to prescribe ϵ_0, L_1, L_2, L_3 a priori, independent of k, so that if ρ^k and $\mathbf{v}^k$ satisfy the estimates in (a)-(c), then ρ^{k+1} and $\mathbf{v}^{k+1}$ also satisfy the same estimates. And we will show that ϵ_0 will be small if $\|D\mathbf{v}_0\|_{s-1}$ is sufficiently small. In the estimates below, we use C to denote a generic constant whose value may change from one instance to the next, but is independent of ϵ_0, L_1, L_2, and L_3.

Estimate for $\nabla\rho^{k+1}$: Applying the divergence operator to equation (3.1), we obtain

$$c\Delta^2\rho^{k+1} - \nabla\cdot((\rho^k)^{-1}\hat{p}'(\rho^k)\nabla\rho^{k+1}) = \nabla\cdot(\mathbf{v}^k\cdot\nabla\mathbf{v}^{k+1}) \tag{3.3}$$

where we used the fact that $\nabla\cdot\mathbf{v}^{k+1} = 0$, and where c is a positive constant. Applying Lemma 2.7 we obtain from the above equation the following estimate for $\nabla\rho^{k+1}$, for $s > \frac{N}{2} + 3$

$$\begin{aligned}\|\nabla\rho^{k+1}\|_{s+1}^2 &\le C(\|\Delta\rho^{k+1}\|_s^2 + \|\nabla\rho^{k+1}\|_s^2)\\ &\le C[1 + \sum_{j=1}^{s}\|D((\rho^k)^{-1}\hat{p}'(\rho^k))\|_{s_1}^{2j}]\|D\mathbf{v}^k\|_{s_2+1}^2\|D\mathbf{v}^{k+1}\|_{s-1}^2\\ &\le C_1\|D\mathbf{v}^k\|_{s-1}^2\|D\mathbf{v}^{k+1}\|_{s-1}^2\end{aligned} \tag{3.4}$$

where $C_1 = \hat{C}_1(L_1)$. Here, $s_1 = \max\{s-1, s_0\} = s-1$, $s_2 = \max\{s-2, s_0\} = s-2$, where $s_0 = [\frac{N}{2}] + 1 = 2$, and $s > \frac{N}{2} + 3$, so $s \ge 5$ for $N = 2, 3$. And we used the induction hypothesis for ρ^k, $\mathbf{v}^k$.

Estimate for ρ^{k+1}: To obtain an L^2 estimate for ρ^{k+1}, we apply Lemma 2.4, where we define $u = \rho^{k+1}$ and we define $w = \rho^0$, to be used for the functions u, w appearing in Lemma 2.4. Note that since by hypothesis we have $\rho^{k+1}(\mathbf{x}_0, t) = \rho^0(\mathbf{x}_0, t)$ at a single fixed point $\mathbf{x}_0 \in \Omega$, the hypotheses of Lemma 2.4 are satisfied, and so we obtain the estimate

$$\begin{aligned}\|\rho^{k+1}\|_0^2 &\le C_0\|\rho^0\|_0^2 + C_0\|\nabla\rho^0\|_2^2 + C_0\|\nabla\rho^{k+1}\|_2^2\\ &\le L_0^2 + C_2\|D\mathbf{v}^k\|_{s_0+1}^2\|D\mathbf{v}^{k+1}\|_1^2\\ &\le L_0^2 + C_2\|D\mathbf{v}^k\|_{s-1}^2\|D\mathbf{v}^{k+1}\|_1^2\end{aligned} \tag{3.5}$$

where C_0 is the constant from Lemma 2.4 (recall C_0 depends only on Ω), and where $C_2 = \hat{C}_2(L_1)$. Here we used the estimate for $\|\Delta\rho^{k+1}\|_2^2 + \|\nabla\rho^{k+1}\|_2^2$ from applying Lemma 2.7 to (3.3) with $r = 2$. We also used the hypothesis that $C_0\|\rho^0\|_0^2 + C_0\|\nabla\rho^0\|_2^2 \le L_0^2$. Adding the estimates (3.4), (3.5) yields the following estimate for ρ^{k+1},

$$\begin{aligned}\|\rho^{k+1}\|_{s+2}^2 &\le \|\rho^{k+1}\|_0^2 + C\|\nabla\rho^{k+1}\|_{s+1}^2\\ &\le L_0^2 + C_3\|D\mathbf{v}^k\|_{s-1}^2\|D\mathbf{v}^{k+1}\|_{s-1}^2\end{aligned} \tag{3.6}$$

where $C_3 = \hat{C}_3(L_1)$.

Estimate for $\mathbf{v}^{k+1}$: Applying Lemma 2.3 to equations (3.1), (3.2), we obtain for $D\mathbf{v}^{k+1}$ the estimate

$$\begin{aligned}
\|D\mathbf{v}^{k+1}\|_{s-1}^2 &\le Ce^{2T}(1+Te^{2T}e^{\beta(T)}\|D\mathbf{v}^k\|_{s-1,T}^2)(\|D\mathbf{v}_0\|_{s-1}^2 \\
&\quad + \int_0^t (\|D((\rho^k)^{-1}\hat{p}'(\rho^k))\|_{s_1}^2 \|\nabla\rho^{k+1}\|_{s-1}^2)d\tau) \\
&\le Ce^{2T}(1+Te^{2T}e^{Te^{2T}})(\|D\mathbf{v}_0\|_{s-1}^2 + C_4\int_0^t \|\nabla\rho^{k+1}\|_{s-1}^2 d\tau) \qquad (3.7)\\
&\le C_5\|D\mathbf{v}_0\|_{s-1}^2 + C_5\int_0^t \|D\mathbf{v}^k\|_{s-1,T}^2\|D\mathbf{v}^{k+1}\|_{s-1}^2 d\tau \\
&\le C_5\|D\mathbf{v}_0\|_{s-1}^2 + C_5\int_0^t \|D\mathbf{v}^{k+1}\|_{s-1}^2 d\tau
\end{aligned}$$

where $\beta(T) = Te^{2T}\|D\mathbf{v}^k\|_{s-1,T}^2$. Here, we used (3.4) to estimate $\|\nabla\rho^{k+1}\|_{s-1}^2$, and we used the fact that by the induction hypothesis, $\|D\mathbf{v}^k\|_{s-1,T}^2 \le \epsilon_0$, where $0<\epsilon_0<1$. And here $C_4 = \hat{C}_4(L_1,T)$, $C_5 = \hat{C}_5(L_1,T)$.

Applying Gronwall's inequality yields the estimate

$$\|D\mathbf{v}^{k+1}\|_{s-1}^2 \le C_6\|D\mathbf{v}_0\|_{s-1}^2 \qquad (3.8)$$

where $C_6 = \hat{C}_6(L_1,T)$. We now choose ϵ_0 to satisfy $\epsilon_0 = C_6\|D\mathbf{v}_0\|_{s-1}^2$. It follows that $\|D\mathbf{v}^{k+1}\|_{s-1,T}^2 \le \epsilon_0$.

Substituting the estimate (3.8) for $\|D\mathbf{v}^{k+1}\|_{s-1}^2$ into the right-hand side of (3.4), (3.6) yields the following estimates for $\|\nabla\rho^{k+1}\|_{s+1}^2$ and $\|\rho^{k+1}\|_{s+2}^2$:

$$\begin{gathered}
\|\nabla\rho^{k+1}\|_{s+1}^2 \le C_1\|D\mathbf{v}^k\|_{s-1}^2\|D\mathbf{v}^{k+1}\|_{s-1}^2 \le \epsilon_0 C_1C_6\|D\mathbf{v}_0\|_{s-1}^2,\\
\|\rho^{k+1}\|_{s+2}^2 \le L_0^2 + C_3\|D\mathbf{v}^k\|_{s-1}^2\|D\mathbf{v}^{k+1}\|_{s-1}^2 \le L_0^2 + \epsilon_0 C_3C_6\|D\mathbf{v}_0\|_{s-1}^2
\end{gathered}$$

Therefore, we have $\|\rho^{k+1}\|_{s+2,T} \le L_1$ and we have $\|\nabla\rho^{k+1}\|_{s+1,T}^2 \le \epsilon_0$, provided that we choose L_1 large enough so that $\frac{1}{2}L_1^2 > L_0^2$, and provided that we choose $\|D\mathbf{v}_0\|_{s-1}$ small enough so that $\epsilon_0C_3C_6\|D\mathbf{v}_0\|_{s-1}^2 < \frac{1}{2}L_1^2$, and provided that we choose $\|D\mathbf{v}_0\|_{s-1}$ small enough so that $C_1C_6\|D\mathbf{v}_0\|_{s-1}^2 < 1$. We also choose $\|D\mathbf{v}_0\|_{s-1}$ small enough so that $C_6\|D\mathbf{v}_0\|_{s-1}^2 < 1$; therefore, we have $0<\epsilon_0<1$, since $\epsilon_0 = C_6\|D\mathbf{v}_0\|_{s-1}^2$. (Note that ϵ_0 will be small if $\|D\mathbf{v}_0\|_{s-1}$ is sufficiently small.) This completes the proof of part (a).

Next, by applying Lemma 2.3 to (3.1), (3.2), and using the estimates (3.4), (3.7), (3.8), we have the estimate

$$\begin{aligned}
\|\mathbf{v}^{k+1}\|_s^2 &\le e^{2T}\|\mathbf{v}_0\|_0^2 + Ce^{2T}(1+Te^{2T}e^{\beta(T)}\|D\mathbf{v}^k\|_{s-1,T}^2)(\|D\mathbf{v}_0\|_{s-1}^2 \\
&\quad + \int_0^t (\|D((\rho^k)^{-1}\hat{p}'(\rho^k))\|_{s_1}^2\|\nabla\rho^{k+1}\|_{s-1}^2)d\tau) \\
&\le L_0^2 + C_7\|D\mathbf{v}_0\|_{s-1}^2 + C_7\int_0^t \|D\mathbf{v}^{k+1}\|_{s-1}^2 d\tau \\
&\le L_0^2 + C_7\|D\mathbf{v}_0\|_{s-1}^2 + C_8\|D\mathbf{v}_0\|_{s-1}^2
\end{aligned}$$

with $\beta(T) = Te^{2T}\|D\mathbf{v}^k\|_{s-1,T}^2$. Here $s_1 = \max\{s-1, s_0\} = s-1$, $s > \frac{N}{2}+3$, and $s_0 = [\frac{N}{2}]+1 = 2$ for $N=2,3$. We also used the hypothesis that $e^{2T}\|\mathbf{v}_0\|_0^2 \le L_0^2$, and that $\|D\mathbf{v}^k\|_{s-1,T}^2 \le \epsilon_0$, where $0 < \epsilon_0 < 1$. And here $C_7 = \hat{C}_7(L_1, T)$, $C_8 = \hat{C}_8(L_1, T)$.

Therefore, we have $\|\mathbf{v}^{k+1}\|_{s,T} \le L_2$ provided that we choose L_2 large enough so that $\frac{1}{2}L_2^2 > L_0^2$, and so that $\frac{1}{2}L_2^2 > (C_7 + C_8)\|D\mathbf{v}_0\|_{s-1}^2$. This completes the proof of part (b). We next consider part (c).

Estimate for $\mathbf{v}_t^{k+1}$: Directly from equation (3.1) for $\mathbf{v}_t^{k+1}$, and the estimates already derived for ρ^{k+1} and $\mathbf{v}^{k+1}$, we deduce that $\|\mathbf{v}_t^{k+1}\|_{s-1}^2 \le C_9(\|D\mathbf{v}^{k+1}\|_{s-1}^2 + \|\nabla\rho^{k+1}\|_{s+1}^2) \le 2\epsilon_0 C_9$, where $C_9 = \hat{C}_9(L_1, L_2, T)$. Thus $\|\mathbf{v}_t^{k+1}\|_{s-1,T}^2 \le \epsilon_0 L_3$ provided we choose L_3 sufficiently large so that $L_3 \ge 2C_9$.

Summarizing, if we choose $\|D\mathbf{v}_0\|_{s-1}$ to be sufficiently small, then ρ^k and $\mathbf{v}^k$ satisfy the estimates in (a)-(c) for all $k \ge 1$. This completes the proof of Proposition 3.2. □

Next, we give the proof of contraction in low Sobolev norm.

Proposition 3.3. *Assume that the hypotheses of Theorem 3.1 hold. Then we have*

$$\|\rho^{k+1} - \rho^k\|_{2,T}^2 + \|\mathbf{v}^{k+1} - \mathbf{v}^k\|_{2,T}^2 \le \zeta\Big(\|\rho^k - \rho^{k-1}\|_{2,T}^2 + \|\mathbf{v}^k - \mathbf{v}^{k-1}\|_{2,T}^2\Big)$$

for $k = 1, 2, 3 \dots$. Here ζ is a constant such that $0 < \zeta < 1$. And $|\rho^{k+1} - \rho^0|_{L^\infty} \le \delta$; hence, $\rho^k \in G_1$, for all k.

Proof. Subtracting the equations (3.1), (3.2) for ρ^k and $\mathbf{v}^k$ from the equations (3.1), (3.2) for ρ^{k+1} and $\mathbf{v}^{k+1}$ yields equations which we write in the form

$$\begin{aligned} &\frac{D^k(\mathbf{v}^{k+1} - \mathbf{v}^k)}{Dt} + (\rho^k)^{-1}\hat{p}'(\rho^k)\nabla(\rho^{k+1} - \rho^k) \\ &= c\nabla\Delta(\rho^{k+1} - \rho^k) - (\mathbf{v}^k - \mathbf{v}^{k-1}) \cdot \nabla\mathbf{v}^k \\ &\quad - \big(\big((\rho^k)^{-1}\hat{p}'(\rho^k) - (\rho^{k-1})^{-1}\hat{p}'(\rho^{k-1})\big)\nabla\rho^k, \end{aligned} \tag{3.9}$$

$$\nabla \cdot (\mathbf{v}^{k+1} - \mathbf{v}^k) = 0, \tag{3.10}$$

where $D^k/Dt = \partial/\partial t + \mathbf{v}^k \cdot \nabla$, and where $(\mathbf{v}^{k+1} - \mathbf{v}^k)(\mathbf{x}, 0) = 0$. Using a proof by induction on k, we assume that $\rho^k \in G_1$, and then we will show that $\rho^{k+1} \in G_1$. First, we obtain estimates for $\rho^{k+1} - \rho^k$ and $\mathbf{v}^{k+1} - \mathbf{v}^k$.

Estimate for $\rho^{k+1} - \rho^k$: Applying the divergence operator to (3.9) yields

$$\begin{aligned} &c\Delta^2(\rho^{k+1} - \rho^k) - \nabla \cdot ((\rho^k)^{-1}p'(\rho^k)\nabla(\rho^{k+1} - \rho^k)) \\ &= \nabla \cdot (\mathbf{v}^k \cdot \nabla(\mathbf{v}^{k+1} - \mathbf{v}^k)) + \nabla \cdot ((\mathbf{v}^k - \mathbf{v}^{k-1}) \cdot \nabla\mathbf{v}^k) \\ &\quad + \nabla \cdot (((\rho^k)^{-1}p'(\rho^k) - (\rho^{k-1})^{-1}p'(\rho^{k-1}))\nabla\rho^k) \end{aligned} \tag{3.11}$$

Applying Lemma 2.7 to equation (3.11) with $r = 2$ yields the estimate

$$\begin{aligned} &\|\Delta(\rho^{k+1} - \rho^k)\|_2^2 + \|\nabla(\rho^{k+1} - \rho^k\|_2^2 \\ &\le C[1 + \sum_{j=1}^{2} \|D((\rho^k)^{-1}p'(\rho^k))\|_2^{2j}](\|D\mathbf{v}^k\|_3^2\|D(\mathbf{v}^{k+1} - \mathbf{v}^k)\|_1^2 \\ &\quad + \|(\mathbf{v}^k - \mathbf{v}^{k-1}) \cdot \nabla\mathbf{v}^k\|_1^2 + \|((\rho^k)^{-1}p'(\rho^k) - (\rho^{k-1})^{-1}p'(\rho^{k-1}))\nabla\rho^k\|_1^2) \\ &\le C_1\|D\mathbf{v}^k\|_3^2\|\mathbf{v}^{k+1} - \mathbf{v}^k\|_2^2 + C_1\|D\mathbf{v}^k\|_2^2\|\mathbf{v}^k - \mathbf{v}^{k-1}\|_2^2 + C_1\|\nabla\rho^k\|_2^2\|\rho^k - \rho^{k-1}\|_2^2 \end{aligned} \tag{3.12}$$

where $C_1 = \hat{C}_1(L_1)$ from Proposition 3.2. Here we used the Sobolev calculus inequality $\|fg\|_r \leq C\|f\|_r\|g\|_r$ for $r = 2 > \frac{N}{2}$.

To obtain an L^2 estimate for ρ^{k+1}, we apply Lemma 2.4, where we define $u = \rho^{k+1}$ and we define $w = \rho^k$, to be used for the functions u, w appearing in Lemma 2.4. Note that since by hypothesis we have $\rho^{k+1}(\mathbf{x}_0, t) = \rho^k(\mathbf{x}_0, t)$ for all $k \geq 0$, at a single fixed point $\mathbf{x}_0 \in \Omega$, the hypotheses of Lemma 2.4 are satisfied, and so we obtain the estimate

$$\begin{aligned}\|\rho^{k+1} - \rho^k\|_0^2 &\leq C\|\nabla(\rho^{k+1} - \rho^k)\|_2^2 \\ &\leq C_2\|D\mathbf{v}^k\|_3^2\|\mathbf{v}^{k+1} - \mathbf{v}^k\|_2^2 + C_2\|D\mathbf{v}^k\|_2^2\|\mathbf{v}^k - \mathbf{v}^{k-1}\|_2^2 \\ &\quad + C_2\|\nabla\rho^k\|_2^2\|\rho^k - \rho^{k-1}\|_2^2\end{aligned} \tag{3.13}$$

where we used the estimate for $\|\Delta(\rho^{k+1} - \rho^k)\|_2^2 + \|\nabla(\rho^{k+1} - \rho^k)\|_2^2$ from (3.12). Here $C_2 = \hat{C}_2(L_1)$ from Proposition 3.2.

Now, adding (3.12), (3.13), we obtain the estimate

$$\begin{aligned}\|\rho^{k+1} - \rho^k\|_2^2 &\leq \|\rho^{k+1} - \rho^k\|_0^2 + C\|\nabla(\rho^{k+1} - \rho^k)\|_1^2 \\ &\leq \|\rho^{k+1} - \rho^k\|_0^2 + C(\|\nabla(\rho^{k+1} - \rho^k)\|_2^2 + \|\Delta(\rho^{k+1} - \rho^k)\|_2^2) \\ &\leq C_3\|D\mathbf{v}^k\|_3^2\|\mathbf{v}^{k+1} - \mathbf{v}^k\|_2^2 + C_3\|D\mathbf{v}^k\|_2^2\|\mathbf{v}^k - \mathbf{v}^{k-1}\|_2^2 \\ &\quad + C_3\|\nabla\rho^k\|_2^2\|\rho^k - \rho^{k-1}\|_2^2 \\ &\leq C_3\left(\|\mathbf{v}^{k+1} - \mathbf{v}^k\|_2^2 + \epsilon_0\|\mathbf{v}^k - \mathbf{v}^{k-1}\|_2^2 + \epsilon_0\|\rho^k - \rho^{k-1}\|_2^2\right)\end{aligned} \tag{3.14}$$

where $C_3 = \hat{C}_3(L_1)$. Here, we used the Proposition 3.2 estimates $\|D\mathbf{v}^k\|_2^2 \leq \|D\mathbf{v}^k\|_3^2 \leq \|D\mathbf{v}^k\|_{s-1,T}^2 \leq \epsilon_0$, and $\|\nabla\rho^k\|_2^2 \leq \|\nabla\rho^k\|_{s+1,T}^2 \leq \epsilon_0$, where $0 < \epsilon_0 < 1$, and where ϵ_0 is as small as we like if $\|D\mathbf{v}_0\|_{s-1}$ is sufficiently small.

Estimate for $\mathbf{v}^{k+1} - \mathbf{v}^k$: After applying Lemma 2.3 to (3.9), using $r = 2$, we obtain

$$\begin{aligned}\|\mathbf{v}^{k+1} - \mathbf{v}^k\|_2^2 &\leq C_4\int_0^t \|D\mathbf{v}^k\|_2^2\|\mathbf{v}^k - \mathbf{v}^{k-1}\|_2^2 d\tau + C_4\int_0^t \|\nabla(\rho^{k+1} - \rho^k)\|_1^2 d\tau \\ &\quad + C_4\int_0^t \|\nabla\rho^k\|_2^2\|(\rho^k)^{-1}p'(\rho^k) - (\rho^{k-1})^{-1}p'(\rho^{k-1})\|_2^2 d\tau \\ &\leq C_5\int_0^t \left(\|\mathbf{v}^{k+1} - \mathbf{v}^k\|_2^2 + \epsilon_0\|\mathbf{v}^k - \mathbf{v}^{k-1}\|_2^2 + \epsilon_0\|\rho^k - \rho^{k-1}\|_2^2\right) d\tau\end{aligned} \tag{3.15}$$

with $C_4 = \hat{C}_4(L_1, L_2, T)$ and $C_5 = \hat{C}_5(L_1, L_2, T)$, and where we substituted the estimate for $\|\nabla(\rho^{k+1} - \rho^k)\|_1^2$ from (3.14). Here, we used the estimates $\|D\mathbf{v}^k\|_2^2 \leq \|D\mathbf{v}^k\|_{s-1,T}^2 \leq \epsilon_0$, and $\|\nabla\rho^k\|_2^2 \leq \|\nabla\rho^k\|_{s+1,T}^2 \leq \epsilon_0$ from Proposition 3.2.

Next, we apply Gronwall's inequality to (3.15), which yields

$$\begin{aligned}\|\mathbf{v}^{k+1} - \mathbf{v}^k\|_2^2 &\leq C_6\int_0^t (\epsilon_0\|\mathbf{v}^k - \mathbf{v}^{k-1}\|_2^2 + \epsilon_0\|\rho^k - \rho^{k-1}\|_2^2) d\tau \\ &\leq C_7\Big(\epsilon_0\|\mathbf{v}^k - \mathbf{v}^{k-1}\|_{2,T}^2 + \epsilon_0\|\rho^k - \rho^{k-1}\|_{2,T}^2\Big)\end{aligned} \tag{3.16}$$

where $C_6 = \hat{C}_6(L_1, L_2, T)$, $C_7 = \hat{C}_7(L_1, L_2, T)$.

Substituting (3.16) into the right-hand side of (3.14) yields

$$\begin{aligned}\|\rho^{k+1}-\rho^k\|_2^2 &\le C_3C_7\left(\epsilon_0\|\mathbf{v}^k-\mathbf{v}^{k-1}\|_{2,T}^2+\epsilon_0\|\rho^k-\rho^{k-1}\|_{2,T}^2\right)\\ &\quad + C_3\left(\epsilon_0\|\mathbf{v}^k-\mathbf{v}^{k-1}\|_2^2+\epsilon_0\|\rho^k-\rho^{k-1}\|_2^2\right)\end{aligned}$$

Adding the above estimate for $\|\rho^{k+1}-\rho^k\|_2^2$ to the estimate (3.16) yields

$$\begin{aligned}\|\rho^{k+1}-\rho^k\|_2^2+\|\mathbf{v}^{k+1}-\mathbf{v}^k\|_2^2 &\le C_3C_7\left(\epsilon_0\|\mathbf{v}^k-\mathbf{v}^{k-1}\|_{2,T}^2+\epsilon_0\|\rho^k-\rho^{k-1}\|_{2,T}^2\right)\\ &\quad + C_3\left(\epsilon_0\|\mathbf{v}^k-\mathbf{v}^{k-1}\|_2^2+\epsilon_0\|\rho^k-\rho^{k-1}\|_2^2\right)\\ &\quad + C_7(\epsilon_0\|\mathbf{v}^k-\mathbf{v}^{k-1}\|_{2,T}^2+\epsilon_0\|\rho^k-\rho^{k-1}\|_{2,T}^2)\end{aligned}$$

Finally, after choosing ϵ_0 sufficiently small, we get

$$\|\rho^{k+1}-\rho^k\|_{2,T}^2+\|\mathbf{v}^{k+1}-\mathbf{v}^k\|_{2,T}^2 \le \zeta\left(\|\rho^k-\rho^{k-1}\|_{2,T}^2+\|\mathbf{v}^k-\mathbf{v}^{k-1}\|_{2,T}^2\right) \tag{3.17}$$

where $0<\zeta<1$. This completes the proof of the first part of Proposition 3.3. It only remains to prove that $|\rho^{k+1}-\rho^0|_{L^\infty}\le\delta$, so that $\rho^{k+1}\in G_1$.

Proof that $|\rho^{k+1}-\rho^0|_{L^\infty}\le\delta$: Repeated application of the estimate (3.17) yields

$$\|\mathbf{v}^{k+1}-\mathbf{v}^k\|_{2,T}^2+\|\rho^{k+1}-\rho^k\|_{2,T}^2 \le \zeta^k\left(\|\mathbf{v}^1-\mathbf{v}^0\|_{2,T}^2+\|\rho^1-\rho^0\|_{2,T}^2\right) \tag{3.18}$$

where $0<\zeta<1$. Next, we estimate

$$\begin{aligned}|\rho^{k+1}-\rho^0|^2 &\le C\sum_{j=0}^k|\rho^{j+1}-\rho^j|_{L^\infty}^2\\ &\le C\sum_{j=0}^k\|\rho^{j+1}-\rho^j\|_{s_0}^2\le C\sum_{j=0}^k\|\rho^{j+1}-\rho^j\|_{2,T}^2\\ &\le C\sum_{j=0}^k\zeta^j\left(\|\rho^1-\rho^0\|_{2,T}^2+\|\mathbf{v}^1-\mathbf{v}^0\|_{2,T}^2\right)\end{aligned} \tag{3.19}$$

where we used (3.18) in the last step. We also used the triangle inequality and Cauchy's inequality and Sobolev's inequality $|h|_{L^\infty}\le C\|h\|_{s_0}$. Here $s_0=[\frac{N}{2}]+1=2$ for $N=2,3$. Thus, we have $|\rho^{k+1}-\rho^0|\le\delta$, provided that $\|\rho^1-\rho^0\|_{2,T}$ and $\|\mathbf{v}^1-\mathbf{v}^0\|_{2,T}$ are sufficiently small. We now proceed to show that $\|\rho^1-\rho^0\|_{2,T}$ and $\|\mathbf{v}^1-\mathbf{v}^0\|_{2,T}$ can be made sufficiently small.

From equations (3.1), (3.2), we obtain the following equations for $\mathbf{v}^1-\mathbf{v}^0$ and $\rho^1-\rho^0$,

$$\begin{gathered}\frac{D^0(\mathbf{v}^1-\mathbf{v}^0)}{Dt}+(\rho^0)^{-1}\hat{p}'(\rho^0)\nabla(\rho^1-\rho^0)\\ =c\nabla\Delta(\rho^1-\rho^0)-\mathbf{v}^0\cdot\nabla\mathbf{v}^0-(\rho^0)^{-1}\hat{p}'(\rho^0)\nabla\rho^0+c\nabla\Delta\rho^0,\\ \nabla\cdot(\mathbf{v}^1-\mathbf{v}^0)=0,\end{gathered} \tag{3.20}$$

where $D^0/Dt=\partial/\partial t+\mathbf{v}^0\cdot\nabla$, and where $(\mathbf{v}^1-\mathbf{v}^0)(\mathbf{x},0)=0$. Here, we used the fact that $\mathbf{v}^0(\mathbf{x},t)=\mathbf{v}_0(\mathbf{x})$. Now we obtain estimates for $\rho^1-\rho^0$ and $\mathbf{v}^1-\mathbf{v}^0$.

Estimate for $\rho^1-\rho^0$: Applying the divergence operator to (3.20), we obtain

$$\begin{aligned}&c\Delta^2(\rho^1-\rho^0)-\nabla\cdot((\rho^0)^{-1}\hat{p}'(\rho^0)\nabla(\rho^1-\rho^0))\\ &=\nabla\cdot(\mathbf{v}^0\cdot\nabla(\mathbf{v}^1-\mathbf{v}^0))+\nabla\cdot(\mathbf{v}^0\cdot\nabla\mathbf{v}^0)\\ &\quad+\nabla\cdot((\rho^0)^{-1}\hat{p}'(\rho^0)\nabla\rho^0)-c\Delta^2\rho^0\end{aligned} \tag{3.21}$$

Applying Lemma 2.7 to (3.21) with $r = 2$, we obtain the estimate

$$\begin{aligned}\|\Delta(\rho^1-\rho^0)\|_2^2 + \|\nabla(\rho^1-\rho^0)\|_2^2 \le C_8(&\|D\mathbf{v}^0\|_3^2\|D(\mathbf{v}^1-\mathbf{v}^0)\|_1^2 + \|\mathbf{v}^0\cdot\nabla\mathbf{v}^0\|_1^2 \\ &+ \|(\rho^0)^{-1}\hat{p}'(\rho^0)\nabla\rho^0\|_1^2 + \|\nabla\Delta\rho^0\|_1^2)\end{aligned} \tag{3.22}$$

where $C_8 = \hat{C}_8(L_1)$. Next, we apply Lemma 2.4 and obtain the L^2 estimate

$$\begin{aligned}\|\rho^1-\rho^0\|_0^2 &\le C\|\nabla(\rho^1-\rho^0)\|_2^2 \\ &\le C_9(\|D\mathbf{v}^0\|_3^2\|D(\mathbf{v}^1-\mathbf{v}^0)\|_1^2 + \|\mathbf{v}^0\cdot\nabla\mathbf{v}^0\|_1^2 \\ &\quad + \|(\rho^0)^{-1}\hat{p}'(\rho^0)\nabla\rho^0\|_1^2 + \|\nabla\Delta\rho^0\|_1^2)\end{aligned} \tag{3.23}$$

where $C_9 = \hat{C}_9(L_1)$. And we used estimate (3.22) for $\|\nabla(\rho^1-\rho^0)\|_2^2$.

Adding the estimates (3.22), (3.23) yields the following estimate for $\rho^1-\rho^0$:

$$\begin{aligned}\|\rho^1-\rho^0\|_2^2 &\le C\left(\|\rho^1-\rho^0\|_0^2 + \|\nabla(\rho^1-\rho^0)\|_1^2\right) \\ &\le C_{10}(\|D\mathbf{v}^0\|_3^2\|D(\mathbf{v}^1-\mathbf{v}^0)\|_1^2 + \|\mathbf{v}^0\cdot\nabla\mathbf{v}^0\|_2^2 \\ &\quad + \|(\rho^0)^{-1}\hat{p}'(\rho^0)\nabla\rho^0\|_2^2 + \|\nabla\Delta\rho^0\|_1^2) \\ &\le C_{11}(\|D\mathbf{v}^0\|_3^2\|\mathbf{v}^1-\mathbf{v}^0\|_2^2 + \|\mathbf{v}^0\|_2^2\|D\mathbf{v}^0\|_2^2 \\ &\quad + \|(\rho^0)^{-1}\hat{p}'(\rho^0)\|_2^2\|\nabla\rho^0\|_2^2 + \|\nabla\rho^0\|_3^2)\end{aligned} \tag{3.24}$$

where $C_{10} = \hat{C}_{10}(L_1)$, $C_{11} = \hat{C}_{11}(L_1)$. Here we used the Sobolev calculus inequality $\|fg\|_r \le C\|f\|_r\|g\|_r$ for $r = 2 > \frac{N}{2}$.

Estimate for $\mathbf{v}^1-\mathbf{v}^0$: After applying Lemma 2.3 to (3.20), using $r = 2$, we obtain

$$\begin{aligned}\|\mathbf{v}^1-\mathbf{v}^0\|_2^2 &\le C_{12}\int_0^t \left(\|\mathbf{v}^0\cdot\nabla\mathbf{v}^0\|_2^2 + \|(\rho^0)^{-1}\hat{p}'(\rho^0)\nabla\rho^0\|_2^2\right)d\tau \\ &\quad + C_{12}\int_0^t \left(\|\nabla\Delta\rho^0\|_2^2 + \|\nabla(\rho^1-\rho^0)\|_1^2\right)d\tau \\ &\le C_{13}\int_0^t \left(\|\mathbf{v}^0\|_2^2\|D\mathbf{v}^0\|_2^2 + \|(\rho^0)^{-1}\hat{p}'(\rho^0)\|_2^2\|\nabla\rho^0\|_2^2\right)d\tau \\ &\quad + C_{13}\int_0^t \left(\|\nabla\rho^0\|_4^2 + \|D\mathbf{v}^0\|_3^2\|\mathbf{v}^1-\mathbf{v}^0\|_2^2\right)d\tau\end{aligned} \tag{3.25}$$

where $C_{12} = \hat{C}_{12}(L_1, L_2)$, $C_{13} = \hat{C}_{13}(L_1, L_2)$, and where we substituted the estimate for $\|\nabla(\rho^1-\rho^0)\|_1^2$ from (3.24). Next, we apply Gronwall's inequality to (3.25), which yields

$$\begin{aligned}\|\mathbf{v}^1-\mathbf{v}^0\|_2^2 &\le C_{14}\int_0^t \left(\|\mathbf{v}^0\|_2^2\|D\mathbf{v}^0\|_2^2 + \|(\rho^0)^{-1}\hat{p}'(\rho^0)\|_2^2\|\nabla\rho^0\|_2^2 + \|\nabla\rho^0\|_4^2\right)d\tau \\ &\le C_{15}\left(\|D\mathbf{v}_0\|_{s-1}^2 + \|\nabla\rho^0\|_{4,T}^2\right) \le 2C_{15}\|D\mathbf{v}_0\|_{s-1}^2\end{aligned} \tag{3.26}$$

where we used the fact that $\mathbf{v}^0(\mathbf{x}, t) = \mathbf{v}_0(\mathbf{x})$, and we used the fact that $\|\nabla\rho^0\|_{4,T}^2 \le \|\nabla\rho^0\|_{s+1,T}^2 \le \|D\mathbf{v}_0\|_{s-1}^2$, by hypothesis. Here $C_{14} = \hat{C}_{14}(L_1, L_2, T)$ and $C_{15} = \hat{C}_{15}(L_1, L_2, T)$. Finally, we substitute the above estimate (3.26) into the right-hand side of the estimate (3.24) for $\|\rho^1-\rho^0\|_2^2$, which yields

$$\|\rho^1-\rho^0\|_2^2 \le C_{16}\|D\mathbf{v}_0\|_{s-1}^2 \tag{3.27}$$

where $C_{16} = \hat{C}_{16}(L_1, L_2, T)$ and where we used the fact that $\|\nabla\rho^0\|_2^2 \le \|\nabla\rho^0\|_3^2 \le \|\nabla\rho^0\|_{s+1,T}^2 \le \|D\mathbf{v}_0\|_{s-1}^2$, by hypothesis. From (3.19), (3.26), and (3.27), we see that $|\rho^{k+1} - \rho^0|_{L^\infty} \le \delta$, for $\|D\mathbf{v}_0\|_{s-1}$ sufficiently small. And so $\rho^{k+1} \in G_1$. It thus follows from the proof by induction on k that $\rho^k \in G_1$ for all k. This completes the proof of Proposition 3.3. □

Using Propositions 3.2 and 3.3, we now complete the proof of Theorem 3.1. From (3.18), it follows that

$$\sum_{k=1}^{\infty} \Big(\|\mathbf{v}^{k+1} - \mathbf{v}^k\|_{2,T}^2 + \|\rho^{k+1} - \rho^k\|_{2,T}^2\Big) < \infty.$$

Hence, $\|\mathbf{v}^{k+1} - \mathbf{v}^k\|_{2,T}^2 \to 0$ and $\|\rho^{k+1} - \rho^k\|_{2,T}^2 \to 0$ as $k \to \infty$. Therefore, we conclude that there exist $\mathbf{v} \in C([0,T], H^2(\Omega))$ and $\rho \in C([0,T], H^2(\Omega))$ so that $\|\mathbf{v}^k - \mathbf{v}\|_{2,T} \to 0$, and $\|\rho^k - \rho\|_{2,T} \to 0$, as $k \to \infty$. Using the interpolation inequalities $\|f\|_{r'+2} \le C\|f\|_2^{\alpha}\|f\|_{r+2}^{1-\alpha}$, with $\alpha = (r - r')/r$, and $\|g\|_{r'} \le C\|g\|_2^{\beta}\|g\|_r^{1-\beta}$, with $\beta = (r - r')/(r-2)$, with $r' < r$, where $r \ge 5$, and using Proposition 3.2, we can conclude that $\|\rho^k - \rho\|_{s'+2,T} \to 0$ and $\|\mathbf{v}^k - \mathbf{v}\|_{s',T} \to 0$, as $k \to \infty$ for any $s' < s$. For $s' > \frac{N}{2} + 3$, Sobolev's lemma implies that $\mathbf{v}^k \to \mathbf{v}$ in $C([0,T], C^3)$ and $\rho^k \to \rho$ in $C([0,T], C^5)$. From the linear system of equations (3.1), (3.2) it follows that $\|\mathbf{v}_t^k - \mathbf{v}_t\|_{s'-1,T} \to 0$, as $k \to \infty$, so that $\mathbf{v}_t^k \to \mathbf{v}_t$ in $C([0,T], C^2)$, and ρ, $\mathbf{v}$, is a classical solution of the system of equations (1.2), (1.3). The additional facts that $\mathbf{v} \in L^\infty([0,T], H^s)$, and $\rho \in L^\infty([0,T], H^{s+2})$ can be deduced using boundedness in high norm and a standard compactness argument (see, for example, [5, 10]).

To prove uniqueness, let ρ^1, $\mathbf{v}^1$, and let ρ^2, $\mathbf{v}^2$, be any solutions of (1.2), (1.3) having the regularity of solutions from Theorem 3.1, and satisfying $\rho^1(\mathbf{x}_0, t) = \rho^2(\mathbf{x}_0, t) = \rho^0(\mathbf{x}_0, t)$ at a single fixed point $\mathbf{x}_0 \in \Omega$. Here, ρ^0 is the initial iterate from Theorem 3.1. And we assume that $\|D\mathbf{v}_0\|_{s-1}$ is sufficiently small. We next show that $\rho^1 = \rho^2$ and $\mathbf{v}^1 = \mathbf{v}^2$.

Let $\rho = \lim_{k\to\infty} \rho^{k+1}$, $\mathbf{v} = \lim_{k\to\infty} \mathbf{v}^{k+1}$, where $\{\rho^{k+1},\ \mathbf{v}^{k+1}\}$, $k \ge 0$, is a sequence of solutions to the linear equations (3.1), (3.2), so that ρ, $\mathbf{v}$ is a solution of (1.2), (1.3) from Theorem 3.1. We next show that $\rho^1 = \rho$ and $\mathbf{v}^1 = \mathbf{v}$. It then follows that the solution to (1.2), (1.3) is unique, because repeating the proof given below will also show that $\rho^2 = \rho$ and $\mathbf{v}^2 = \mathbf{v}$, and so we will obtain

$$\begin{aligned}
&\|\rho^2 - \rho^1\|_{2,T} + \|\mathbf{v}^2 - \mathbf{v}^1\|_{2,T} \\
&\le \|\rho^2 - \rho\|_{2,T} + \|\rho^1 - \rho\|_{2,T} + \|\mathbf{v}^2 - \mathbf{v}\|_{2,T} + \|\mathbf{v}^1 - \mathbf{v}\|_{2,T} = 0
\end{aligned}$$

Therefore, we now show that $\rho^1 = \rho$ and $\mathbf{v}^1 = \mathbf{v}$.

Subtracting the equations (1.2), (1.3) for ρ^1 and $\mathbf{v}^1$ from the equations (1.2), (1.3) for ρ and $\mathbf{v}$ yields equations which we write in the form

$$\begin{gathered}
\begin{aligned}
&\frac{D^1(\mathbf{v} - \mathbf{v}^1)}{Dt} + (\rho^1)^{-1}\hat{p}'(\rho^1)\nabla(\rho - \rho^1) \\
&= c\nabla\Delta(\rho - \rho^1) - (\mathbf{v} - \mathbf{v}^1)\cdot\nabla\mathbf{v} - ((\rho)^{-1}\hat{p}'(\rho) - (\rho^1)^{-1}\hat{p}'(\rho^1))\nabla\rho,
\end{aligned} \\
\nabla\cdot(\mathbf{v} - \mathbf{v}^1) = 0,
\end{gathered}$$

where $D^1/Dt = \partial/\partial t + \mathbf{v}^1\cdot\nabla$, where $(\mathbf{v} - \mathbf{v}^1)(\mathbf{x}, 0) = 0$. Repeating the proof of Proposition 3.3 yields the estimate

$$\|\rho - \rho^1\|_{2,T}^2 + \|\mathbf{v} - \mathbf{v}^1\|_{2,T}^2 \le \zeta\left(\|\rho - \rho^1\|_{2,T}^2 + \|\mathbf{v} - \mathbf{v}^1\|_{2,T}^2\right)$$

where $0 < \zeta < 1$. Note that in repeating the proof of the estimates (3.14), (3.15) in the proof of Proposition 3.3, we use the fact that $\|D\mathbf{v}\|^2_{s-1,T} \le \epsilon_0$ and $\|\nabla\rho\|^2_{s+1,T} \le \epsilon_0$, which holds since ρ, $\mathbf{v}$ is a solution from Theorem 3.1. Finally, after moving the right-hand side of the above inequality to the left-hand side, we obtain $\|\rho - \rho^1\|^2_{2,T} + \|\mathbf{v} - \mathbf{v}^1\|^2_{2,T} = 0$ and so the solution is unique. □

A Final Remark. We sketch a proof that $\partial\rho/\partial t$ is small. Then, since we already know that $\nabla\rho$ is small, it follows that the conservation of mass equation is approximately satisfied. Therefore, the model equations (1.2), (1.3) might be useful as an approximation in the case of almost constant density, and nearly incompressible fluid flow with a small velocity gradient.

Proceeding formally, we differentiate equation (1.3) with respect to t, obtaining

$$\frac{\partial^2\mathbf{v}}{\partial t^2} + \mathbf{v}\cdot\nabla\mathbf{v}_t + \mathbf{v}_t\cdot\nabla\mathbf{v} + \rho^{-1}\hat{p}'(\rho)\nabla\rho_t + \frac{\partial}{\partial\rho}(\rho^{-1}\hat{p}'(\rho))\rho_t\nabla\rho = c\nabla\Delta\rho_t$$

Applying the divergence operator to this equation yields

$$c\Delta^2\rho_t - \nabla\cdot((\rho)^{-1}\hat{p}'(\rho)\nabla\rho_t) = \nabla\cdot(\frac{\partial}{\partial\rho}(\rho^{-1}\hat{p}'(\rho))\rho_t\nabla\rho) + \nabla\cdot(\mathbf{v}\cdot\nabla\mathbf{v}_t) + \nabla\cdot(\mathbf{v}_t\cdot\nabla\mathbf{v})$$

where we used the fact that $\nabla\cdot\mathbf{v} = 0$, and where c is a positive constant. Applying Lemma 2.7 we obtain from the above equation the following estimate for $\nabla\rho_t$,

$$\begin{aligned}
\|\nabla\rho_t\|^2_3 &\le C(\|\Delta\rho_t\|^2_2 + \|\nabla\rho_t\|^2_2) \\
&\le C[1 + \sum_{j=1}^{2}\|D((\rho)^{-1}\hat{p}'(\rho))\|^{2j}_2](\|D\mathbf{v}\|^2_3\|D\mathbf{v}_t\|^2_1 + \|\mathbf{v}_t\cdot\nabla\mathbf{v}\|^2_1 \\
&\quad + \|\frac{\partial}{\partial\rho}(\rho^{-1}\hat{p}'(\rho))\rho_t\nabla\rho\|^2_1) \\
&\le C_{17}(\|D\mathbf{v}\|^2_3\|\mathbf{v}_t\|^2_2 + \|\rho_t\|^2_2\|\nabla\rho\|^2_2)
\end{aligned} \tag{3.28}$$

where $C_{17} = \hat{C}_{17}(L_1, T)$. Next, by Lemma 2.4 we have the L^2 estimate

$$\begin{aligned}
\|\rho_t\|^2_0 &\le C_0\|\rho^0_t\|^2_0 + C_0\|\nabla\rho^0_t\|^2_2 + C_0\|\nabla\rho_t\|^2_2 \\
&\le C_{18}(\|\rho^0_t\|^2_0 + \|\nabla\rho^0_t\|^2_2 + \|D\mathbf{v}\|^2_3\|\mathbf{v}_t\|^2_2 + \|\rho_t\|^2_2\|\nabla\rho\|^2_2)
\end{aligned} \tag{3.29}$$

where $C_{18} = \hat{C}_{18}(L_1, T)$ and where we used the estimate for $\|\Delta\rho_t\|^2_2 + \|\nabla\rho_t\|^2_2$ from (3.28). Here, ρ^0 is the initial iterate as defined in Theorem 3.1. Note that by definition of ρ^0, we have $\rho_t(\mathbf{x}_0, t) = \rho^0_t(\mathbf{x}_0, t)$ at the single fixed point $\mathbf{x}_0 \in \Omega$. Also, we now specify that $\|\rho^0_t\|_{0,T} \le \|D\mathbf{v}_0\|_{s-1}$ and $\|\nabla\rho^0_t\|_{2,T} \le \|D\mathbf{v}_0\|_{s-1}$, as an additional hypothesis for ρ^0. Recall that $\epsilon_0 = C_6\|D\mathbf{v}_0\|^2_{s-1}$ where $C_6 = \hat{C}_6(L_1, T)$, from Proposition 3.2. Then adding the estimates (3.28), (3.29) yields the following estimate for ρ_t:

$$\begin{aligned}
\|\rho_t\|^2_3 &\le C(\|\rho_t\|^2_0 + \|\nabla\rho_t\|^2_2) \le C_{19}(\|\rho^0_t\|^2_0 + \|\nabla\rho^0_t\|^2_2 + \|D\mathbf{v}\|^2_3\|\mathbf{v}_t\|^2_2 + \|\rho_t\|^2_2\|\nabla\rho\|^2_2) \\
&\le C_{20}(\epsilon_0 + \epsilon_0\|\mathbf{v}_t\|^2_2 + \epsilon_0\|\rho_t\|^2_3)
\end{aligned}$$

where $C_{19} = \hat{C}_{19}(L_1, T)$, $C_{20} = \hat{C}_{20}(L_1, T)$ and where we used the facts that $\|D\mathbf{v}\|^2_3 \le \epsilon_0$, and $\|\nabla\rho\|^2_2 \le \epsilon_0$ from Proposition 3.2. After moving the term $\epsilon_0\|\rho_t\|^2_3$ to the left-hand side, and substituting the estimate for $\mathbf{v}_t$ from Proposition 3.2, we then obtain the estimate

$$\|\rho_t\|^2_3 \le \epsilon_0 C_{21}$$

where $C_{21} = \hat{C}_{21}(L_1, L_3, T)$. Therefore, ρ_t is small, since ϵ_0 will be small if $\|D\mathbf{v}_0\|_{s-1}$ is sufficiently small.

4. Existence for the linear case

In this section, we sketch the proof of the existence of a solution to the linear equations (3.1), (3.2). Before going into the existence proof, which appears in Lemma 4.5, we first collect some preliminary technical facts. We recall that every vector field $\mathbf{u} \in L^2(\mathbb{T}^N)$ admits a unique orthogonal decomposition in terms of a solenoidal vector field $\mathbf{w}$ and a potential $\nabla\phi$, so that $\mathbf{u} = \mathbf{w} + \nabla\phi$, where $\mathbf{w} = P\mathbf{u}$ and $\nabla\phi = Q\mathbf{u}$. Moreover, if $\mathbf{u} \in H^s(\mathbb{T}^N)$, with $s \geq 1$, then $\mathbf{w}$ and ϕ satisfy $\nabla\cdot\mathbf{w} = 0$, $\Delta\phi = \nabla \cdot \mathbf{u}$. The next lemma, due to Embid [5], provides us with an estimate that will be useful for the existence proof.

Lemma 4.1. *If $\mathbf{u}$, $\mathbf{v} \in H^r(\Omega)$, $r > \frac{N}{2} + 3$, $\Omega = \mathbb{T}^N$, then $Q(\mathbf{v}\cdot\nabla P\mathbf{u}) \in H^r(\Omega)$ and $\|Q(\mathbf{v}\cdot\nabla P\mathbf{u})\|_r \leq C\|\mathbf{v}\|_r\|\mathbf{u}\|_r$.*

For a proof of this lemma, see Embid [5]. To prove existence of a solution to (3.1), (3.2), we need the following lemmas.

Lemma 4.2. *Given $a \in H^r(\Omega)$, $f \in H^r(\Omega)$, $r > \frac{N}{2} + 3$, $\Omega = \mathbb{T}^N$, then $P(a\nabla f) \in H^r(\Omega)$, and $\|P(a\nabla f)\|_r \leq C\|Da\|_{r-1}\|\nabla f\|_{r-1}$, where P is the projection onto the solenoidal vector field.*

Proof. First, from the orthogonality property of the projection P, we obtain an L^2 estimate as follows:

$$\|P(a\nabla f)\|_0 = \|P(\bar{f}\nabla a)\|_0 \leq \|\bar{f}\nabla a\|_0 \leq C|Da|_{L^\infty}\|\bar{f}\|_0 \leq C|Da|_{L^\infty}\|\nabla f\|_0.$$

where $\bar{f} = f - \frac{1}{|\Omega|}\int_\Omega f dx$. Here, we used Poincaré's inequality to estimate $\|\bar{f}\|_0 \leq C\|\nabla f\|_0$.

Next, from the orthogonality property of the projection P, and using the commutator estimate from Lemma 2.2, as well as using the triangle inequality, we obtain for $|\alpha| \geq 1$:

$$\begin{aligned}
\|P(a\nabla f)_\alpha\|_0 &\leq \|P(a\nabla f_\alpha)\|_0 + \|P\big[(a\nabla f)_\alpha - a\nabla f_\alpha\big]\|_0 \\
&= \|P(\bar{f}_\alpha\nabla a)\|_0 + \|P[(a\nabla f)_\alpha - a\nabla f_\alpha]\|_0 \\
&\leq C|Da|_{L^\infty}\|\bar{f}\|_k + C\|Da\|_{k_1}\|\nabla f\|_{k-1} \\
&\leq C\|Da\|_{k_1}\|\nabla f\|_{k-1},
\end{aligned}$$

where $|\alpha| = k$, where $k_1 = \max(k-1, s_0)$, and where $s_0 = [\frac{N}{2}] + 1$. And we used the facts that $\|\bar{f}\|_k \leq C(\|\bar{f}\|_0 + \|\nabla f\|_{k-1})$, and $\|\bar{f}\|_0 \leq C\|\nabla f\|_0$, by Poincaré's inequality.

Finally, by adding the above estimates over all $0 \leq |\alpha| \leq r$, we obtain the estimate $\|P(a\nabla f)\|_r \leq C\|Da\|_{r-1}\|\nabla f\|_{r-1}$. □

Lemma 4.3. *Given $\mathbf{v}$ in $C([0,T], H^0) \cap L^\infty([0,T], H^r)$, and $\mathbf{f}$ in $C([0,T], H^0) \cap L^\infty([0,T], H^r)$, with $r > \frac{N}{2} + 3$, for $\mathbf{x} \in \Omega$, $\Omega = \mathbb{T}^N$, $0 \leq t \leq T$, there is a unique, classical solution $\mathbf{u} \in C([0,T], C^3) \cap L^\infty([0,T], H^r)$ of*

$$D\mathbf{u}/Dt = \mathbf{f},$$

with $\mathbf{u}(\mathbf{x},0)=\mathbf{u}_0(\mathbf{x})\in H^r$, *where* $D/Dt=\partial/\partial t+\mathbf{v}\cdot\nabla$. *Moreover,* $\mathbf{u}$ *satisfies the estimate* $\|\mathbf{u}\|_r^2\le e^{\alpha(t)}\Big(\|\mathbf{u}_0\|_r^2+\int_0^t C\|\mathbf{f}\|_r^2 d\tau\Big)$, *where* $\alpha(t)=\int_0^t C(1+\|D\mathbf{v}\|_{r-1})d\tau$ *and* C *depends on* r.

The proof of the above lemma is standard; see, for example, [5].

Lemma 4.4. *Given* a *in* $C([0,T],H^0)\cap L^\infty([0,T],H^r)$ *and* $\mathbf{f}$ *in* $C([0,T],H^0)\cap L^\infty([0,T],H^{r-1})$, *where* $r>\frac{N}{2}+3$, $a(\mathbf{x},t)\ge c_1$, *with* $c_1>0$ *for* $\mathbf{x}\in\Omega$, $\Omega=\mathbb{T}^N$, $0\le t\le T$, *there is a classical solution* $\rho\in C([0,T],H^0)\cap L^\infty([0,T],H^{r+2})$ *of*

$$c\Delta^2\rho-\nabla\cdot(a\nabla\rho)=\nabla\cdot\mathbf{f} \tag{4.1}$$

Here, c *is a positive constant. And* ρ *satisfies the estimate*

$$\|\nabla\rho\|_{r+1}^2\le C\left(\|\Delta\rho\|_r^2+\|\nabla\rho\|_r^2\right)\le C\Big[1+\sum_{j=1}^r\|Da\|_{r-1}^{2j}\Big]\|\mathbf{f}\|_{r-1}^2\le C\|\mathbf{f}\|_{r-1}^2$$

Proof. The operator in (4.1) is linear with $a(\mathbf{x},t)\ge c_1$, where $c_1>0$ for $\mathbf{x}\in\Omega$, $\Omega=\mathbb{T}^N$, and c is a positive constant, and the compatibility condition $\int_\Omega\nabla\cdot\mathbf{f}dx=0$ is satisfied. The existence of a solution ρ follows from the standard theory for elliptic equations, specifically, the Lax-Milgram Lemma (see, for example, [6]). The a priori estimate follows from Lemma 2.7. □

Now we prove the existence of a solution to (3.1), (3.2).

Lemma 4.5. *Given* $\mathbf{v}$ *in* $C([0,T],H^0)\cap L^\infty([0,T],H^r)$ *and* a *in* $C([0,T],H^0)\cap L^\infty([0,T],H^r)$, *and given* $r>\frac{N}{2}+3$, $a(\mathbf{x},t)\ge c_1$, *with* $c_1>0$ *for* $\mathbf{x}\in\Omega$, $\Omega=\mathbb{T}^N$, $0\le t\le T$, *there exists a classical solution* ρ, $\mathbf{w}$ *of*

$$\frac{D\mathbf{w}}{Dt}=-a\nabla\rho+c\nabla\Delta\rho, \tag{4.2}$$

$$\nabla\cdot\mathbf{w}=0, \tag{4.3}$$

$$\mathbf{w}(\mathbf{x},0)=\mathbf{w}_0(\mathbf{x}),\quad \nabla\cdot\mathbf{w}_0=0, \tag{4.4}$$

with $\mathbf{w}\in C([0,T],C^3)\cap L^\infty([0,T],H^r)$, *and* $\rho\in C([0,T],C^5)\cap L^\infty([0,T],H^{r+2})$. *Here,* c *is a positive constant, and* $D/Dt=\partial/\partial t+\mathbf{v}\cdot\nabla$, *and* $\nabla\cdot\mathbf{v}=0$.

Proof. First, we reduce the equations to an equivalent system by employing the projections P and $Q=I-P$, where P is the orthogonal projection of L^2 onto the solenoidal vector field and Q is the orthogonal projection of L^2 onto the gradient field. Applying the operator P to (4.2), and using the fact that $P\mathbf{w}=\mathbf{w}$, we obtain the equation

$$\frac{D\mathbf{w}}{Dt}=J:=Q(\mathbf{v}\cdot\nabla P\mathbf{w})-P(a\nabla\rho). \tag{4.5}$$

Applying the operator Q to (4.2), we obtain the equation

$$Q(\mathbf{v}\cdot\nabla P\mathbf{w}+a\nabla\rho-c\nabla\Delta\rho)=0. \tag{4.6}$$

This equation is equivalent to the equation

$$Q(Q(\mathbf{v}\cdot\nabla P\mathbf{w})+a\nabla\rho-c\nabla\Delta\rho)=0. \tag{4.7}$$

From the definition of Q, we observe that equation (4.7) is equivalent to

$$c\Delta^2\rho-\nabla\cdot(a\nabla\rho)=\nabla\cdot Q(\mathbf{v}\cdot\nabla P\mathbf{w}). \tag{4.8}$$

With the given initial data (4.4), the two systems of equations (4.2), (4.3), and (4.5), (4.8) are equivalent. (The proof is standard; see, for example, [5]).

Next, we construct the solution of the system (4.5), (4.8) using the method of successive approximation as follows: Set $\mathbf{w}^0 = \mathbf{w}_0(\mathbf{x})$ and set ρ^0 to be the initial iterate from Theorem 3.1. For $k = 0, 1, 2, \dots$ define $\mathbf{w}^{k+1}$, ρ^{k+1} as the solution of the equations

$$\frac{D\mathbf{w}^{k+1}}{Dt} = J^k, \tag{4.9}$$

$$c\Delta\rho^{k+1} - \nabla\cdot(a\nabla\rho^{k+1}) = \nabla\cdot Q(\mathbf{v}\cdot\nabla(P\mathbf{w}^{k+1})), \tag{4.10}$$

where $J^k = Q(\mathbf{v}\cdot\nabla P\mathbf{w}^k) - P(a\nabla\rho^k)$. And $\mathbf{w}^{k+1}(\mathbf{x}, 0) = \mathbf{w}_0 \in H^r$, with $\nabla\cdot\mathbf{w}_0 = 0$.

The first step is to solve (4.9) for $\mathbf{w}^{k+1}$. By Lemma 4.1 and by the induction hypothesis we have $Q(\mathbf{v}\cdot\nabla P\mathbf{w}^k) \in C([0,T], H^0) \cap L^\infty([0,T], H^r)$. Furthermore, from Lemma 4.2 and the induction hypothesis we have $P(a\nabla\rho^k) \in C([0,T], H^0) \cap L^\infty([0,T], H^r)$. Therefore, the existence of a solution $\mathbf{w}^{k+1} \in C([0,T], H^0) \cap L^\infty([0,T], H^r)$ to (4.9) follows from Lemma 4.3.

The next step is, given the solution $\mathbf{w}^{k+1}$ just obtained, to solve (4.10) for ρ^{k+1}. From Lemma 4.1 we have $Q(\mathbf{v}\cdot\nabla(P\mathbf{w}^{k+1})) \in C([0,T], H^0) \cap L^\infty([0,T], H^r)$, and therefore it follows from Lemma 4.4 that there is a solution $\rho^{k+1} \in C([0,T], H^0) \cap L^\infty([0,T], H^{r+2})$ to (4.10). And since ρ^{k+1} is unique up to an arbitrary function of t, we specify that $\rho^{k+1}(\mathbf{x}_0, t) = \rho^k(\mathbf{x}_0, t)$ for all $k \ge 0$, at the single fixed point $\mathbf{x}_0 \in \Omega$, so $\rho^{k+1}(\mathbf{x}_0, t) = \rho^0(\mathbf{x}_0, t)$.

Next, we derive estimates for $\|\rho^{k+1} - \rho^k\|_{r+2,T}$ and $\|\mathbf{w}^{k+1} - \mathbf{w}^k\|_{r,T}$. We see that $\rho^{k+1} - \rho^k$ and $\mathbf{w}^{k+1} - \mathbf{w}^k$ solve the system of equations

$$\frac{D(\mathbf{w}^{k+1} - \mathbf{w}^k)}{Dt} = J^k - J^{k-1}, \tag{4.11}$$

$$c\Delta^2(\rho^{k+1} - \rho^k) - \nabla\cdot(a\nabla(\rho^{k+1} - \rho^k)) = \nabla\cdot Q(\mathbf{v}\cdot\nabla(P(\mathbf{w}^{k+1} - \mathbf{w}^k))), \tag{4.12}$$

where $J^k - J^{k-1} = Q(\mathbf{v}\cdot\nabla P(\mathbf{w}^k - \mathbf{w}^{k-1})) - P(a\nabla(\rho^k - \rho^{k-1}))$. Initially we have $(\mathbf{w}^{k+1} - \mathbf{w}^k)(0) = 0$. First, we derive the estimate for $\|\mathbf{w}^{k+1} - \mathbf{w}^k\|_{r,T}$. From (4.11) and from Lemmas 4.1, 4.2, and 4.3, we obtain the estimate

$$\begin{aligned} \|\mathbf{w}^{k+1} - \mathbf{w}^k\|_r^2 &\le Ce^{\alpha(t)} \int_0^t \|Q(\mathbf{v}\cdot\nabla P(\mathbf{w}^k - \mathbf{w}^{k-1}))\|_r^2 + \|P(a\nabla(\rho^k - \rho^{k-1}))\|_r^2 d\tau \\ &\le C_1 \int_0^t \|\mathbf{w}^k - \mathbf{w}^{k-1}\|_r^2 + \|\nabla(\rho^k - \rho^{k-1})\|_{r-1}^2 d\tau \end{aligned} \tag{4.13}$$

Here $\alpha(t) = C\int_0^t (1 + \|D\mathbf{v}\|_{r-1}) d\tau$, C depends on r, and C_1 depends on r, $\|\mathbf{v}\|_{r,T}$ and $\|a\|_{r,T}$. Next, from (4.12), and from Lemmas 4.1 and 4.4, we obtain the estimate

$$\begin{aligned} \|\nabla(\rho^{k+1} - \rho^k)\|_{r+1}^2 &\le C[1 + \sum_{j=1}^r \|Da\|_{r-1}^{2j}] \|Q(\mathbf{v}\cdot\nabla(P(\mathbf{w}^{k+1} - \mathbf{w}^k)))\|_{r-1}^2 \\ &\le C_2 \|\mathbf{w}^{k+1} - \mathbf{w}^k\|_r^2 \\ &\le C_3 \int_0^t \left(\|\mathbf{w}^k - \mathbf{w}^{k-1}\|_r^2 + \|\nabla(\rho^k - \rho^{k-1})\|_{r-1}^2\right) d\tau \end{aligned} \tag{4.14}$$

where C_2 and C_3 depend on r, $\|a\|_{r,T}$, and $\|\mathbf{v}\|_{r,T}$. Here, we used the estimate (4.13) in the last step. From Lemma 2.4 and (4.14), we have the estimate

$$\begin{aligned}\|\rho^{k+1}-\rho^k\|_0^2 &\le C\|\nabla(\rho^{k+1}-\rho^k)\|_2^2 \\ &\le C_4\|\mathbf{w}^{k+1}-\mathbf{w}^k\|_r^2 \\ &\le C_5\int_0^t \big(\|\mathbf{w}^k-\mathbf{w}^{k-1}\|_r^2+\|\nabla(\rho^k-\rho^{k-1})\|_{r-1}^2\big)\,d\tau\end{aligned} \tag{4.15}$$

where C_4 and C_5 depend on r, $\|a\|_{r,T}$, and $\|\mathbf{v}\|_{r,T}$, and where $\rho^{k+1}(\mathbf{x}_0,t)=\rho^k(\mathbf{x}_0,t)$ for all $k\ge 0$, at the single fixed point $\mathbf{x}_0\in\Omega$. Adding the estimates (4.13), (4.14), (4.15), we obtain

$$\|\mathbf{w}^{k+1}-\mathbf{w}^k\|_r^2+\|\rho^{k+1}-\rho^k\|_{r+2}^2 \le C_6\int_0^t \big(\|\mathbf{w}^k-\mathbf{w}^{k-1}\|_r^2+\|\rho^k-\rho^{k-1}\|_{r+2}^2\big)\,d\tau \tag{4.16}$$

where C_6 depends on r, $\|\mathbf{v}\|_{r,T}$, $\|a\|_{r,T}$. Repeated application of (4.16) yields

$$\|\mathbf{w}^{k+1}-\mathbf{w}^k\|_{r,T}^2+\|\rho^{k+1}-\rho^k\|_{r+2,T}^2 \le \frac{(C_6T)^k}{k!}\big(\|\mathbf{w}^1-\mathbf{w}^0\|_{r,T}^2+\|\rho^1-\rho^0\|_{r+2,T}^2\big)$$

from which it follows that

$$\sum_{k=1}^{\infty}\big(\|\mathbf{w}^{k+1}-\mathbf{w}^k\|_{r,T}^2+\|\rho^{k+1}-\rho^k\|_{r+2,T}^2\big)<\infty.$$

Hence, $\|\mathbf{w}^{k+1}-\mathbf{w}^k\|_{r,T}^2\to 0$ as $k\to\infty$ and $\|\rho^{k+1}-\rho^k\|_{r+2,T}^2\to 0$ as $k\to\infty$. Therefore, we deduce that there exist $\mathbf{w}\in C([0,T],H^0)\cap L^\infty([0,T],H^r)$ such that $\mathbf{w}^k\to\mathbf{w}$ as $k\to\infty$ strongly in $C([0,T],H^0)\cap L^\infty([0,T],H^r)$, and there exists $\rho\in C([0,T],H^0)\cap L^\infty([0,T],H^{r+2})$ such that $\rho^k\to\rho$ as $k\to\infty$ strongly in $C([0,T],H^0)\cap L^\infty([0,T],H^{r+2})$. The fact that $\mathbf{w}$ and ρ is a solution to the system of equations (4.5), (4.8) follows by a standard argument. And therefore $\mathbf{w}$, ρ is a solution to the equivalent system (4.2), (4.3). □

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