Electronic Journal of Differential Equations, Vol. 2007(2007), No. 90, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

INFINITELY MANY WEAK SOLUTIONS FOR A p-LAPLACIAN EQUATION WITH NONLINEAR BOUNDARY CONDITIONS

JI-HONG ZHAO, PEI-HAO ZHAO

Abstract. We study the following quasilinear problem with nonlinear boundary conditions

$$-\Delta_p u + a(x)|u|^{p-2}u = f(x,u) \quad \text{in } \Omega,$$
$$|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = g(x,u) \quad \text{on } \partial\Omega,$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary and $\frac{\partial}{\partial\nu}$ is the outer normal derivative, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian with $1 < p < N$. We consider the above problem under several conditions on f and g, where f and g are both Carathéodory functions. If f and g are both superlinear and subcritical with respect to u, then we prove the existence of infinitely many solutions of this problem by using "fountain theorem" and "dual fountain theorem" respectively. In the case, where g is superlinear but subcritical and f is critical with a subcritical perturbation, namely $f(x,u) = |u|^{p^*-2}u + \lambda|u|^{r-2}u$, we show that there exists at least a nontrivial solution when $p < r < p^*$ and there exist infinitely many solutions when $1 < r < p$, by using "mountain pass theorem" and "concentration-compactness principle" respectively.

1. Introduction

Consider a quasilinear elliptic problem

$$\begin{gathered}-\Delta_p u + a(x)|u|^{p-2}u = f(x,u) \quad \text{in } \Omega,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = g(x,u) \quad \text{on } \partial\Omega,\end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary and $\frac{\partial}{\partial\nu}$ is the outer normal derivative, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian with $1 < p < N$,

$$a(x) \in L^\infty(\Omega) \text{ satisfying } \operatorname*{ess\,inf}_{x\in\overline{\Omega}} a(x) > 0. \tag{1.2}$$

The study of nonlinear elliptic boundary value problem about p-Laplacian of the form (1.1) is an interesting topic in recent years. Many results have been obtained

2000 *Mathematics Subject Classification.* 35J20, 35J25.
Key words and phrases. p-Laplacian; nonlinear boundary conditions; weak solutions; critical exponent; variational principle.

Submitted March 26, 2007. Published June 15, 2007.
This work was partly supported by the Fundamental Research Fund for Physics and Mathematics of Lanzhou University.

on this kind of problem, for example see [19, 7, 4, 13, 17, 18, 2] and the references therein. Such problem appear naturally in the study of optimal constants for Sobolev trace embedding and it arises in various applications, e.g. non-Newtonian fluids, reaction-diffusion problems, glaciology, biology etc(see [8, 1, 5, 3, 2]). The first paper that analyzed (1.1) is [4]. In that paper, the authors systematically studied the existence of nontrivial solutions of (1.1) under $f(u) = |u|^{p-2}u$ and g are subcritical, critical with a subcritical perturbation and supercritical with respect to u. Using the ideas from [12], they established the existence results, nonexistence result, especially the result of nonlinear eigenvalue problem. In [2], the author proved the existence of at least three nontrivial solutions for (1.1) under adequate assumptions on the source terms f and g. On the other hand, when Ω is unbounded, we can see [19, 7, 13] for some existence and multiplicity results of solutions to problem (1.1) in some weighted Sobolev spaces. Our aim in this paper is to prove that the infinitely many solutions results for the problem (1.1) under various assumptions on nonlinear terms f and g. If f and g are both superlinear and subcritical with respect to u, then we prove the existence of infinitely many solutions of problem (1.1) by using "fountain theorem" and "dual fountain theorem" respectively. In the case, where g is superlinear but subcritical and f is critical with a subcritical perturbation, namely $f(x,u) = |u|^{p^*-2}u+\lambda|u|^{r-2}u$, we show that there exists at least a nontrivial solution when $p < r < p^*$ and there exist infinitely many solutions when $1 < r < p$, by using "mountain pass theorem" and "concentration-compactness principle" respectively. The main ideas of our paper is from [20, 4].

Throughout this paper the following hypotheses are assumed.

(F1) $f(x,u)$ is a Carathéodory function and for some $p < q < p^* = \frac{Np}{N-p}$, there exists a constant $C_1 > 0$ such that
$$|f(x,u)| \le C_1(1+|u|^{q-1}) \quad \text{for all } x \in \Omega,\ u \in \mathbb{R}.$$
(G1) $g(x,u)$ is a Carathéodory function and for some $p < z < p_* = \frac{(N-1)p}{N-p}$, there exists a constant $C_2 > 0$ such that
$$|g(x,u)| \le C_2(1+|u|^{z-1}) \quad \text{for all } x \in \partial\Omega,\ u \in \mathbb{R}.$$
(F2) There exists $\alpha_1 > p$ and $R > 0$ such that
$$|u| \ge R \Longrightarrow 0 < \alpha_1 F(x,u) \le uf(x,u) \quad \text{for all } x \in \Omega,$$
where $F(x,u) = \int_0^u f(x,t)dt$ is the primitive function of $f(x,u)$.

(G2) There exists $\alpha_2 > p$ and $R > 0$ such that
$$|u| \ge R \Longrightarrow 0 < \alpha_2 G(x,u) \le ug(x,u) \quad \text{for all } x \in \partial\Omega,$$
where $G(x,u) = \int_0^u g(x,t)dt$ is the primitive function of $g(x,u)$.

(F3) $f(x,u)$ is an odd function with respect to u, that is,
$$f(x,-u) = -f(x,u) \quad \text{for all } x \in \Omega.$$
(G3) $g(x,u)$ is an odd function with respect to u, that is,
$$g(x,-u) = -g(x,u) \quad \text{for all } x \in \Omega.$$
(G4) $\lim_{u\to 0} \frac{g(x,u)}{|u|^{p-1}} = 0$.

Define $W^{1,p}(\Omega) = \{u \in L^p(\Omega) : \int_\Omega |\nabla u|^p dx < \infty\}$ with the norm
$$\|u\|_{1,p} := (\int_\Omega (|\nabla u|^p + a(x)|u|^p)dx)^{\frac{1}{p}}. \tag{1.3}$$

Then $W^{1,p}(\Omega)$ is a Banach space. For a variational approach, the functional associated to the problem (1.1) is

$$\varphi(u)=\frac{1}{p}\int_\Omega(|\nabla u|^p+a(x)|u|^p)\mathrm{d}x-\int_\Omega F(x,u)\mathrm{d}x-\int_{\partial\Omega}G(x,u)\mathrm{d}S, \tag{1.4}$$

where $u\in W^{1,p}(\Omega)$ and $\mathrm{d}S$ is the measure on the boundary. Since (F1) and (G1) we can easily to obtain $\varphi\in C^1(W^{1,p}(\Omega),\mathbb{R})$ and

$$\begin{aligned}\langle\varphi'(u);v\rangle=&\int_\Omega(|\nabla u|^{p-2}\nabla u\nabla v+a(x)|u|^{p-2}uv)\mathrm{d}x\\&-\int_\Omega f(x,u)v\mathrm{d}x-\int_{\partial\Omega}g(x,u)v\mathrm{d}S\end{aligned}$$

for all $u,v\in W^{1,p}(\Omega)$. We say that u is a weak solution of the problem (1.1) if u is the critical point of the functional φ on $W^{1,p}(\Omega)$.

Remark 1.1. According to the regularity theorem of [14], if $\partial\Omega$ is of class $C^{1,\alpha}(0<\alpha\le 1)$ and g satisfies

$$|g(x,u)-g(y,v)|\le C(|x-y|^\alpha+|u-v|^\alpha),\quad |g(x,u)|\le C$$

for all $x,y\in\Omega,u,v\in\mathbb{R}$, then the regularity up to the boundary of [14, Theorem 2] shows that every weak solution of (1.1) belongs to $C^{1,\beta}_{\mathrm{loc}}(\overline{\Omega})$ for some $0<\beta\le 1$.

Remark 1.2. Under the assumption (1.2) it is easy to check that the norm (1.3) is equivalent to the usual one, that is the norm with $a(x)\equiv 1$ in (1.3).

Our main results are as follows.

Theorem 1.3. *Under the assumptions (F1)–(F3) and (G1)–(G3), problem* (1.1) *has a sequence of solutions $u_k\in W^{1,p}(\Omega)$ such that $\varphi(u_k)\to\infty$ as $k\to\infty$.*

For a special f, we obtain a sequence of weak solutions with negative energy.

Theorem 1.4. *Let $f(x,u)=\mu|u|^{r-2}u+\lambda|u|^{s-2}u$, where $1<r<p<s<p^*$ and assume $(G_1)\sim(G_4)$ are satisfied. Then*

(1) *for every $\lambda>0$, $\mu\in\mathbb{R}$, problem* (1.1) *has a sequence of solutions $u_k\in W^{1,p}(\Omega)$ such that $\varphi(u_k)\to\infty$ as $k\to\infty$,*
(2) *for every $\mu>0$, $\lambda\ge 0$, problem* (1.1) *has a sequence of solutions $v_k\in W^{1,p}(\Omega)$ such that $\varphi(v_k)<0$, $\varphi(v_k)\to 0$ as $k\to\infty$.*

Next we consider the critical growth on f. In this case, the compactness of the embedding $W^{1,p}(\Omega)\hookrightarrow L^{p^*}(\Omega)$ fails, so to recover some sort of compactness, in spirit of [6], we consider a perturbation of the critical power, that is, $f(x,u)=|u|^{p^*-2}u+\lambda|u|^{r-2}u$. We also need much more assumptions on g around about the origin.

(G2') there exists $\alpha_2>p$ such that

$$0<\alpha_2G(x,u)\le ug(x,u)\quad\text{for all } x\in\partial\Omega,\ u\in\mathbb{R}\setminus\{0\}.$$

Here we use the "concentration-compactness principle" introduced in [15, 16]. We prove the following two theorems.

Theorem 1.5. *Let $f(x,u)=|u|^{p^*-2}u+\lambda|u|^{r-2}u$ with $p<r<p^*$ and assume (G1), (G2), (G3) and (G4) are satisfied. Then there exists a constant $\lambda_0>0$ depending on p, r, N and $|\Omega|$ such that if $\lambda>\lambda_0$, problem* (1.1) *has at least a nontrivial solution in $W^{1,p}(\Omega)$.*

Theorem 1.6. *Let $f(x,u) = |u|^{p^*-2}u + \lambda |u|^{r-2}u$ with $1 < r < p$ and assume (G1), (G2), (G3) and (G4) are satisfied. Then their exists a constant $\widetilde{\lambda} > 0$ depending on p, r, N and $|\Omega|$ such that if $0 < \lambda < \widetilde{\lambda}$, problem (1.1) has infinitely many nontrivial solutions $u_k \in W^{1,p}(\Omega)$ such that $\varphi(u_k) < 0$, $\varphi(u_k) \to 0$ as $k \to \infty$.*

This paper is organized as follows. In the second section, we recall some definitions and preliminary theorems, including the well-known "fountain theorem" and "dual fountain theorem". The $(PS)_c$ condition and $(PS)_c^*$ condition are also introduced. In the third section, we consider the subcritical case and give the proof of Theorem 1.3 and Theorem 1.4. In the last section. We consider the critical case and give the proof of Theorems 1.5 and 1.6.

2. Preliminaries

First we introduced some notations: X denotes Banach space with the norm $\|\cdot\|_X$, X^* denotes the conjugate space with X, $L^p(\Omega)$ denotes Lebesgue space with the usual norm $|\cdot|_p$, $W^{1,p}(\Omega)$ denotes Sobolev space with the norm $\|\cdot\|_{1,p}$ defined by (1.3), $\langle \cdot ; \cdot \rangle$ is the dual paring of the space X^* and X, $|\Omega|$ denotes the Lebesgue measure of the set $\Omega \subset \mathbb{R}^N$, $C_1, C_2, \dots,$ denote (possibly different) positive constants.

One important aspect of applying the standard methods of variational theory is to show that the functional φ satisfies the $(PS)_c$ or $(PS)_c^*$ condition which is introduced the following definition.

Definition 2.1. Let $\varphi \in C^1(X, \mathbb{R})$ and $c \in \mathbb{R}$. The function φ satisfies the $(PS)_c$ condition if any sequence $\{u_n\} \subset X$ such that

$$\varphi(u_n) \to c, \quad \varphi'(u_n) \to 0 \quad \text{in } X^* \text{ as } n \to \infty$$

contains a subsequence converging to a critical point of φ.

Let X be a reflexive and separable Banach space, then there are $e_j \in X$ and $e_j^* \in X^*$ such that

$$X = \overline{\operatorname{span}\{e_j | j = 1, 2, \dots\}}, \quad X^* = \overline{\operatorname{span}\{e_j^* | j = 1, 2, \dots\}},$$

$$\langle e_i^*; e_j \rangle = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

For convenience, we write $X_j := \operatorname{span}\{e_j\}$, $Y_k := \oplus_{j=1}^k X_j$, $Z_k := \overline{\oplus_{j=k}^\infty X_j}$. And let $B_k := \{u \in Y_k : \|u\|_X \le \rho_k\}$, $N_k := \{u \in Z_k : \|u\|_X = \gamma_k\}$, where $\rho_k > \gamma_k > 0$.

Definition 2.2. Let $\varphi \in C^1(X, \mathbb{R})$ and $c \in \mathbb{R}$. The function φ satisfies the $(PS)_c^*$ condition (with respect to (Y_n)) if any sequence $\{u_{n_j}\} \subset Y_{n_j}$ such that

$$\varphi(u_{n_j}) \to c, \quad \varphi|'_{Y_{n_j}}(u_{n_j}) \to 0 \quad \text{in } X^* \text{ as } n_j \to \infty$$

contains a subsequence converging to a critical point of φ.

Theorem 2.3 (Fountain theorem, [20, Thm. 3.6]). *Let $\varphi \in C^1(X, \mathbb{R})$ be an even functional. If , for every $k \in \mathbb{N}$, there exists $\rho_k > \gamma_k > 0$ such that*

- (A1) $a_k := \sup_{u \in Y_k, \|u\|_X = \rho_k} \varphi(u) \le 0$,
- (A2) $b_k := \inf_{u \in Z_k, \|u\|_X = \gamma_k} \varphi(u) \to \infty$ *as* $k \to \infty$,
- (A3) φ *satisfies the* $(PS)_c$ *condition for every* $c > 0$.

Then φ has an unbounded sequence of critical values.

Theorem 2.4 (Dual fountain theorem, [20, Theorem 3.18])**.** *Let $\varphi \in C^1(X, \mathbb{R})$ be an even functional. If , for every $k \geq k_0$, there exists $\rho_k > \gamma_k > 0$ such that*

- (B1) $a_k := \inf_{u \in Z_k,\, \|u\|_X = \rho_k} \varphi(u) \geq 0$,
- (B2) $b_k := \sup_{u \in Y_k,\, \|u\|_X = \gamma_k} \varphi(u) < 0$,
- (B3) $d_k := \inf_{u \in Z_k,\, \|u\|_X \leq \rho_k} \varphi(u) \to 0$ *as* $k \to \infty$,
- (B4) φ *satisfies the* $(PS)_c^*$ *condition for every* $c \in [d_{k_0}, 0[$.

Then φ has a sequence of negative critical values converging to 0.

3. Proof of Theorem 1.3

Proof of the $(PS)_c$ condition. Let us introduce the following lemmas which will be helpful in the proof.

Lemma 3.1 ([18, Lemma 2.1])**.** *Let $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be the function given by $\langle A(u); v\rangle := \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v \mathrm{d}x + \int_\Omega a(x)|u|^{p-2}uv\mathrm{d}x$. Then A is continuous, odd, $(p-1)$-homogeneous, and continuously invertible.*

Lemma 3.2 ([18, Lemma 2.2])**.** *Let $B : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be the function given by $\langle B(u); v\rangle := \int_{\partial\Omega} g(x,u)v\mathrm{d}S$, where $g(x,u)$ be a Carathéodory function with subcritical growth. Then B is continuous and compact.*

Lemma 3.3 ([18, Lemma 2.3])**.** *Let $C : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be the function given by $\langle C(u); v\rangle = \int_\Omega f(x,u)v\mathrm{d}x$, where $f(x,u)$ is a Carathéodory function with subcritical growth. Then C is continuous and compact.*

Lemma 3.4. *Under the hypotheses of Theorem 1.3, φ satisfies the $(PS)_c$ condition with $c > 0$.*

Proof. Suppose that $\{u_n\} \subset W^{1,p}(\Omega)$, for every $c > 0$,
$$\varphi(u_n) \to c,\ \varphi'(u_n) \to 0 \text{ in } W^{1,p}(\Omega)^* \text{ as } n \to \infty.$$

First we prove the boundness of $\{u_n\}$. After integrating, we obtain from the assumptions (F2) and (G2) that there exist C_1, $C_2 > 0$ such that
$$C_1(|u|^{\alpha_1} - 1) \leq F(x,u) \quad \text{for all } x \in \Omega,\ u \in \mathbb{R}, \tag{3.1}$$
$$C_2(|u|^{\alpha_2} - 1) \leq G(x,u) \quad \text{for all } x \in \partial\Omega,\ u \in \mathbb{R}. \tag{3.2}$$
Set $\alpha = \min\{\alpha_1, \alpha_2\}$ and choose $\frac{1}{\beta} \in (\frac{1}{\alpha}, \frac{1}{p})$, and from (3.1) and (3.2), we obtain for n sufficiently large,
$$\begin{aligned}
&c + 1 + \|u_n\|_{1,p}\\
&\geq \varphi(u_n) - \frac{1}{\beta}\langle \varphi'(u_n), u_n\rangle\\
&\geq (\frac{1}{p} - \frac{1}{\beta})\|u_n\|_{1,p}^p + (\frac{\alpha_1}{\beta} - 1)\int_\Omega F(x,u_n)\mathrm{d}x - (\frac{\alpha_2}{\beta} - 1)\int_{\partial\Omega} G(x,u_n)\mathrm{d}S\\
&\geq (\frac{1}{p} - \frac{1}{\beta})\|u_n\|_{1,p}^p + C_1(\frac{\alpha_1}{\beta} - 1)|u_n|_{\alpha_1}^{\alpha_1} + C_2(\frac{\alpha_2}{\beta} - 1)|u_n|_{L^{\alpha_2}(\partial\Omega)}^{\alpha_2} - C_3.
\end{aligned}$$
Note that $\frac{\alpha_i}{\beta} - 1 > 0 (i = 1, 2)$, then $\{u_n\}$ is bounded in $W^{1,p}(\Omega)$.

Next we show that the strongly convergence of $\{u_n\}$ in $W^{1,p}(\Omega)$. Since $\{u_n\}$ is bounded, up to a subsequence (which we still denote by $\{u_n\}$), we may assume that there exists $u \in W^{1,p}(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$ as $n \to \infty$. Note that $\varphi'(u_n) = A(u_n) - B(u_n) - C(u_n) \to 0$. By the compactness of B, C and the continuity of A^{-1}, we have

$$u_n \to A^{-1}(a(x)B(u) - C(u)) \text{ in } W^{1,p}(\Omega) \text{ as } n \to \infty.$$

Thus $u_n \to u$ in $W^{1,p}(\Omega)$. □

To prove Theorem 1.3 we also need the following two lemmas.

Lemma 3.5 ([11])**.** *If $1 \le q < p^* = \frac{Np}{N-p}$, then*

$$\beta_k := \sup_{u \in Z_k,\, \|u\|_{1,p}=1} |u|_q \to 0 \text{ as } k \to \infty.$$

Lemma 3.6 ([11])**.** *If $1 \le z < p_* = \frac{(N-1)p}{N-p}$, then*

$$\sigma_k := \sup_{u \in Z_k,\, \|u\|_{1,p}=1} |u|_{L^z(\partial\Omega)} \to 0 \text{ as } k \to \infty.$$

Remark 3.7. In [11], the authors gave a more general form of two above lemmas. Here the key step of the proof of these two lemmas is that the Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact for $1 \le q < p^*$ and the Sobolev trace embedding $W^{1,p}(\Omega) \hookrightarrow L^z(\partial\Omega)$ is compact for $1 \le z < p_*$.

The proof of Theorem 1.3.

Assumptions (F1) and (G1) and the Lemma 3.4 imply that φ is continuously differentiable on $W^{1,p}(\Omega)$ and satisfies the $(PS)_c$ condition for every $c > 0$. So from Theorem 2.3, we need only to verify ϕ satisfying the condition (A1) and (A2).

As for (3.1) and (3.2) in Lemma 3.4, we have

$$\varphi(u) \le \frac{\|u_n\|_{1,p}^p}{p} - C_1|u|_{\alpha_1}^{\alpha_1} - C_2|u|_{L^{\alpha_2}(\partial\Omega)}^{\alpha_2} - C_1|\Omega| - C_2|\partial\Omega|.$$

Since on the finite-dimensional space Y_k all norms are equivalent, so $\alpha_i > p$ $(i = 1, 2)$ implies that (A1) is satisfied for $\rho_k > 0$ large enough.

After integrating, we obtain from the assumptions (F1) and (G1) that there exist constants $C_1, C_2 > 0$ such that

$$F(x, u) \le C_1(1 + |u|^q), \quad G(x, u) \le C_2(1 + |u|^z).$$

Let us define

$$\beta_k := \sup_{u \in Z_k,\, |u\|_{1,p}=1} |u|_q, \quad d\sigma_k := \sup_{u \in Z_k,\, \|u\|_{1,p}=1} |u|_{L^z(\partial\Omega)}.$$

On Z_k, we have

$$\begin{aligned}
\varphi(u) &= \frac{1}{p}\|u\|_{1,p}^p - \int_\Omega F(x, u)\mathrm{d}x - \int_{\partial\Omega} G(x, u)\mathrm{d}S \\
&\ge \frac{1}{p}\|u\|_{1,p}^p - C_1|u|_q^q - C_1|\Omega| - C_2|u|_{L^z(\partial\Omega)}^z - C_2|\partial\Omega| \\
&\ge \frac{1}{p}\|u\|_{1,p}^p - C_1\beta_k^q\|u\|_{1,p}^q - C_2\sigma_k^z\|u\|_{1,p}^z - C_3.
\end{aligned}$$

Let

$$\begin{aligned}
&\frac{1}{4p}\rho^p - C_1\beta_k^q\rho^q = 0, \\
&\frac{1}{4p}\rho^p - C_2\sigma_k^z\rho^z = 0.
\end{aligned}$$

From these two equations, we have $\rho_k := (4pC_1\beta_k^q)^{\frac{1}{p-q}}$, $\rho_k' := (4pC_2\sigma_k^z)^{\frac{1}{p-z}}$. From Lemmas 3.5 and 3.6 we know that $\beta_k \to 0$, $\sigma_k \to 0$ as $k \to \infty$. So, we know

$$\rho_k \to \infty \text{ as } k \to \infty, \quad \rho_k' \to \infty \text{ as } k \to \infty. \tag{3.3}$$

Let

$$\gamma_k = \min\{\rho_k, \rho_k'\}, \tag{3.4}$$

we obtain, if $u \in Z_k$ and $\|u\|_{1,p} = \gamma_k$, then $\varphi(u) \geq \frac{1}{2p}\gamma_k^p - C_3$. From (3.3) and (3.4), so (A2) is proved. It suffices then to use the fountain theorem to complete the proof.

Here, we show two examples for readers for special cases of f to understand our theorem.

Example 3.8. Let $p < q < p^*$ and assumed (G1)–(G3) are satisfied. We consider the quasilinear elliptic equation

$$\begin{gathered} -\Delta_p u + a(x)|u|^{p-2}u = |u|^{q-2}u \quad \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = g(x,u) \quad \text{on } \partial\Omega. \end{gathered}$$

This problem has a sequence of solutions $\{u_k\}$ such that $\varphi(u_k) \to \infty$ as $k \to \infty$.

Example 3.9. Let $1 < r < p < s < p^*$ and assumed (G1)–(G3) are satisfied. We consider the quasilinear elliptic equation

$$\begin{gathered} -\Delta_p u + a(x)|u|^{p-2}u = \mu|u|^{r-2}u + \lambda|u|^{s-2}u \quad \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = g(x,u) \quad \text{on } \partial\Omega. \end{gathered}$$

Then for every $\lambda > 0$, $\mu \in \mathbb{R}$, this problem has a sequence of solutions $\{u_k\}$ such that $\varphi(u_k) \to \infty$ as $k \to \infty$.

Proof of Theorem 1.4. The first conclusion of Theorem 1.4 is just example 3.9. We shall prove Theorem 1.4 by using Theorem 2.4, so we need to verify the condition (B1)–(B4). Now we assume that $\mu > 0$.

To verify (B1), we define $\beta_k := \sup_{\substack{u \in Z_k \\ \|u\|_{1,p}=1}} |u|_r$. From the assumptions (G1) and (G4), we have $G(x,u) \leq \varepsilon|u|^p + C|u|^z$, where $\varepsilon \to 0$ as $|u| \to 0$. So from the Sobolev trace embedding, we have

$$\begin{aligned} \varphi(u) &= \frac{1}{p}\|u\|_{1,p}^p - \frac{\mu}{r}|u|_r^r - \frac{\lambda}{s}|u|_s^s - \int_{\partial\Omega} G(x,u)\mathrm{d}S \\ &\geq \frac{1}{p}\|u\|_{1,p}^p - \frac{\mu}{r}\beta_k^r\|u\|_{1,p}^r - \frac{\lambda}{s}C_1\|u\|_{1,p}^s - \varepsilon|u|_{L^p(\partial\Omega)}^p - C_2|u|_{L^z(\partial\Omega)}^z \\ &\geq \frac{1}{p}\|u\|_{1,p}^p - \frac{\mu}{r}\beta_k^r\|u\|_{1,p}^r - \frac{\lambda}{s}C_1\|u\|_{1,p}^s - \varepsilon C_3\|u\|_{1,p}^p - C_4\|u\|_{1,p}^z. \\ &\geq (\frac{1}{p} - \varepsilon C_3)\|u\|_{1,p}^p - \frac{\mu}{r}\beta_k^r\|u\|_{1,p}^r - \frac{\lambda}{s}C_1\|u\|_{1,p}^s - C_4\|u\|_{1,p}^z. \end{aligned}$$

Since $p < s$ and $p < z$, there exists $R > 0$ such that $\|u\|_{1,p} \leq R$. We have

$$\begin{gathered} \frac{1}{4p}\|u\|_{1,p}^p - \frac{\lambda}{s}C_1\|u\|_{1,p}^s \geq 0, \\ \frac{1}{4p}\|u\|_{1,p}^p - C_4\|u\|_{1,p}^z \geq 0. \end{gathered}$$

From these two inequalities, it follows that

$$\varphi(u) \geq (\frac{1}{2p} - \varepsilon C_3)\|u\|_{1,p}^p - \frac{\mu}{r}\beta_k^r\|u\|_{1,p}^r. \tag{3.5}$$

Choose ε so small such that $\frac{1}{2p} - \varepsilon C_3 > 0$ and let $\rho_k := [(\frac{1}{2p} - \varepsilon C_3)^{-1}\frac{\mu}{r}\beta_k^r]^{\frac{1}{p-r}}$, by Lemma 3.5, $\beta_k \to 0$ as $k \to \infty$, it follows that $\rho_k \to 0$ as $k \to \infty$. So there exists k_0 such that $\rho_k \leq R$ when $k \geq k_0$. Thus, for $k \geq k_0$, $u \in Z_k$ and $\|u\|_{1,p} = \rho_k$, we have $\varphi(u) \geq 0$ and (B1) is proved.

From (G2) we know there exists $C > 0$ such that $C(|u|^{\alpha_2} - 1) \leq G(x,u)$. Then, we have

$$\begin{aligned}
\varphi(u) &= \frac{1}{p}\|u\|_{1,p}^p - \frac{\mu}{r}|u|_r^r - \frac{\lambda}{s}|u|_s^s - \int_{\partial\Omega} G(x,u)\mathrm{d}S \\
&\leq \frac{1}{p}\|u\|_{1,p}^p - \frac{\mu}{r}|u|_r^r - \frac{\lambda}{s}|u|_s^s - C|u|_{L^{\alpha_2}(\partial\Omega)}^{\alpha_2} - C|\partial\Omega|.
\end{aligned}$$

Since on the finite dimensional space Y_k all norms are equivalent, as $r < p$, so if $\mu > 0$ then (B2) is satisfied for every $r_k > 0$ small enough.

We obtain from (3.5), for $k \geq k_0$, $u \in Z_k, \|u\|_{1,p} \leq \rho_k$, $\varphi(u) \geq -\frac{\mu}{r}\beta_k^r\rho_k^r$, since $\beta_k \to 0$ and $\rho_k \to 0$ as $k \to \infty$, (B3) is also satisfied.

Finally we proved the $(PS)_c^*$ condition. Consider a sequence $u_{n_j} \in Y_{n_j}$ such that

$$\varphi(u_{n_j}) \to c,\ \varphi|'_{Y_{n_j}}(u_{n_j}) \to 0 \text{ in } W^{1,p}(\Omega)^* \text{ as } n_j \to \infty.$$

For n_j big enough, let $\zeta = \min\{s, \alpha_2\}$ and choose $\frac{1}{\beta} \in (\frac{1}{\zeta}, \frac{1}{p})$. Now as $\lambda \geq 0$ we have

$$\begin{aligned}
c + 1 + \|u_{n_j}\|_{1,p} &\geq \varphi(u_{n_j}) - \frac{1}{\beta}\langle\varphi'(u_{n_j}); u_{n_j}\rangle \\
&\geq (\frac{1}{p} - \frac{1}{\beta})\|u_{n_j}\|_{1,p}^p - \mu(\frac{1}{r} - \frac{1}{\beta})|u_{n_j}|_r^r.
\end{aligned}$$

We can obtain the boundness of (u_{n_j}) in $W^{1,p}(\Omega)$ since $1 < r < p$. Going if necessary to a subsequence, we can assume that $u_{n_j} \rightharpoonup u$ in $W^{1,p}(\Omega)$. As in Lemma 3.4, it is easy to conclude, $u_{n_j} \to u$ in $W^{1,p}(\Omega)$ and $\varphi'(u) = 0$.

To obtain much more general conclusion of Theorem 1.4, we make the following assumption.

(G2") There exists $\alpha_2 \geq s$ such that

$$|u| \geq R \Longrightarrow 0 < \alpha_2 G(x,u) \leq ug(x,u) \quad \text{for all } x \in \partial\Omega.$$

Corollary 3.10. *Let $1 < r < p < s < p^*$ and under the assumptions (G1), (G2"), (G3) and (G4). We consider the quasilinear elliptic equation*

$$\begin{gathered}
-\Delta_p u + a(x)|u|^{p-2}u = \mu|u|^{r-2}u + \lambda|u|^{s-2}u \quad \text{in } \Omega, \\
|\nabla u|^{p-2}\frac{\partial u}{\partial\nu} = g(x,u) \quad \text{on } \partial\Omega.
\end{gathered} \tag{3.6}$$

Then

(1) *for every $\lambda > 0$, $\mu \in \mathbb{R}$, problem (3.6) has a sequence of solutions $u_k \in W^{1,p}(\Omega)$ such that $\varphi(u_k) \to \infty$ as $k \to \infty$,*
(2) *for every $\mu > 0$, $\lambda \in \mathbb{R}$, problem (3.6) has a sequence of solutions $v_k \in W^{1,p}(\Omega)$ such that $\varphi(v_k) < 0$, $\varphi(v_k) \to 0$ as $k \to \infty$.*

Proof. We need only to prove the boundness of $\{u_{n_j}\}$ in $(PS)_c^*$ sequence. Consider a sequence $u_{n_j} \in Y_{n_j}$ such that

$$\varphi(u_{n_j}) \to c,\ \varphi|'_{Y_{n_j}}(u_{n_j}) \to 0 \text{ in } W^{1,p}(\Omega)^* \text{ as } n_j \to \infty.$$

For n_j big enough, from (G2") we have

$$\begin{aligned}
&c + 1 + \|u_{n_j}\|_{1,p} \\
&\geq \varphi(u_{n_j}) - \frac{1}{s}\langle \varphi'(u_{n_j}); u_{n_j}\rangle \\
&= (\frac{1}{p} - \frac{1}{s})\|u_{n_j}\|_{1,p}^p - \mu(\frac{1}{r} - \frac{1}{s})|u_{n_j}|_r^r + \int_{\partial\Omega}(\frac{1}{s}g(x,u_{n_j})u_{n_j} - G(x,u_{n_j}))\mathrm{d}S \\
&\geq (\frac{1}{p} - \frac{1}{s})\|u_{n_j}\|_{1,p}^p - \mu(\frac{1}{r} - \frac{1}{s})|u_{n_j}|_r^r.
\end{aligned}$$

We obtain the boundness of $\{u_{n_j}\}$ in $W^{1,p}(\Omega)$ since $1 < r < p$. □

4. Critical cases

Critical case 1. In this subsection and next subsection, we study that f has the critical growth with superlinear or sublinear perturbation in problem (1.1). In these cases, we know all of conditions of Theorem 1.4 are satisfied and we need only to prove the $(PS)_c$ condition. However, noticing that the inclusion $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ is only continuous but not compact, we can no longer expect the $(PS)_c$ condition to be hold. Thanks to the concentration-compactness principle in [15, 16]. We can prove a local $(PS)_c$ condition that will hold true for $\varphi(u)$ below a certain value of energy. The proof of Theorem 1.5 and Theorem 1.6 is similar to [4]. We write here for the readers convenience. Now, let $f(x,u) = |u|^{p^*-2}u + \lambda|u|^{r-2}u$ with $p < r < p^*$ and λ is a positive parameter.

Lemma 4.1 ([15, 16])**.** *Let $\{u_j\}$ be a weakly convergent sequence in $W^{1,p}(\Omega)$ with weak limit u such that $|\nabla u_j|^p \rightharpoonup d\mu$, $|u_j|^{p^*} \rightharpoonup d\sigma$ weakly convergent in the sense of measures. Then there exist $x_1, x_2, \dots, x_l \in \Omega$ such that*

$$d\sigma = |u|^{p^*} + \sum_{j=1}^{l} \sigma_j \delta_{x_j}, \quad \sigma_j > 0,$$

$$d\mu \geq |\nabla u|^p + \sum_{j=1}^{l} \mu_j \delta_{x_j}, \quad \mu_j > 0,$$

$$(\sigma_j)^{\frac{p}{p^*}} \leq \frac{\mu_j}{S}.$$

Now, we can prove a local $(PS)_c$ condition by using Lemma 4.1.

Lemma 4.2. *Let $\{u_j\} \subset W^{1,p}(\Omega)$ be a $(PS)_c$ sequence for φ with energy level c. If $c < (\frac{1}{p} - \frac{1}{p^*})S^{\frac{p^*}{p^*-p}}$, where S is the best constant in the Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$, then there exists a subsequence that converges strongly in $W^{1,p}(\Omega)$.*

Proof. Let $\{u_j\}$ be a $(PS)_c$ sequence, it follows that $\{u_j\}$ is bounded in $W^{1,p}(\Omega)$(see Lemma 3.4). By Lemma 4.1, there exists a subsequence, that we still denote $\{u_j\}$

such that

$$
\begin{gathered}
u_j \rightharpoonup u \text{ weakly in } W^{1,p}(\Omega), \\
u_j \to u \text{ strongly in } L^r(\Omega), \quad 1 < r < p^*, \\
u_j \to u \text{ a.e. in } \Omega,
\end{gathered}
$$

$$
|\nabla u_j|^p \rightharpoonup d\mu \geq |\nabla u|^p + \sum_{k=1}^{l} \mu_k \delta_{x_k}, \quad \mu_k > 0, \tag{4.1}
$$

$$
|u_j|^{p^*} \rightharpoonup d\sigma = |u|^{p^*} + \sum_{k=1}^{l} \sigma_k \delta_{x_k}, \quad \sigma_k > 0. \tag{4.2}
$$

Choose $\phi \in C_0^\infty(\mathbb{R}^{\mathbb{N}})$ such that

$$
\phi \equiv 1 \text{ in } B(x_k, \varepsilon), \quad \phi \equiv 0 \text{ in } B(x_k, 2\varepsilon)^c, \quad |\nabla \phi| \leq \frac{2}{\varepsilon},
$$

where x_k belongs to the support of $d\sigma$. Considering $\{u_j \phi\}$, it is easy to see this sequence is bounded in $W^{1,p}(\Omega)$. Since $\varphi'(u_j) \to 0$ in $W^{1,p}(\Omega)^*$ as $j \to \infty$, we obtain that

$$
\lim_{j\to\infty} \langle \varphi'(u_j); \phi u_j \rangle = 0. \tag{4.3}
$$

Then from (4.1) and (4.2), we obtain

$$
\begin{aligned}
&\lim_{j\to\infty} \int_\Omega |\nabla u_j|^{p-2} \nabla u_j \nabla \phi u_j \mathrm{d}x \\
&= \int_\Omega \phi \, d\sigma + \lambda \int_\Omega |u|^r \phi \mathrm{d}x + \int_{\partial\Omega} u g(x,u) \phi \mathrm{d}S - a(x) \int_\Omega |u|^p \phi \mathrm{d}x - \int_\Omega \phi d\mu.
\end{aligned}
$$

Now, by Hölder inequality and weak convergence, we obtain

$$
\begin{aligned}
0 &\leq \lim_{j\to\infty} \Big| \int_\Omega |\nabla u_j|^{p-2} \nabla u_j \nabla \phi u_j \mathrm{d}x \Big| \\
&\leq \lim_{j\to\infty} \Big(\int_\Omega |\nabla u_j|^p \mathrm{d}x \Big)^{\frac{p-1}{p}} \Big(\int_\Omega |\nabla \phi|^p |u_j|^p \mathrm{d}x \Big)^{\frac{1}{p}} \\
&\leq C \Big(\int_{B(x_k, 2\varepsilon)\cap\Omega} |\nabla \phi|^p |u|^p \mathrm{d}x \Big)^{\frac{1}{p}} \\
&\leq C \Big(\int_{B(x_k, 2\varepsilon)\cap\Omega} |\nabla \phi|^N \mathrm{d}x \Big)^{\frac{1}{N}} \Big(\int_{B(x_k, 2\varepsilon)\cap\Omega} |u|^{\frac{Np}{N-p}} \mathrm{d}x \Big)^{\frac{N-p}{Np}} \\
&\leq C \Big(\int_{B(x_k, 2\varepsilon)\cap\Omega} |u|^{\frac{Np}{N-p}} \mathrm{d}x \Big)^{\frac{N-p}{Np}} \to 0 \quad \text{as} \quad \varepsilon \to 0.
\end{aligned}
$$

Then from (4.3) we have

$$
\begin{aligned}
&\lim_{\varepsilon\to 0} \Big[\int_\Omega \phi \, d\sigma + \lambda \int_\Omega |u|^r \phi \mathrm{d}x + \int_{\partial\Omega} u g(x,u) \phi \mathrm{d}S - a(x) \int_\Omega |u|^p \phi \mathrm{d}x - \int_\Omega \phi d\mu \Big] \\
&= \sigma_k - \mu_k = 0.
\end{aligned}
$$

Then from Lemma 4.1, we have that $(\sigma_k)^{p/p^*} S \leq \mu_k$. Therefore by the above equality,

$$
(\sigma_k)^{p/p^*} S \leq \sigma_k.
$$

Then, either $\sigma_k = 0$ or

$$
\sigma_k \geq S^{\frac{p*}{p*-p}}.
$$

If this inequality occurs for some k_0, then, from the fact that $\{u_j\}$ is a $(PS)_c$ sequence and from (G2) we obtain

$$\begin{aligned}
c &= \lim_{j\to\infty} \varphi(u_j) = \lim_{j\to\infty} \varphi(u_j) - \frac{1}{p}\langle \varphi'(u_j); u_j\rangle \\
&\ge (\frac{1}{p} - \frac{1}{p^*}) \int_\Omega |u|^{p^*} \mathrm{d}x + (\frac{1}{p} - \frac{1}{p^*}) S^{\frac{p*}{p*-p}} + \lambda(\frac{1}{p} - \frac{1}{r}) \int_\Omega |u|^r \mathrm{d}x \\
&\ge (\frac{1}{p} - \frac{1}{p^*}) S^{\frac{p*}{p*-p}},
\end{aligned}$$

which contradicts our hypothesis. Since $c < (\frac{1}{p} - \frac{1}{p^*})S^{\frac{p*}{p*-p}}$, it follows that

$$\int_\Omega |u_j|^{p^*} \mathrm{d}x \to \int_\Omega |u|^{p^*} \mathrm{d}x,$$

so we have $u_j \to u$ in $L^{p^*}(\Omega)$. Now the proof is complete with the continuity of the operator A^{-1}. □

Proof of Theorem 1.5. We want to obtain our result by using mountain pass theorem. First from the assumption (G1) and (G4), we have

$$G(x,u) \le \varepsilon |u|^p + C|u|^z, \tag{4.4}$$

where $\varepsilon \to 0$ as $|u| \to 0$. From the Sobolev embedding theorem and Sobolev trace inequality, we have

$$\begin{aligned}
\varphi(u) &\ge \frac{1}{p}\|u\|_{1,p}^p - \frac{1}{p^*}\int_\Omega |u|^{p^*}\mathrm{d}x - \frac{\lambda}{r}\int_\Omega |u|^r \mathrm{d}x - \varepsilon \int_{\partial\Omega} |u|^p \mathrm{d}S - C_1 \int_{\partial\Omega} |u|^z \mathrm{d}S \\
&\ge \frac{1}{p}\|u\|_{1,p}^p - \frac{1}{p^*}\int_\Omega |u|^{p^*}\mathrm{d}x - \frac{\lambda}{r}\int_\Omega |u|^r \mathrm{d}x - \varepsilon C_2 \|u\|_{1,p}^p - C_3\|u\|_{1,p}^z \\
&\ge (\frac{1}{p} - \varepsilon C_2)\|u\|_{1,p}^p - \frac{1}{p^*}\int_\Omega |u|^{p^*}\mathrm{d}x - \frac{\lambda}{r}\int_\Omega |u|^r \mathrm{d}x - C_3\|u\|_{1,p}^z \\
&\ge (\frac{1}{p} - \varepsilon C_2)\|u\|_{1,p}^p - \frac{1}{p^*} S^{p^*}\|u\|_{1,p}^{p^*} - \frac{\lambda}{r} C_4 \|u\|_{1,p}^r - C_3\|u\|_{1,p}^z.
\end{aligned}$$

Choose $\varepsilon > 0$ sufficiently small such that $\frac{1}{p} - \varepsilon C_2 > 0$ and let

$$g(t) = (\frac{1}{p} - \varepsilon C_2)t^p - \frac{1}{p^*}S^{p^*}t^{p^*} - \frac{\lambda}{r}C_4 t^r - C_3 t^z, \tag{4.5}$$

it is easy to check that $g(R) > r > 0$ for some R sufficiently small since $p < \min\{r, p^*, z\}$. On the other hand, since $p < \min\{r, p^*, z\}$, so for fixed $\omega \in W^{1,p}(\Omega)$ with $\omega|_\Omega \ne 0$, we have $\lim_{t\to\infty} \varphi(t\omega) = -\infty$. Then there exists $v_0 \in W^{1,p}(\Omega)$ such that $\|v_0\|_{1,p} > R$ and $\varphi(v_0) < r$. So according to the mountain pass Theorem, we know the critical value is $c := \inf_{\phi\in\Gamma} \sup_{t\in[0,1]} \varphi(\phi(t))$, where $\Gamma = \{\phi : [0,1] \to W^{1,p}(\Omega)$ is continuous and $\phi(0) = 0$, $\phi(1) = v_0\}$. Now the problem is to show that $c < (\frac{1}{p} - \frac{1}{p^*})S^{\frac{p^*}{p^*-p}}$ and we want to apply the local $(PS)_c$ condition. For this purpose we fix $\omega \in W^{1,p}(\Omega)$ with $\|\omega\|_{p*} = 1$ and define $h(t) = \varphi(t\omega)$. Let us calculate the maximum of h. Since $\lim\limits_{t\to\infty} h(t) = -\infty$, it follows that there exists a $t_\lambda > 0$ such that $\sup_{t>0} \varphi(t\omega) = h(t_\lambda)$. Differentiating h, we obtain

$$0 = h'(t_\lambda) = t_\lambda^{p-1}\|\omega\|_{1,p}^p - t_\lambda^{p^*-1} - t_\lambda^{r-1}\lambda|\omega|_r^r - \int_{\partial\Omega} g(x, t_\lambda\omega)\omega \mathrm{d}S. \tag{4.6}$$

From assumptions (G1) and (G4), we obtain

$$\begin{aligned}|\int_{\partial\Omega} g(x,t_\lambda\omega)\omega \mathrm{d}S| &\le \int_{\partial\Omega} |g(x,t_\lambda\omega)||\omega|\mathrm{d}S\\ &\le \varepsilon t_\lambda^{p-1}\int_{\partial\Omega}|\omega|^p\mathrm{d}S + C_1 t_\lambda^{z-1}\int_{\partial\Omega}|\omega|^z\mathrm{d}S\\ &= \varepsilon t_\lambda^{p-1}|\omega|^p_{L^p(\partial\Omega)} + C_1 t_\lambda^{z-1}|\omega|^z_{L^z(\partial\Omega)}.\end{aligned}$$

From (4.6),

$$t_\lambda^{p-1}\|\omega\|^p_{1,p} - t_\lambda^{p^*-1} - t_\lambda^{r-1}\lambda|\omega|^r_r - \varepsilon t_\lambda^{p-1}|\omega|^p_{L^p(\partial\Omega)} - C_1 t_\lambda^{z-1}|\omega|^z_{L^z(\partial\Omega)} \le 0.$$

Then

$$t_\lambda^{p^*-p} + t_\lambda^{r-p}\lambda|\omega|^r_r + C_3 t_\lambda^{z-p}\|\omega\|^z_{1,p} \le (1-\varepsilon C_2)\|\omega\|^p_{1,p}. \tag{4.7}$$

Hence, $t_\lambda \le C\|\omega\|_{1,p}^{\frac{p}{p^*-p}}$. So from (4.7), $t_\lambda^{p^*-r}+\lambda|\omega|^r_r+C_3 t_\lambda^{z-r}\|\omega\|^z_{1,p} \to \infty$ as $\lambda\to\infty$. we obtain

$$\lim_{\lambda\to\infty} t_\lambda = 0. \tag{4.8}$$

On the other hand, it is easy to check that if $\lambda > \overline{\lambda}$ we could have $\varphi(t_{\overline{\lambda}}\omega) \ge \varphi(t_\lambda\omega)$. So by (4.8), we get $\lim_{\lambda\to\infty}\varphi(t_\lambda\omega) = 0$. But this equality means that there exists a constant $\lambda_0 > 0$ such that if $\lambda > \lambda_0$, then $\sup\limits_{t\ge 0}\varphi(t\omega) < (\frac{1}{p} - \frac{1}{p^*})S^{\frac{p^*}{p^*-p}}$. We choose $v_0 = t_0\omega$ with t_0 sufficiently large to have $\varphi(t_0\omega) < 0$. This completes the proof.

4.1. **Critical case 2.** In this subsection we study f has critical and sublinear terms in problem (1.1), that is, $f(x,u) = |u|^{p^*-2}u + \lambda|u|^{r-2}u$ with $1 < r < p$ and λ a positive parameter. By applying the variational approach, we will show the existence of infinitely many nontrivial critical points of the associated functional φ when λ is small enough. First we use Lemma 4.1 to prove local $(PS)_c$ condition.

Lemma 4.3. *Let $\{u_j\} \subset W^{1,p}(\Omega)$ be a $(PS)_c$ sequence for φ with energy level c. If $c < (\frac{1}{p} - \frac{1}{p^*})S^{\frac{p^*}{p^*-p}} - K\lambda^{\frac{p^*}{p^*-r}}$, where K depend only on p, r, N and $|\Omega|$, then there exists a subsequence that converges strongly in $W^{1,p}(\Omega)$.*

Proof. Let $\{u_j\}$ be a $(PS)_c$ sequence; that is,

$$\varphi(u_j) \to c,\ \varphi'(u_j) \to 0 \text{ in } W^{1,p}(\Omega)^* \text{ as } j\to\infty.$$

From Lemma 3.4 it follows immediately that $\{u_j\}$ is bounded in $W^{1,p}(\Omega)$. Then there exists a subsequence (we still denote $\{u_j\}$) which is weakly convergent to u in $W^{1,p}(\Omega)$. We need to prove $\{u_j\}$ is strongly convergent to u in $L^{p^*}(\Omega)$. For this purpose, suppose $\{u_j\}$ is not strongly convergent to u in $L^{p^*}(\Omega)$, then from (G2) we have

$$\begin{aligned} c &= \lim_{j\to\infty} \varphi(u_j) = \lim_{j\to\infty} \varphi(u_j) - \frac{1}{p}\langle \varphi'(u_j); u_j\rangle \\ &= \lim_{j\to\infty} \Big((\frac{1}{p}\int_\Omega |\nabla u_j|^p + a(x)|u_j|^p \mathrm{d}x - \frac{1}{p^*}\int_\Omega |u_j|^{p^*}\mathrm{d}x - \frac{\lambda}{r}\int_\Omega |u_j|^r \mathrm{d}x \\ &\quad - \int_{\partial\Omega} G(x,u_j)\mathrm{d}S) - (\frac{1}{p}\int_\Omega |\nabla u_j|^p + a(x)|u_j|^p\mathrm{d}x \\ &\quad - \frac{1}{p}\int_\Omega |u_j|^{p^*}\mathrm{d}x - \frac{\lambda}{p}\int_\Omega |u_j|^r\mathrm{d}x - \int_{\partial\Omega}\frac{1}{p} g(x,u_j u_j)\mathrm{d}S)\Big) \\ &\ge (\frac{1}{p} - \frac{1}{p^*})\int_\Omega |u|^{p^*}\mathrm{d}x + (\frac{1}{p} - \frac{1}{p^*}) S^{\frac{p*}{p*-p}} + \lambda(\frac{1}{p} - \frac{1}{r})\int_\Omega |u|^r \mathrm{d}x. \end{aligned}$$

Now, applying Hölder inequality, we find

$$\begin{aligned} c &\ge (\frac{1}{p} - \frac{1}{p^*})\int_\Omega |u|^{p^*}\mathrm{d}x + (\frac{1}{p} - \frac{1}{p^*}) S^{\frac{p*}{p*-p}} + \lambda(\frac{1}{p} - \frac{1}{r})|\Omega|^{1-\frac{r}{p^*}}(\int_\Omega |u|^{p^*}\mathrm{d}x)^{\frac{r}{p^*}} \\ &= (\frac{1}{p} - \frac{1}{p^*}) S^{\frac{p*}{p*-p}} + (\frac{1}{p} - \frac{1}{p^*})\|u\|^{p^*}_{L^{p^*}(\Omega)} + \lambda(\frac{1}{p} - \frac{1}{r})|\Omega|^{1-\frac{r}{p^*}}(\int_\Omega |u|^{p^*}\mathrm{d}x)^{\frac{r}{p^*}}. \end{aligned}$$

Let

$$g(x) = C_1 x^{p^*} - \lambda C_2 x^r.$$

This function reaches its absolute minimum at $x_0 = (\frac{\lambda r C_2}{p^* C_1})^{\frac{1}{p^*-r}}$, that is

$$g(x) \ge g(x_0) = -K\lambda^{\frac{p^*}{p^*-r}}, \text{ where } K = K(p, r, N, |\Omega|).$$

Hence, $c \ge (\frac{1}{p} - \frac{1}{p^*}) S^{\frac{p^*}{p^*-p}} - K\lambda^{\frac{p^*}{p^*-r}}$ which contradicts our hypothesis. So we know $\int_\Omega |u_j|^{p^*}\mathrm{d}x \to \int_\Omega |u|^{p^*}\mathrm{d}x$, and therefore $u_j \to u$ in $L^{p^*}(\Omega)$. Now the rest of the proof is as that of Lemma 4.2. $\square$

Lemma 4.4. *There exists $\widetilde{\lambda} > 0$ such that if $0 < \lambda < \widetilde{\lambda}$, then φ satisfies a local $(PS)_c$ condition for $c \le 0$.*

Proof. We need only to check the local $(PS)_c$ condition. Obviously observe that every $(PS)_c$ sequence for φ with energy level $c \le 0$ must be bounded. Therefore by Lemma 4.3 if λ verifies

$$0 < \lambda < (\frac{1}{p} - \frac{1}{p^*}) S^{\frac{p^*}{p^*-p}} - K\lambda^{\frac{p^*}{p^*-r}},$$

then their exists a convergent subsequence. $\square$

Proof of Theorem 1.6. The proof is analogous to that of Theorem 1.4. Here we use Lemma 4.3 and Lemma 4.4 respectively to work with the functional φ and complete the proof.

References

[1] D. Arcoya and J. I. Diaz; *S-shaped bifurcation branch in a quasilinear multivalued model arising in climatology,* J. Differential Equations. 150(1998), 215-225.

[2] J. F. Bonder; *Multiple solutions for the p-Laplacian equation with nonlinear boundary conditions,* Electron. J. Differential Equations. 2006(2006) no. 37, 1-7.

[3] J. F. Bonder, S. Martínez and J. D. Rossi; *The behavior of the best Sobolev trace constant and extremals in thin domains,* J. Differential Equations. 198(2004), no.1, 129-148.

[4] J. F. Bonder, J. D. Rossi; *Existence results for the p-Laplacian with nonlinear boundary conditions,* J. Math. Anal. Appl. 263(1)(2001), 195-223.

[5] J. F. Bonder, J. D. Rossi; *Asymptotic behavior of the best Sobolev trace constant in expanding and contracting domains,* Comm. Pure. Appl. 1(2002), no.3, 359-378.
[6] H. Brezis and L. Nirenberg; *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponent,* Comm. Pure. Appl. 36(1983), 437-477.
[7] F.-C. St. Círstea and V. Radulescu; *Existence and non-existence results for a quasilinear problem with nonlinear boundary conditions,* J. Math. Anal. Appl. 244(2000), 169-183.
[8] J. I. Diaz; *Nonlinear partial differential equations and free boundaries, Elliptic equations.* Pitman Adv. Publ. Boston. 1986.
[9] P. Drábek and S. B. Robinson; *Resonance problem for the p-Laplacian,* J. Funct. Anal. 169(1999), 189-200.
[10] I. Ekeland and N. Ghoussoub; *Selected new aspects of the caculus of variations in the large,* Bull. Amer. Math. Soc. 39:2(2002), 207-265.
[11] X. L. Fan and X. Y. Han; *Existence and multiplicity of solutions for p(x)-Laplacian equations in* $\mathbb{R}^N$*,* Nonlinear Analysis: TMA, 59(2004), 173-188.
[12] J. Garcia-Azorero and I. Peral; *Existence and non-uniqueness for the p-Laplacian: Nonlinear eigenvalue,* Comm.Partial Differential Equations 12(1987), 1389-1430.
[13] D. A. Kandilakis; *A multiplicity result for quasilinear problems with convex and concave nonlinearities and nonlinear boundary conditions in unbounded domains,* Electron. J. Differential Equations 57(2005), 1-12.
[14] G. M. Lieberman; *Boundary regularity for solutions of degenerate elliptic equations,* Nonlinear Analysis: TMA 12(1988), 1203-1219.
[15] P. L. Lions; *The concentration-compactness principle in the calculus of variations. The limit case part 1,* Rev. Mat. Iberoamericana **1** No.1 (1985), 145-201.
[16] P. L. Lions; *The concentration-compactness principle in the calculus of variations. The limit case part 2,* Rev. Mat. Iberoamericana **1** No.2 (1985), 45-121.
[17] S. Martínez and J. D. Rossi; *Isolation and simplicity for the first eigenvalue of the p-Laplacian with nonlinear boundary condition,* Abstr. Appl. Anal. 7(5)(2002), 287-293.
[18] S. Martínez and J. D. Rossi; *Weak solutions for the p-Laplacian with a nonlinear boundary condition at resonance,* Electron. J. Differential Equations. 2003(2003) no. 27, 1-14.
[19] K. Pflüger; *Existence and multiplicity of solutions to a p-Laplacian equation with nonlinear boundary condition,* Electron. J. Differential Equations 1998(1998) no. 10, 1-13.
[20] M. Willem; *Minimax theorem,* Birkhäuser Boston, Inc, Boston, 1996.

Department of Mathematics, Lanzhou University, Lanzhou, 730000, China
E-mail address, Ji-Hong Zhao: zhaojihong2007@yahoo.com.cn
E-mail address, Pei-Hao Zhao: zhaoph@lzu.edu.cn