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THREE POSITIVE SOLUTIONS FOR P-LAPLACIAN FUNCTIONAL DYNAMIC EQUATIONS ON TIME SCALES

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Abstract. In this paper, we establish the existence of three positive solutions to the following p-Laplacian functional dynamic equation on time scales,

$$\begin{gathered}[\Phi_p(u^\Delta(t))]^\nabla + a(t)f(u(t),u(\mu(t))) = 0, \quad t \in (0,T)_{\mathbf{T}}, \\ u_0(t) = \varphi(t), \quad t \in [-r,0]_{\mathbf{T}}, \\ u(0) - B_0(u^\Delta(\eta)) = 0, \quad u^\Delta(T) = 0,.\end{gathered}$$

using the fixed-point theorem due to Avery and Peterson [8]. An example is given to illustrate the main result.

1. Introduction

Let $\mathbf{T}$ be a time scale; i.e., $\mathbf{T}$ is a nonempty closed subset of R. Let $0, T$ be points in $\mathbf{T}$, an interval $(0,T)_{\mathbf{T}}$ denotes time scales interval, that is, $(0,T)_{\mathbf{T}} := (0,T) \cap \mathbf{T}$. Other types of intervals are defined similarly.

The theory of dynamic equations on time scales has been a new important mathematical branch (see, for example, [1, 2, 9, 10, 17]) since it was initiated by Hilger [16]. At the same time, boundary value problems (BVPs) for dynamic equation on time scales have received considerable attention [3, 4, 5, 6, 11, 12, 13, 14, 15, 18, 19, 20, 21, 22]. However, to the best of our knowledge, few papers can be found in the literature on bvps of p-Laplacian dynamic equations on time scales [5, 14, 15, 19, 20, 22], especially for p-Laplacian functional dynamic equations on time scales [19].

This paper concerns the existence of positive solutions for the p-Laplacian functional dynamic equation on time scale,

$$\begin{aligned}[\Phi_p(u^\Delta(t))]^\nabla + a(t)f(u(t),u(\mu(t))) = 0, \quad t \in (0,T)_{\mathbf{T}}, \\ u_0(t) = \varphi(t), \quad t \in [-r,0]_{\mathbf{T}}, \\ u(0) - B_0(u^\Delta(\eta)) = 0, \quad u^\Delta(T) = 0,\end{aligned} \tag{1.1}$$

where $\Phi_p(s)$ is p-Laplacian operator, i.e., $\Phi_p(s) = |s|^{p-2}s$, $p > 1$, $(\Phi_p)^{-1} = \Phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$, $\eta \in (0, \rho(T))_{\mathbf{T}}$ and

(C1) $f : (\mathbb{R}^+)^2 \to \mathbb{R}^+$ is continuous;

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(C2) $a : \mathbf{T} \to \mathbb{R}^+$ is left dense continuous (i.e., $a \in C_{\mathbf{ld}}(\mathbf{T}, \mathbb{R}^+)$) and does not vanish identically on any closed subinterval of $[0, T]$, where $C_{\mathbf{ld}}(\mathbf{T}, \mathbb{R}^+)$ denotes the set of all left dense continuous functions from $\mathbf{T}$ to $\mathbb{R}^+$;
(C3) $\varphi : [-r, 0]_{\mathbf{T}} \to \mathbb{R}^+$ is continuous and $r > 0$;
(C4) $\mu : [0, T]_{\mathbf{T}} \to [-r, T]_{\mathbf{T}}$ is continuous, $\mu(t) \leq t$ for all t;
(C5) $B_0 : R \to R$ is continuous and there exist constant $A \geq 1$, $B > 0$ such that

$$Bv \leq B_0(v) \leq Av, \text{ for all } v \geq 0.$$

In [19], by using a double-fixed-point theorem due to Avery et al. [7] in a cone, Song and Xiao considered the problem (1.1) and obtained the existence of two positive solutions.

In paper [15], Hong studied the problem (1.1) when $\varphi(t) = 0, t \in [-r, 0]_{\mathbf{T}}$ and the nonlinear term is not involved $u(\mu(t))$. He imposed conditions on f to yield at least three positive solutions to the problem (1.1), by applying the fixed-point theorem due to Avery and Peterson [8].

Motivated by [15, 19], we shall show that the problem (1.1), has at least three positive solutions by means of the fixed point theorem due to Avery and Peterson.

In the remainder of this section we list the following well known definitions which can be found in [2, 6, 9, 10].

Definition 1.1. For $t < \sup \mathbf{T}$ and $r > \inf \mathbf{T}$, define the forward jump operator σ and the backward jump operator ρ,

$$\sigma(t) = \inf\{\tau \in \mathbf{T} | \tau > t\} \in \mathbf{T}, \quad \rho(r) = \sup\{\tau \in \mathbf{T} | \tau < r\} \in \mathbf{T}$$

for all $t, r \in \mathbf{T}$. If $\sigma(t) > t$, t is said to be right scattered, and if $\rho(r) < r$, r is sad to be left scattered. If $\sigma(t) = t$, t is said to be right dense, and if $\rho(r) = r$, r is said to be left dense. If $\mathbf{T}$ has a right scattered minimum m, define $\mathbf{T}_k = \mathbf{T} - \{m\}$; otherwise set $\mathbf{T}_k = \mathbf{T}$. If $\mathbf{T}$ has a left scattered maximum M, define $\mathbf{T}^k = \mathbf{T} - \{M\}$; otherwise set $\mathbf{T}^k = \mathbf{T}$.

Definition 1.2. For $x : \mathbf{T} \to R$ and $t \in \mathbf{T}^k$, we define the delta derivative of $x(t)$, $x^\Delta(t)$, to be the number (when it exists), with the property that, for any $\varepsilon > 0$, there is a neighborhood U of t such that

$$|[x(\sigma(t)) - x(s)] - x^\Delta(t)[\sigma(t) - s]| < \varepsilon|\sigma(t) - s|,$$

for all $s \in U$. For $x : \mathbf{T} \to R$ and $t \in \mathbf{T}_k$, we define the nabla derivative of $x(t)$, $x^\nabla(t)$, to be the number (when it exists), with the property that, for any $\varepsilon > 0$, there is a neighborhood V of t such that

$$|[x(\rho(t)) - x(s)] - x^\nabla(t)[\rho(t) - s]| < \varepsilon|\rho(t) - s|,$$

for all $s \in V$. If $\mathbf{T} = R$, then $x^\Delta(t) = x^\nabla(t) = x'(t)$. If $\mathbf{T} = Z$, then $x^\Delta(t) = x(t+1) - x(t)$ is the forward difference operator while $x^\nabla(t) = x(t) - x(t-1)$ is the backward difference operator.

Definition 1.3. If $F^\Delta(t) = f(t)$, then we define the delta integral by

$$\int_a^t f(s)\Delta s = F(t) - F(a).$$

If $\Phi^\nabla(t) = f(t)$, then we define the nabla integral by

$$\int_a^t f(s)\nabla s = \Phi(t) - \Phi(a).$$

Throughout this papers, we assume $\mathbf{T}$ is closed subset of $\mathbb{R}$ with $0 \in \mathbf{T}_k$ and $T \in \mathbf{T}^k$.

Lemma 1.4 ([6]). *The following formulas hold:*

(i) $\left(\int_a^t f(s)\Delta s\right)^{\Delta} = f(t)$,
(ii) $\left(\int_a^t f(s)\Delta s\right)^{\nabla} = f(\rho(t))$,
(iii) $\left(\int_a^t f(s)\nabla s\right)^{\Delta} = f(\sigma(t))$,
(iv) $\left(\int_a^t f(s)\nabla s\right)^{\nabla} = f(t)$.

2. Preliminaries

In this section, we provide some background materials from the theory of cones in Banach spaces and we then state the fixed-point theorem due to Avery and Peterson.

Definition 2.1. Let E be a real Banach space. A nonempty, closed, convex set $P \subset E$ is said to be a cone provided the following conditions are satisfied:

(i) if $x \in P$ and $\lambda \geq 0$, then $\lambda x \in P$;
(ii) if $x \in P$ and $-x \in P$, then $x = 0$.

Every cone $P \subset E$ induces an ordering in E given by

$$x \leq y \quad \text{if } y - x \in P.$$

Definition 2.2. Given a cone P in a real Banach space E, the map $\varsigma : P \to [0, \infty)$ is called a nonnegative continuous concave function on cone P provided that ς is continuous and

$$\varsigma(tx + (1-t)y) \geq t\varsigma(x) + (1-t)\varsigma(y), \quad \text{for } x,\ y \in P \text{ and } 0 \leq t \leq 1.$$

Dual to this, we call the map $\tau : P \to [0, \infty)$ is called a nonnegative continuous convex function on cone P provided that τ is continuous and

$$\tau(tx + (1-t)y) \leq t\tau(x) + (1-t)\tau(y), \quad \text{for } x,\ y \in P \text{ and } 0 \leq t \leq 1.$$

Let γ and θ be nonnegative continuous convex functions on P, α be a nonnegative continuous concave function on P and ψ be a nonnegative continuous function on P. Then, for positive real numbers a, b, c and d, we define the following convex sets

$$\begin{gathered}
P(\gamma, d) = \{x \in P : \gamma(x) < d\}, \\
P(\gamma, \alpha, b, d) = \{x \in P : b \leq \alpha(x),\ \gamma(x) \leq d\}, \\
P(\gamma, \theta, \alpha, b, c, d) = \{x \in P : b \leq \alpha(x),\ \theta(x) \leq c,\ \gamma(x) \leq d\},
\end{gathered}$$

and a closed set

$$R(\gamma, \psi, a, d) = \{x \in P : a \leq \psi(x),\ \gamma(x) \leq d\}.$$

To prove our main results, we need the following fixed-point theorem due to Avery and Peterson in [8].

Theorem 2.3. *Let P be a cone in a real Banach space E. Let γ and θ be nonnegative continuous convex functionals on P, α be a nonnegative continuous concave functionals on P and ψ be a nonnegative continuous functional on P satisfying $\psi(\lambda x) \le \lambda\psi(x)$ for $0 \le \lambda \le 1$, such that for some positive numbers h and d,*
$$\alpha(x) \le \psi(x) \quad \text{and} \quad \|x\| \le h\gamma(x),$$
for all $x \in \overline{P(\gamma, d)}$. Suppose that
$$F : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$$
is completely continuous and there exist positive numbers a, b and c with $a < b$ such that:

- (i) *$\{x \in P(\gamma, \theta, \alpha, b, c, d) : \alpha(x) > b\} \neq \emptyset$ and $\alpha(Fx) > b$ for x in the set $P(\gamma, \theta, \alpha, b, c, d)$;*
- (ii) *$\alpha(Fx) > b$ for $x \in P(\gamma, \alpha, b, d)$ with $\theta(Fx) > c$;*
- (iv) *$0 \notin R(\gamma, \psi, a, d)$ and $\psi(Fx) < a$ for $x \in R(\gamma, \psi, a, d)$ with $\psi(x) = a$.*

Then F has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, d)}$ such that $\gamma(x_i) \le d$ for $i = 1, 2, 3$, and $b < \alpha(x_1)$, $a < \psi(x_2)$ with $\alpha(x_2) < b$ and $\psi(x_3) < a$.

3. Existence of Three Positive Solutions

We note that $u(t)$ is a solution of (1.1) if and only if
$$u(t) = \begin{cases} B_0(\Phi_q(\int_\eta^T a(r) f(u(r), u(\mu(r)))\nabla r)) \\ + \int_0^t \Phi_q(\int_s^T a(r) f(u(r), u(\mu(r)))\nabla r)\Delta s, & t \in [0, T]_{\mathbf{T}}, \\ \varphi(t), & t \in [-r, 0]_{\mathbf{T}}. \end{cases}$$
Let $E = C^\Delta_{\mathbf{ld}}([0, T]_{\mathbf{T}}, R)$ be endowed with the norm
$$\|u\| = \max\big\{\max_{t \in [0,T]_{\mathbf{T}}} |u(t)|, \max_{t \in [0,T]_{\mathbf{T}^k}} |u^\Delta(t)|\big\},$$
so E is a Banach space. Define cone $P \subset E$ by
$$P = \big\{u \in E : u \text{ is concave and nonnegative valued on } [0, T]_{\mathbf{T}}, \text{ and } u^\Delta(T) = 0\big\}.$$
For each $u \in E$, extend $u(t)$ to $[-r, T]_{\mathbf{T}}$ with $u(t) = \varphi(t)$ for $t \in [-r, 0]_{\mathbf{T}}$. Define a completely continuous operator F: $P \to E$ by
$$\begin{aligned}(Fu)(t) =& B_0(\Phi_q(\int_\eta^T a(r) f(u(r), u(\mu(r)))\nabla r)) \\ &+ \int_0^t \Phi_q(\int_s^T a(r) f(u(r), u(\mu(r)))\nabla r)\Delta s, \quad t \in [0, T]_{\mathbf{T}}.\end{aligned}$$
We seek for a point, u_1, of F in the cone P. Define
$$u(t) = \begin{cases} u_1(t), & t \in [0, T]_{\mathbf{T}}, \\ \varphi(t), & t \in [-r, 0]_{\mathbf{T}}. \end{cases}$$
Then $u(t)$ denotes a positive solution of (1.1).

Lemma 3.1. *If $u \in P$, then*

- (i) *$u(t) \ge \frac{t}{T} \max_{t \in [0,T]_{\mathbf{T}}} |u(t)|$, $t \in [0, T]_{\mathbf{T}}$.*
- (ii) *$u(t)$ is increasing on $t \in [0, T]_{\mathbf{T}}$.*
- (iii) *$u^\Delta(t)$ is decreasing on $t \in [0, T]_{\mathbf{T}^k}$.*

Proof. Part (i) is of [14, Lemma 3.1]. Parts (ii) and (iii) are easy, so we omit them here. □

Lemma 3.2. *The operator F maps P into P.*

Proof. For each $u \in P$, $Fu \in E$ and $(Fu)(t) \geq 0$, for all $t \in [0,T]_{\mathbf{T}}$. It follows from Lemma 1.4 that

$$(Fu)^{\Delta}(t) = \Phi_q\Big(\int_t^T a(r) f(u(r),\varphi(r))\nabla r\Big).$$

Obviously $(Fu)^{\Delta}(t)$ is a continuous function and $(Fu)^{\Delta}(t) \geq 0$, that is $(Fu)(t)$ is increasing on $[0,T]_{\mathbf{T}}$. Note that Φ_q is increasing, we have that $(Fu)^{\Delta}(t)$ is decreasing.

If $t \in [0,T]_{\mathbf{T}^k \cap \mathbf{T}_k}$, then from [6, Theorem 2.3] it follows that $(Fu)^{\Delta\nabla}(t) \leq 0$; i.e., Fu is concave on $[0,T]_{\mathbf{T}}$. This implies that $Fu \in P$ and $F : P \to P$. □

Let $l \in \mathbf{T}$ be fixed such that $0 < \eta < l < T$, and set

$$Y_1 = \big\{t \in [0,T]_{\mathbf{T}} : \mu(t) \leq 0\big\}; \quad Y_2 = \big\{t \in [0,T]_{\mathbf{T}} : \mu(t) > 0\big\}; \quad Y_3 = Y_1 \cap [l,T]_{\mathbf{T}}.$$

Throughout this paper, we assume $Y_3 \neq \phi$ and $\int_{Y_3} a(r)\nabla r > 0$.

Define the nonnegative continuous concave functionals α, the nonnegative continuous convex functionals θ, γ, and the nonnegative continuous functionals ψ on the cone P respectively as

$$\gamma(u) = \|u\|, \quad \theta(u) = \max_{t \in [l,T]_{\mathbf{T}^k}} u^{\Delta}(t),$$
$$\alpha(u) = \min_{t \in [\eta,l]_{\mathbf{T}}} u(t), \quad \psi(u) = \min_{t \in [\eta,T]_{\mathbf{T}}} u(t).$$

In addition, by Lemma 3.1, we have $\alpha(u) = \psi(u) = u(\eta)$, $\theta(u) = u^{\Delta}(l)$ for each $u \in P$. For convenience, we define

$$\rho = (A+T)\Phi_q\Big(\int_0^T a(r)\nabla r\Big), \quad \delta = (B+\eta)\Phi_q\Big(\int_{Y_3} a(r)\nabla r\Big),$$
$$\lambda = (A+\eta)\Phi_q\Big(\int_0^T a(r)\nabla r\Big).$$

We now state growth conditions on f so that (1.1) has at least three positive solutions.

Theorem 3.3. *Let $0 < \frac{T}{\eta}a < b < d$, $\rho b < \delta d$, and suppose that f satisfies the following conditions:*

- (H1) *$f(u,\varphi(s)) \leq \Phi_p(\frac{d}{\rho})$, if $0 \leq u \leq d$, uniformly in $s \in [-r,0]_{\mathbf{T}}$; $f(u_1,u_2) \leq \Phi_p(\frac{d}{\rho})$, if $0 \leq u_i \leq d$, $i = 1,2$,*
- (H2) *$f(u,\varphi(s)) > \Phi_p(\frac{b}{\delta})$, if $b \leq u \leq d$, uniformly in $s \in [-r,0]_{\mathbf{T}}$,*
- (H3) *$f(u,\varphi(s)) < \Phi_p(\frac{a}{\lambda})$, if $0 \leq u \leq \frac{T}{\eta}a$, uniformly in $s \in [-r,0]_{\mathbf{T}}$; $f(u_1,u_2) < \Phi_p(\frac{a}{\lambda})$, if $0 \leq u_i \leq \frac{T}{\eta}a$, $i = 1,2$.*

Then (1.1) has at least three positive solutions of the form

$$u(t) = \begin{cases} u_i(t), & t \in [0,T]_{\mathbf{T}}, \quad i = 1,2,3, \\ \varphi(t), & t \in [-r,0]_{\mathbf{T}}, \end{cases}$$

where $\gamma(u_i) \le d$ *for* $i = 1, 2, 3$, $b < \alpha(u_1)$, $a < \psi(u_2)$ *with* $\alpha(u_2) < b$ *and* $\psi(u_3) < a$.

Proof. We first assert that $F : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$. Let $u \in \overline{P(\gamma, d)}$, then $\gamma(u) = \|u\| \le d$, consequently, $0 \le u(t) \le d$ for $t \in [0, T]_{\mathbf{T}}$. From (H1), we have

$$
\begin{aligned}
&|(Fu)(t)| \\
&= B_0\Big(\Phi_q\Big(\int_\eta^T a(r)f(u(r), u(\mu(r)))\nabla r\Big)\Big) \\
&\quad + \int_0^t \Phi_q\Big(\int_s^T a(r)f(u(r), u(\mu(r)))\nabla r\Big)\Delta s \\
&\le A\Phi_q\Big(\int_0^T a(r)f(u(r), u(\mu(r)))\nabla r\Big) + T\Phi_q\Big(\int_0^T a(r)f(u(r), u(\mu(r)))\nabla r\Big) \\
&= (A+T)\Phi_q\Big[\int_{Y_1} a(r)f(u(r), \varphi(\mu(r)))\nabla r + \int_{Y_2} a(r)f(u(r), u(\mu(r)))\nabla r\Big] \\
&\le (A+T)\Phi_q\Big(\int_0^T a(r)\nabla r\Big)\frac{d}{\rho} = d,
\end{aligned}
$$

and

$$
\begin{aligned}
|(Fu)^\Delta(t)| &= \Phi_q\Big(\int_t^T a(r)f(u(r), u(\mu(r)))\nabla r\Big) \\
&\le \Phi_q\Big(\int_0^T a(r)f(u(r), u(\mu(r)))\nabla r\Big) \\
&= \Phi_q\Big[\int_{Y_1} a(r)f(u(r), \varphi(\mu(r)))\nabla r + \int_{Y_2} a(r)f(u(r), u(\mu(r)))\nabla r\Big] \\
&\le \Phi_q\Big(\int_0^T a(r)\nabla r\Big)\frac{d}{\rho} \\
&= \frac{d}{(A+T)} \le d.
\end{aligned}
$$

Therefore $F(u) \in \overline{P(\gamma, d)}$, i.e., $F : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$.

Secondly, we assert that $\{u \in P(\gamma, \theta, \alpha, b, c, d) : \quad \alpha(u) > b\} \ne \phi$ and $\alpha(Fu) > b$ for $u \in P(\gamma, \theta, \alpha, b, c, d)$.

Let $u(t) = kb$ with $k = \frac{\rho}{\delta} > 1$, then $u(t) = kb > b$ and $\theta(u) = 0 < b$. Furthermore, by $\rho b < \delta d$ we have $\gamma(u) \le d$. Let $c = kb$, then

$$
\{u \in P(\gamma, \theta, \alpha, b, c, d) : \quad \alpha(u) > b\} \ne \emptyset.
$$

Moreover, for all $u \in P(\gamma, \theta, \alpha, b, kb, d)$, we have $b \le u(t) \le d$, $t \in [\eta, T]_{\mathbf{T}}$. From (H2), we see that

$$
\begin{aligned}
&\alpha(Fu) \\
&= (Fu)(\eta) \\
&= B_0(\Phi_q(\int_\eta^T a(r)f(u(r), u(\mu(r)))\nabla r)) + \int_0^\eta \Phi_q(\int_s^T a(r)f(u(r), u(\mu(r)))\nabla r)\Delta s \\
&\ge B\Phi_q(\int_\eta^T a(r)f(u(r), u(\mu(r)))\nabla r) + \eta\Phi_q(\int_\eta^T a(r)f(u(r), u(\mu(r)))\nabla r)
\end{aligned}
$$

$$
\begin{aligned}
&\geq (B+\eta)\Phi_q(\int_l^T a(r)f(u(r),u(\mu(r)))\nabla r)\\
&\geq (B+\eta)\Phi_q(\int_{Y_3} a(r)f(u(r),\varphi(\mu(r)))\nabla r)\\
&> (B+\eta)\Phi_q(\int_{Y_3} a(r)\nabla r)\frac{b}{\delta} = b,
\end{aligned}
$$

as required.

Thirdly, we assert that $\alpha(Fu) > b$ for $u \in P(\gamma,\alpha,b,d)$ with $\theta(Fu) > c$. For all $u \in P(\gamma,\alpha,b,d)$ with $\theta(Fu) > kb$, from Lemma 3.1 we have

$$
\theta(Fu) = (Fu)^{\Delta}(l) = \Phi_q\Big(\int_l^T a(r)f(u(r),u(\mu(r)))\nabla r\Big) > kb.
$$

So,

$$
\begin{aligned}
&\alpha(Fu)\\
&= (Fu)(\eta)\\
&= B_0\Big(\Phi_q\Big(\int_\eta^T a(r)f(u(r),u(\mu(r)))\nabla r\Big)\Big) + \int_0^\eta \Phi_q(\int_s^T a(r)f(u(r),u(\mu(r)))\nabla r)\Delta s\\
&\geq B\Phi_q\Big(\int_\eta^T a(r)f(u(r),u(\mu(r)))\nabla r\Big) + \eta\Phi_q(\int_\eta^T a(r)f(u(r),u(\mu(r)))\nabla r)\\
&\geq (B+\eta)\Phi_q\Big(\int_l^T a(r)f(u(r),u(\mu(r)))\nabla r\Big)\\
&> (B+\eta)kb = (B+\eta)\frac{\rho}{\delta}b\\
&\geq (A+T)b > b.
\end{aligned}
$$

This implies that $\alpha(Fu) > b$ for $u \in P(\gamma,\alpha,b,d)$ with $\theta(Fu) > c$.

Finally, we assert that $0 \notin R(\gamma,\psi,a,d)$ and $\psi(Fu) < a$ for $u \in R(\gamma,\psi,a,d)$ with $\psi(u) = a$.

Since $\psi(0) = 0 < a$, we have $0 \notin R(\gamma,\psi,a,d)$. For all $u \in R(\gamma,\psi,a,d)$ with $\psi(u) = \min_{t\in[\eta,T]_{\mathbf{T}}} u(t) = u(\eta) = a$, by Lemma 3.1 we have $0 \leq u(t) \leq \frac{T}{\eta}a$, for $t \in [0,T]_{\mathbf{T}}$. From (H3), we have

$$
\begin{aligned}
\psi(Fu) &= (Fu)(\eta)\\
&= B_0\Big(\Phi_q\Big(\int_\eta^T a(r)f(u(r),u(\mu(r)))\nabla r\Big)\Big)\\
&\quad + \int_0^\eta \Phi_q\Big(\int_s^T a(r)f(u(r),u(\mu(r)))\nabla r\Big)\Delta s\\
&\leq A\Phi_q\Big(\int_0^T a(r)f(u(r),u(\mu(r)))\nabla r\Big) + \eta\Phi_q\Big(\int_0^T a(r)f(u(r),u(\mu(r)))\nabla r\Big)\\
&= (A+\eta)\Phi_q\Big[\int_{Y_1} a(r)f(u(r),\varphi(\mu(r)))\nabla r + \int_{Y_2} a(r)f(u(r),u(\mu(r)))\nabla r\Big]\\
&< (A+\eta)\Phi_q\Big(\int_0^T a(r)\nabla r\Big)\frac{a}{\lambda} = a,
\end{aligned}
$$

which shows that condition (iv) of Theorem 2.3 is fulfilled.

Thus, all the conditions of Theorem 2.3 are satisfied. Hence, F has at least three fixed points u_1, u_2, u_3 satisfying $\gamma(u_i) \le d$ for $i = 1,2,3$, $b < \alpha(u_1)$, $a < \psi(u_2)$ with $\alpha(u_2) < b$ and $\psi(u_3) < a$. Let

$$u(t) = \begin{cases} u_i(t), & t \in [0,T]_{\mathbf{T}}, \quad i = 1,2,3, \\ \varphi(t), & t \in [-r,0]_{\mathbf{T}}, \end{cases}$$

which are three positive solutions of (1.1). $\square$

4. Example

Let $\mathbf{T} = [-\frac{3}{4}, -\frac{1}{4}] \cup \{0, \frac{3}{4}\} \cup \{(\frac{1}{2})^{\mathbb{N}_0}\}$, where $\mathbb{N}_0$ denotes the set of all nonnegative integers. Consider the following p-Laplacian functional dynamic equation on time scale $\mathbf{T}$,

$$\begin{gathered} [\Phi_p(u^{\Delta}(t))]^{\nabla} + \frac{8u^3(t)}{u^3(t) + u^3(t-\frac{3}{4}) + 1} = 0, \quad t \in (0,1)_{\mathbf{T}}, \\ u_0(t) = \varphi(t) \equiv 0, \quad t \in [-\frac{3}{4}, 0]_{\mathbf{T}}, \\ u(0) - B_0(u^{\Delta}(\frac{1}{4})) = 0, \quad u^{\Delta}(1) = 0, \end{gathered} \tag{4.1}$$

where $T = 1$, $p = \frac{3}{2}$, $a(t) \equiv 1, B = 1$, $A = 1$, $\mu : [0,1]_{\mathbf{T}} \to [-\frac{3}{4}, 1]_{\mathbf{T}}$ and $\mu(t) = t - \frac{3}{4}$, $r = \frac{3}{4}$, $\eta = \frac{1}{4}$, $l = \frac{1}{2}$ and $f(u, \varphi(s)) = \frac{8u^3}{u^3+1}$, $f(u_1, u_2) = \frac{8u_1^3}{u_1^3+u_2^3+1}$. We deduce that $Y_1 = [0, \frac{3}{4}]_{\mathbf{T}}$, $Y_2 = (\frac{3}{4}, 1]_{\mathbf{T}}$, $Y_3 = [\frac{1}{2}, \frac{3}{4}]_{\mathbf{T}}$.

Thus it is easy to see by calculating that $\rho = 2$, $\delta = \frac{5}{64}$, $\lambda = \frac{5}{4}$. Choose $a = \frac{1}{40}$, $b = 1$, $d = 140$, then we have $0 < \frac{T}{\eta} a < b \ < d$, $\rho b < \delta d$, then

$$\begin{gathered} f(u, \varphi(s)) < 8 < \Phi_p(\frac{d}{\rho}) = \sqrt{\frac{140}{2}} \approx 8.3666, \quad 0 \le u \le 140; \\ f(u_1, u_2) < 8 < \Phi_p(\frac{d}{\rho}) = \sqrt{\frac{140}{2}} \approx 8.3666, \quad 0 \le u \le 140, \\ f(u, \varphi(s)) \ge 4 > \Phi_p(\frac{b}{\delta}) = \sqrt{\frac{64}{5}} \approx 3.5777, \quad 1 \le u \le 140, \\ f(u, \varphi(s)) \le \frac{8}{1001} \approx 0.008 < \Phi_p(\frac{a}{\lambda}) = \sqrt{\frac{1}{50}} \approx 0.1414, \quad 0 \le u \le \frac{1}{10}; \\ f(u_1, u_2) \le \frac{8}{1002} \approx 0.008 < \Phi_p(\frac{a}{\lambda}) = \sqrt{\frac{1}{50}} \approx 0.1414, \quad 0 \le u \le \frac{1}{10}, \end{gathered}$$

Thus by Theorem 3.3, the (4.1) has at least three positive solutions of the form

$$u(t) = \begin{cases} u_i(t), & t \in [0,1]_{\mathbf{T}}, \quad i = 1,2,3, \\ \varphi(t), & t \in [-\frac{3}{4},0]_{\mathbf{T}}, \end{cases}$$

where $\gamma(u_i) \le 140$ for $i = 1,2,3$, $1 < \alpha(u_1)$, $\frac{1}{40} < \psi(u_2)$ with $\alpha(u_2) < 1$ and $\psi(u_3) < \frac{1}{40}$.

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