Electronic Journal of Differential Equations, Vol. 2007(2007), No. 106, pp. 1–35.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

BEHAVIOR OF THE SOLUTIONS TO SECOND ORDER LINEAR AUTONOMOUS DELAY DIFFERENTIAL EQUATIONS

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Abstract. A wide class of second order linear autonomous delay differential equations with distributed type delays is considered. An asymptotic result, a useful exponential estimate of the solutions, a stability criterion, and a result on the behavior of the solutions are established.

1. Introduction and preliminaries

The theory of delay differential equations is of both theoretical and practical interest. For the basic theory of delay differential equations, the reader is referred to the books by Diekmann *et al.* [2], Driver [7], Hale [12], and Hale and Verduyn Lunel [13].

The old but very interesting asymptotic and stability results for delay differential equations due to Driver [5, 6] and to Driver, Sasser and Slater [8] gave rise to the publication of a number of articles concerning the asymptotic behavior (and, more generally, the behavior) and the stability for delay differential equations, neutral delay differential equations and (neutral or non-neutral) integrodifferential equations with unbounded delay during the last few years. See [1, 3, 4, 9, 10, 14]-[24]; for some related results see [11]. Moreover, in the last few years, a number of articles dealing with the asymptotic behavior (and, more generally, the behavior) and the stability of delay, and neutral delay, difference equations (with discrete or continuous variable) and of (neutral or non-neutral) Volterra difference equations with infinite delay (see [25] and the references cited therein) appeared in the literature. Very recently, Yeniçerioğlu [26] obtained some results on the qualitative behavior of the solutions of a second order linear autonomous delay differential equation with a single delay. The main idea in [26] is that of transforming the second order delay differential equation into a first order delay differential equation, by the use of a real root of the corresponding characteristic equation. The same idea will be used in this paper to obtain some general results including the results in [26] as particular cases.

In the present paper, a wide class of second order linear autonomous delay differential equations with distributed type delays is considered. An asymptotic result for the solutions is obtained. Also, an estimate of the solutions and a stability

2000 *Mathematics Subject Classification.* 34K20, 34K25.
Key words and phrases. Delay differential equation; asymptotic behavior; stability; behavior of solutions; characteristic equation.

Submitted February 16, 2007. Published July 30, 2007.

criterion for the trivial solution are established. Moreover, a result on the behavior of the solutions is given.

Consider the delay differential equation

$$x''(t) + \int_{-r}^{0} x'(t+s)d\zeta(s) = \int_{-r}^{0} x(t+s)d\eta(s), \tag{1.1}$$

where *r is a positive real constant, ζ and η are real-valued functions of bounded variation on the interval $[-r, 0]$, and the integrals are Riemann-Stieltjes integrals. It will be assumed that η is not constant on $[-r, 0]$.*

By a *solution* of the delay differential equation (1.1), we mean a continuously differentiable real-valued function x defined on the interval $[-r, \infty)$, which is twice continuously differentiable on $[0, \infty)$ and satisfies (1.1) for all $t \geq 0$.

Together with the delay differential equation (1.1), it is customary to specify an *initial condition* of the form

$$x(t) = \phi(t) \quad \text{for } -r \leq t \leq 0, \tag{1.2}$$

where the initial function ϕ is a given continuously differentiable real-valued function on the initial interval $[-r, 0]$.

Equations (1.1) and (1.2) constitute an *initial value problem* (IVP, for short). It is well-known (see, for example, Diekmann *et al.* [2], Driver [7], Hale [12], or Hale and Verduyn Lunel [13]) that there exists a unique solution x of the delay differential equation (1.1) which satisfies the initial condition (1.2); this unique solution x will be called the *solution* of the initial value problem (1.1) and (1.2) or, more briefly, the *solution* of the IVP (1.1) and (1.2).

Along with the delay differential equation (1.1), we associate its *characteristic equation*

$$\lambda^2 + \lambda \int_{-r}^{0} e^{\lambda s} d\zeta(s) = \int_{-r}^{0} e^{\lambda s} d\eta(s), \tag{1.3}$$

which is obtained from (1.1) by looking for solutions of the form $x(t) = e^{\lambda t}$ for $t \geq -r$.

For a given real root λ_0 of the characteristic equation (1.3), we consider the (first order) delay differential equation

$$\begin{aligned} &z'(t) + 2\lambda_0 z(t) + \int_{-r}^{0} e^{\lambda_0 s} z(t+s) d\zeta(s) \\ &= \lambda_0 \int_{-r}^{0} e^{\lambda_0 s} \left[\int_{s}^{0} z(t+u)du \right] d\zeta(s) - \int_{-r}^{0} e^{\lambda_0 s} \left[\int_{s}^{0} z(t+u)du \right] d\eta(s). \end{aligned} \tag{1.4}$$

A *solution* of the delay differential equation (1.4) is a continuous real-valued function z defined on the interval $[-r, \infty)$, which is continuously differentiable on $[0, \infty)$ and satisfies (1.4) for all $t \geq 0$.

The *characteristic equation* of the delay differential equation (1.4) is

$$\begin{aligned} &\mu + 2\lambda_0 + \int_{-r}^{0} e^{(\lambda_0+\mu)s} d\zeta(s) \\ &= \lambda_0 \int_{-r}^{0} e^{\lambda_0 s} \left(\int_{s}^{0} e^{\mu u} du \right) d\zeta(s) - \int_{-r}^{0} e^{\lambda_0 s} \left(\int_{s}^{0} e^{\mu u} du \right) d\eta(s). \end{aligned} \tag{1.5}$$

This equation is obtained from (1.4) by seeking solutions of the form $z(t) = e^{\mu t}$ for $t \geq -r$.

For our convenience, we introduce some notations. For a given real root λ_0 of the characteristic equation (1.3), we set

$$\beta(\lambda_0) = 2\lambda_0 + \int_{-r}^{0} [1 - \lambda_0(-s)] e^{\lambda_0 s} d\zeta(s) + \int_{-r}^{0} (-s) e^{\lambda_0 s} d\eta(s) \tag{1.6}$$

and, also, we define

$$\begin{aligned} K(\lambda_0;\phi) = \phi'(0) + \lambda_0\phi(0) &+ \int_{-r}^{0} \Big[\phi(s) - \lambda_0 e^{\lambda_0 s} \int_{s}^{0} e^{-\lambda_0 u}\phi(u)du\Big] d\zeta(s) \\ &+ \int_{-r}^{0} e^{\lambda_0 s} \Big[\int_{s}^{0} e^{-\lambda_0 u}\phi(u)du\Big] d\eta(s); \end{aligned} \tag{1.7}$$

in addition, provided that $\beta(\lambda_0) \neq 0$, we define

$$\Phi(\lambda_0;\phi)(t) = e^{-\lambda_0 t}\phi(t) - \frac{K(\lambda_0;\phi)}{\beta(\lambda_0)} \quad \text{for } -r \le t \le 0. \tag{1.8}$$

We will now give a proposition, which plays a crucial role in obtaining our main results.

Proposition 1.1. *Let λ_0 be a real root of the characteristic equation* (1.3)*, and let $\beta(\lambda_0)$ and $K(\lambda_0;\phi)$ be defined by* (1.6) *and* (1.7)*, respectively. Suppose that $\beta(\lambda_0) \neq 0$, and define $\Phi(\lambda_0;\phi)$ by* (1.8)*.*

Then a continuous real-valued function x defined on the interval $[-r,\infty)$ is the solution of the IVP (1.1) *and* (1.2) *if and only if the function z defined by*

$$z(t) = e^{-\lambda_0 t}x(t) - \frac{K(\lambda_0;\phi)}{\beta(\lambda_0)} \quad \textit{for } t \ge -r \tag{1.9}$$

is the solution of the delay differential equation (1.4) *which satisfies the initial condition*

$$z(t) = \Phi(\lambda_0;\phi)(t) \quad \textit{for } -r \le t \le 0. \tag{1.10}$$

Proof. Let x be the solution of the IVP (1.1) and (1.2), and define

$$y(t) = e^{-\lambda_0 t}x(t) \quad \text{for } t \ge -r. \tag{1.11}$$

Then, by taking into account the fact that λ_0 is a real root of the characteristic equation (1.3), we get, for every $t \ge 0$,

$$\begin{aligned} &x''(t) + \int_{-r}^{0} x'(t+s)d\zeta(s) - \int_{-r}^{0} x(t+s)d\eta(s) \\ &= e^{\lambda_0 t}\Big\{\big[y''(t) + 2\lambda_0 y'(t) + \lambda_0^2 y(t)\big] + \int_{-r}^{0} e^{\lambda_0 s}\left[y'(t+s) + \lambda_0 y(t+s)\right] d\zeta(s) \\ &\quad - \int_{-r}^{0} e^{\lambda_0 s} y(t+s)d\eta(s)\Big\} \\ &= e^{\lambda_0 t}\Big\{\Big[y''(t) + 2\lambda_0 y'(t) + \int_{-r}^{0} e^{\lambda_0 s} y'(t+s)d\zeta(s)\Big] + \lambda_0^2 y(t) \\ &\quad + \lambda_0 \int_{-r}^{0} e^{\lambda_0 s} y(t+s)d\zeta(s) - \int_{-r}^{0} e^{\lambda_0 s} y(t+s)d\eta(s)\Big\} \\ &= e^{\lambda_0 t}\Big\{\Big[y'(t) + 2\lambda_0 y(t) + \int_{-r}^{0} e^{\lambda_0 s} y(t+s)d\zeta(s)\Big]' \end{aligned}$$

$$
\begin{aligned}
&+\Big[-\lambda_0\int_{-r}^{0}e^{\lambda_0 s}d\zeta(s)+\int_{-r}^{0}e^{\lambda_0 s}d\eta(s)\Big]y(t)\\
&+\lambda_0\int_{-r}^{0}e^{\lambda_0 s}y(t+s)d\zeta(s)-\int_{-r}^{0}e^{\lambda_0 s}y(t+s)d\eta(s)\Big\}\\
&=e^{\lambda_0 t}\Big\{\Big[y'(t)+2\lambda_0 y(t)+\int_{-r}^{0}e^{\lambda_0 s}y(t+s)d\zeta(s)\Big]'\\
&-\lambda_0\int_{-r}^{0}e^{\lambda_0 s}\left[y(t)-y(t+s)\right]d\zeta(s)+\int_{-r}^{0}e^{\lambda_0 s}\left[y(t)-y(t+s)\right]d\eta(s)\Big\}.
\end{aligned}
$$

Hence, the fact that x is a solution of the delay differential equation (1.1) is equivalent to the fact that y satisfies

$$
\begin{aligned}
&\Big[y'(t)+2\lambda_0 y(t)+\int_{-r}^{0}e^{\lambda_0 s}y(t+s)d\zeta(s)\Big]'\\
&=\lambda_0\int_{-r}^{0}e^{\lambda_0 s}\left[y(t)-y(t+s)\right]d\zeta(s)-\int_{-r}^{0}e^{\lambda_0 s}\left[y(t)-y(t+s)\right]d\eta(s)
\end{aligned}
\tag{1.12}
$$

for all $t\geq 0$. On the other hand, x satisfies the initial condition (1.2) if and only if y satisfies the initial condition

$$
y(t)=e^{-\lambda_0 t}\phi(t)\quad\text{for }-r\leq t\leq 0. \tag{1.13}
$$

Furthermore, we see that y satisfies (1.12) for $t\geq 0$ if and only if

$$
\begin{aligned}
&y'(t)+2\lambda_0 y(t)+\int_{-r}^{0}e^{\lambda_0 s}y(t+s)d\zeta(s)\\
&=\lambda_0\int_{-r}^{0}e^{\lambda_0 s}\Big[\int_{t+s}^{t}y(u)du\Big]d\zeta(s)-\int_{-r}^{0}e^{\lambda_0 s}\Big[\int_{t+s}^{t}y(u)du\Big]d\eta(s)+\Theta
\end{aligned}
$$

or, equivalently,

$$
\begin{aligned}
&y'(t)+2\lambda_0 y(t)+\int_{-r}^{0}e^{\lambda_0 s}y(t+s)d\zeta(s)\\
&=\lambda_0\int_{-r}^{0}e^{\lambda_0 s}\Big[\int_{s}^{0}y(t+u)du\Big]d\zeta(s)-\int_{-r}^{0}e^{\lambda_0 s}\Big[\int_{s}^{0}y(t+u)du\Big]d\eta(s)+\Theta
\end{aligned}
$$

for all $t\geq 0$, where Θ is some real constant. By using the initial condition (1.13) and taking into account the definition of $K(\lambda_0;\phi)$ by (1.7), we have

$$
\begin{aligned}
\Theta&=y'(0)+2\lambda_0 y(0)+\int_{-r}^{0}e^{\lambda_0 s}y(s)d\zeta(s)\\
&\quad-\lambda_0\int_{-r}^{0}e^{\lambda_0 s}\Big[\int_{s}^{0}y(u)du\Big]d\zeta(s)+\int_{-r}^{0}e^{\lambda_0 s}\Big[\int_{s}^{0}y(u)du\Big]d\eta(s)\\
&=[\phi'(0)-\lambda_0\phi(0)]+2\lambda_0\phi(0)+\int_{-r}^{0}\phi(s)d\zeta(s)\\
&\quad-\lambda_0\int_{-r}^{0}e^{\lambda_0 s}\Big[\int_{s}^{0}e^{-\lambda_0 u}\phi(u)du\Big]d\zeta(s)+\int_{-r}^{0}e^{\lambda_0 s}\Big[\int_{s}^{0}e^{-\lambda_0 u}\phi(u)du\Big]d\eta(s)\\
&=\phi'(0)+\lambda_0\phi(0)+\int_{-r}^{0}\Big[\phi(s)-\lambda_0 e^{\lambda_0 s}\int_{s}^{0}e^{-\lambda_0 u}\phi(u)du\Big]d\zeta(s)
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{-r}^{0} e^{\lambda_0 s}\left[\int_{s}^{0} e^{-\lambda_0 u}\phi(u)du\right] d\eta(s)\\
&\equiv K(\lambda_0;\phi).
\end{aligned}
$$

So, the fact that y satisfies (1.12) for $t \geq 0$ is equivalent to the fact that y satisfies

$$
\begin{aligned}
&y'(t) + 2\lambda_0 y(t) + \int_{-r}^{0} e^{\lambda_0 s} y(t+s) d\zeta(s)\\
&= \lambda_0 \int_{-r}^{0} e^{\lambda_0 s}\left[\int_{s}^{0} y(t+u)du\right] d\zeta(s)\\
&\quad - \int_{-r}^{0} e^{\lambda_0 s}\left[\int_{s}^{0} y(t+u)du\right] d\eta(s) + K(\lambda_0;\phi)
\end{aligned}
\tag{1.14}
$$

for all $t \geq 0$.

Now, we take into account the assumption $\beta(\lambda_0) \neq 0$ and we define

$$
z(t) = y(t) - \frac{K(\lambda_0;\phi)}{\beta(\lambda_0)} \quad \text{for } t \geq -r. \tag{1.15}
$$

Then, because of the definition of $\beta(\lambda_0)$ by (1.6), it is a matter of elementary calculations to show that y satisfies (1.14) for $t \geq 0$ if and only if z satisfies (1.4) for all $t \geq 0$, i.e., if and only if z is a solution of the delay differential equation (1.4). Moreover, we see that the initial condition (1.13) is equivalently written as follows

$$
z(t) = e^{-\lambda_0 t}\phi(t) - \frac{K(\lambda_0;\phi)}{\beta(\lambda_0)} \quad \text{for } -r \leq t \leq 0. \tag{1.16}
$$

We have thus proved that x is the solution of the IVP (1.1) and (1.2) if and only if z is the solution of the delay differential equation (1.4) which satisfies the initial condition (1.16). By (1.11), we see that (1.15) coincides with (1.9). Also, by taking into account the definition of $\Phi(\lambda_0;\phi)$ by (1.8), we observe that (1.16) coincides with the initial condition (1.10). The proof of our proposition is complete. □

Let $C([-r,0],\mathbb{R})$ be the Banach space of all continuous real-valued functions on the interval $[-r,0]$, endowed with the usual sup-norm

$$
\|\psi\| = \max_{-r\leq t\leq 0} |\psi(t)| \quad \text{for } \psi \in C([-r,0],\mathbb{R}).
$$

Moreover, let $C^1([-r,0],\mathbb{R})$ be the set of all continuously differentiable real-valued functions on the interval $[-r,0]$. This set is a Banach space with the norm

$$
\nparallel \omega \nparallel = \max\{\|\omega\|, \|\omega'\|\} \quad \text{for } \omega \in C^1([-r,0],\mathbb{R}).
$$

As it concerns the IVP (1.1) and (1.2) studied in this paper, the initial function ϕ belongs to $C^1([-r,0],\mathbb{R})$. So, the notation $\nparallel \phi \nparallel$ used in Section 3 is defined by

$$
\nparallel \phi \nparallel = \max\{\|\phi\|, \|\phi'\|\} \equiv \max\Big\{ \max_{-r\leq t\leq 0}|\phi(t)|, \max_{-r\leq t\leq 0}|\phi'(t)| \Big\}.
$$

It will be considered that the reader is familiar with the notions of *stability, uniform stability, asymptotic stability,* and *uniform asymptotic stability* of the *trivial solution* of a linear delay differential system. (We choose to refer to the book by Driver [7].) It is known (see, for example, [7]) that, in the case of *autonomous* linear delay differential systems, the trivial solution is uniformly stable if and only if it is stable (at 0), and the trivial solution is uniformly asymptotically stable if and only if it is asymptotically stable (at 0).

The substitution
$$x_1 = x, \quad x_2 = x'$$
transforms the (second order) linear delay differential equation (1.1) into the following equivalent (first order) linear delay differential system
$$x_1'(t) = x_2(t), \quad x_2'(t) = \int_{-r}^{0} x_1(t+s)d\eta(s) - \int_{-r}^{0} x_2(t+s)d\zeta(s). \tag{1.17}$$
The delay differential system (1.17) is considered in conjunction with the initial condition
$$x_1(t) = \phi(t) \quad \text{for } -r \le t \le 0, \quad x_2(t) = \phi'(t) \quad \text{for } -r \le t \le 0. \tag{1.18}$$
So, the IVP (1.1) and (1.2) is transformed into the equivalent initial value problem (1.17) and (1.18).

On the basis of the above transformation of the delay differential equation (1.1) into the equivalent delay differential system (1.17), one can formulate the definitions of the notions of the stability, uniform stability, asymptotic stability, and uniform asymptotic stability of the trivial solution of (1.1). As the delay differential equation (1.1) is autonomous, the trivial solution of (1.1) is uniformly stable (respectively, uniformly asymptotically stable) if and only if it is stable (at 0) (respectively, asymptotically stable (at 0)). We restrict ourselves to giving here the definitions of the stability (at 0) and the asymptotic stability (at 0) of the trivial solution of (1.1). The trivial solution of the delay differential equation (1.1) is said to be *stable* (*at* 0) if, for each $\epsilon > 0$, there exists a $\delta \equiv \delta(\epsilon) > 0$ such that, for any $\phi \in C^1([-r, 0], \mathbb{R})$ with $\nparallel \phi \nparallel < \delta$, the solution x of the IVP (1.1) and (1.2) satisfies
$$\max\{|x(t)|, |x'(t)|\} < \epsilon \quad \text{for all } t \ge -r.$$
Moreover, the *trivial solution* of (1.1) is called *asymptotically stable* (at 0) if it is stable (at 0) in the above sense and, in addition, there exists a $\delta_0 > 0$ such that, for any $\phi \in C^1([-r, 0], \mathbb{R})$ with $\nparallel \phi \nparallel < \delta_0$, the solution x of the IVP (1.1) and (1.2) satisfies
$$\lim_{t\to\infty} [\max\{|x(t)|, |x'(t)|\}] = 0; \qquad \text{i.e., } \lim_{t\to\infty} x(t) = \lim_{t\to\infty} x'(t) = 0.$$

Let us consider the special case of the delay differential equation
$$x''(t) = \int_{-r}^{0} x(t+s)d\eta(s). \tag{1.19}$$
This equation is obtained from (1.1), by considering that ζ is any constant real -valued function on the interval $[-r, 0]$.

The *characteristic equation* of the delay differential equation (1.19) is
$$\lambda^2 = \int_{-r}^{0} e^{\lambda s} d\eta(s). \tag{1.20}$$

For a given real root λ_0 of the characteristic equation (1.20), we consider the (first order) delay differential equation
$$z'(t) + 2\lambda_0 z(t) = -\int_{-r}^{0} e^{\lambda_0 s} \left[\int_{s}^{0} z(t+u)du\right] d\eta(s). \tag{1.21}$$
The equation
$$\mu + 2\lambda_0 = -\int_{-r}^{0} e^{\lambda_0 s} \left(\int_{s}^{0} e^{\mu u} du\right) d\eta(s) \tag{1.22}$$

is the *characteristic equation* of the delay differential equation (1.21). For a given real root λ_0 of the characteristic equation (1.20), we define

$$\widetilde{\beta}(\lambda_0) = 2\lambda_0 + \int_{-r}^{0} (-s)e^{\lambda_0 s} d\eta(s), \tag{1.23}$$

$$\widetilde{K}(\lambda_0; \phi) = \phi'(0) + \lambda_0\phi(0) + \int_{-r}^{0} e^{\lambda_0 s} \left[\int_{s}^{0} e^{-\lambda_0 u}\phi(u)du\right] d\eta(s) \tag{1.24}$$

and, provided that $\widetilde{\beta}(\lambda_0) \neq 0$,

$$\widetilde{\Phi}(\lambda_0; \phi)(t) = e^{-\lambda_0 t}\phi(t) - \frac{\widetilde{K}(\lambda_0; \phi)}{\widetilde{\beta}(\lambda_0)} \quad \text{for } -r \leq t \leq 0. \tag{1.25}$$

Our results given in the next sections (Sections 2−5) are formulated as four theorems (Theorems 2.1, 3.1, 4.2, and 4.4), a corollary (Corollary 3.2), and five lemmas (Lemmas 4.1, 4.3, and 5.1–5.3). Section 2 contains Theorem 2.1, Section 3 is devoted to Theorem 3.1 and Corollary 3.2, Section 4 includes Lemmas 4.1 and 4.3 as well as Theorems 4.2 and 4.4, and Section 5 contains Lemmas 5.1–5.3.

Theorem 2.1 constitutes a basic asymptotic result for the solution of the IVP (1.1) and (1.2) as well as for the first order derivative of this solution. Estimates of the solution of the IVP (1.1) and (1.2) and of the first order derivative of the solution are established by Theorem 3.1. Corollary 3.2 is a stability criterion for the trivial solution of the delay differential equation (1.1). Lemma 4.1 is an auxiliary result about the real roots of the characteristic equation (1.5), where λ_0 is a negative real root of the characteristic equation (1.3). Analogously, Lemma 4.3 is an auxiliary result concerning the real roots of the characteristic equation (1.22), where λ_0 is a nonzero real root of the characteristic equation (1.20). Lemmas 4.1 and 4.3 are used in establishing Theorems 4.2 and 4.4, respectively. Theorem 4.2 is concerned with the behavior of the solution of the IVP (1.1) and (1.2) and of the first order derivative of this solution. Theorem 4.4 concerns the special case of the IVP (1.19) and (1.2), and provides a result on the behavior of the solution and of the first order derivative of this solution. For a given real root λ_0 of the characteristic equation (1.3), Lemma 5.1 establishes sufficient conditions for the characteristic equation (1.5) to have a real root with an appropriate property. Lemma 5.2 is concerned with the real roots of (1.5), where λ_0 is a negative real root of (1.3). Finally, Lemma 5.3 concerns the real roots of the characteristic equation (1.22), where λ_0 is a nonzero real root of the characteristic equation (1.20).

Theorems 2.1 and 3.1 as well as Corollary 3.2 are obtained, by the use of a real root λ_0 of the characteristic equation (1.3) and of a real root μ_0 of the characteristic equation (1.5). Theorem 4.2 is derived, via a negative real root λ_0 of (1.3) and two distinct real roots μ_0 and μ_1 of (1.5). In obtaining Theorem 4.4, a nonzero real root λ_0 of the characteristic equation (1.20) and two distinct real roots μ_0 and μ_1 of the characteristic equation (1.22) are used.

Throughout the paper, we need a notation concerning a real-valued function θ which is of bounded variation on the interval $[-r, 0]$. By $V(\theta)$ we will denote the *total variation function* of θ, which is defined on the interval $[-r, 0]$ as follows: $V(\theta)(-r) = 0$, and $V(\theta)(s)$ is the total variation of θ on $[-r, s]$ for each s in $(-r, 0]$. Note that the function $V(\theta)$ is nonnegative and increasing on the interval $[-r, 0]$. Moreover, it must be noted that $V(\theta)$ is identically zero on $[-r, 0]$ if and only if θ is constant on the interval $[-r, 0]$. So, as η is assumed to be not constant on

$[-r, 0]$, the function $V(\eta)$ is not identically zero on the interval $[-r, 0]$ (and so it is always not constant on $[-r, 0]$). It will be considered that the reader is familiar with the theory of functions of bounded variation and the theory of Riemann-Stieltjes integration.

2. An asymptotic result

Our purpose in this section is to establish the following theorem.

Theorem 2.1. *Let λ_0 be a real root of the characteristic equation* (1.3)*, and let $\beta(\lambda_0)$ and $K(\lambda_0; \phi)$ be defined by* (1.6) *and* (1.7)*, respectively. Suppose that $\beta(\lambda_0) \neq 0$, and define $\Phi(\lambda_0; \phi)$ by* (1.8)*. Furthermore, let μ_0 be a real root of the characteristic equation* (1.5)*, and set*

$$\begin{aligned} \gamma(\lambda_0, \mu_0) = &- \int_{-r}^{0} e^{\lambda_0 s} \left[(-s) e^{\mu_0 s} - \lambda_0 \int_{s}^{0} (-u) e^{\mu_0 u} du \right] d\zeta(s) \\ &- \int_{-r}^{0} e^{\lambda_0 s} \left[\int_{s}^{0} (-u) e^{\mu_0 u} du \right] d\eta(s) \end{aligned} \tag{2.1}$$

and, also, define

$$\begin{aligned} L(\lambda_0, \mu_0; \phi) = \Phi(\lambda_0; \phi)(0) &- \int_{-r}^{0} e^{\lambda_0 s} \Big\{ e^{\mu_0 s} \int_{s}^{0} e^{-\mu_0 u} \Phi(\lambda_0; \phi)(u) du \\ &- \lambda_0 \int_{s}^{0} e^{\mu_0 u} \left[\int_{u}^{0} e^{-\mu_0 v} \Phi(\lambda_0; \phi)(v) dv \right] du \Big\} d\zeta(s) \\ &- \int_{-r}^{0} e^{\lambda_0 s} \left\{ \int_{s}^{0} e^{\mu_0 u} \left[\int_{u}^{0} e^{-\mu_0 v} \Phi(\lambda_0; \phi)(v) dv \right] du \right\} d\eta(s). \end{aligned} \tag{2.2}$$

(Note that, because of $\beta(\lambda_0) \neq 0$, we always have $\mu_0 \neq 0$.) Assume that

$$\begin{aligned} &\int_{-r}^{0} e^{\lambda_0 s} \left[(-s) e^{\mu_0 s} + |\lambda_0| \int_{s}^{0} (-u) e^{\mu_0 u} du \right] dV(\zeta)(s) \\ &\quad + \int_{-r}^{0} e^{\lambda_0 s} \left[\int_{s}^{0} (-u) e^{\mu_0 u} du \right] dV(\eta)(s) < 1. \end{aligned} \tag{2.3}$$

(This assumption guarantees that $1 + \gamma(\lambda_0, \mu_0) > 0$.) Then the solution x of the IVP (1.1) *and* (1.2) *satisfies*

$$\lim_{t \to \infty} \left\{ e^{-\mu_0 t} \left[e^{-\lambda_0 t} x(t) - \frac{K(\lambda_0; \phi)}{\beta(\lambda_0)} \right] \right\} = \frac{L(\lambda_0, \mu_0; \phi)}{1 + \gamma(\lambda_0, \mu_0)} \tag{2.4}$$

and

$$\lim_{t \to \infty} \left\{ e^{-\mu_0 t} \left[e^{-\lambda_0 t} x'(t) - \lambda_0 \frac{K(\lambda_0; \phi)}{\beta(\lambda_0)} \right] \right\} = (\lambda_0 + \mu_0) \frac{L(\lambda_0, \mu_0; \phi)}{1 + \gamma(\lambda_0, \mu_0)}. \tag{2.5}$$

Before we prove the above theorem, we will present some observations, which are concerned with a real root λ_0 of the characteristic equation (1.3) and a real root μ_0 of the characteristic equation (1.5).

We immediately see that zero is a root of (1.5) if and only if

$$2\lambda_0 + \int_{-r}^{0} e^{\lambda_0 s} d\zeta(s) = \lambda_0 \int_{-r}^{0} (-s) e^{\lambda_0 s} d\zeta(s) - \int_{-r}^{0} (-s) e^{\lambda_0 s} d\eta(s);$$

i.e., if and only if $\beta(\lambda_0) = 0$, where $\beta(\lambda_0)$ is defined by (1.6). Hence, if we assume that $\beta(\lambda_0) \neq 0$, then we always have $\mu_0 \neq 0$.

Let us define

$$\begin{aligned}\rho(\lambda_0,\mu_0) &= \int_{-r}^{0} e^{\lambda_0 s}\Big[(-s)e^{\mu_0 s} + |\lambda_0|\int_s^0 (-u)e^{\mu_0 u}du\Big]\, dV(\zeta)(s) \\ &\quad + \int_{-r}^{0} e^{\lambda_0 s}\Big[\int_s^0 (-u)e^{\mu_0 u}du\Big]\, dV(\eta)(s).\end{aligned} \tag{2.6}$$

As η is assumed to be not constant on $[-r,0]$, it is clear that $\rho(\lambda_0,\mu_0)$ is positive. So, (2.3) can equivalently be written as follows

$$0 < \rho(\lambda_0,\mu_0) < 1. \tag{2.7}$$

Furthermore, for the real constant $\gamma(\lambda_0,\mu_0)$ defined by (2.1), we have

$$\begin{aligned}|\gamma(\lambda_0,\mu_0)| &\le \Big|\int_{-r}^{0} e^{\lambda_0 s}\Big[(-s)e^{\mu_0 s} - \lambda_0\int_s^0 (-u)e^{\mu_0 u}du\Big]\, d\zeta(s)\Big| \\ &\quad + \Big|\int_{-r}^{0} e^{\lambda_0 s}\Big[\int_s^0 (-u)e^{\mu_0 u}du\Big]\, d\eta(s)\Big| \\ &\le \int_{-r}^{0} e^{\lambda_0 s}\Big|(-s)e^{\mu_0 s} - \lambda_0\int_s^0 (-u)e^{\mu_0 u}du\Big|\, dV(\zeta)(s) \\ &\quad + \int_{-r}^{0} e^{\lambda_0 s}\Big[\int_s^0 (-u)e^{\mu_0 u}du\Big]\, dV(\eta)(s) \\ &\le \int_{-r}^{0} e^{\lambda_0 s}\Big[(-s)e^{\mu_0 s} + |\lambda_0|\int_s^0 (-u)e^{\mu_0 u}du\Big]\, dV(\zeta)(s) \\ &\quad + \int_{-r}^{0} e^{\lambda_0 s}\Big[\int_s^0 (-u)e^{\mu_0 u}du\Big]\, dV(\eta)(s).\end{aligned}$$

That is,

$$|\gamma(\lambda_0,\mu_0)| \le \rho(\lambda_0,\mu_0). \tag{2.8}$$

Thus, if we assume that (2.3) is satisfied, i.e., that (2.7) holds, then (2.8) gives $|\gamma(\lambda_0,\mu_0)| < 1$. This guarantees, in particular, that

$$1 + \gamma(\lambda_0,\mu_0) > 0. \tag{2.9}$$

Proof of Theorem 2.1. Let x be the solution of the IVP (1.1) and (1.2). Define the function z by (1.9). By Proposition 1.1, the fact that x is the solution of the IVP (1.1) and (1.2) is equivalent to the fact that z is the solution of the delay differential equation (1.4) which satisfies the initial condition (1.10). Set

$$w(t) = e^{-\mu_0 t}z(t) \quad \text{for } t \ge -r. \tag{2.10}$$

Then, using the fact that μ_0 is a real root of the characteristic equation (1.5), we obtain, for every $t \ge 0$,

$$\begin{aligned}&z'(t) + 2\lambda_0 z(t) + \int_{-r}^{0} e^{\lambda_0 s} z(t+s)d\zeta(s) \\ &- \lambda_0\int_{-r}^{0} e^{\lambda_0 s}\Big[\int_s^0 z(t+u)du\Big]\, d\zeta(s) + \int_{-r}^{0} e^{\lambda_0 s}\Big[\int_s^0 z(t+u)du\Big]\, d\eta(s) \\ &= e^{\mu_0 t}\Big\{w'(t) + (\mu_0 + 2\lambda_0)w(t) + \int_{-r}^{0} e^{(\lambda_0+\mu_0)s}w(t+s)d\zeta(s)\end{aligned}$$

$$
\begin{aligned}
&- \lambda_0 \int_{-r}^{0} e^{\lambda_0 s} \left[\int_s^0 e^{\mu_0 u} w(t+u) du \right] d\zeta(s) \\
&+ \int_{-r}^{0} e^{\lambda_0 s} \left[\int_s^0 e^{\mu_0 u} w(t+u) du \right] d\eta(s) \Big\} \\
&= e^{\mu_0 t} \Big\{ w'(t) + \Big[- \int_{-r}^{0} e^{(\lambda_0+\mu_0)s} d\zeta(s) + \lambda_0 \int_{-r}^{0} e^{\lambda_0 s} \Big(\int_s^0 e^{\mu_0 u} du \Big) d\zeta(s) \\
&- \int_{-r}^{0} e^{\lambda_0 s} \Big(\int_s^0 e^{\mu_0 u} du \Big) d\eta(s) \Big] w(t) + \int_{-r}^{0} e^{(\lambda_0+\mu_0)s} w(t+s) d\zeta(s) \\
&- \lambda_0 \int_{-r}^{0} e^{\lambda_0 s} \left[\int_s^0 e^{\mu_0 u} w(t+u) du \right] d\zeta(s) \\
&+ \int_{-r}^{0} e^{\lambda_0 s} \left[\int_s^0 e^{\mu_0 u} w(t+u) du \right] d\eta(s) \Big\} \\
&= e^{\mu_0 t} \Big(w'(t) - \int_{-r}^{0} e^{(\lambda_0+\mu_0)s} [w(t) - w(t+s)] d\zeta(s) \\
&+ \lambda_0 \int_{-r}^{0} e^{\lambda_0 s} \Big\{ \int_s^0 e^{\mu_0 u} [w(t) - w(t+u)] du \Big\} d\zeta(s) \\
&- \int_{-r}^{0} e^{\lambda_0 s} \Big\{ \int_s^0 e^{\mu_0 u} [w(t) - w(t+u)] du \Big\} d\eta(s) \Big).
\end{aligned}
$$

So, z is a solution of the delay differential equation (1.4) if and only if w satisfies

$$
\begin{aligned}
w'(t) = &\int_{-r}^{0} e^{(\lambda_0+\mu_0)s} [w(t) - w(t+s)] d\zeta(s) \\
&- \lambda_0 \int_{-r}^{0} e^{\lambda_0 s} \Big\{ \int_s^0 e^{\mu_0 u} [w(t) - w(t+u)] du \Big\} d\zeta(s) \\
&+ \int_{-r}^{0} e^{\lambda_0 s} \Big\{ \int_s^0 e^{\mu_0 u} [w(t) - w(t+u)] du \Big\} d\eta(s)
\end{aligned} \tag{2.11}
$$

for all $t \geq 0$. Moreover, z satisfies the initial condition (1.10) if and only if w satisfies

$$
w(t) = e^{-\mu_0 t} \Phi(\lambda_0; \phi)(t) \quad \text{for } -r \leq t \leq 0. \tag{2.12}
$$

Furthermore, we see that the fact that w satisfies (2.11) for $t \geq 0$ is equivalent to the fact that w satisfies

$$
\begin{aligned}
w(t) = &\int_{-r}^{0} e^{(\lambda_0+\mu_0)s} \left[\int_{t+s}^{t} w(u) du \right] d\zeta(s) \\
&- \lambda_0 \int_{-r}^{0} e^{\lambda_0 s} \Big\{ \int_s^0 e^{\mu_0 u} \left[\int_{t+u}^{t} w(v) dv \right] du \Big\} d\zeta(s) \\
&+ \int_{-r}^{0} e^{\lambda_0 s} \Big\{ \int_s^0 e^{\mu_0 u} \left[\int_{t+u}^{t} w(v) dv \right] du \Big\} d\eta(s) + \Lambda,
\end{aligned}
$$

i.e.,

$$
w(t) = \int_{-r}^{0} e^{(\lambda_0+\mu_0)s} \left[\int_s^0 w(t+u) du \right] d\zeta(s)
$$

$$
\begin{aligned}
&- \lambda_0 \int_{-r}^{0} e^{\lambda_0 s} \left\{ \int_s^0 e^{\mu_0 u} \left[\int_u^0 w(t+v)dv \right] du \right\} d\zeta(s) \\
&+ \int_{-r}^{0} e^{\lambda_0 s} \left\{ \int_s^0 e^{\mu_0 u} \left[\int_u^0 w(t+v)dv \right] du \right\} d\eta(s) + \Lambda
\end{aligned}
$$

for all $t \geq 0$, where Λ is some real number. But, by taking into account the initial condition (2.12) and the definition of $L(\lambda_0, \mu_0; \phi)$ by (2.2), we have

$$
\begin{aligned}
\Lambda &= w(0) - \int_{-r}^{0} e^{(\lambda_0+\mu_0)s} \left[\int_s^0 w(u)du \right] d\zeta(s) \\
&\quad + \lambda_0 \int_{-r}^{0} e^{\lambda_0 s} \left\{ \int_s^0 e^{\mu_0 u} \left[\int_u^0 w(v)dv \right] du \right\} d\zeta(s) \\
&\quad - \int_{-r}^{0} e^{\lambda_0 s} \left\{ \int_s^0 e^{\mu_0 u} \left[\int_u^0 w(v)dv \right] du \right\} d\eta(s) \\
&= \Phi(\lambda_0; \phi)(0) - \int_{-r}^{0} e^{(\lambda_0+\mu_0)s} \left[\int_s^0 e^{-\mu_0 u} \Phi(\lambda_0; \phi)(u)du \right] d\zeta(s) \\
&\quad + \lambda_0 \int_{-r}^{0} e^{\lambda_0 s} \left\{ \int_s^0 e^{\mu_0 u} \left[\int_u^0 e^{-\mu_0 v} \Phi(\lambda_0; \phi)(v)dv \right] du \right\} d\zeta(s) \\
&\quad - \int_{-r}^{0} e^{\lambda_0 s} \left\{ \int_s^0 e^{\mu_0 u} \left[\int_u^0 e^{-\mu_0 v} \Phi(\lambda_0; \phi)(v)dv \right] du \right\} d\eta(s) \\
&= \Phi(\lambda_0; \phi)(0) - \int_{-r}^{0} e^{\lambda_0 s} \Big\{ e^{\mu_0 s} \int_s^0 e^{-\mu_0 u} \Phi(\lambda_0; \phi)(u)du \\
&\quad -\lambda_0 \int_s^0 e^{\mu_0 u} \left[\int_u^0 e^{-\mu_0 v} \Phi(\lambda_0; \phi)(v)dv \right] du \Big\} d\zeta(s) \\
&\quad - \int_{-r}^{0} e^{\lambda_0 s} \left\{ \int_s^0 e^{\mu_0 u} \left[\int_u^0 e^{-\mu_0 v} \Phi(\lambda_0; \phi)(v)dv \right] du \right\} d\eta(s) \\
&\equiv L(\lambda_0, \mu_0; \phi).
\end{aligned}
$$

Thus, (2.11) is satisfied for $t \geq 0$ if and only if w satisfies

$$
\begin{aligned}
w(t) &= \int_{-r}^{0} e^{(\lambda_0+\mu_0)s} \left[\int_s^0 w(t+u)du \right] d\zeta(s) \\
&\quad - \lambda_0 \int_{-r}^{0} e^{\lambda_0 s} \left\{ \int_s^0 e^{\mu_0 u} \left[\int_u^0 w(t+v)dv \right] du \right\} d\zeta(s) \\
&\quad + \int_{-r}^{0} e^{\lambda_0 s} \left\{ \int_s^0 e^{\mu_0 u} \left[\int_u^0 w(t+v)dv \right] du \right\} d\eta(s) + L(\lambda_0, \mu_0; \phi)
\end{aligned}
\tag{2.13}
$$

for all $t \geq 0$. Next, taking into account (2.9) (which is a consequence of the assumption (2.3)), we define

$$
f(t) = w(t) - \frac{L(\lambda_0, \mu_0; \phi)}{1 + \gamma(\lambda_0, \mu_0)} \quad \text{for } t \geq -r. \tag{2.14}
$$

Then, using the definition of $\gamma(\lambda_0, \mu_0)$ by (2.1), it is not difficult to show that the fact that w satisfies (2.13) for $t \geq 0$ is equivalent to the fact that f satisfies

$$\begin{aligned} f(t) = &\int_{-r}^{0} e^{(\lambda_0+\mu_0)s} \left[\int_s^0 f(t+u)du\right] d\zeta(s) \\ &- \lambda_0 \int_{-r}^{0} e^{\lambda_0 s} \left\{\int_s^0 e^{\mu_0 u} \left[\int_u^0 f(t+v)dv\right] du\right\} d\zeta(s) \\ &+ \int_{-r}^{0} e^{\lambda_0 s} \left\{\int_s^0 e^{\mu_0 u} \left[\int_u^0 f(t+v)dv\right] du\right\} d\eta(s) \end{aligned} \tag{2.15}$$

for all $t \geq 0$. On the other hand, the initial condition (2.12) takes the following equivalent form

$$f(t) = e^{-\mu_0 t}\Phi(\lambda_0; \phi)(t) - \frac{L(\lambda_0, \mu_0; \phi)}{1 + \gamma(\lambda_0, \mu_0)} \quad \text{for } -r \leq t \leq 0. \tag{2.16}$$

Now, we shall prove that

$$\lim_{t\to\infty} f(t) = 0. \tag{2.17}$$

Define

$$M(\lambda_0, \mu_0; \phi) = \max_{-r\leq t\leq 0} \left|e^{-\mu_0 t}\Phi(\lambda_0; \phi)(t) - \frac{L(\lambda_0, \mu_0; \phi)}{1 + \gamma(\lambda_0, \mu_0)}\right|. \tag{2.18}$$

It follows from (2.16) and (2.18) that

$$|f(t)| \leq M(\lambda_0, \mu_0; \phi) \quad \text{for } -r \leq t \leq 0. \tag{2.19}$$

We will show that $M(\lambda_0, \mu_0; \phi)$ is a bound of the function f on the whole interval $[-r, \infty)$, i.e., that

$$|f(t)| \leq M(\lambda_0, \mu_0; \phi) \quad \text{for all } t \geq -r. \tag{2.20}$$

For this purpose, we consider an arbitrary positive real number ϵ. We claim that

$$|f(t)| < M(\lambda_0, \mu_0; \phi) + \epsilon \quad \text{for every } t \geq -r. \tag{2.21}$$

Otherwise, since (2.19) implies that $|f(t)| < M(\lambda_0, \mu_0; \phi) + \epsilon$ for $-r \leq t \leq 0$, there exists a point $t_0 > 0$ so that

$$|f(t)| < M(\lambda_0, \mu_0; \phi) + \epsilon \quad \text{for } -r \leq t < t_0, \quad \text{and} \quad |f(t_0)| = M(\lambda_0, \mu_0; \phi) + \epsilon.$$

Then, by taking into account the definition of $\rho(\lambda_0, \mu_0)$ by (2.6) and using (2.7) (which is equivalent to the assumption (2.3)), from (2.15) we obtain

$$\begin{aligned} &M(\lambda_0, \mu_0; \phi) + \epsilon \\ &= |f(t_0)| \\ &= \Big|\int_{-r}^{0} e^{(\lambda_0+\mu_0)s} \left[\int_s^0 f(t_0+u)du\right] d\zeta(s) \\ &\quad - \lambda_0 \int_{-r}^{0} e^{\lambda_0 s} \left\{\int_s^0 e^{\mu_0 u} \left[\int_u^0 f(t_0+v)dv\right] du\right\} d\zeta(s) \\ &\quad + \int_{-r}^{0} e^{\lambda_0 s} \left\{\int_s^0 e^{\mu_0 u} \left[\int_u^0 f(t_0+v)dv\right] du\right\} d\eta(s)\Big| \\ &\leq \left|\int_{-r}^{0} e^{(\lambda_0+\mu_0)s} \left[\int_s^0 f(t_0+u)du\right] d\zeta(s)\right| \end{aligned}$$

$$
\begin{aligned}
&+ |\lambda_0| \left| \int_{-r}^{0} e^{\lambda_0 s} \left\{ \int_{s}^{0} e^{\mu_0 u} \left[\int_{u}^{0} f(t_0+v)dv \right] du \right\} d\zeta(s) \right| \\
&+ \left| \int_{-r}^{0} e^{\lambda_0 s} \left\{ \int_{s}^{0} e^{\mu_0 u} \left[\int_{u}^{0} f(t_0+v)dv \right] du \right\} d\eta(s) \right| \\
\le& \int_{-r}^{0} e^{(\lambda_0+\mu_0)s} \left| \int_{s}^{0} f(t_0+u)du \right| dV(\zeta)(s) \\
&+ |\lambda_0| \int_{-r}^{0} e^{\lambda_0 s} \left| \int_{s}^{0} e^{\mu_0 u} \left[\int_{u}^{0} f(t_0+v)dv \right] du \right| dV(\zeta)(s) \\
&+ \int_{-r}^{0} e^{\lambda_0 s} \left| \int_{s}^{0} e^{\mu_0 u} \left[\int_{u}^{0} f(t_0+v)dv \right] du \right| dV(\eta)(s) \\
\le& \int_{-r}^{0} e^{(\lambda_0+\mu_0)s} \left[\int_{s}^{0} |f(t_0+u)|\, du \right] dV(\zeta)(s) \\
&+ |\lambda_0| \int_{-r}^{0} e^{\lambda_0 s} \left\{ \int_{s}^{0} e^{\mu_0 u} \left[\int_{u}^{0} |f(t_0+v)|\, dv \right] du \right\} dV(\zeta)(s) \\
&+ \int_{-r}^{0} e^{\lambda_0 s} \left\{ \int_{s}^{0} e^{\mu_0 u} \left[\int_{u}^{0} |f(t_0+v)|\, dv \right] du \right\} dV(\eta)(s) \\
\le& \Big\{ \int_{-r}^{0} e^{(\lambda_0+\mu_0)s} \left(\int_{s}^{0} du \right) dV(\zeta)(s) \\
&+ |\lambda_0| \int_{-r}^{0} e^{\lambda_0 s} \left[\int_{s}^{0} e^{\mu_0 u} \left(\int_{u}^{0} dv \right) du \right] dV(\zeta)(s) \\
&+ \int_{-r}^{0} e^{\lambda_0 s} \left[\int_{s}^{0} e^{\mu_0 u} \left(\int_{u}^{0} dv \right) du \right] dV(\eta)(s) \Big\} [M(\lambda_0,\mu_0;\phi)+\epsilon] \\
=& \Big\{ \int_{-r}^{0} (-s)e^{(\lambda_0+\mu_0)s} dV(\zeta)(s) + |\lambda_0| \int_{-r}^{0} e^{\lambda_0 s} \left[\int_{s}^{0} (-u)e^{\mu_0 u} du \right] dV(\zeta)(s) \\
&+ \int_{-r}^{0} e^{\lambda_0 s} \left[\int_{s}^{0} (-u)e^{\mu_0 u} du \right] dV(\eta)(s) \Big\} [M(\lambda_0,\mu_0;\phi)+\epsilon] \\
=& \Big\{ \int_{-r}^{0} e^{\lambda_0 s} \left[(-s)e^{\mu_0 s} + |\lambda_0| \int_{s}^{0} (-u)e^{\mu_0 u} du \right] dV(\zeta)(s) \\
&+ \int_{-r}^{0} e^{\lambda_0 s} \left[\int_{s}^{0} (-u)e^{\mu_0 u} du \right] dV(\eta)(s) \Big\} [M(\lambda_0,\mu_0;\phi)+\epsilon] \\
=& \rho(\lambda_0,\mu_0) [M(\lambda_0,\mu_0;\phi)+\epsilon] \\
<& M(\lambda_0,\mu_0;\phi)+\epsilon.
\end{aligned}
$$

We have thus arrived at a contradiction, which establishes our claim, i.e., that (2.21) holds true. As (2.21) is satisfied for all real numbers $\epsilon > 0$, it follows that (2.20) is always fulfilled. Furthermore, by using (2.20), from (2.15) we get, for every $t \ge 0$,

$$
\begin{aligned}
|f(t)| \le& \left| \int_{-r}^{0} e^{(\lambda_0+\mu_0)s} \left[\int_{s}^{0} f(t+u)du \right] d\zeta(s) \right| \\
&+ |\lambda_0| \left| \int_{-r}^{0} e^{\lambda_0 s} \left\{ \int_{s}^{0} e^{\mu_0 u} \left[\int_{u}^{0} f(t+v)dv \right] du \right\} d\zeta(s) \right|
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \left| \int_{-r}^{0} e^{\lambda_0 s} \left\{ \int_{s}^{0} e^{\mu_0 u} \left[\int_{u}^{0} f(t+v)dv \right] du \right\} d\eta(s) \right| \\
&\le \int_{-r}^{0} e^{(\lambda_0+\mu_0)s} \left| \int_{s}^{0} f(t+u)du \right| dV(\zeta)(s) \\
&\quad + |\lambda_0| \int_{-r}^{0} e^{\lambda_0 s} \left| \int_{s}^{0} e^{\mu_0 u} \left[\int_{u}^{0} f(t+v)dv \right] du \right| dV(\zeta)(s) \\
&\quad + \int_{-r}^{0} e^{\lambda_0 s} \left| \int_{s}^{0} e^{\mu_0 u} \left[\int_{u}^{0} f(t+v)dv \right] du \right| dV(\eta)(s) \\
&\le \int_{-r}^{0} e^{(\lambda_0+\mu_0)s} \left[\int_{s}^{0} |f(t+u)|\, du \right] dV(\zeta)(s) \\
&\quad + |\lambda_0| \int_{-r}^{0} e^{\lambda_0 s} \left\{ \int_{s}^{0} e^{\mu_0 u} \left[\int_{u}^{0} |f(t+v)|\, dv \right] du \right\} dV(\zeta)(s) \\
&\quad + \int_{-r}^{0} e^{\lambda_0 s} \left\{ \int_{s}^{0} e^{\mu_0 u} \left[\int_{u}^{0} |f(t+v)|\, dv \right] du \right\} dV(\eta)(s) \\
&\le \left\{ \int_{-r}^{0} e^{(\lambda_0+\mu_0)s} \left(\int_{s}^{0} du \right) dV(\zeta)(s) \right. \\
&\quad + |\lambda_0| \int_{-r}^{0} e^{\lambda_0 s} \left[\int_{s}^{0} e^{\mu_0 u} \left(\int_{u}^{0} dv \right) du \right] dV(\zeta)(s) \\
&\quad \left. + \int_{-r}^{0} e^{\lambda_0 s} \left[\int_{s}^{0} e^{\mu_0 u} \left(\int_{u}^{0} dv \right) du \right] dV(\eta)(s) \right\} M(\lambda_0, \mu_0; \phi) \\
&= \left\{ \int_{-r}^{0} (-s) e^{(\lambda_0+\mu_0)s} dV(\zeta)(s) + |\lambda_0| \int_{-r}^{0} e^{\lambda_0 s} \left[\int_{s}^{0} (-u) e^{\mu_0 u} du \right] dV(\zeta)(s) \right. \\
&\quad \left. + \int_{-r}^{0} e^{\lambda_0 s} \left[\int_{s}^{0} (-u) e^{\mu_0 u} du \right] dV(\eta)(s) \right\} M(\lambda_0, \mu_0; \phi) \\
&= \left\{ \int_{-r}^{0} e^{\lambda_0 s} \left[(-s) e^{\mu_0 s} + |\lambda_0| \int_{s}^{0} (-u) e^{\mu_0 u} du \right] dV(\zeta)(s) \right. \\
&\quad \left. + \int_{-r}^{0} e^{\lambda_0 s} \left[\int_{s}^{0} (-u) e^{\mu_0 u} du \right] dV(\eta)(s) \right\} M(\lambda_0, \mu_0; \phi).
\end{aligned}
$$

Thus, by taking into account the definition of $\rho(\lambda_0, \mu_0)$ by (2.6), we have

$$
|f(t)| \le \rho(\lambda_0, \mu_0) M(\lambda_0, \mu_0; \phi) \quad \text{for every } t \ge 0. \tag{2.22}
$$

By using (2.15) and taking into account the definition of $\rho(\lambda_0, \mu_0)$ by (2.6) as well as taking into account (2.20) and (2.22), one can prove, by an easy induction, that the function f satisfies

$$
|f(t)| \le [\rho(\lambda_0, \mu_0)]^{\nu} M(\lambda_0, \mu_0; \phi) \quad \text{for all } t \ge \nu r - r \quad (\nu = 0, 1, 2, \dots). \tag{2.23}
$$

Because of (2.7) (which is equivalent to the assumption (2.3)), we have

$$
\lim_{\nu \to \infty} [\rho(\lambda_0, \mu_0)]^{\nu} = 0. \tag{2.24}
$$

In view of (2.24), it follows from (2.23) that $\lim_{t\to\infty} f(t) = 0$, i.e., (2.17) holds true.

Next, we will establish that

$$
\lim_{t \to \infty} f'(t) = 0. \tag{2.25}
$$

From (2.15) it follows that f' satisfies

$$\begin{aligned} f'(t) = &\int_{-r}^{0} e^{(\lambda_0+\mu_0)s} \left[\int_{s}^{0} f'(t+u)du\right] d\zeta(s) \\ &- \lambda_0 \int_{-r}^{0} e^{\lambda_0 s} \left\{\int_{s}^{0} e^{\mu_0 u} \left[\int_{u}^{0} f'(t+v)dv\right] du\right\} d\zeta(s) \\ &+ \int_{-r}^{0} e^{\lambda_0 s} \left\{\int_{s}^{0} e^{\mu_0 u} \left[\int_{u}^{0} f'(t+v)dv\right] du\right\} d\eta(s) \end{aligned} \tag{2.26}$$

for all $t \geq 0$. Moreover, the initial condition (2.16) gives

$$f'(t) = e^{-\mu_0 t} \left[(\Phi(\lambda_0;\phi))'(t) - \mu_0 \Phi(\lambda_0;\phi)(t)\right] \quad \text{for } -r \leq t \leq 0. \tag{2.27}$$

Set

$$N(\lambda_0,\mu_0;\phi) = \max_{-r\leq t\leq 0} \left|e^{-\mu_0 t} \left[(\Phi(\lambda_0;\phi))'(t) - \mu_0 \Phi(\lambda_0;\phi)(t)\right]\right|. \tag{2.28}$$

It follows from (2.27) and (2.28) that

$$|f'(t)| \leq N(\lambda_0,\mu_0;\phi) \quad \text{for } -r \leq t \leq 0. \tag{2.29}$$

By taking into account the definition of $\rho(\lambda_0,\mu_0)$ by (2.6) and using (2.29), (2.26) and (2.7), we can follow the same arguments applied previously in proving (2.20) to conclude that $N(\lambda_0,\mu_0;\phi)$ is a bound of f' on the whole interval $[-r,\infty)$, i.e., that

$$|f'(t)| \leq N(\lambda_0,\mu_0;\phi) \quad \text{for all } t \geq -r. \tag{2.30}$$

Furthermore, by taking again into account the definition of $\rho(\lambda_0,\mu_0)$ by (2.6) and using (2.30) and (2.26), we may apply the same arguments used above in establishing (2.22) to obtain

$$|f'(t)| \leq \rho(\lambda_0,\mu_0) N(\lambda_0,\mu_0;\phi) \quad \text{for every } t \geq 0. \tag{2.31}$$

Taking into account (2.6) as well as (2.30) and (2.31), one can use (2.26) to show, by induction, that

$$|f'(t)| \leq [\rho(\lambda_0,\mu_0)]^{\nu} N(\lambda_0,\mu_0;\phi) \quad \text{for all } t \geq \nu r - r \quad (\nu = 0,1,2,\dots). \tag{2.32}$$

Because of (2.24), it follows from (2.32) that $\lim_{t\to\infty} f'(t) = 0$. So, (2.25) has been established.

Finally, by (1.9), (2.10) and (2.14), we have

$$f(t) = e^{-\mu_0 t} \left[e^{-\lambda_0 t} x(t) - \frac{K(\lambda_0;\phi)}{\beta(\lambda_0)}\right] - \frac{L(\lambda_0,\mu_0;\phi)}{1+\gamma(\lambda_0,\mu_0)} \quad \text{for } t \geq -r. \tag{2.33}$$

In view of this equality, (2.4) coincides with (2.17). So, the solution x of the IVP (1.1) and (1.2) satisfies (2.4). Furthermore, for $t \geq -r$, we define

$$g(t) = e^{-\mu_0 t} \left[e^{-\lambda_0 t} x'(t) - \lambda_0 \frac{K(\lambda_0;\phi)}{\beta(\lambda_0)}\right] - (\lambda_0+\mu_0) \frac{L(\lambda_0,\mu_0;\phi)}{1+\gamma(\lambda_0,\mu_0)}. \tag{2.34}$$

Then it is a matter of elementary calculations we check that

$$g(t) = f'(t) + (\lambda_0+\mu_0) f(t) \quad \text{for all } t \geq -r. \tag{2.35}$$

In view of (2.17) and (2.25), it follows from (2.35) that

$$\lim_{t\to\infty} g(t) = 0. \tag{2.36}$$

By (2.34), we see that (2.5) coincides with (2.36). Hence, the solution x of the IVP (1.1) and (1.2) satisfies (2.5). The proof of the theorem is complete. □

3. An estimate of the solutions. A stability criterion

Our results in this section are Theorem 3.1 below and its corollary.

Theorem 3.1. *Let λ_0 be a real root of the characteristic equation* (1.3)*, and suppose that $\beta(\lambda_0) \neq 0$, where $\beta(\lambda_0)$ is defined by* (1.6)*. Set*

$$m(\lambda_0) = \max\left\{1, e^{\lambda_0 r}\right\}, \tag{3.1}$$

$$\alpha(\lambda_0) = \int_{-r}^{0} [1 + |\lambda_0|(-s)]\, e^{\lambda_0 s} dV(\zeta)(s) + \int_{-r}^{0} (-s) e^{\lambda_0 s} dV(\eta)(s). \tag{3.2}$$

Furthermore, let μ_0 be a real root of the characteristic equation (1.5)*, and let $\gamma(\lambda_0, \mu_0)$ and $\rho(\lambda_0, \mu_0)$ be defined by* (2.1) *and* (2.6)*, respectively. (Note that, because of $\beta(\lambda_0) \neq 0$, we always have $\mu_0 \neq 0$). Also, set*

$$m(\mu_0) = \max\{1, e^{\mu_0 r}\}. \tag{3.3}$$

Assume that (2.3) *holds. (This assumption guarantees that $1 + \gamma(\lambda_0, \mu_0) > 0$.) Then the solution x of the IVP* (1.1) *and* (1.2) *satisfies*

$$|x(t)| \le \left[P(\lambda_0) e^{\lambda_0 t} + Q(\lambda_0, \mu_0) e^{(\lambda_0 + \mu_0)t}\right] \nparallel \phi \nparallel \quad \text{for all } t \ge 0 \tag{3.4}$$

and

$$|x'(t)| \le \left\{|\lambda_0| P(\lambda_0) e^{\lambda_0 t} + [|\lambda_0 + \mu_0| Q(\lambda_0, \mu_0) + R(\lambda_0, \mu_0)]\, e^{(\lambda_0 + \mu_0)t}\right\} \nparallel \phi \nparallel \tag{3.5}$$

for all $t \ge 0$, where

$$P(\lambda_0) = \frac{1 + |\lambda_0| + \alpha(\lambda_0) m(\lambda_0)}{|\beta(\lambda_0)|}, \tag{3.6}$$

$$\begin{aligned} Q(\lambda_0, \mu_0) = &\left\{\rho(\lambda_0, \mu_0) m(\mu_0) + [1 + \rho(\lambda_0, \mu_0)] \frac{1 + \rho(\lambda_0, \mu_0) m(\mu_0)}{1 + \gamma(\lambda_0, \mu_0)}\right\} \\ &\times \left[m(\lambda_0) + \frac{1 + |\lambda_0| + \alpha(\lambda_0) m(\lambda_0)}{|\beta(\lambda_0)|}\right] \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} &R(\lambda_0, \mu_0) \\ &= \rho(\lambda_0, \mu_0) m(\mu_0) \left[(1 + |\lambda_0| + |\mu_0|)\, m(\lambda_0) + |\mu_0| \frac{1 + |\lambda_0| + \alpha(\lambda_0) m(\lambda_0)}{|\beta(\lambda_0)|}\right]. \end{aligned} \tag{3.8}$$

The constant $Q(\lambda_0, \mu_0)$ is greater than 1.

Corollary 3.2. *Let λ_0 be a real root of the characteristic equation* (1.3)*, and suppose that $\beta(\lambda_0) \neq 0$, where $\beta(\lambda_0)$ is defined by* (1.6)*. Furthermore, let μ_0 be a real root of the characteristic equation* (1.5)*. (Note that, because of $\beta(\lambda_0) \neq 0$, we always have $\mu_0 \neq 0$.)*

Assume that (2.3) *holds. Then the trivial solution of the delay differential equation* (1.1) *is uniformly stable if $\lambda_0 \le 0$ and $\lambda_0 + \mu_0 \le 0$, and it is uniformly asymptotically stable if $\lambda_0 < 0$ and $\lambda_0 + \mu_0 < 0$.*

Proof of Theorem 3.1. First of all, we observe that, for any real number c, it holds $\max_{-r\le t\le 0} e^{-ct} = \max\{1, e^{cr}\}$. So, by taking into account the definitions of $m(\lambda_0)$ and $m(\mu_0)$ by (3.1) and (3.3), respectively, we immediately see that

$$e^{-\lambda_0 t} \le m(\lambda_0) \quad \text{for } -r \le t \le 0, \tag{3.9}$$

$$e^{-\mu_0 t} \le m(\mu_0) \quad \text{for } -r \le t \le 0. \tag{3.10}$$

These inequalities will be frequently used in the sequel.

Define $K(\lambda_0;\phi)$ by (1.7). Then

$$\begin{aligned}
&|K(\lambda_0;\phi)| \\
&\le |\phi'(0)| + |\lambda_0|\,|\phi(0)| + \left|\int_{-r}^{0}\left[\phi(s) - \lambda_0 e^{\lambda_0 s}\int_s^0 e^{-\lambda_0 u}\phi(u)du\right] d\zeta(s)\right| \\
&\quad + \left|\int_{-r}^{0} e^{\lambda_0 s}\left[\int_s^0 e^{-\lambda_0 u}\phi(u)du\right] d\eta(s)\right| \\
&= |\phi'(0)| + |\lambda_0|\,|\phi(0)| + \left|\int_{-r}^{0}\left[e^{-\lambda_0 s}\phi(s) - \lambda_0 \int_s^0 e^{-\lambda_0 u}\phi(u)du\right] e^{\lambda_0 s} d\zeta(s)\right| \\
&\quad + \left|\int_{-r}^{0}\left[\int_s^0 e^{-\lambda_0 u}\phi(u)du\right] e^{\lambda_0 s} d\eta(s)\right| \\
&\le |\phi'(0)| + |\lambda_0|\,|\phi(0)| + \int_{-r}^{0}\left|e^{-\lambda_0 s}\phi(s) - \lambda_0 \int_s^0 e^{-\lambda_0 u}\phi(u)du\right| e^{\lambda_0 s} dV(\zeta)(s) \\
&\quad + \int_{-r}^{0}\left|\int_s^0 e^{-\lambda_0 u}\phi(u)du\right| e^{\lambda_0 s} dV(\eta)(s) \\
&\le |\phi'(0)| + |\lambda_0|\,|\phi(0)| + \int_{-r}^{0}\left[e^{-\lambda_0 s}\,|\phi(s)| + |\lambda_0| \int_s^0 e^{-\lambda_0 u}\,|\phi(u)|\,du\right] e^{\lambda_0 s} dV(\zeta)(s) \\
&\quad + \int_{-r}^{0}\left[\int_s^0 e^{-\lambda_0 u}\,|\phi(u)|\,du\right] e^{\lambda_0 s} dV(\eta)(s) \\
&\le \|\phi'\| + \Big[|\lambda_0| + \int_{-r}^{0}\left(e^{-\lambda_0 s} + |\lambda_0|\int_s^0 e^{-\lambda_0 u}du\right) e^{\lambda_0 s} dV(\zeta)(s) \\
&\quad + \int_{-r}^{0}\left(\int_s^0 e^{-\lambda_0 u}du\right) e^{\lambda_0 s} dV(\eta)(s)\Big]\,\|\phi\|\,.
\end{aligned}$$

In view of (3.9), we have

$$e^{-\lambda_0 s} \le m(\lambda_0), \qquad \int_s^0 e^{-\lambda_0 u}du \le (-s)m(\lambda_0)$$

for every $s \in [-r, 0]$. Thus, we obtain

$$\begin{aligned}
|K(\lambda_0;\phi)| \le \|\phi'\| + \Big(|\lambda_0| + \Big\{\int_{-r}^{0}\left[1 + |\lambda_0|\,(-s)\right] e^{\lambda_0 s} dV(\zeta)(s) \\
+ \int_{-r}^{0}(-s)e^{\lambda_0 s} dV(\eta)(s)\Big\}\, m(\lambda_0)\Big)\,\|\phi\|
\end{aligned}$$

and consequently, because of the definition of $\alpha(\lambda_0)$ by (3.2), we get

$$|K(\lambda_0;\phi)| \le \|\phi'\| + \left[|\lambda_0| + \alpha(\lambda_0)m(\lambda_0)\right]\|\phi\|\,.$$

This gives

$$|K(\lambda_0;\phi)| \le [1 + |\lambda_0| + \alpha(\lambda_0)m(\lambda_0)] \nparallel \phi \nparallel . \tag{3.11}$$

Consider the function $\Phi(\lambda_0;\phi)$ defined by (1.8). Then, by (3.9), we have

$$\|\Phi(\lambda_0;\phi)\| \le m(\lambda_0)\, \|\phi\| + \frac{|K(\lambda_0;\phi)|}{|\beta(\lambda_0)|}$$

and so, in view of (3.11),

$$\|\Phi(\lambda_0;\phi)\| \le m(\lambda_0)\, \|\phi\| + \frac{1 + |\lambda_0| + \alpha(\lambda_0)m(\lambda_0)}{|\beta(\lambda_0)|} \nparallel \phi \nparallel .$$

Therefore,

$$\|\Phi(\lambda_0;\phi)\| \le \left[m(\lambda_0) + \frac{1 + |\lambda_0| + \alpha(\lambda_0)m(\lambda_0)}{|\beta(\lambda_0)|} \right] \nparallel \phi \nparallel . \tag{3.12}$$

Let us consider the constant $L(\lambda_0, \mu_0;\phi)$ defined by (2.2). Then

$$\begin{aligned}
&|L(\lambda_0, \mu_0;\phi)| \\
&\le |\Phi(\lambda_0;\phi)(0)| + \Big| \int_{-r}^{0} e^{\lambda_0 s} \Big\{ e^{\mu_0 s} \int_s^0 e^{-\mu_0 u} \Phi(\lambda_0;\phi)(u) du \\
&\quad - \lambda_0 \int_s^0 e^{\mu_0 u} \Big[\int_u^0 e^{-\mu_0 v} \Phi(\lambda_0;\phi)(v) dv \Big] du \Big\} d\zeta(s) \Big| \\
&\quad + \Big| \int_{-r}^{0} e^{\lambda_0 s} \Big\{ \int_s^0 e^{\mu_0 u} \Big[\int_u^0 e^{-\mu_0 v} \Phi(\lambda_0;\phi)(v) dv \Big] du \Big\} d\eta(s) \Big| \\
&\le |\Phi(\lambda_0;\phi)(0)| + \int_{-r}^{0} e^{\lambda_0 s} \Big| e^{\mu_0 s} \int_s^0 e^{-\mu_0 u} \Phi(\lambda_0;\phi)(u) du \\
&\quad - \lambda_0 \int_s^0 e^{\mu_0 u} \Big[\int_u^0 e^{-\mu_0 v} \Phi(\lambda_0;\phi)(v) dv \Big] du \Big| dV(\zeta)(s) \\
&\quad + \int_{-r}^{0} e^{\lambda_0 s} \Big| \int_s^0 e^{\mu_0 u} \Big[\int_u^0 e^{-\mu_0 v} \Phi(\lambda_0;\phi)(v) dv \Big] du \Big| dV(\eta)(s) \\
&\le |\Phi(\lambda_0;\phi)(0)| + \int_{-r}^{0} e^{\lambda_0 s} \Big\{ e^{\mu_0 s} \int_s^0 e^{-\mu_0 u} |\Phi(\lambda_0;\phi)(u)| du \\
&\quad + |\lambda_0| \int_s^0 e^{\mu_0 u} \Big[\int_u^0 e^{-\mu_0 v} |\Phi(\lambda_0;\phi)(v)| dv \Big] du \Big\} dV(\zeta)(s) \\
&\quad + \int_{-r}^{0} e^{\lambda_0 s} \Big\{ \int_s^0 e^{\mu_0 u} \Big[\int_u^0 e^{-\mu_0 v} |\Phi(\lambda_0;\phi)(v)| dv \Big] du \Big\} dV(\eta)(s) \\
&\le \Big\{ 1 + \int_{-r}^{0} e^{\lambda_0 s} \Big[e^{\mu_0 s} \int_s^0 e^{-\mu_0 u} du + |\lambda_0| \int_s^0 e^{\mu_0 u} \Big(\int_u^0 e^{-\mu_0 v} dv \Big) du \Big] dV(\zeta)(s) \\
&\quad + \int_{-r}^{0} e^{\lambda_0 s} \Big[\int_s^0 e^{\mu_0 u} \Big(\int_u^0 e^{-\mu_0 v} dv \Big) du \Big] dV(\eta)(s) \Big\} \|\Phi(\lambda_0;\phi)\| .
\end{aligned}$$

By (3.10), we have

$$\int_s^0 e^{-\mu_0 u} du \le (-s)m(\mu_0),$$

$$\int_u^0 e^{-\mu_0 v} dv \le (-u)m(\mu_0) \quad \text{for } u \in [s, 0]$$

for every $s \in [-r, 0]$. Hence, we derive

$$\begin{aligned}|L(\lambda_0, \mu_0; \phi)| \le \Big(1 + \Big\{ \int_{-r}^{0} e^{\lambda_0 s} \Big[(-s)e^{\mu_0 s} + |\lambda_0| \int_{s}^{0} (-u)e^{\mu_0 u} du\Big] dV(\zeta)(s) \\ + \int_{-r}^{0} e^{\lambda_0 s} \Big[\int_{s}^{0} (-u)e^{\mu_0 u} du\Big] dV(\eta)(s) \Big\} m(\mu_0)\Big) \|\Phi(\lambda_0; \phi)\| .\end{aligned}$$

Because of the definition of $\rho(\lambda_0, \mu_0)$ by (2.6), the last inequality is written as

$$|L(\lambda_0, \mu_0; \phi)| \le [1 + \rho(\lambda_0, \mu_0)m(\mu_0)] \|\Phi(\lambda_0; \phi)\| .$$

A combination of this inequality and (3.12) leads to

$$|L(\lambda_0, \mu_0; \phi)| \le [1 + \rho(\lambda_0, \mu_0)m(\mu_0)] \Big[m(\lambda_0) + \frac{1 + |\lambda_0| + \alpha(\lambda_0)m(\lambda_0)}{|\beta(\lambda_0)|}\Big] \nparallel \phi \nparallel . \tag{3.13}$$

Let $\gamma(\lambda_0, \mu_0)$ be defined by (2.1). Take into account (2.9) (which is a consequence of the assumption (2.3)), and define $M(\lambda_0, \mu_0; \phi)$ by (2.18). Then, by using (3.10), we have

$$M(\lambda_0, \mu_0; \phi) \le m(\mu_0) \|\Phi(\lambda_0; \phi)\| + \frac{|L(\lambda_0, \mu_0; \phi)|}{1 + \gamma(\lambda_0, \mu_0)}.$$

So, by virtue of (3.12) and (3.13),

$$\begin{aligned}&M(\lambda_0, \mu_0; \phi) \\ &\le \Big[m(\mu_0) + \frac{1 + \rho(\lambda_0, \mu_0)m(\mu_0)}{1 + \gamma(\lambda_0, \mu_0)}\Big] \Big[m(\lambda_0) + \frac{1 + |\lambda_0| + \alpha(\lambda_0)m(\lambda_0)}{|\beta(\lambda_0)|}\Big] \nparallel \phi \nparallel .\end{aligned} \tag{3.14}$$

Consider the constant $N(\lambda_0, \mu_0; \phi)$ defined by (2.28). Then, by (3.10), we have

$$N(\lambda_0, \mu_0; \phi) \le m(\mu_0) \left[\left\|(\Phi(\lambda_0; \phi))'\right\| + |\mu_0| \|\Phi(\lambda_0; \phi)\|\right] .$$

From the definition of $\Phi(\lambda_0; \phi)$ by (1.18) it follows that

$$(\Phi(\lambda_0; \phi))'(t) = e^{-\lambda_0 t} [\phi'(t) - \lambda_0\phi(t)] \quad \text{for } -r \le t \le 0$$

and consequently, in view of (3.9),

$$\left\|(\Phi(\lambda_0; \phi))'\right\| \le m(\lambda_0) (\|\phi'\| + |\lambda_0| \|\phi\|) ,$$

which gives

$$\left\|(\Phi(\lambda_0; \phi))'\right\| \le (1 + |\lambda_0|) m(\lambda_0) \nparallel \phi \nparallel .$$

By using the last inequality and inequality (3.12), we find

$$\begin{aligned}&N(\lambda_0, \mu_0; \phi) \\ &\le m(\mu_0) \Big[(1 + |\lambda_0| + |\mu_0|) m(\lambda_0) + |\mu_0| \frac{1 + |\lambda_0| + \alpha(\lambda_0)m(\lambda_0)}{|\beta(\lambda_0)|}\Big] \nparallel \phi \nparallel .\end{aligned} \tag{3.15}$$

Now, let x be the solution of the IVP (1.1) and (1.2), and define the function z by (1.9). Also, we define the functions w and f by (2.10) and (2.14), respectively. Note that (2.9) (which is a consequence of the assumption (2.3)) states that $1 + \gamma(\lambda_0, \mu_0) > 0$. Then, as in the proof of Theorem 2.1, we show that (2.22), (2.31) and (2.33) are satisfied. Moreover, we consider the function g defined by (2.34); the function g satisfies (2.35). We shall prove that x satisfies (3.4) and (3.5), where the constants $P(\lambda_0)$, $Q(\lambda_0, \mu_0)$ and $R(\lambda_0, \mu_0)$ are defined by (3.6), (3.7) and (3.8), respectively.

From (2.33) it follows that

$$x(t)=\frac{K(\lambda_0;\phi)}{\beta(\lambda_0)}e^{\lambda_0 t}+\Big[f(t)+\frac{L(\lambda_0,\mu_0;\phi)}{1+\gamma(\lambda_0,\mu_0)}\Big]e^{(\lambda_0+\mu_0)t}\quad\text{for } t\ge 0$$

and consequently

$$|x(t)|\le\frac{|K(\lambda_0;\phi)|}{|\beta(\lambda_0)|}e^{\lambda_0 t}+\Big[|f(t)|+\frac{|L(\lambda_0,\mu_0;\phi)|}{1+\gamma(\lambda_0,\mu_0)}\Big]e^{(\lambda_0+\mu_0)t}\quad\text{for } t\ge 0.$$

Thus, using (2.22), we obtain

$$|x(t)|\le\frac{|K(\lambda_0;\phi)|}{|\beta(\lambda_0)|}e^{\lambda_0 t}+\Big[\rho(\lambda_0,\mu_0)M(\lambda_0,\mu_0;\phi)+\frac{|L(\lambda_0,\mu_0;\phi)|}{1+\gamma(\lambda_0,\mu_0)}\Big]e^{(\lambda_0+\mu_0)t}\tag{3.16}$$

for $t\ge 0$. In view of (3.6), inequality (3.11) can equivalently be written as

$$\frac{|K(\lambda_0;\phi)|}{|\beta(\lambda_0)|}\le P(\lambda_0)\nparallel\phi\nparallel.\tag{3.17}$$

Moreover, by the use of (3.13) and (3.14), we get

$$\begin{aligned}
&\rho(\lambda_0,\mu_0)M(\lambda_0,\mu_0;\phi)+\frac{|L(\lambda_0,\mu_0;\phi)|}{1+\gamma(\lambda_0,\mu_0)}\\
&\le\Big\{\rho(\lambda_0,\mu_0)\Big[m(\mu_0)+\frac{1+\rho(\lambda_0,\mu_0)m(\mu_0)}{1+\gamma(\lambda_0,\mu_0)}\Big]\Big[m(\lambda_0)+\frac{1+|\lambda_0|+\alpha(\lambda_0)m(\lambda_0)}{|\beta(\lambda_0)|}\Big]\\
&\quad+\frac{1+\rho(\lambda_0,\mu_0)m(\mu_0)}{1+\gamma(\lambda_0,\mu_0)}\Big[m(\lambda_0)+\frac{1+|\lambda_0|+\alpha(\lambda_0)m(\lambda_0)}{|\beta(\lambda_0)|}\Big]\Big\}\nparallel\phi\nparallel\\
&=\Big\{\rho(\lambda_0,\mu_0)m(\mu_0)+[1+\rho(\lambda_0,\mu_0)]\frac{1+\rho(\lambda_0,\mu_0)m(\mu_0)}{1+\gamma(\lambda_0,\mu_0)}\Big\}\\
&\quad\times\Big[m(\lambda_0)+\frac{1+|\lambda_0|+\alpha(\lambda_0)m(\lambda_0)}{|\beta(\lambda_0)|}\Big]\nparallel\phi\nparallel.
\end{aligned}$$

So, because of (3.7), we have

$$\rho(\lambda_0,\mu_0)M(\lambda_0,\mu_0;\phi)+\frac{|L(\lambda_0,\mu_0;\phi)|}{1+\gamma(\lambda_0,\mu_0)}\le Q(\lambda_0,\mu_0)\nparallel\phi\nparallel.\tag{3.18}$$

Using (3.17) and (3.18), we immediately see that (3.16) implies (3.4). Hence, (3.4) has been proved.

Next, we see that (2.34) gives

$$x'(t)=\lambda_0\frac{K(\lambda_0;\phi)}{\beta(\lambda_0)}e^{\lambda_0 t}+\Big[g(t)+(\lambda_0+\mu_0)\frac{L(\lambda_0,\mu_0;\phi)}{1+\gamma(\lambda_0,\mu_0)}\Big]e^{(\lambda_0+\mu_0)t}\quad\text{for } t\ge 0$$

and hence, by (2.35), we have

$$x'(t)=\lambda_0\frac{K(\lambda_0;\phi)}{\beta(\lambda_0)}e^{\lambda_0 t}+\Big\{(\lambda_0+\mu_0)\Big[f(t)+\frac{L(\lambda_0,\mu_0;\phi)}{1+\gamma(\lambda_0,\mu_0)}\Big]+f'(t)\Big\}e^{(\lambda_0+\mu_0)t}$$

for $t\ge 0$. Consequently,

$$\begin{aligned}
|x'(t)|&\le|\lambda_0|\,\frac{|K(\lambda_0;\phi)|}{|\beta(\lambda_0)|}e^{\lambda_0 t}\\
&\quad+\Big\{|\lambda_0+\mu_0|\Big[|f(t)|+\frac{|L(\lambda_0,\mu_0;\phi)|}{1+\gamma(\lambda_0,\mu_0)}\Big]+|f'(t)|\Big\}e^{(\lambda_0+\mu_0)t}\quad\text{for } t\ge 0.
\end{aligned}$$

So, in view of (2.22) and (2.31), we derive

$$
\begin{aligned}
|x'(t)| \le{}& |\lambda_0| \frac{|K(\lambda_0;\phi)|}{|\beta(\lambda_0)|} e^{\lambda_0 t} \\
&+ \Big\{ |\lambda_0+\mu_0| \Big[\rho(\lambda_0,\mu_0) M(\lambda_0,\mu_0;\phi) + \frac{|L(\lambda_0,\mu_0;\phi)|}{1+\gamma(\lambda_0,\mu_0)} \Big] \\
&+ \rho(\lambda_0,\mu_0) N(\lambda_0,\mu_0;\phi) \Big\} e^{(\lambda_0+\mu_0)t} \quad \text{for } t \ge 0.
\end{aligned} \tag{3.19}
$$

But, because of (3.8), it follows from (3.15) that

$$
\rho(\lambda_0,\mu_0) N(\lambda_0,\mu_0;\phi) \le R(\lambda_0,\mu_0) \nparallel \phi \nparallel . \tag{3.20}
$$

By (3.17), (3.18) and (3.20), we see that (3.5) can be obtained from (3.19). Thus, we have shown that (3.5) holds true.

Finally, we will establish that the constant $Q(\lambda_0,\mu_0)$ is greater than 1. By (2.8) and (2.9), we have

$$
0 < 1+\gamma(\lambda_0,\mu_0) \le 1 + |\gamma(\lambda_0,\mu_0)| \le 1+\rho(\lambda_0,\mu_0)
$$

and so, as $m(\mu_0) \ge 1$,

$$
0 < 1+\gamma(\lambda_0,\mu_0) \le 1+\rho(\lambda_0,\mu_0) m(\mu_0),
$$

which ensures that

$$
\frac{1+\rho(\lambda_0,\mu_0)m(\mu_0)}{1+\gamma(\lambda_0,\mu_0)} \ge 1.
$$

Thus, since $\rho(\lambda_0,\mu_0) > 0$, we obtain

$$
[1+\rho(\lambda_0,\mu_0)] \frac{1+\rho(\lambda_0,\mu_0)m(\mu_0)}{1+\gamma(\lambda_0,\mu_0)} > 1
$$

and consequently

$$
\rho(\lambda_0,\mu_0)m(\mu_0) + [1+\rho(\lambda_0,\mu_0)] \frac{1+\rho(\lambda_0,\mu_0)m(\mu_0)}{1+\gamma(\lambda_0,\mu_0)} > 1.
$$

Moreover, as $m(\lambda_0) \ge 1$, we have

$$
m(\lambda_0) + \frac{1+|\lambda_0| + \alpha(\lambda_0)m(\lambda_0)}{|\beta(\lambda_0)|} > 1.
$$

Hence, it follows from the definition of $Q(\lambda_0,\mu_0)$ by (3.7) that $Q(\lambda_0,\mu_0)$ is always greater than 1. The proof of the theorem is now complete. $\square$

Proof of Corollary 3.2. Define $m(\lambda_0)$, $\alpha(\lambda_0)$, $\gamma(\lambda_0,\mu_0)$, $\rho(\lambda_0,\mu_0)$ and $m(\mu_0)$ by (3.1), (3.2), (2.1), (2.6) and (3.3), respectively. Note that assumption (2.3) guarantees that $1+\gamma(\lambda_0,\mu_0) > 0$. Let x be the solution of the IVP (1.1) and (1.2). By Theorem 3.1, the solution x satisfies (3.4) and (3.5), where $P(\lambda_0)$, $Q(\lambda_0,\mu_0)$ and $R(\lambda_0,\mu_0)$ are defined by (3.6), (3.7) and (3.8), respectively. The constant $Q(\lambda_0,\mu_0)$ is greater than 1.

Assume first that $\lambda_0 \le 0$ and $\lambda_0+\mu_0 \le 0$. Then (3.4) and (3.5) give

$$
|x(t)| \le [P(\lambda_0) + Q(\lambda_0,\mu_0)] \nparallel \phi \nparallel \quad \text{for } t \ge 0,
$$

$$
|x'(t)| \le [|\lambda_0|\, P(\lambda_0) + |\lambda_0+\mu_0|\, Q(\lambda_0,\mu_0) + R(\lambda_0,\mu_0)] \nparallel \phi \nparallel \quad \text{for } t \ge 0,
$$

respectively. So, if we set

$$
\begin{aligned}
&S(\lambda_0,\mu_0) \\
&= \max\left\{ P(\lambda_0) + Q(\lambda_0,\mu_0), |\lambda_0|\, P(\lambda_0) + |\lambda_0+\mu_0|\, Q(\lambda_0,\mu_0) + R(\lambda_0,\mu_0) \right\},
\end{aligned}
$$

then we have

$$\max\{|x(t)|\,,|x'(t)|\} \le S(\lambda_0,\mu_0)\,\|\phi\| \quad \text{for every } t \ge 0.$$

Since $Q(\lambda_0,\mu_0) > 1$, we always have $S(\lambda_0,\mu_0) > 1$. Thus, we obtain

$$\max\{|x(t)|\,,|x'(t)|\} \le S(\lambda_0,\mu_0)\,\|\phi\| \quad \text{for all } t \ge -r.$$

Using this inequality, we can immediately verify that the trivial solution of (1.1) is stable (at 0). Because of the autonomous character of (1.1), the trivial solution of (1.1) is uniformly stable.

Next, let us suppose that $\lambda_0 < 0$ and $\lambda_0 + \mu_0 < 0$. Then the trivial solution of (1.1) is stable (at 0). Furthermore, we see that it follows from (3.4) and (3.5) that the solution x satisfies

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} x'(t) = 0.$$

Hence, the trivial solution of (1.1) is asymptotically stable (at 0). As (1.1) is autonomous, we conclude that the trivial solution of (1.1) is uniformly asymptotically stable. The proof is complete. $\square$

4. A result on the behavior of the solutions

We begin this section with the following lemma.

Lemma 4.1. *Suppose that*

$$\zeta \text{ and } \eta \text{ are increasing on } [-r,0]. \tag{4.1}$$

Let λ_0 be a negative real root of the characteristic equation (1.3)*. Furthermore, let μ_0 be a real root of the characteristic equation* (1.5)*, and define $\gamma(\lambda_0,\mu_0)$ by* (2.1)*. Then $1+\gamma(\lambda_0,\mu_0) > 0$ if* (1.5) *has another real root less than μ_0, and $1+\gamma(\lambda_0,\mu_0) < 0$ if* (1.5) *has another real root greater than μ_0.*

Before we proceed to the proof of Lemma 4.1, we remark that: *if η is increasing on* $[-r,0]$, then, as η is also assumed to be not constant on $[-r,0]$, we always have $\int_{-r}^{0} d\eta(s) > 0$ and so *the zero is not a root of the characteristic equation* (1.3).

Proof of Lemma 4.1. Consider the real-valued function Ω defined by

$$\begin{aligned}\Omega(\mu) = \mu + 2\lambda_0 + \int_{-r}^{0} e^{(\lambda_0+\mu)s} d\zeta(s) - \lambda_0 \int_{-r}^{0} e^{\lambda_0 s}\Big(\int_{s}^{0} e^{\mu u} du\Big) d\zeta(s)\\ + \int_{-r}^{0} e^{\lambda_0 s}\Big(\int_{s}^{0} e^{\mu u} du\Big) d\eta(s) \quad \text{for } \mu \in \mathbb{R}.\end{aligned} \tag{4.2}$$

We obtain immediately

$$\begin{aligned}\Omega'(\mu) = 1 - \int_{-r}^{0} e^{\lambda_0 s}\Big[(-s)e^{\mu s} - \lambda_0 \int_{s}^{0} (-u)e^{\mu u} du\Big] d\zeta(s)\\ - \int_{-r}^{0} e^{\lambda_0 s}\Big[\int_{s}^{0} (-u)e^{\mu u} du\Big] d\eta(s) \quad \text{for } \mu \in \mathbb{R}.\end{aligned} \tag{4.3}$$

Furthermore,

$$\begin{aligned}\Omega''(\mu) = \int_{-r}^{0} e^{\lambda_0 s}\Big[s^2 e^{\mu s} + (-\lambda_0) \int_{s}^{0} u^2 e^{\mu u} du\Big] d\zeta(s)\\ + \int_{-r}^{0} e^{\lambda_0 s}\Big(\int_{s}^{0} u^2 e^{\mu u} du\Big) d\eta(s) \quad \text{for } \mu \in \mathbb{R}.\end{aligned}$$

So, taking into account (4.1) and the fact that η is not constant on $[-r,0]$ and using the hypothesis that $\lambda_0<0$, we conclude that

$$\Omega''(\mu)>0 \quad \text{for all } \mu\in\mathbb{R}. \tag{4.4}$$

Now, assume that (1.5) has another real root μ_1 with $\mu_1<\mu_0$ (respectively, $\mu_1>\mu_0$). From the definition of the function Ω by (4.2) it follows that $\Omega(\mu_0)=\Omega(\mu_1)=0$, and consequently Rolle's Theorem guarantees the existence of a point ξ with $\mu_1<\xi<\mu_0$ (resp., $\mu_0<\xi<\mu_1$) such that $\Omega'(\xi)=0$. But, (4.4) implies that Ω' is strictly increasing on $\mathbb{R}$ and hence, as $\Omega'(\xi)=0$, we conclude that Ω' is positive on (ξ,∞) (resp., Ω' is negative on $(-\infty,\xi)$). Thus, we must have $\Omega'(\mu_0)>0$ (resp., $\Omega'(\mu_0)<0$). By taking into account the definition of $\gamma(\lambda_0,\mu_0)$ by (2.1), from (4.3) we obtain

$$\Omega'(\mu_0)=1+\gamma(\lambda_0,\mu_0)$$

and so the proof of the lemma is complete. □

Now, we will establish the following theorem.

Theorem 4.2. *Suppose that statement* (4.1) *is true. Let λ_0 be a negative real root of the characteristic equation* (1.3)*, and let $\beta(\lambda_0)$ and $K(\lambda_0;\phi)$ be defined by* (1.6) *and* (1.7)*, respectively. Suppose that $\beta(\lambda_0)\neq 0$, and define $\Phi(\lambda_0;\phi)$ by* (1.8)*. Furthermore, let μ_0 be a real root of the characteristic equation* (1.5)*, and let $\gamma(\lambda_0,\mu_0)$ and $L(\lambda_0,\mu_0;\phi)$ be defined by* (2.1) *and* (2.2)*, respectively. Also, let μ_1 be a real root of* (1.5) *with $\mu_1\neq\mu_0$. (Note that, because of $\beta(\lambda_0)\neq 0$, we always have $\mu_0\neq 0$ and $\mu_1\neq 0$; moreover, note that Lemma 4.1 guarantees that $1+\gamma(\lambda_0,\mu_0)\neq 0$.)*

Then the solution x of the IVP (1.1) *and* (1.2) *satisfies*

$$\begin{aligned} C_1(\lambda_0,\mu_0,\mu_1;\phi) &\le e^{-\mu_1 t}\Big[e^{-\lambda_0 t}x(t)-\frac{K(\lambda_0;\phi)}{\beta(\lambda_0)}-\frac{L(\lambda_0,\mu_0;\phi)}{1+\gamma(\lambda_0,\mu_0)}e^{\mu_0 t}\Big]\\ &\le C_2(\lambda_0,\mu_0,\mu_1;\phi) \quad \text{for all } t\ge 0 \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} &D_1(\lambda_0,\mu_0,\mu_1;\phi)\\ &\le e^{-\mu_1 t}\Big[e^{-\lambda_0 t}x'(t)-\lambda_0\frac{K(\lambda_0;\phi)}{\beta(\lambda_0)}-(\lambda_0+\mu_0)\frac{L(\lambda_0,\mu_0;\phi)}{1+\gamma(\lambda_0,\mu_0)}e^{\mu_0 t}\Big]\\ &\le D_2(\lambda_0,\mu_0,\mu_1;\phi) \quad \text{for all } t\ge 0, \end{aligned} \tag{4.6}$$

where

$$C_1(\lambda_0,\mu_0,\mu_1;\phi)=\min_{-r\le t\le 0}\Big\{e^{-\mu_1 t}\Big[e^{-\lambda_0 t}\phi(t)-\frac{K(\lambda_0;\phi)}{\beta(\lambda_0)}-\frac{L(\lambda_0,\mu_0;\phi)}{1+\gamma(\lambda_0,\mu_0)}e^{\mu_0 t}\Big]\Big\}, \tag{4.7}$$

$$C_2(\lambda_0,\mu_0,\mu_1;\phi)=\max_{-r\le t\le 0}\Big\{e^{-\mu_1 t}\Big[e^{-\lambda_0 t}\phi(t)-\frac{K(\lambda_0;\phi)}{\beta(\lambda_0)}-\frac{L(\lambda_0,\mu_0;\phi)}{1+\gamma(\lambda_0,\mu_0)}e^{\mu_0 t}\Big]\Big\} \tag{4.8}$$

and

$$\begin{aligned} &D_1(\lambda_0,\mu_0,\mu_1;\phi)\\ &=\min_{-r\le t\le 0}\Big\{e^{-\mu_1 t}\Big[e^{-\lambda_0 t}\phi'(t)-\lambda_0\frac{K(\lambda_0;\phi)}{\beta(\lambda_0)}-(\lambda_0+\mu_0)\frac{L(\lambda_0,\mu_0;\phi)}{1+\gamma(\lambda_0,\mu_0)}e^{\mu_0 t}\Big]\Big\}, \end{aligned} \tag{4.9}$$

$$
\begin{aligned}
&D_2(\lambda_0,\mu_0,\mu_1;\phi) \\
&= \max_{-r\le t\le 0}\Big\{e^{-\mu_1 t}\Big[e^{-\lambda_0 t}\phi'(t)-\lambda_0\frac{K(\lambda_0;\phi)}{\beta(\lambda_0)}-(\lambda_0+\mu_0)\frac{L(\lambda_0,\mu_0;\phi)}{1+\gamma(\lambda_0,\mu_0)}e^{\mu_0 t}\Big]\Big\}.
\end{aligned}
\tag{4.10}
$$

We see immediately that inequalities (4.5) and (4.6) can equivalently be written as follows

$$
\begin{aligned}
C_1(\lambda_0,\mu_0,\mu_1;\phi)e^{(\mu_1-\mu_0)t} &\le e^{-\mu_0 t}\Big[e^{-\lambda_0 t}x(t)-\frac{K(\lambda_0;\phi)}{\beta(\lambda_0)}\Big]-\frac{L(\lambda_0,\mu_0;\phi)}{1+\gamma(\lambda_0,\mu_0)} \\
&\le C_2(\lambda_0,\mu_0,\mu_1;\phi)e^{(\mu_1-\mu_0)t}\quad \text{for all } t\ge 0
\end{aligned}
$$

and

$$
\begin{aligned}
&D_1(\lambda_0,\mu_0,\mu_1;\phi)e^{(\mu_1-\mu_0)t} \\
&\le e^{-\mu_0 t}\Big[e^{-\lambda_0 t}x'(t)-\lambda_0\frac{K(\lambda_0;\phi)}{\beta(\lambda_0)}\Big]-(\lambda_0+\mu_0)\frac{L(\lambda_0,\mu_0;\phi)}{1+\gamma(\lambda_0,\mu_0)} \\
&\le D_2(\lambda_0,\mu_0,\mu_1;\phi)e^{(\mu_1-\mu_0)t}\quad \text{for all } t\ge 0,
\end{aligned}
$$

respectively. Hence, if $\mu_1<\mu_0$, then the solution x of the IVP (1.1) and (1.2) satisfies (2.4) and (2.5).

Also, we observe that (4.5) and (4.6) are, respectively, equivalent to

$$
\begin{aligned}
&e^{\lambda_0 t}\Big[C_1(\lambda_0,\mu_0,\mu_1;\phi)e^{\mu_1 t}+\frac{K(\lambda_0;\phi)}{\beta(\lambda_0)}+\frac{L(\lambda_0,\mu_0;\phi)}{1+\gamma(\lambda_0,\mu_0)}e^{\mu_0 t}\Big] \\
&\le x(t) \\
&\le e^{\lambda_0 t}\Big[C_2(\lambda_0,\mu_0,\mu_1;\phi)e^{\mu_1 t}+\frac{K(\lambda_0;\phi)}{\beta(\lambda_0)}+\frac{L(\lambda_0,\mu_0;\phi)}{1+\gamma(\lambda_0,\mu_0)}e^{\mu_0 t}\Big]\quad \text{for all } t\ge 0
\end{aligned}
$$

and

$$
\begin{aligned}
&e^{\lambda_0 t}\Big[D_1(\lambda_0,\mu_0,\mu_1;\phi)e^{\mu_1 t}+\lambda_0\frac{K(\lambda_0;\phi)}{\beta(\lambda_0)}+(\lambda_0+\mu_0)\frac{L(\lambda_0,\mu_0;\phi)}{1+\gamma(\lambda_0,\mu_0)}e^{\mu_0 t}\Big] \\
&\le x'(t) \\
&\le e^{\lambda_0 t}\Big[D_2(\lambda_0,\mu_0,\mu_1;\phi)e^{\mu_1 t}+\lambda_0\frac{K(\lambda_0;\phi)}{\beta(\lambda_0)}+(\lambda_0+\mu_0)\frac{L(\lambda_0,\mu_0;\phi)}{1+\gamma(\lambda_0,\mu_0)}e^{\mu_0 t}\Big]
\end{aligned}
$$

for all $t\ge 0$.

Proof of Theorem 4.2. Let x be the solution of the IVP (1.1) and (1.2), and consider the function z defined by (1.9). Consider also the functions w and f defined by (2.10) and (2.14), respectively. Note that, by Lemma 4.1, we necessarily have $1+\gamma(\lambda_0,\mu_0)\ne 0$. As it has been shown in the proof of Theorem 2.1, the fact that x satisfies (1.1) for $t\ge 0$ is equivalent to the fact that f satisfies (2.15) for all $t\ge 0$. Moreover, as in the proof of Theorem 2.1, we see that f' satisfies (2.26) for all $t\ge 0$, and that (2.33) is valid. Furthermore, we consider the function g defined by (2.34). As in the proof of Theorem 2.1, equality (2.35) holds true. Because of (2.35), we can use (2.15) and (2.26) to conclude that the function g satisfies, for all

$t \geq 0$,

$$
\begin{aligned}
g(t) = &\int_{-r}^{0} e^{(\lambda_0+\mu_0)s}\left[\int_{s}^{0} g(t+u)du\right] d\zeta(s) \\
&- \lambda_0 \int_{-r}^{0} e^{\lambda_0 s}\left\{\int_{s}^{0} e^{\mu_0 u}\left[\int_{u}^{0} g(t+v)dv\right] du\right\} d\zeta(s) \\
&+ \int_{-r}^{0} e^{\lambda_0 s}\left\{\int_{s}^{0} e^{\mu_0 u}\left[\int_{u}^{0} g(t+v)dv\right] du\right\} d\eta(s).
\end{aligned}
\tag{4.11}
$$

Now, we define

$$h(t) = e^{(\mu_0-\mu_1)t} f(t) \quad \text{for } t \geq -r, \tag{4.12}$$

$$k(t) = e^{(\mu_0-\mu_1)t} g(t) \quad \text{for } t \geq -r. \tag{4.13}$$

Then we see that (2.15) holds for $t \geq 0$ if and only if h satisfies

$$
\begin{aligned}
h(t) = &\int_{-r}^{0} e^{(\lambda_0+\mu_0)s}\left[\int_{s}^{0} e^{-(\mu_0-\mu_1)u} h(t+u)du\right] d\zeta(s) \\
&- \lambda_0 \int_{-r}^{0} e^{\lambda_0 s}\left\{\int_{s}^{0} e^{\mu_0 u}\left[\int_{u}^{0} e^{-(\mu_0-\mu_1)v} h(t+v)dv\right] du\right\} d\zeta(s) \\
&+ \int_{-r}^{0} e^{\lambda_0 s}\left\{\int_{s}^{0} e^{\mu_0 u}\left[\int_{u}^{0} e^{-(\mu_0-\mu_1)v} h(t+v)dv\right] du\right\} d\eta(s)
\end{aligned}
\tag{4.14}
$$

for all $t \geq 0$, and that (4.11) is fulfilled for $t \geq 0$ if and only if k satisfies

$$
\begin{aligned}
k(t) = &\int_{-r}^{0} e^{(\lambda_0+\mu_0)s}\left[\int_{s}^{0} e^{-(\mu_0-\mu_1)u} k(t+u)du\right] d\zeta(s) \\
&- \lambda_0 \int_{-r}^{0} e^{\lambda_0 s}\left\{\int_{s}^{0} e^{\mu_0 u}\left[\int_{u}^{0} e^{-(\mu_0-\mu_1)v} k(t+v)dv\right] du\right\} d\zeta(s) \\
&+ \int_{-r}^{0} e^{\lambda_0 s}\left\{\int_{s}^{0} e^{\mu_0 u}\left[\int_{u}^{0} e^{-(\mu_0-\mu_1)v} k(t+v)dv\right] du\right\} d\eta(s)
\end{aligned}
\tag{4.15}
$$

for all $t \geq 0$. By combining (2.33) and (4.12), we have

$$h(t) = e^{-\mu_1 t}\left[e^{-\lambda_0 t}x(t) - \frac{K(\lambda_0;\phi)}{\beta(\lambda_0)} - \frac{L(\lambda_0,\mu_0;\phi)}{1+\gamma(\lambda_0,\mu_0)} e^{\mu_0 t}\right] \quad \text{for } t \geq -r, \tag{4.16}$$

while a combination of (2.34) and (4.13) leads to

$$k(t) = e^{-\mu_1 t}\left[e^{-\lambda_0 t}x'(t) - \lambda_0\frac{K(\lambda_0;\phi)}{\beta(\lambda_0)} - (\lambda_0+\mu_0)\frac{L(\lambda_0,\mu_0;\phi)}{1+\gamma(\lambda_0,\mu_0)} e^{\mu_0 t}\right] \tag{4.17}$$

for $t \geq -r$. As the solution x satisfies the initial condition (1.2), we can use (4.16) as well as the definitions of $C_1(\lambda_0,\mu_0,\mu_1;\phi)$ and $C_2(\lambda_0,\mu_0,\mu_1;\phi)$ by (4.7) and (4.8), respectively, to see that

$$C_1(\lambda_0,\mu_0,\mu_1;\phi) = \min_{-r\leq t\leq 0} h(t) \quad \text{and} \quad C_2(\lambda_0,\mu_0,\mu_1;\phi) = \max_{-r\leq t\leq 0} h(t). \tag{4.18}$$

Moreover, from the initial condition (1.2) we obtain

$$x'(t) = \phi'(t) \quad \text{for } -r \leq t \leq 0.$$

So, by taking into account (4.17) as well as the definitions of $D_1(\lambda_0,\mu_0,\mu_1;\phi)$ and $D_2(\lambda_0,\mu_0,\mu_1;\phi)$ by (4.9) and (4.10), respectively, we have

$$D_1(\lambda_0,\mu_0,\mu_1;\phi)=\min_{-r\le t\le 0}k(t)\quad\text{and}\quad D_2(\lambda_0,\mu_0,\mu_1;\phi)=\max_{-r\le t\le 0}k(t). \tag{4.19}$$

In view of (4.16) and (4.18), the double inequality (4.5) can equivalently written as follows

$$\min_{-r\le s\le 0}h(s)\le h(t)\le \max_{-r\le s\le 0}h(s)\quad\text{for all } t\ge 0. \tag{4.20}$$

Also, by (4.17) and (4.19), the double inequality (4.6) takes the following equivalent form

$$\min_{-r\le s\le 0}k(s)\le k(t)\le \max_{-r\le s\le 0}k(s)\quad\text{for all } t\ge 0. \tag{4.21}$$

All we have to prove is that (4.20) and (4.21) hold. We will use the fact that h satisfies (4.14) for all $t\ge 0$ in order to show that (4.20) is valid. By a similar way, one can use the fact that k satisfies (4.15) for all $t\ge 0$ to establish (4.21). So, the proof of (4.21) will be omitted. We restrict ourselves to proving that

$$h(t)\ge \min_{-r\le s\le 0}h(s)\quad\text{for every } t\ge 0. \tag{4.22}$$

The proof of the inequality

$$h(t)\le \max_{-r\le s\le 0}h(s)\quad\text{for every } t\ge 0$$

can be obtained in a similar way, and so it is omitted. In the rest of the proof we will establish (4.22). In order to so, we consider an arbitrary real number A with $A<\min_{-r\le s\le 0}h(s)$, i.e., with

$$h(t)>A\quad\text{for } -r\le t\le 0. \tag{4.23}$$

We will show that

$$h(t)>A\quad\text{for all } t\ge 0. \tag{4.24}$$

To this end, let us assume that (4.24) fails to hold. Then, because of (4.23), there exists a point $t_0>0$ so that

$$h(t)>A\quad\text{for } -r\le t<t_0,\quad\text{and}\quad h(t_0)=A.$$

Thus, by using (4.1) and the fact that η is not constant on $[-r,0]$ and taking into account the hypothesis that $\lambda_0<0$, from (4.14) we obtain

$$\begin{aligned}
A&=h(t_0)\\
&=\int_{-r}^{0}e^{(\lambda_0+\mu_0)s}\Big[\int_s^0 e^{-(\mu_0-\mu_1)u}h(t_0+u)du\Big]d\zeta(s)\\
&\quad-\lambda_0\int_{-r}^{0}e^{\lambda_0 s}\Big\{\int_s^0 e^{\mu_0 u}\Big[\int_u^0 e^{-(\mu_0-\mu_1)v}h(t_0+v)dv\Big]du\Big\}d\zeta(s)\\
&\quad+\int_{-r}^{0}e^{\lambda_0 s}\Big\{\int_s^0 e^{\mu_0 u}\Big[\int_u^0 e^{-(\mu_0-\mu_1)v}h(t_0+v)dv\Big]du\Big\}d\eta(s)\\
&>A\Big(\int_{-r}^{0}e^{(\lambda_0+\mu_0)s}\Big[\int_s^0 e^{-(\mu_0-\mu_1)u}du\Big]d\zeta(s)\\
&\quad-\lambda_0\int_{-r}^{0}e^{\lambda_0 s}\Big\{\int_s^0 e^{\mu_0 u}\Big[\int_u^0 e^{-(\mu_0-\mu_1)v}dv\Big]du\Big\}d\zeta(s)
\end{aligned}$$

$$
\begin{aligned}
&+\int_{-r}^{0} e^{\lambda_0 s}\left\{\int_{s}^{0} e^{\mu_0 u}\left[\int_{u}^{0} e^{-(\mu_0-\mu_1)v}dv\right]du\right\}d\eta(s)\Big) \\
&= A\Big(\int_{-r}^{0} e^{(\lambda_0+\mu_0)s}\left(-\frac{1}{\mu_0-\mu_1}\right)\left[1-e^{-(\mu_0-\mu_1)s}\right]d\zeta(s) \\
&\quad -\lambda_0\int_{-r}^{0} e^{\lambda_0 s}\left\{\int_{s}^{0} e^{\mu_0 u}\left(-\frac{1}{\mu_0-\mu_1}\right)\left[1-e^{-(\mu_0-\mu_1)u}\right]du\right\}d\zeta(s) \\
&\quad +\int_{-r}^{0} e^{\lambda_0 s}\left\{\int_{s}^{0} e^{\mu_0 u}\left(-\frac{1}{\mu_0-\mu_1}\right)\left[1-e^{-(\mu_0-\mu_1)u}\right]du\right\}d\eta(s)\Big) \\
&= \frac{A}{\mu_0-\mu_1}\Big\{-\int_{-r}^{0} e^{\lambda_0 s}\left(e^{\mu_0 s}-e^{\mu_1 s}\right)d\zeta(s) \\
&\quad +\lambda_0\int_{-r}^{0} e^{\lambda_0 s}\left[\int_{s}^{0}\left(e^{\mu_0 u}-e^{\mu_1 u}\right)du\right]d\zeta(s) \\
&\quad -\int_{-r}^{0} e^{\lambda_0 s}\left[\int_{s}^{0}\left(e^{\mu_0 u}-e^{\mu_1 u}\right)du\right]d\eta(s)\Big\} \\
&= \frac{A}{\mu_0-\mu_1}\Big\{\Big[-\int_{-r}^{0} e^{(\lambda_0+\mu_0)s}d\zeta(s)+\lambda_0\int_{-r}^{0} e^{\lambda_0 s}\left(\int_{s}^{0} e^{\mu_0 u}du\right)d\zeta(s) \\
&\quad -\int_{-r}^{0} e^{\lambda_0 s}\left(\int_{s}^{0} e^{\mu_0 u}du\right)d\eta(s)\Big] \\
&\quad -\Big[-\int_{-r}^{0} e^{(\lambda_0+\mu_1)s}d\zeta(s)+\lambda_0\int_{-r}^{0} e^{\lambda_0 s}\left(\int_{s}^{0} e^{\mu_1 u}du\right)d\zeta(s) \\
&\quad -\int_{-r}^{0} e^{\lambda_0 s}\left(\int_{s}^{0} e^{\mu_1 u}du\right)d\eta(s)\Big]\Big\} \\
&= \frac{A}{\mu_0-\mu_1}\left[(\mu_0+2\lambda_0)-(\mu_1+2\lambda_0)\right]=A.
\end{aligned}
$$

We have thus arrived at a contradiction and so (4.24) is true. Since (4.24) is satisfied for all real numbers A with $A<\min_{-r\le s\le 0} h(s)$, it follows that (4.22) is always fulfilled. The proof of the theorem is complete. □

Now, let us concentrate on the special case of the delay differential equation (1.19). In this case, the hypothesis $\lambda_0<0$ posed in Lemma 4.1 and Theorem 4.2 can be removed without damage. More precisely, we have the following results.

Lemma 4.3. *Suppose that*

$$\eta \text{ is increasing on } [-r,0]. \tag{4.25}$$

Let $\lambda_0\neq 0$ be a real root of the characteristic equation (1.20)*. Furthermore, let μ_0 be a real root of the characteristic equation* (1.22)*, and set*

$$\widetilde{\gamma}(\lambda_0,\mu_0)=-\int_{-r}^{0} e^{\lambda_0 s}\left[\int_{s}^{0}(-u)e^{\mu_0 u}du\right]d\eta(s). \tag{4.26}$$

Then $1+\widetilde{\gamma}(\lambda_0,\mu_0)>0$ if (1.22) *has another real root less than μ_0, and $1+\widetilde{\gamma}(\lambda_0,\mu_0)<0$ if* (1.22) *has another real root greater than μ_0.*

Theorem 4.4. *Suppose that* (4.25) *holds. Let $\lambda_0\neq 0$ be a real root of the characteristic equation* (1.20)*, and let $\widetilde{\beta}(\lambda_0)$ and $\widetilde{K}(\lambda_0;\phi)$ be defined by* (1.23) *and* (1.24)*,* *respectively. Suppose that* $\widetilde{\beta}(\lambda_0) \neq 0$*, and define* $\widetilde{\Phi}(\lambda_0;\phi)$ *by* (1.25)*. Furthermore, let* μ_0 *be a real root of the characteristic equation* (1.22)*, and let* $\widetilde{\gamma}(\lambda_0,\mu_0)$ *and* $\widetilde{L}(\lambda_0,\mu_0;\phi)$ *be defined by* (4.26) *and*

$$\widetilde{L}(\lambda_0,\mu_0;\phi) = \widetilde{\Phi}(\lambda_0;\phi)(0) - \int_{-r}^{0} e^{\lambda_0 s}\left\{\int_s^0 e^{\mu_0 u}\left[\int_u^0 e^{-\mu_0 v}\widetilde{\Phi}(\lambda_0;\phi)(v)dv\right]du\right\}d\eta(s),$$

respectively. Also, let μ_1 *be a real root of* (1.22) *with* $\mu_1 \neq \mu_0$*. (Note that, because of* $\widetilde{\beta}(\lambda_0) \neq 0$*, we always have* $\mu_0 \neq 0$ *and* $\mu_1 \neq 0$*; moreover, note that Lemma 4.3 guarantees that* $1+\widetilde{\gamma}(\lambda_0,\mu_0) \neq 0$*.)*

Then the solution x *of the IVP* (1.19) *and* (1.2) *satisfies*

$$\begin{aligned}\widetilde{C}_1(\lambda_0,\mu_0,\mu_1;\phi) &\le e^{-\mu_1 t}\left[e^{-\lambda_0 t}x(t) - \frac{\widetilde{K}(\lambda_0;\phi)}{\widetilde{\beta}(\lambda_0)} - \frac{\widetilde{L}(\lambda_0,\mu_0;\phi)}{1+\widetilde{\gamma}(\lambda_0,\mu_0)}e^{\mu_0 t}\right]\\ &\le \widetilde{C}_2(\lambda_0,\mu_0,\mu_1;\phi) \quad \text{for all } t \ge 0\end{aligned}$$

and

$$\begin{aligned}\widetilde{D}_1(\lambda_0,\mu_0,\mu_1;\phi) &\le e^{-\mu_1 t}\left[e^{-\lambda_0 t}x'(t) - \lambda_0\frac{\widetilde{K}(\lambda_0;\phi)}{\widetilde{\beta}(\lambda_0)} - (\lambda_0+\mu_0)\frac{\widetilde{L}(\lambda_0,\mu_0;\phi)}{1+\widetilde{\gamma}(\lambda_0,\mu_0)}e^{\mu_0 t}\right]\\ &\le \widetilde{D}_2(\lambda_0,\mu_0,\mu_1;\phi) \quad \text{for all } t \ge 0,\end{aligned}$$

where

$$\widetilde{C}_1(\lambda_0,\mu_0,\mu_1;\phi) = \min_{-r\le t\le 0}\left\{e^{-\mu_1 t}\left[e^{-\lambda_0 t}\phi(t) - \frac{\widetilde{K}(\lambda_0;\phi)}{\widetilde{\beta}(\lambda_0)} - \frac{\widetilde{L}(\lambda_0,\mu_0;\phi)}{1+\widetilde{\gamma}(\lambda_0,\mu_0)}e^{\mu_0 t}\right]\right\},$$

$$\widetilde{C}_2(\lambda_0,\mu_0,\mu_1;\phi) = \max_{-r\le t\le 0}\left\{e^{-\mu_1 t}\left[e^{-\lambda_0 t}\phi(t) - \frac{\widetilde{K}(\lambda_0;\phi)}{\widetilde{\beta}(\lambda_0)} - \frac{\widetilde{L}(\lambda_0,\mu_0;\phi)}{1+\widetilde{\gamma}(\lambda_0,\mu_0)}e^{\mu_0 t}\right]\right\}$$

and

$$\begin{aligned}&\widetilde{D}_1(\lambda_0,\mu_0,\mu_1;\phi)\\ &= \min_{-r\le t\le 0}\left\{e^{-\mu_1 t}\left[e^{-\lambda_0 t}\phi'(t) - \lambda_0\frac{\widetilde{K}(\lambda_0;\phi)}{\widetilde{\beta}(\lambda_0)} - (\lambda_0+\mu_0)\frac{\widetilde{L}(\lambda_0,\mu_0;\phi)}{1+\widetilde{\gamma}(\lambda_0,\mu_0)}e^{\mu_0 t}\right]\right\},\end{aligned}$$

$$\begin{aligned}&\widetilde{D}_2(\lambda_0,\mu_0,\mu_1;\phi)\\ &= \max_{-r\le t\le 0}\left\{e^{-\mu_1 t}\left[e^{-\lambda_0 t}\phi'(t) - \lambda_0\frac{\widetilde{K}(\lambda_0;\phi)}{\widetilde{\beta}(\lambda_0)} - (\lambda_0+\mu_0)\frac{\widetilde{L}(\lambda_0,\mu_0;\phi)}{1+\widetilde{\gamma}(\lambda_0,\mu_0)}e^{\mu_0 t}\right]\right\}.\end{aligned}$$

Note that the observations presented after the statement of Theorem 4.2 can also be formulated in connection with Theorem 4.4, i.e., in the special case of the delay differential equation (1.19).

Proof of Lemma 4.3. The proof will be omitted since it is similar to that of Lemma 4.1. We restrict ourselves only to noting that, here, we have the real-valued function Ω_0 defined by

$$\Omega_0(\mu) = \mu + 2\lambda_0 + \int_{-r}^{0} e^{\lambda_0 s}\left(\int_s^0 e^{\mu u}du\right)d\eta(s) \quad \text{for } \mu \in \mathbb{R} \tag{4.27}$$

instead of Ω. We note that

$$\Omega_0''(\mu) > 0 \quad \text{for all } \mu \in \mathbb{R}.$$

This inequality holds true without the hypothesis that λ_0 is negative. $\square$

Proof of Theorem 4.4. The need for assuming, in Theorem 4.2, that the root λ_0 of the characteristic equation (1.3) is negative is due only to the existence of the term

$$-\lambda_0 \int_{-r}^{0} e^{\lambda_0 s} \left\{ \int_{s}^{0} e^{\mu_0 u} \left[\int_{u}^{0} e^{-(\mu_0-\mu_1)v} h(t+v) dv \right] du \right\} d\zeta(s)$$

in (4.14) as well as to the existence of the term

$$-\lambda_0 \int_{-r}^{0} e^{\lambda_0 s} \left\{ \int_{s}^{0} e^{\mu_0 u} \left[\int_{u}^{0} e^{-(\mu_0-\mu_1)v} k(t+v) dv \right] du \right\} d\zeta(s)$$

in (4.15). These terms do not appear in the special case of the delay differential equation (1.19). In this special case, h satisfies

$$h(t) = \int_{-r}^{0} e^{\lambda_0 s} \left\{ \int_{s}^{0} e^{\mu_0 u} \left[\int_{u}^{0} e^{-(\mu_0-\mu_1)v} h(t+v) dv \right] du \right\} d\eta(s)$$

for all $t \geq 0$, and k satisfies

$$k(t) = \int_{-r}^{0} e^{\lambda_0 s} \left\{ \int_{s}^{0} e^{\mu_0 u} \left[\int_{u}^{0} e^{-(\mu_0-\mu_1)v} k(t+v) dv \right] du \right\} d\eta(s)$$

for all $t \geq 0$. After these observations, we omit the proof of the theorem. $\square$

5. Additional Lemmas

We have already obtained two lemmas (Lemmas 4.1 and 4.3); Lemma 4.1 is concerned with the real roots of the characteristic equation (1.5), while Lemma 4.3 concerns the real roots of the characteristic equation (1.22). Lemmas 4.1 and 4.3 have been used in order to establish Theorems 4.2 and 4.4, respectively. Here, we will give two lemmas about the real roots of (1.5) and a lemma concerning the real roots of (1.22).

Lemma 5.1. *Let λ_0 be a real root of the characteristic equation* (1.3)*. Assume that*

$$\begin{aligned} &\int_{-r}^{0} e^{\lambda_0 s} \left[e^{-\left(2\lambda_0 + \frac{1}{r}\right)s} - \lambda_0 \int_{s}^{0} e^{-\left(2\lambda_0 + \frac{1}{r}\right)u} du \right] d\zeta(s) \\ &\quad + \int_{-r}^{0} e^{\lambda_0 s} \left[\int_{s}^{0} e^{-\left(2\lambda_0 + \frac{1}{r}\right)u} du \right] d\eta(s) < \frac{1}{r} \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} &\int_{-r}^{0} e^{\lambda_0 s} \left[(-s) e^{-\left(2\lambda_0 + \frac{1}{r}\right)s} + |\lambda_0| \int_{s}^{0} (-u) e^{-\left(2\lambda_0 + \frac{1}{r}\right)u} du \right] dV(\zeta)(s) \\ &\quad + \int_{-r}^{0} e^{\lambda_0 s} \left[\int_{s}^{0} (-u) e^{-\left(2\lambda_0 + \frac{1}{r}\right)u} du \right] dV(\eta)(s) \leq 1. \end{aligned} \tag{5.2}$$

Then, in the interval $\left(-2\lambda_0 - \frac{1}{r}, \infty\right)$, the characteristic equation (1.5) *has a unique root μ_0; this root satisfies* (2.3)*, and the root μ_0 is less than $-2\lambda_0 + \frac{1}{r}$, provided* *that*

$$\begin{aligned}\int_{-r}^{0} e^{\lambda_0 s}\Big[e^{\left(-2\lambda_0+\frac{1}{r}\right)s}-\lambda_0\int_s^0 e^{\left(-2\lambda_0+\frac{1}{r}\right)u}du\Big]d\zeta(s)&\\ +\int_{-r}^{0} e^{\lambda_0 s}\Big[\int_s^0 e^{\left(-2\lambda_0+\frac{1}{r}\right)u}du\Big]d\eta(s)>-\frac{1}{r}.&\end{aligned}\tag{5.3}$$

Proof. Consider the real-valued function Ω defined by (4.2). The derivative Ω' of Ω is given by (4.3). It follows from (4.2) that

$$\begin{aligned}&\Omega\Big(-2\lambda_0-\frac{1}{r}\Big)\\ &=\Big(-2\lambda_0-\frac{1}{r}\Big)+2\lambda_0+\int_{-r}^{0} e^{\lambda_0 s}e^{-\left(2\lambda_0+\frac{1}{r}\right)s}d\zeta(s)\\ &\quad-\lambda_0\int_{-r}^{0} e^{\lambda_0 s}\Big[\int_s^0 e^{-\left(2\lambda_0+\frac{1}{r}\right)u}du\Big]d\zeta(s)+\int_{-r}^{0} e^{\lambda_0 s}\Big[\int_s^0 e^{-\left(2\lambda_0+\frac{1}{r}\right)u}du\Big]d\eta(s)\\ &=-\frac{1}{r}+\int_{-r}^{0} e^{\lambda_0 s}\Big[e^{-\left(2\lambda_0+\frac{1}{r}\right)s}-\lambda_0\int_s^0 e^{-\left(2\lambda_0+\frac{1}{r}\right)u}du\Big]d\zeta(s)\\ &\quad+\int_{-r}^{0} e^{\lambda_0 s}\Big[\int_s^0 e^{-\left(2\lambda_0+\frac{1}{r}\right)u}du\Big]d\eta(s)\end{aligned}$$

and consequently, by (5.1), it holds

$$\Omega\Big(-2\lambda_0-\frac{1}{r}\Big)<0.\tag{5.4}$$

Moreover, from (4.2) we obtain, for $\mu\geq -2\lambda_0-\frac{1}{r}$,

$$\begin{aligned}\Omega(\mu)&=\mu+2\lambda_0+\int_{-r}^{0} e^{\lambda_0 s}\Big(e^{\mu s}-\lambda_0\int_s^0 e^{\mu u}du\Big)d\zeta(s)\\ &\quad+\int_{-r}^{0} e^{\lambda_0 s}\Big(\int_s^0 e^{\mu u}du\Big)d\eta(s)\\ &\geq \mu+2\lambda_0-\Big|\int_{-r}^{0} e^{\lambda_0 s}\Big(e^{\mu s}-\lambda_0\int_s^0 e^{\mu u}du\Big)d\zeta(s)\Big|\\ &\quad-\Big|\int_{-r}^{0} e^{\lambda_0 s}\Big(\int_s^0 e^{\mu u}du\Big)d\eta(s)\Big|\\ &\geq \mu+2\lambda_0-\int_{-r}^{0} e^{\lambda_0 s}\Big|e^{\mu s}-\lambda_0\int_s^0 e^{\mu u}du\Big|dV(\zeta)(s)\\ &\quad-\int_{-r}^{0} e^{\lambda_0 s}\Big(\int_s^0 e^{\mu u}du\Big)dV(\eta)(s)\\ &\geq \mu+2\lambda_0-\int_{-r}^{0} e^{\lambda_0 s}\Big(e^{\mu s}+|\lambda_0|\int_s^0 e^{\mu u}du\Big)dV(\zeta)(s)\\ &\quad-\int_{-r}^{0} e^{\lambda_0 s}\Big(\int_s^0 e^{\mu u}du\Big)dV(\eta)(s)\\ &\geq \mu+2\lambda_0-\int_{-r}^{0} e^{\lambda_0 s}\Big[e^{-\left(2\lambda_0+\frac{1}{r}\right)s}+|\lambda_0|\int_s^0 e^{-\left(2\lambda_0+\frac{1}{r}\right)u}du\Big]dV(\zeta)(s)\end{aligned}$$

$$- \int_{-r}^{0} e^{\lambda_0 s} \left[\int_{s}^{0} e^{-\left(2\lambda_0 + \frac{1}{r}\right)u} du \right] dV(\eta)(s).$$

Therefore,

$$\Omega(\infty) = \infty. \tag{5.5}$$

Furthermore, using (4.3), we have, for every $\mu > -2\lambda_0 - \frac{1}{r}$,

$$\begin{aligned}
\Omega'(\mu) &\geq 1 - \left| \int_{-r}^{0} e^{\lambda_0 s} \left[(-s)e^{\mu s} - \lambda_0 \int_{s}^{0} (-u)e^{\mu u} du \right] d\zeta(s) \right| \\
&\quad - \left| \int_{-r}^{0} e^{\lambda_0 s} \left[\int_{s}^{0} (-u)e^{\mu u} du \right] d\eta(s) \right| \\
&\geq 1 - \int_{-r}^{0} e^{\lambda_0 s} \left| (-s)e^{\mu s} - \lambda_0 \int_{s}^{0} (-u)e^{\mu u} du \right| dV(\zeta)(s) \\
&\quad - \int_{-r}^{0} e^{\lambda_0 s} \left[\int_{s}^{0} (-u)e^{\mu u} du \right] dV(\eta)(s) \\
&\geq 1 - \int_{-r}^{0} e^{\lambda_0 s} \left[(-s)e^{\mu s} + |\lambda_0| \int_{s}^{0} (-u)e^{\mu u} du \right] dV(\zeta)(s) \\
&\quad - \int_{-r}^{0} e^{\lambda_0 s} \left[\int_{s}^{0} (-u)e^{\mu u} du \right] dV(\eta)(s) \\
&> 1 - \int_{-r}^{0} e^{\lambda_0 s} \left[(-s)e^{-\left(2\lambda_0 + \frac{1}{r}\right)s} + |\lambda_0| \int_{s}^{0} (-u)e^{-\left(2\lambda_0 + \frac{1}{r}\right)u} du \right] dV(\zeta)(s) \\
&\quad - \int_{-r}^{0} e^{\lambda_0 s} \left[\int_{s}^{0} (-u)e^{-\left(2\lambda_0 + \frac{1}{r}\right)u} du \right] dV(\eta)(s).
\end{aligned}$$

Consequently, in view of (5.2), it holds

$$\Omega'(\mu) > 0 \quad \text{for all } \mu > -2\lambda_0 - \frac{1}{r},$$

which implies that Ω is strictly increasing on $\left(-2\lambda_0 - \frac{1}{r}, \infty\right)$. By using this fact as well as (5.4) and (5.5), we conclude that, in the interval $\left(-2\lambda_0 - \frac{1}{r}, \infty\right)$, the equation $\Omega(\mu) = 0$ (which coincides with (1.5)) has a unique root μ_0. This root satisfies (2.3). Indeed, by using again (5.2), we have

$$\begin{aligned}
&\int_{-r}^{0} e^{\lambda_0 s} \left[(-s)e^{\mu_0 s} + |\lambda_0| \int_{s}^{0} (-u)e^{\mu_0 u} du \right] dV(\zeta)(s) \\
&+ \int_{-r}^{0} e^{\lambda_0 s} \left[\int_{s}^{0} (-u)e^{\mu_0 u} du \right] dV(\eta)(s) \\
&< \int_{-r}^{0} e^{\lambda_0 s} \left[(-s)e^{-\left(2\lambda_0 + \frac{1}{r}\right)s} + |\lambda_0| \int_{s}^{0} (-u)e^{-\left(2\lambda_0 + \frac{1}{r}\right)u} du \right] dV(\zeta)(s) \\
&\quad + \int_{-r}^{0} e^{\lambda_0 s} \left[\int_{s}^{0} (-u)e^{-\left(2\lambda_0 + \frac{1}{r}\right)u} du \right] dV(\eta)(s) \\
&\leq 1.
\end{aligned}$$

Finally, let us assume that (5.3) holds. Then it follows from (4.2) that

$$\Omega\left(-2\lambda_0 + \frac{1}{r}\right)$$

$$
\begin{aligned}
&= \Big(-2\lambda_0 + \frac{1}{r}\Big) + 2\lambda_0 + \int_{-r}^{0} e^{\lambda_0 s} e^{\left(-2\lambda_0+\frac{1}{r}\right)s} d\zeta(s) \\
&\quad - \lambda_0 \int_{-r}^{0} e^{\lambda_0 s}\Big[\int_{s}^{0} e^{\left(-2\lambda_0+\frac{1}{r}\right)u}du\Big] d\zeta(s) + \int_{-r}^{0} e^{\lambda_0 s}\Big[\int_{s}^{0} e^{\left(-2\lambda_0+\frac{1}{r}\right)u}du\Big] d\eta(s) \\
&= \frac{1}{r} + \int_{-r}^{0} e^{\lambda_0 s}\Big[e^{\left(-2\lambda_0+\frac{1}{r}\right)s} - \lambda_0 \int_{s}^{0} e^{\left(-2\lambda_0+\frac{1}{r}\right)u}du\Big] d\zeta(s) \\
&\quad + \int_{-r}^{0} e^{\lambda_0 s}\Big[\int_{s}^{0} e^{\left(-2\lambda_0+\frac{1}{r}\right)u}du\Big] d\eta(s) > 0.
\end{aligned}
$$

As $\Omega\big(-2\lambda_0 + \frac{1}{r}\big) > 0$, we see that μ_0 must be less than $-2\lambda_0 + \frac{1}{r}$. This completes the proof of the lemma. $\square$

Lemma 5.1 can be applied to the special case of the characteristic equation (1.22), where λ_0 is a real root of the characteristic equation (1.20). In this particular case, conditions (5.1), (5.2) and (5.3) become

$$\int_{-r}^{0} e^{\lambda_0 s}\Big[\int_{s}^{0} e^{-\left(2\lambda_0+\frac{1}{r}\right)u}du\Big] d\eta(s) < \frac{1}{r}, \tag{5.6}$$

$$\int_{-r}^{0} e^{\lambda_0 s}\Big[\int_{s}^{0} (-u)e^{-\left(2\lambda_0+\frac{1}{r}\right)u}du\Big] dV(\eta)(s) \le 1, \tag{5.7}$$

$$\int_{-r}^{0} e^{\lambda_0 s}\Big[\int_{s}^{0} e^{\left(-2\lambda_0+\frac{1}{r}\right)u}du\Big] d\eta(s) > -\frac{1}{r}, \tag{5.8}$$

respectively. It is remarkable that conditions (5.6), (5.7) and (5.8) are satisfied if the following stronger condition holds:

$$\int_{-r}^{0} e^{\lambda_0 s}\Big[\int_{s}^{0} e^{-\left(2\lambda_0+\frac{1}{r}\right)u}du\Big] dV(\eta)(s) < \frac{1}{r}. \tag{5.9}$$

In fact, we have

$$
\begin{aligned}
\int_{-r}^{0} e^{\lambda_0 s}\Big[\int_{s}^{0} e^{-\left(2\lambda_0+\frac{1}{r}\right)u}du\Big] d\eta(s) &\le \Big|\int_{-r}^{0} e^{\lambda_0 s}\Big[\int_{s}^{0} e^{-\left(2\lambda_0+\frac{1}{r}\right)u}du\Big] d\eta(s)\Big| \\
&\le \int_{-r}^{0} e^{\lambda_0 s}\Big[\int_{s}^{0} e^{-\left(2\lambda_0+\frac{1}{r}\right)u}du\Big] dV(\eta)(s),
\end{aligned}
$$

$$
\begin{aligned}
&\int_{-r}^{0} e^{\lambda_0 s}\Big[\int_{s}^{0} (-u)e^{-\left(2\lambda_0+\frac{1}{r}\right)u}du\Big] dV(\eta)(s) \\
&\le r\int_{-r}^{0} e^{\lambda_0 s}\Big[\int_{s}^{0} e^{-\left(2\lambda_0+\frac{1}{r}\right)u}du\Big] dV(\eta)(s)
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{-r}^{0} e^{\lambda_0 s}\Big[\int_{s}^{0} e^{\left(-2\lambda_0+\frac{1}{r}\right)u}du\Big] d\eta(s) &\ge -\Big|\int_{-r}^{0} e^{\lambda_0 s}\Big[\int_{s}^{0} e^{\left(-2\lambda_0+\frac{1}{r}\right)u}du\Big] d\eta(s)\Big| \\
&\ge -\int_{-r}^{0} e^{\lambda_0 s}\Big[\int_{s}^{0} e^{\left(-2\lambda_0+\frac{1}{r}\right)u}du\Big] dV(\eta)(s) \\
&\ge -\int_{-r}^{0} e^{\lambda_0 s}\Big[\int_{s}^{0} e^{-\left(2\lambda_0+\frac{1}{r}\right)u}du\Big] dV(\eta)(s).
\end{aligned}
$$

So, condition (5.9) implies each one of the conditions (5.6), (5.7) and (5.8).

Lemma 5.2. *Suppose that statement* (4.1) *is true. Let λ_0 be a negative real root of the characteristic equation* (1.3). *Then we have:*

(I) *In the interval $[-2\lambda_0, \infty)$, the characteristic equation* (1.5) *has no roots.*

(II) *Assume that* (5.1) *holds. Then: (i) $\mu = -2\lambda_0 - \frac{1}{r}$ is not a root of the characteristic equation* (1.5). *(ii) In the interval $\left(-2\lambda_0 - \frac{1}{r}, -2\lambda_0\right)$,* (1.5) *has a unique root. (iii) In the interval $\left(-\infty, -2\lambda_0 - \frac{1}{r}\right)$,* (1.5) *has a unique root.*

Proof. (I) Let $\widetilde{\mu}$ be a real root of the characteristic equation (1.5). By taking into account (4.1) as well as the fact that η is not constant on $[-r, 0]$ and using the hypothesis that $\lambda_0 < 0$, we can immediately see that

$$-\int_{-r}^{0} e^{(\lambda_0+\widetilde{\mu})s} d\zeta(s) + \lambda_0 \int_{-r}^{0} e^{\lambda_0 s}\left(\int_{s}^{0} e^{\widetilde{\mu}u} du\right) d\zeta(s) - \int_{-r}^{0} e^{\lambda_0 s}\left(\int_{s}^{0} e^{\widetilde{\mu}u} du\right) d\eta(s)$$
$$< 0.$$

Hence, from (1.5) it follows that $\widetilde{\mu} + 2\lambda_0 < 0$, i.e., $\widetilde{\mu} < -2\lambda_0$. We have thus proved that every real root of (1.5) is always less than $-2\lambda_0$.

(II) Consider the real-valued function Ω defined by (4.2). As in the proof of Lemma 4.1, we see that (4.4) holds and consequently

$$\Omega \text{ is convex on } \mathbb{R}. \tag{5.10}$$

Next, we observe that, as in the proof of Lemma 5.1, assumption (5.1) means that (5.4) holds true. Inequality (5.4) implies, in particular, that $\mu = -2\lambda_0 - \frac{1}{r}$ is not a root of the characteristic equation (1.5). From (4.2) we obtain

$$\begin{aligned}\Omega(-2\lambda_0) = &\int_{-r}^{0} e^{-\lambda_0 s} d\zeta(s) - \lambda_0 \int_{-r}^{0} e^{\lambda_0 s}\left(\int_{s}^{0} e^{-2\lambda_0 u} du\right) d\zeta(s)\\ &+ \int_{-r}^{0} e^{\lambda_0 s}\left(\int_{s}^{0} e^{-2\lambda_0 u} du\right) d\eta(s).\end{aligned}$$

So, by using (4.1) and taking into account the facts that η is not constant on $[-r, 0]$ and that λ_0 is negative, we conclude that

$$\Omega(-2\lambda_0) > 0. \tag{5.11}$$

Furthermore, as $\lambda_0 < 0$ and ζ is increasing on $[-r, 0]$, from (4.2) we get

$$\Omega(\mu) \geq \mu + 2\lambda_0 + \int_{-r}^{0} e^{\lambda_0 s}\left(\int_{s}^{0} e^{\mu u} du\right) d\eta(s) \quad \text{for } \mu \in \mathbb{R}.$$

Using this inequality and the fact that η is increasing and not constant on $[-r, 0]$, it is not difficult to show that

$$\Omega(-\infty) = \infty. \tag{5.12}$$

From (5.10), (5.4) and (5.11) it follows that, in the interval $\left(-2\lambda_0 - \frac{1}{r}, -2\lambda_0\right)$, the characteristic equation (1.5) has a unique root. Moreover, (5.10), (5.4) and (5.12) guarantee that, in the interval $\left(-\infty, -2\lambda_0 - \frac{1}{r}\right)$, (1.5) has also a unique root. The proof of the lemma is complete. $\square$

If someone reads carefully the proof of Lemma 5.2, he/she may see that the assumption that the root λ_0 of the characteristic equation (1.3) is negative is a necessity because of the presence of the term

$$\lambda_0 \int_{-r}^{0} e^{\lambda_0 s} \left(\int_{s}^{0} e^{\mu u} du \right) d\zeta(s)$$

in the characteristic equation (1.5). As this term does not exist in the characteristic equation (1.22), using the function Ω_0 defined by

$$\Omega_0(\mu) = \mu + 2\lambda_0 + \int_{-r}^{0} e^{\lambda_0 s} \left(\int_{s}^{0} e^{\mu u} du \right) d\eta(s) \quad \text{for } \mu \in \mathbb{R}$$

instead of the function Ω and following the steps of the proof of Lemma 5.2, we can prove the next lemma valid in the special case of the characteristic equation (1.22), without the restriction that the root λ_0 of the characteristic equation (1.20) is necessarily negative.

Lemma 5.3. *Suppose that* (4.25) *holds. Let* $\lambda_0 \neq 0$ *be a real root of the characteristic equation* (1.20)*. Then we have:*

(I) *In the interval* $[-2\lambda_0, \infty)$*, the characteristic equation* (1.20) *has no roots.*
(II) *Assume that* (5.6) *holds. Then: (i)* $\mu = -2\lambda_0 - \frac{1}{r}$ *is not a root of the characteristic equation* (1.22)*. (ii) In the interval* $\left(-2\lambda_0 - \frac{1}{r}, -2\lambda_0\right)$*,* (1.22) *has a unique root. (iii) In the interval* $\left(-\infty, -2\lambda_0 - \frac{1}{r}\right)$*,* (1.22) *has a unique root.*

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