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SECTORIAL OSCILLATION OF LINEAR DIFFERENTIAL EQUATIONS AND ITERATED ORDER

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Abstract. In the present paper, we investigate higher order linear differential equations with entire coefficients of iterated order. Using value distribution theory of transcendental meromorphic functions and covering surface theory, we extend a result on the order of growth of solutions published by Bank and Langley [2].

1. Introduction and main results

In 1982, Bank and Laine [1] investigated the exponent of convergence of zeros of the solutions for the differential equation

$$f'' + A(z)f = 0, \tag{1.1}$$

where $A(z)$ is a transcendental entire function and E is the product of normalized linearly independent solutions f_1, f_2 for (1.1). They proved that

$$\sigma(E) = \max\{\sigma(A), \lambda(E)\}.$$

A considerable number of research results concerning (1.1) have been proved. We refer the reader to the book by Laine [7] for a summary of those results. We assume that the reader is familiar with the basic results and notation of the Nevanlinna's value distribution theory of meromorphic functions (see [13],[5]), such as $\sigma(f), \lambda(f)$ to denote, respectively the order and exponent of convergence of meromorphic function f.

For $k \geq 2$, we consider a linear differential equation

$$f^{(k)} + A_{k-2}f^{(k-2)} + \cdots + A_0 f = 0, \tag{1.2}$$

where $A_0, \dots, A_{k-2}$ are entire functions with $A_0 \not\equiv 0$. It is well known that all solutions of (1.2) are entire functions, and if some of the coefficients of (1.2) are transcendental, then (1.2) has at least one solution with order $\sigma(f) = \infty$. Now there exists a question: How to describe precisely the properties of growth of solutions of infinite order of (1.2)? It is to make use of iterated order of entire functions, see Laine [7]. Let us define inductively (see e.g. [3]), for $r \in [0, +\infty), \exp^{[1]} r = e^r$ and

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$\exp^{[n+1]} r = \exp(\exp^{[n]} r), n \in \mathbb{N}$. For all r sufficiently large, we define $\log^{[1]} r = \log r$ and $\log^{[n+1]} r = \log(\log^{[n]} r), n \in \mathbb{N}$. We also denote $\exp^{[0]} r = r = \log^{[0]} r$, $\log^{[-1]} r = \exp^{[1]} r$ and $\exp^{[-1]} r = \log^{[1]} r$. We recall the following definitions and remarks (see [6, 9, 4]).

Definition 1.1. The iterated p-order $\sigma_p(f)$ of a meromorphic function $f(z)$ is defined by

$$\sigma_p(f) = \limsup_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log r} \quad (p \in \mathbb{N}).$$

Remark 1.2. (1) If $p = 1$, then we denote $\sigma_1(f) = \sigma(f)$. (2) If $p = 2$, then we denote by $\sigma_2(f)$ the so-called hyper order (see [14]). (3) If $f(z)$ is an entire function, then

$$\sigma_p(f) = \limsup_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log r}.$$

Definition 1.3. The growth index of the iterated order of a meromorphic function $f(z)$ is defined by

$$i(f) = \begin{cases} 0 & \text{if } f \text{ is rational,} \\ \min\{n \in \mathbb{N} : \sigma_n(f) < \infty\} & \text{if } f \text{ is transcendental and} \\ & \sigma_n(f) < \infty \text{ for some } n \in \mathbb{N}, \\ \infty & \text{if } \sigma_n(f) = \infty \text{ for all } n \in \mathbb{N}. \end{cases}$$

Definition 1.4. The iterated convergence exponent of the sequence of a-points ($a \in \mathbb{C} \cup \{\infty\}$) is defined by

$$\lambda_n(f-a) = \lambda_n(f,a) = \limsup_{r\to\infty} \frac{\log^{[n]} N(r, \frac{1}{f-a})}{\log r} \quad (n \in \mathbb{N}),$$

and $\overline{\lambda}_n(f-a)$, the iterated convergence exponent of the sequence of distinct a-points is defined by

$$\overline{\lambda}_n(f-a) = \overline{\lambda}_n(f,a) = \limsup_{r\to\infty} \frac{\log^{[n]} \overline{N}(r, \frac{1}{f-a})}{\log r} \quad (n \in \mathbb{N}).$$

Remark 1.5. (1) $\lambda_1(f-a) = \lambda(f-a)$. (2) $\overline{\lambda}_1(f-a) = \overline{\lambda}(f-a)$.

For the sake of convenience, we also make the following definitions and remarks.

Definition 1.6. The iterated sectorial convergence exponent of the sequence of a-points ($a \in \mathbb{C} \cup \{\infty\}$) is defined by

$$\lambda_{n,\alpha,\beta}(f-a) = \lambda_{n,\alpha,\beta}(f,a) = \limsup_{r\to\infty} \frac{\log^{[n]} n(r, X(\alpha,\beta), \frac{1}{f-a})}{\log r} \quad (n \in \mathbb{N}),$$

and $\overline{\lambda}_n(f-a)$, the iterated sectorial convergence exponent of the sequence of distinct a-points is defined by

$$\overline{\lambda}_n(f-a) = \overline{\lambda}_n(f,a) = \limsup_{r\to\infty} \frac{\log^{[n]} \overline{n}(r, X(\alpha,\beta), \frac{1}{f-a})}{\log r} \quad (n \in \mathbb{N}).$$

where $X(\alpha,\beta) = \{z | \alpha < \arg z < \beta\}, 0 < \beta - \alpha \le \pi$ and $n(r, X(\alpha,\beta), f = a)$ is the roots of $f(z) - a = 0$ in $\Omega(\alpha,\beta) \cap \{|z| < r\}$, counting multiplicities, and $\overline{n}(r, X(\alpha,\beta), f = a)$ is the corresponding notion ignoring multiplicities.

Remark 1.7 ([11])**.** (1) $\lambda_{1,\alpha,\beta}(f-a)=\lambda_{\alpha,\beta}(f-a)$. (2) $\overline{\lambda}_{1,\alpha,\beta}(f-a)=\overline{\lambda}_{\alpha,\beta}(f-a)$.

Definition 1.8. The iterated radial convergence exponent of the sequence of a-points ($a\in\mathbb{C}\cup\{\infty\}$) is defined by

$$\lambda_{n,\theta}(f-a)=\lambda_{n,\theta}(f,a)=\lim_{\varepsilon\to 0^+}\lambda_{n,\theta-\varepsilon,\theta+\varepsilon}(f,a). \quad (n\in\mathbb{N}),$$

Remark 1.9 ([11])**.** (1) $\lambda_{1,\theta}(f-a)=\lambda_{\theta}(f-a)$. (2) $\overline{\lambda}_{1,\theta}(f-a)=\overline{\lambda}_{\theta}(f-a)$.

In 1991, Bank and Langley considered the higher order linear differential equations and obtained the following result.

Theorem 1.10 ([2])**.** *Let $A_0,\dots,A_{k-2}$ be entire functions of finite order, and assume that* (1.2) *possesses a solution base $f_1,f_2,\dots,f_n$ such that $\lambda(f_i)<+\infty$ for $i=1,2,\dots,n$. Then the product $E=f_1\dots f_n$ is of finite order of growth, $\sigma(E)<\infty$.*

In this paper, we extend Theorem 1.10 by using value distribution theory of a transcendental meromorphic function due to Nevanlinna [8] and the covering surface theory (see e.g. [10]). In fact, we shall prove the following theorem.

Theorem 1.11. *Assume that some (or all) of $A_0,\dots,A_{k-2}$ are transcendental entire functions, and $p=\max\{i(A_j),j=1,\dots,k-2\}<\infty$. Suppose that* (1.2) *possesses a solution base $f_1,f_2,\dots,f_n$. If $E:=f_1\dots f_n$ is of infinite iterated p-order growth, i.e. $\sigma_p(E)=\infty$, then there at least exists a ray $L:\arg z=\theta$ such that $\lambda_{p,\theta}(E)=\infty$.*

From Theorem 1.11, we can deduce the following result.

Corollary 1.12. *Under the conditions of Theorem 1.11, we assume that* (1.2) *possesses a solution base $f_1,f_2,\dots,f_n$ such that $\lambda_p(f_i)<+\infty$ for $i=1,2,\dots,n$. Then the product $E=f_1\dots f_n$ is of finite iterated p-order growth, i.e. $\sigma_p(E)<+\infty$.*

When $p=1$, Corollary 1.12 becomes Theorem 1.10.

2. Auxiliary Lemmas

Our proof requires the Nevanlinna's theory in an angular domain. Let $f(z)$ be a meromorphic function and $X(\alpha,\beta)=\{z|\alpha\le\arg z\le\beta\}$ be an angular domain, where $0<\beta-\alpha\le 2\pi$. Nevanlinna defined the following notation ([8]),

$$A_{\alpha,\beta}(r,f)=\frac{k}{\pi}\int_1^r(\frac{1}{t^k}-\frac{t^k}{r^{2k}})\{\log^+|f(te^{i\alpha})|+\log^+|f(te^{i\beta})|\}\frac{dt}{t};$$

$$B_{\alpha,\beta}(r,f)=\frac{2k}{\pi r^k}\int_\alpha^\beta\log^+|f(re^{i\theta})|\sin k(\theta-\alpha)d\theta;$$

$$C_{\alpha,\beta}(r,f)=2\sum_{b\in\Delta}(\frac{1}{|b_v|^k}-\frac{|b_v|^k}{r^{2k}})\sin k(\beta_v-\alpha),$$

where $k=\frac{\pi}{\beta-\alpha}, 1\le r<\infty$ and the summation $\sum_{b\in\Delta}$ is taken over all poles $b=|b|e^{i\theta}$ of the function $f(z)$ in the sector $\Delta: 1<|z|<r,\ \alpha<\arg z<\beta$, counting multiplicity. The corresponding notation $\overline{C}(r,f)$ then applies to distinct poles. Furthermore, for $r>1$, we define

$$D_{\alpha,\beta}(r,f)=A_{\alpha,\beta}(r,f)+B_{\alpha,\beta}(r,f),\quad S_{\alpha,\beta}(r,f)=C_{\alpha,\beta}(r,f)+D_{\alpha,\beta}(r,f).$$

For the sake of simplicity, we omit the subscript of all the notation and use the notation $A(r,f)$, $B(r,f)$, $C(r,f)$, $D(r,f)$ and $S(r,f)$ instead of $A_{\alpha,\beta}(r,f)$, $B_{\alpha,\beta}(r,f)$, $C_{\alpha,\beta}(r,f)$, $D_{\alpha,\beta}(r,f)$ and $S_{\alpha,\beta}(r,f)$.

Lemma 2.1 ([12]). *Suppose that $f(z)$ is a meromorphic function and $\Omega(\alpha,\beta)$ be an angular domain, where $0<\beta-\alpha\le 2\pi$. Then,*

(i) *for any value $a\in\mathbb{C}$, we have*

$$S(r,\frac{1}{f-a}) = S(r,f)+O(1),$$

holds for any $r>1$.

(ii) *for any $r<R$,*

$$A(r,\frac{f'}{f}) \le k\{(\frac{R}{r})^k\int_1^R \frac{\log T(t,f)}{t^{1+k}}dt + \log\frac{r}{R-r} + \log\frac{R}{r} + 1\},$$
$$B(r,\frac{f'}{f}) \le \frac{4k}{r^k}m(r,\frac{f'}{f}).$$

We also need the Ahlfors' theory in an angular domain. We firstly recall some notation (see e.g. Tsuji [10]).

Let $f(z)$ be a meromorphic function in an angular domain $\Delta(\theta,\alpha_0)=\{z:|\arg z-\theta|\le\alpha_0\}$ and $\Delta(\theta,\alpha)=\{z:|\arg z-\theta|\le\alpha\}$ be an angular domain which was contained in $\Delta(\theta,\alpha_0)$, where $\theta\in[0,2\pi)$ and $\alpha\le\alpha_0$. Let $\Delta_0(r)$, $\Delta(r)$ be the part of $\Delta(\theta,\alpha_0)$, $\Delta(\theta,\alpha)$, which is contained in $|z|\le r$, respectively. We put

$$S_0(r,\Delta(\theta,\alpha)) = \frac{1}{\pi}\iint_{\Delta(r)}(\frac{|f^{'}(z)|}{(1+|f(z)|^2)})^2 rd\theta dr, \quad z=re^{i\theta},$$
$$T_0(r,\Delta(\theta,\alpha)) = \int_0^r \frac{S_0(t,\Delta(\theta,\alpha))}{t}dt,$$

which is called as Ahlfors-Shimizu characteristics. We denote the above characteristic functions of $f(z)$ in the whole complex plane by $S_0(r,f), T_0(r,f)$. From [5, Theorem 1.4], we have

$$|T(r,f)-T_0(r,f)-\log|f(0)||\le\frac{1}{2}\log 2. \tag{2.1}$$

Let $n(r,\theta,\alpha,a)$ be the number of zeros of $f(z)-a$ contained in $\Delta(r)$, counting multiplicities. We can assume that $f(0)\ne a$ and put

$$N(r,\theta,\alpha,a) = \int_0^r \frac{n(t,\theta,\alpha,a)}{t}dt.$$

If not, then the definition has to be modified, in a well known manner. Now, we give the following lemmas.

Lemma 2.2 ([10]). *Let $f(z)$ be meromorphic in the complex plane, then*

$$S_0(r,\Delta(\theta,\alpha)) \le 3\sum_{i=1}^{3} n(2r,\theta,\alpha_0,a_i)+O(\log r),$$
$$T_0(r,\Delta(\theta,\alpha)) \le 3\sum_{i=1}^{3} N(2r,\theta,\alpha_0,a_i)+O(\log^2 r).$$

where a_1,a_2,a_3 be any three distinct points in $\mathbb{C}_\infty$.

3. Proof of main results

Proof of Theorem 1.11. The Wronskian determinant $W(f_1, f_2, \dots, f_n)$ of the fundamental system of solutions $\{f_1, f_2, \dots, f_n\}$ is given by

$$W = W(f_1, f_2, \dots, f_n) = \det \begin{bmatrix} 1 & 1 & \dots & 1 \\ \frac{f_1'}{f_1} & \frac{f_2'}{f_2} & \dots & \frac{f_n'}{f_n} \\ & \dots & \dots & \\ \frac{f_1^{(n-1)}}{f_1} & \frac{f_2^{(n-1)}}{f_2} & \dots & \frac{f_n^{(n-1)}}{f_n} \end{bmatrix}$$

Apply the [7, Proposition 1.4.8 pp.16], we can derive that W is a positive constant denoted by K. Hence

$$\frac{1}{E} = \frac{1}{K}\frac{W}{E} = \frac{1}{K}\sum_{1\le i_l \ne i_t \le n} (-1)^\tau \Pi_{l=1}^{n-1} \frac{f_{i_l}^{(l)}}{f_{i_l}}.$$

Let $f \not\equiv 0$ be a solution of (1.2). It follows from [3, Theorem 4 (i)] that the iterated p-order of $\log T(r, f)$ is at most σ, where $\sigma < \infty$ is a constant.

For any $\theta \in \mathbb{R}$, if $\varepsilon > 0$ is sufficiently small, we deduce from Lemma 2.1 (ii) in which $R = 2r$ that

$$A_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{f_i'}{f_i}) = \begin{cases} O(1) & \text{if } p = 1, \\ O(\int_1^{2r} \frac{\log^+ T(t, f_i)}{t^{1+\frac{\pi}{2\varepsilon}}} dt) & \\ = O(\int_1^{2r} \frac{e^{[p-1]} t^{\sigma+1}}{t^{1+\frac{\pi}{2\varepsilon}}} dt) = O(e^{[p-1]} r^{\sigma+1}). & \text{if } p \ge 2. \end{cases} \tag{3.1}$$

Since

$$m(r, \frac{f_i'}{f_i}) = O(\log rT(r, f_i)) = O(e^{[p-1]} r^{\sigma+1}), \quad r \notin F,$$

where F is a set of finite linear measure, we can deduce from lemma 2.1 (ii) that

$$B_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{f_i'}{f_i}) = \begin{cases} O(1) & \text{if } p = 1, \\ O(e^{[p-1]} r^{\sigma+1}). & \text{if } p \ge 2. \end{cases} \tag{3.2}$$

holds for any $r \notin F$. Since

$$D_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{f_i^{(h)}}{f_i}) \le \sum_{i=1}^{h} D_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{f_i^{(l)}}{f_i^{(l-1)}}) + O(1),$$

where $i = 1, 2, \dots, n$, $h = 2, 3, \dots, n-1$. Therefore we have

$$D_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{f_i'}{f_i}) = \begin{cases} O(1) & \text{if } p = 1, \\ O(e^{[p-1]} r^{\sigma+1}). & \text{if } p \ge 2. \end{cases}$$

By the definition and Lemma 2.1 (i), we can deduce that for any $\theta \in \mathbb{R}$ and any sufficiently small $\varepsilon > 0$,

$$S(r, E) \le C(r, \frac{1}{E}) + O(e^{[p-1]} r^{\sigma+1}), \quad r \notin F \tag{3.3}$$

holds in the angular domain $\{z | \theta - \varepsilon < \arg z < \theta + \varepsilon\}$.

In the following, we shall prove that there exists a ray $L : \arg z = \theta$ such that for any $0 < \varepsilon < \frac{\pi}{2}$, we have

$$\limsup_{r\to\infty} \frac{\log^{[p]} S(r, E)}{\log r} = \infty \tag{3.4}$$

holds in the angular domain $\{z|\theta-\varepsilon < \arg z < \theta+\varepsilon\}$. Otherwise, for any $\theta \in [0, 2\pi)$, we have a $\varepsilon_\theta \in (0, \frac{\pi}{2})$, such that

$$\limsup_{r\to\infty} \frac{\log^{[p]} S(r,E)}{\log r} < \infty. \tag{3.5}$$

holds in the angular domain $\{z|\theta - \varepsilon_\theta < \arg z < \theta + \varepsilon_\theta\}$. We deduce from Lemma 2.1 (i) that for any finite value a, we have $S(r, \frac{1}{E-a}) = S(r,E) + O(1)$. Since $C(r,a) \le S(r, \frac{1}{E-a})$, then

$$C(r, \frac{1}{E-a}) \le S(r, \frac{1}{E-a}) = S(r,E) + O(1). \tag{3.6}$$

On the other hand, it follows from $\theta - \frac{\varepsilon_\theta}{2} < \beta_v < \theta + \frac{\varepsilon_\theta}{2}$ that $\sin k(\beta_v - \theta + \frac{\varepsilon_\theta}{2}) \ge \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, where $k = \frac{\pi}{2\varepsilon_\theta}$. Hence

$$\begin{aligned}
C(2r, \frac{1}{E-a}) &\ge C_{\theta-\frac{\varepsilon_\theta}{2},\theta+\frac{\varepsilon_\theta}{2}}(2r, \frac{1}{E-a})\\
&\ge 2 \sum_{1<|b_v|<r, \theta-\frac{\varepsilon_\theta}{2}<\beta_v<\theta+\frac{\varepsilon_\theta}{2}} (\frac{1}{|b_v|^k} - \frac{|b_v|^k}{(2r)^{2k}}) \sin k(\beta_v - \theta + \frac{\varepsilon_\theta}{2})\\
&\ge \sqrt{2} \sum_{1<|b_v|<r, \theta-\frac{\varepsilon_\theta}{2}<\beta_v<\theta+\frac{\varepsilon_\theta}{2}} (\frac{1}{|b_v|^k} - \frac{|b_v|^k}{(2r)^{2k}})\\
&\ge \sqrt{2}[\int_1^r \frac{1}{t^k} dn(t) + \frac{1}{(2r)^{2k}} \int_1^r t^k dn(t)]\\
&\ge \sqrt{2}[k\int_1^r \frac{1}{t^{k+1}} n(t)dt + \frac{n(r)}{r^k} - \frac{r^k n(r)}{r^{2k}} + \frac{k}{(2r)^{2k}} \int_1^r t^{k-1} n(t)dt]\\
&\ge \sqrt{2}[\frac{n(r)}{r^k} - \frac{r^k n(r)}{(2r)^{2k}}]\\
&\ge \sqrt{2}(1 - \frac{1}{2^{2k}})\frac{n(r)}{r^k},
\end{aligned}$$

where $n(t) = n(t, \theta, \frac{\varepsilon_\theta}{2}, a)$. From (3.5), (3.6) and the above equation,

$$\limsup_{r\to\infty} \frac{\log^{[p]} n(r, \theta, \frac{\varepsilon_\theta}{2}, a)}{\log r} < \infty. \tag{3.7}$$

Because $[0, 2\pi]$ is compact and $[0, 2\pi] \subset \cup\{(\theta - \frac{\varepsilon_\theta}{4}, \theta - \frac{\varepsilon_\theta}{4}), \theta \in [0, 2\pi)\}$, then we can choose finitely many $(\theta_i - \frac{\varepsilon_{\theta_i}}{4}, \theta_i - \frac{\varepsilon_{\theta_i}}{4})(i = 1, 2, \dots, T)$, such that $[0, 2\pi] \subset \cup\{(\theta_i - \frac{\varepsilon_{\theta_i}}{4}, \theta_i - \frac{\varepsilon_{\theta_i}}{4}), i = 1, 2, \dots, T\}$.

By using Lemma 2.2, for any three distinct complex numbers a_j, $j = 1, 2, 3$, we have

$$\begin{aligned}
S_0(r, f) &\le \sum_{i=1}^T S_0(r, \Delta(\theta_i, \frac{\varepsilon_{\theta_i}}{4}))\\
&\le \sum_{i=1}^T \{3\sum_{i=j}^3 n(2r, \theta_i, \frac{\varepsilon_{\theta_i}}{2}, a_j)\} + O(\log r)
\end{aligned}$$

From (2.1), (3.7) and the definition of $T_0(r,f)$ and the above equation, we can get that E is of finite p-iterated order. This contradicts with the hypothesis and so (3.4) follows.

From (3.3), (3.4) and definition 1.1, we know that there exists a ray $L : \arg z = \theta$ such that for any $0 < \varepsilon < \frac{\pi}{2}$, we have

$$\limsup_{r\to\infty} \frac{\log^{[p]} C(r, \frac{1}{E})}{\log r} = \infty \tag{3.8}$$

holds in the angular domain $\{z|\theta-\varepsilon < \arg z < \theta+\varepsilon\}$. Since $C(r, \frac{1}{E}) \le 2n(r,\theta,\varepsilon,E = 0)$, then $\lambda_{p,\theta-\varepsilon,\theta+\varepsilon}(E) = \infty$. Since ε is arbitrary, we have $\lambda_{p,\theta}(E) = \infty$. Therefore, we can deduce that Theorem 1.11. □

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