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UNIFORM REGULARITY OF FULLY COMPRESSIBLE HALL-MHD SYSTEMS

JISHAN FAN, YONG ZHOU

Dedicated to Prof. Jiaxing Hong on his 80th birthday

Abstract. In this article we study a fully compressible Hall-MHD system. These equations include shear viscosity, bulk viscosity of the flow, and heat conductivity and resistivity coefficients. We prove uniform regularity estimates.

1. Introduction

In this article we consider the fully compressible Hall-MHD system [17],

$$\partial_t\rho + \operatorname{div}(\rho u) = 0, \tag{1.1}$$

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p - \mu\Delta u - (\lambda+\mu)\nabla \operatorname{div} u = \operatorname{rot} b \times b + \rho\frac{\partial w}{\partial t}, \tag{1.2}$$

$$\partial_t(\rho e) + \operatorname{div}(\rho u e) - \operatorname{div}(\kappa\nabla\theta) = \frac{\mu}{2}|\nabla u + \nabla u^T|^2 + \lambda(\operatorname{div} u)^2 + \eta|\operatorname{rot} b|^2, \tag{1.3}$$

$$\partial_t b + \operatorname{rot}(b\times u) + \xi \operatorname{rot}\Big(\frac{\operatorname{rot} b \times b}{\rho}\Big) = \eta\Delta b, \tag{1.4}$$

$$\operatorname{div} b = 0 \quad \text{in } \mathbb{T}^3 \times (0,\infty), \tag{1.5}$$

$$(\rho,u,\theta,b)(\cdot,0) = (\rho_0,u_0,\theta_0,b_0) \quad \text{in } \mathbb{T}^3. \tag{1.6}$$

Here ρ, u, p, e, θ and b denote the density, velocity, pressure, internal energy, temperature, and magnetic field, respectively. The physical constants μ and λ are the shear viscosity and bulk viscosity of the flow and satisfy $\mu > 0$ and $\lambda + \frac{2}{3}\mu \geq 0$. $\kappa > 0$ is the heat conductivity. $\eta > 0$ is the resistivity coefficient. w is a given function. ξ is a Hall-constant. ∇u^T is the transpose of the ∇u. For simplicity, we consider the case that the fluid is a polytropic ideal gas; that is

$$e := C_V\theta, \quad p := R\rho\theta$$

with $C_V > 0$ and R being the specific heat at constant volume and the generic gas constant, respectively.

Applications of the Hall-MHD system cover a very wide range of physical objects, for example, magnetic reconnection in space plasmas, star formation, neutron stars, and geo-dynamo. For well-posedness, regularity and decay properties, and related

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incompressible models, we refer to works [5, 6, 7, 10, 11, 20, 21, 22, 23] and references therein.

When the Hall effect term rot $\left(\frac{\operatorname{rot} b\times b}{\rho}\right)$ is neglected, the system (1.1)-(1.5) reduces to the well-known fully compressible MHD system, which has been studied in [2, 4, 8, 9, 12, 13, 14]. The existence of local strong solution was proved by Fan-Yu [9]. Fan-Yu [8], Ducomet-Feireisl [4] and Hu-Wang [12, 13] established the global weak solutions. The low Mach number limit problem was studied by Jiang-Ju-Li-Xin [14] in $\mathbb{R}^3$ and Cui-Ou-Ren [2] in a bounded domain.

Before stating our main results, we recall the existence of local smooth solutions to (1.1)-(1.6). Since the system (1.1)-(1.6) is parabolic-hyperbolic, we have the following result.

Proposition 1.1 ([19]). *Let $\rho_0, u_0, \theta_0, b_0 \in H^3$ and $1/C_0 \le \rho_0,\ \theta_0$ for a positive constant C_0. Then (1.1)-(1.6) has a unique smooth solution (ρ, u, θ, b) satisfying $\rho \in C^\ell([0,T); H^{3-\ell})$, $u, \theta, b \in C^\ell([0,T); H^{3-2\ell})$, $\ell = 0, 1$, and $1/C \le \rho,\ \theta$ for some $0 < T \le \infty$.*

The aim of this article is to prove uniform regularity estimates in $(\lambda, \mu, \kappa, \eta)$, as stated in the following theorem.

Theorem 1.2. *Let $\xi^2 \le C\eta$ and $w \in C([0,1]; H^4)$, $0 < \mu < 1$, $0 < \lambda + \mu < 1$, $0 < \eta < 1$, $0 < \kappa < 1$, $0 < \frac{1}{C_0} \le \rho_0,\ \theta_0 \le C_0$, $\rho_0, u_0, b_0, \theta_0 \in H^3(\mathbb{T}^3)$ with $\operatorname{div} b_0 = 0$ in $\mathbb{T}^3$. Let (ρ, u, b, θ) be the unique local smooth solutions to (1.1)-(1.5). Then*

$$\|(\rho, u, b, \theta)(\cdot, t)\|_{H^3} \le C \quad \text{in } [0, T] \tag{1.7}$$

holds for some positive constants C and T_0 $(\le T)$ independent of λ, μ, η and k.

Remark 1.3. By the uniform estimates, one can easily take the limits of λ, μ, η and k to zero, hence we omit the details here.

Our estimates are uniform in a with $a := (\lambda, \mu, \eta, k)$ while the ones in existence results of Hall-MHD depend on a.

We define

$$\begin{aligned} M(t) :=& 1 + \|w\|_{C([0,1];H^4)} + \sup_{0\le\tau\le t}\Big\{\|(\rho, u, b, \theta)(\cdot, \tau)\|_{H^3} + \|\partial_t v(\cdot, \tau)\|_{L^2} \\ &+ \|\partial_t\theta(\cdot, \tau)\|_{L^2} + \|\frac{1}{\rho}(\cdot, \tau)\|_{L^\infty} + \|\frac{1}{\theta}(\cdot, \tau)\|_{L^\infty}\Big\}. \end{aligned} \tag{1.8}$$

Here we note that $v := u - w$.

Theorem 1.4. *For any $t \in [0, 1]$, we have*

$$M(t) \le C_0(M_0)\exp(tC(M)) \tag{1.9}$$

for some nondecreasing continuous functions $C_0(\cdot)$ and $C(\cdot)$.

From (1.9) it follows that [1, 3, 16]

$$M(t) \le C. \tag{1.10}$$

In the following proofs, we will use the bilinear commutator and product estimates due to Kato-Ponce [15],

$$\|D^s(fg) - fD^s g\|_{L^p} \le C(\|\nabla f\|_{L^{p_1}}\|D^{s-1}g\|_{L^{q_1}} + \|g\|_{L^{p_2}}\|D^s f\|_{L^{q_2}}), \tag{1.11}$$

$$\|D^s(fg)\|_{L^p} \le C(\|f\|_{L^{p_1}}\|D^s g\|_{L^{q_1}} + \|D^s f\|_{L^{p_2}}\|g\|_{L^{q_2}}), \tag{1.12}$$

with $s>0$ and $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{q_1}=\frac{1}{p_2}+\frac{1}{q_2}$.

We only need to show Theorem 1.4, which is given in the next section. Our proof consists of two steps. In step 1, we give the lower order estimates and in step 2, we show the higher order estimates.

2. Proof of Theorem 1.4

Step 1. Lower order estimates. First, testing (1.1) by ρ^{q-1}, we see that

$$\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho^q \mathrm{d}x = -\big(1-\frac{1}{q}\big)\int \rho^q \operatorname{div} u \mathrm{d}x \le \|\operatorname{div} u\|_{L^\infty}\int \rho^q \mathrm{d}x,$$

and thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho\|_{L^q} \le \|\operatorname{div} u\|_{L^\infty}\|\rho\|_{L^q},$$

which gives

$$\|\rho\|_{L^q} \le \|\rho_0\|_{L^q}\exp\Big(\int_0^t \|\operatorname{div} u\|_{L^\infty}\mathrm{d}\tau\Big). \tag{2.1}$$

In the limit as $q\to+\infty$, we obtain

$$\|\rho\|_{L^\infty} \le \|\rho_0\|_{L^\infty}\exp(tC(M)). \tag{2.2}$$

It follows from (1.1) that

$$\partial_t\frac{1}{\rho} + u\cdot\nabla\frac{1}{\rho} - \frac{1}{\rho}\operatorname{div} u = 0. \tag{2.3}$$

Testing (2.3) by $\big(\frac{1}{\rho}\big)^{q-1}$, we find that

$$\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\int \big(\frac{1}{\rho}\big)^q \mathrm{d}x = \big(1+\frac{1}{q}\big)\int \big(\frac{1}{\rho}\big)^q \operatorname{div} u \mathrm{d}x \le \big(1+\frac{1}{q}\big)\|\frac{1}{\rho}\|_{L^q}^q\|\operatorname{div} u\|_{L^\infty},$$

and therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\frac{1}{\rho}\|_{L^q} \le \big(1+\frac{1}{q}\big)\|\frac{1}{\rho}\|_{L^q}\|\operatorname{div} u\|_{L^\infty},$$

which gives

$$\|\frac{1}{\rho}\|_{L^q} \le \|\frac{1}{\rho_0}\|_{L^q}\exp\Big(\big(1+\frac{1}{q}\big)\int_0^t \|\operatorname{div} u\|_{L^\infty}\mathrm{d}\tau\Big)$$

and we have

$$\|\frac{1}{\rho}\|_{L^\infty} \le \|\frac{1}{\rho_0}\|_{L^\infty}\exp(tC(M)) \tag{2.4}$$

by letting $q\to+\infty$.

Testing (1.3) by θ^{q-1} and using (1.1) and denoting Q to be the right-hand side of (1.3), we obtain

$$\begin{aligned}
&\frac{C_V}{q}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho\theta^q \mathrm{d}x + \kappa\int \nabla\theta\cdot\nabla\theta^{q-1}\mathrm{d}x\\
&= \int Q\theta^{q-1}\mathrm{d}x - \int p\theta^{q-1}\operatorname{div} u\mathrm{d}x\\
&\le C(M)\|Q\|_{L^q}\|\rho^{1/q}\theta\|_{L^q}^{q-1} + C\|\operatorname{div} u\|_{L^\infty}\|\rho^{1/q}\theta\|_{L^q}^q,
\end{aligned}$$

and therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho^{1/q}\theta\|_{L^q} \le C(M)\|Q\|_{L^q} + C\|\operatorname{div} u\|_{L^\infty}\|\rho^{1/q}\theta\|_{L^q},$$

which, similarly to (2.2), implies

$$\|\theta\|_{L^\infty} \le C_0(M_0)\exp(tC(M)). \tag{2.5}$$

Multiplying (1.3) by $\frac{1}{\theta^2}\cdot\frac{1}{C_V}$, we deduce that

$$\rho\partial_t\frac{1}{\theta} + \rho u\cdot\nabla\frac{1}{\theta} + \frac{\kappa}{C_V}\cdot\frac{1}{\theta^2}\Delta\theta = \frac{R}{C_V}\rho\frac{\operatorname{div} u}{\theta} - \frac{Q}{\theta^2 C_V} \le \frac{R}{C_V}\rho\frac{1}{\theta}\operatorname{div} u. \tag{2.6}$$

Similarly to (2.5), testing (2.6) by $\left(\frac{1}{\theta}\right)^{q-1}$, we have

$$\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\int\rho\big(\frac{1}{\theta}\big)^q\mathrm{d}x \le \frac{R}{C_V}\int\rho\big(\frac{1}{\theta}\big)^q\operatorname{div} u\mathrm{d}x \le \frac{R}{C_V}\|\operatorname{div} u\|_{L^\infty}\int\rho\big(\frac{1}{\theta}\big)^q\mathrm{d}x,$$

and thus

$$\|\frac{1}{\theta}\|_{L^\infty} \le C_0(M_0)\exp(tC(M)). \tag{2.7}$$

It is easy to verify that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int|v|^2\mathrm{d}x = 2\int v\partial_t v\mathrm{d}x \le 2\|v\|_{L^2}\|\partial_t v\|_{L^2} \le C(M),$$

which implies

$$\|v\|_{L^2} \le C_0(M_0)\exp(tC(M)). \tag{2.8}$$

Testing (1.3) by b, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|b|^2\mathrm{d}x + \eta\int|\nabla b|^2\mathrm{d}x &= -\int(u\cdot\nabla b - b\cdot\nabla u + b\operatorname{div} u)b\mathrm{d}x\\ &= -\int\Big(\frac{1}{2}|b|^2\operatorname{div} u - b\cdot\nabla u\cdot b\Big)\mathrm{d}x\\ &\le C\|\nabla u\|_{L^\infty}\|b\|_{L^2}^2 \le C(M),\end{aligned}$$

which leads to

$$\|b\|_{L^2}^2 + \eta\int_0^t\int|\nabla b|^2\mathrm{d}x\mathrm{d}\tau \le C_0(M_0)\exp(tC(M)). \tag{2.9}$$

Step 2. Higher order estimates. The equation (1.1)-(1.3) can be rewritten in the symmetric form

$$\frac{\theta}{\rho}\partial_t\rho + \frac{\theta}{\rho}u\cdot\nabla\rho + \theta\operatorname{div} u = 0, \tag{2.10}$$

$$\rho\partial_t v + \rho u\cdot\nabla v + \rho\nabla\theta + \theta\nabla\rho - \mu\Delta v - (\lambda+\mu)\nabla\operatorname{div} v = r + \operatorname{rot} b\times b, \tag{2.11}$$

$$\frac{\rho}{\theta}\partial_t\theta + \frac{\rho}{\theta}u\cdot\nabla\theta - \frac{\kappa}{\theta}\Delta\theta + \rho\operatorname{div} u = \frac{1}{\theta}Q, \tag{2.12}$$

where we have taken $R = C_V = 1$ for simplicity, and

$$r := \mu\Delta w + (\lambda+\mu)\nabla\operatorname{div} w - \rho u\cdot\nabla w. \tag{2.13}$$

Taking D^3 on (2.10), testing by $D^3\rho$, using (1.1), (1.11) and (1.12), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{\theta}{\rho}(D^3\rho)^2\mathrm{d}x + \int \theta D^3 \operatorname{div} u D^3\rho\mathrm{d}x \\
&= -\int \Big[D^3\Big(\frac{\theta}{\rho}\partial_t\rho\Big) - \frac{\theta}{\rho}D^3\partial_t\rho\Big]D^3\rho\mathrm{d}x \\
&\quad - \int \Big[D^3\Big(\frac{\theta}{\rho}u\cdot\nabla\rho\Big) - \frac{\theta}{\rho}u\cdot\nabla D^3\rho\Big]D^3\rho\mathrm{d}x \\
&\quad - \int (D^3(\theta \operatorname{div} u) - \theta D^3 \operatorname{div} u)D^3\rho\mathrm{d}x \\
&\quad + \frac{1}{2}\int \partial_t\Big(\frac{\theta}{\rho}\Big)(D^3\rho)^2\mathrm{d}x - \int \Big(\frac{\theta}{\rho}u\cdot\nabla D^3\rho\Big)D^3\rho\mathrm{d}x \\
&\le C\Big(\|\nabla\frac{\theta}{\rho}\|_{L^\infty}\|D^2\partial_t\rho\|_{L^2} + \|\partial_t\rho\|_{L^\infty}\|D^3\frac{\theta}{\rho}\|_{L^2}\Big)\|D^3\rho\|_{L^2} \\
&\quad + C\Big(\|\nabla\frac{\theta u}{\rho}\|_{L^\infty}\|D^3\rho\|_{L^2} + \|\nabla\rho\|_{L^\infty}\|D^3\frac{\theta u}{\rho}\|_{L^2}\Big)\|D^3\rho\|_{L^2} \\
&\quad + C(\|\nabla\theta\|_{L^\infty}\|D^3u\|_{L^2} + \|\nabla u\|_{L^\infty}\|D^3\theta\|_{L^2})\|D^3\rho\|_{L^2} \\
&\quad + C\|\partial_t\frac{\theta}{\rho}\|_{L^\infty}\|D^3\rho\|_{L^2}^2 + C\|\nabla\frac{\theta u}{\rho}\|_{L^\infty}\|D^3\rho\|_{L^2}^2 \\
&\le C(M)(\|D^2\partial_t\rho\|_{L^2} + \|\partial_t\rho\|_{L^\infty}) + C(M) + C(M)\|\partial_t\frac{\theta}{\rho}\|_{L^\infty} \\
&\le C(M) + \frac{\kappa}{4}\int \frac{1}{\theta}(D^4\theta)^2\mathrm{d}x.
\end{aligned}
\tag{2.14}
$$

Here we have used the estimate [18]:

$$
\|D^3\frac{1}{\rho}\|_{L^2} \le C(M)\|D^3\rho\|_{L^2} \le C(M).
$$

Applying D^2 to (2.11), testing by $D^2\partial_t v$, using (1.11) and (1.12), we obtain

$$
\begin{aligned}
&\frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |D^3v|^2\mathrm{d}x + \frac{\lambda+\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int (D^2 \operatorname{div} v)^2\mathrm{d}x + \int \rho|D^2\partial_t v|^2\mathrm{d}x \\
&= -\int D^2\nabla p\cdot D^2\partial_t v\mathrm{d}x - \int D^2(\rho u\cdot\nabla v)\cdot D^2\partial_t v\mathrm{d}x \\
&\quad - \int [D^2(\rho\partial_t v) - \rho D^2\partial_t v]D^2\partial_t v\mathrm{d}x + \int D^2 r\cdot D^2\partial_t v\mathrm{d}x \\
&\quad + \int D^2\Big(b\cdot\nabla b - \frac{1}{2}\nabla|b|^2\Big)D^2\partial_t v\mathrm{d}x \\
&\le C\|D^3p\|_{L^2}\|D^2\partial_t v\|_{L^2} + C\|\rho\|_{H^2}\|u\|_{H^2}\|v\|_{H^3}\|D^2\partial_t v\|_{L^2} \\
&\quad + C(\|\nabla\rho\|_{L^\infty}\|D\partial_t v\|_{L^2} + \|\partial_t v\|_{L^\infty}\|D^2\rho\|_{L^2})\|D^2\partial_t v\|_{L^2} \\
&\quad + \|D^2 r\|_{L^2}\|D^2\partial_t v\|_{L^2} + \|D^2\Big(b\cdot\nabla b - \frac{1}{2}\nabla|b|^2\Big)\|_{L^2}\|D^2\partial_t v\|_{L^2} \\
&\le C(M)\|D^2\partial_t v\|_{L^2} + C(M)(\|D\partial_t v\|_{L^2} + \|\partial_t v\|_{L^\infty})\|D^2\partial_t v\|_{L^2} \\
&\le C(M)\|D^2\partial_t v\|_{L^2} + C(M)(\|\partial_t v\|_{L^2}^{1/2}\|D^2\partial_t v\|_{L^2}^{1/2} \\
&\quad + \|\partial_t v\|_{L^2} + \|\partial_t v\|_{L^2}^{1/4}\|D^2\partial_t v\|_{L^2}^{3/4})\|D^2\partial_t v\|_{L^2}
\end{aligned}
$$

$$\begin{aligned}
&\le C(M)\|D^2\partial_t v\|_{L^2} + C(M)(\|D^2\partial_t v\|_{L^2}^{1/2} + \|D^2\partial_t v\|_{L^2}^{3/4})\|D^2\partial_t v\|_{L^2} \\
&\le \frac{1}{2}\int \rho|D^2\partial_t v|^2 \mathrm{d}x + C(M),
\end{aligned}$$

which gives

$$\int_0^t \int |D^2\partial_t v|^2 \mathrm{d}x\mathrm{d}\tau \le C_0(M_0)\exp(tC(M)). \tag{2.15}$$

Applying D^3 on (2.11), testing by $D^3 v$, using (1.1), (1.11) and (1.12), we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|D^3 v|^2\mathrm{d}x + \mu\int |D^4 v|^2\mathrm{d}x + (\lambda+\mu)\int (D^3 \operatorname{div} v)^2\mathrm{d}x \\
&+ \int \rho D^3\nabla\theta\cdot D^3 v\mathrm{d}x + \int \theta\nabla D^3\rho\cdot D^3 v\mathrm{d}x + \int (b\times D^3\operatorname{rot} b)D^3 v\mathrm{d}x \\
&= -\int (D^3(\rho\partial_t v) - \rho D^3\partial_t v)D^3 v\mathrm{d}x - \int (D^3(\rho u\cdot\nabla v) - \rho u\cdot\nabla D^3 v)D^3 v\mathrm{d}x \\
&\quad - \int (D^3(\rho\nabla\theta) - \rho\nabla D^3\theta)D^3 v\mathrm{d}x - \int (D^3(\theta\nabla\rho) - \theta\nabla D^3\rho)D^3 v\mathrm{d}x \\
&\quad + \int D^3 r D^3 v\mathrm{d}x - \int (D^3(b\times\operatorname{rot} b) - b\times D^3\operatorname{rot} b)D^3 v\mathrm{d}x \\
&\le C(\|\nabla\rho\|_{L^\infty}\|D^2\partial_t v\|_{L^2} + \|\partial_t v\|_{L^\infty}\|D^3\rho\|_{L^2})\|D^3 v\|_{L^2} \\
&\quad + C(\|\nabla v\|_{L^\infty}\|D^3(\rho u)\|_{L^2} + \|\nabla(\rho u)\|_{L^\infty}\|D^3 v\|_{L^2})\|D^3 v\|_{L^2} \\
&\quad + C(\|\nabla\rho\|_{L^\infty}\|D^3\theta\|_{L^2} + \|\nabla\theta\|_{L^\infty}\|D^3\rho\|_{L^2})\|D^3 v\|_{L^2} \\
&\quad + C(\|\nabla\theta\|_{L^\infty}\|D^3\rho\|_{L^2} + \|\nabla\rho\|_{L^\infty}\|D^3\theta\|_{L^2})\|D^3 v\|_{L^2} + C(M) \\
&\quad + \frac{\mu}{16}\|D^4 v\|_{L^2}^2 + C\|\nabla b\|_{L^\infty}\|D^3 b\|_{L^2}\|D^3 v\|_{L^2} \\
&\le C(M) + C(M)(\|D^2\partial_t v\|_{L^2} + \|\partial_t v\|_{L^\infty}) + \frac{\mu}{16}\|D^4 v\|_{L^2}^2 \\
&\le C(M) + \|D^2\partial_t v\|_{L^2}^2 + \frac{\mu}{16}\|D^4 v\|_{L^2}^2.
\end{aligned} \tag{2.16}$$

Applying D^3 on (1.4), testing by $D^3 b$, using (1.11) and (1.12), we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |D^3 b|^2\mathrm{d}x + \eta\int |D^4 b|^2\mathrm{d}x + \int (b\times D^3 u)D^3\operatorname{rot} b\mathrm{d}x \\
&= -\int (D^3(b\times u) - D^3 b\times u - b\times D^3 u)D^3\operatorname{rot} b\mathrm{d}x \\
&\quad - \int (D^3 b\times u)D^3\operatorname{rot} b\mathrm{d}x + \xi\int D^3\Big(\frac{b}{\rho}\times\operatorname{rot} b\Big)D^3\operatorname{rot} b\mathrm{d}x \\
&= -\int \operatorname{rot}(D^3(b\times u) - D^3 b\times u - b\times D^3 u)D^3 b\mathrm{d}x + \int (D^3 b\times D^3\operatorname{rot} b)u\mathrm{d}x \\
&= -\int \operatorname{rot}(D^3(b\times u) - D^3 b\times u - b\times D^3 u)D^3 b\mathrm{d}x \\
&\quad + \int \big[\frac{1}{2}\nabla|D^3 b|^2 - (D^3 b\cdot\nabla)D^3 b\big]u\mathrm{d}x \\
&\quad + \xi\int \Big(D^3\Big(\frac{b}{\rho}\times\operatorname{rot} b\Big) - \frac{b}{\rho}\times D^3\operatorname{rot} b\Big)D^3\operatorname{rot} b\mathrm{d}x \\
&= -\int \operatorname{rot}(D^3(b\times u) - D^3 b\times u - b\times D^3 u)D^3 b\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{2}\int |D^3 b|^2 \operatorname{div} u \mathrm{d}x + \int D^3 b \otimes D^3 b : \nabla u \mathrm{d}x \\
&+ \xi \int \Big(D^3 \Big(\frac{b}{\rho} \times \operatorname{rot} b\Big) - \frac{b}{\rho} \times D^3 \operatorname{rot} b\Big) D^3 \operatorname{rot} b \mathrm{d}x \\
\le{}& \|\operatorname{rot}(D^3(b\times u) - D^3 b \times u - b \times D^3 u)\|_{L^2} \|D^3 b\|_{L^2} \\
&+ \frac{1}{2}\|D^3 b\|_{L^2}^2 \|\operatorname{div} u\|_{L^\infty} + \|D^3 b\|_{L^2}^2 \|\nabla u\|_{L^\infty} \\
&+ C\sqrt{\eta}\Big(\|\frac{b}{\rho}\|_{L^\infty}\|D^3 b\|_{L^2} + \|\nabla b\|_{L^\infty}\|D^3\big(\frac{b}{\rho}\big)\|_{L^2}\Big)\|D^4 b\|_{L^2} \\
\le{}& C(M) + \frac{\eta}{16}\|D^4 b\|_{L^2}^2.
\end{aligned}
$$

Here we have used that $a \cdot \nabla a + a \times \operatorname{rot} a = \frac{1}{2}\nabla |a|^2$.

Applying D^2 on (1.3), testing by $D^2\partial_t\theta$, using (1.11) and (1.12), we have

$$
\begin{aligned}
&\frac{\kappa}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int (D^3\theta)^2 \mathrm{d}x + \int \rho |D^2\partial_t\theta|_{L^2}^2 \mathrm{d}x \\
&= -\int D^2(p \operatorname{div} u) D^2\partial_t\theta \mathrm{d}x - \int D^2(\rho u \cdot \nabla\theta) D^2\partial_t\theta \mathrm{d}x \\
&\quad - \int [D^2(\rho\partial_t\theta) - \rho D^2\partial_t\theta] D^2\partial_t\theta \mathrm{d}x + \int D^2 Q \cdot D^2\partial_t\theta \mathrm{d}x \quad \text{with } C_V = 1) \\
&\le \|D^2(p\operatorname{div} u)\|_{L^2}\|D^2\partial_t\theta\|_{L^2} + \|D^2(\rho u \cdot \nabla\theta)\|_{L^2}\|D^2\partial_t\theta\|_{L^2} \\
&\quad + C(\|\nabla\rho\|_{L^\infty}\|D\partial_t\theta\|_{L^2} + \|\partial_t\theta\|_{L^\infty}\|D^2\rho\|_{L^2})\|D^2\partial_t\theta\|_{L^2} + \|D^2 Q\|_{L^2}\|D^2\partial_t\theta\|_{L^2} \\
&\le C(M)\|D^2\partial_t\theta\|_{L^2} + C(M)(\|D\partial_t\theta\|_{L^2} + \|\partial_t\theta\|_{L^\infty})\|D^2\partial_t\theta\|_{L^2} \\
&\le C(M)\|D^2\partial_t\theta\|_{L^2} + C(M)(\|\partial_t\theta\|_{L^2}^{1/2}\|D^2\partial_t\theta\|_{L^2}^{1/2} + \|\partial_t\theta\|_{L^2} \\
&\quad + \|\partial_t\theta\|_{L^2}^{1/4}\|D^2\partial_t\theta\|_{L^2}^{3/4})\|D^2\partial_t\theta\|_{L^2} \\
&\le \frac{1}{2}\int \rho|D^2\partial_t\theta|^2\mathrm{d}x + C(M),
\end{aligned}
$$

which leads to

$$
\int_0^t \int |D^2\partial_t\theta|^2 \mathrm{d}x\mathrm{d}\tau \le C_0(M_0)\exp(tC(M)). \tag{2.17}
$$

Taking D^3 on (2.12), testing by $D^3\theta$, using (1.11) and (1.12), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{\rho}{\theta}(D^3\theta)^2\mathrm{d}x + \kappa\int \frac{1}{\theta}(D^4\theta)^2\mathrm{d}x + \int \rho D^3 \operatorname{div} u D^3\theta \mathrm{d}x \\
&= \kappa\int \frac{\nabla\theta}{\theta^2} D^3\theta \nabla D^3\theta \mathrm{d}x + \kappa\int \big[D^3\big(\frac{1}{\theta}\Delta\theta\big) - \frac{1}{\theta}\Delta D^3\theta\big] D^3\theta \mathrm{d}x \\
&\quad - \int \big[D^3\big(\frac{\rho}{\theta}\partial_t\theta\big) - \frac{\rho}{\theta}D^3\partial_t\theta\big] D^3\theta \mathrm{d}x + \frac{1}{2}\int \partial_t\big(\frac{\rho}{\theta}\big)(D^3\theta)^2\mathrm{d}x \\
&\quad - \int \Big[D^3\Big(\frac{\rho u}{\theta}\nabla\theta\Big) - \frac{\rho u}{\theta}\nabla D^3\theta\Big] D^3\theta\mathrm{d}x - \int \frac{\rho u}{\theta}\nabla D^3\theta \cdot D^3\theta \mathrm{d}x \\
&\quad - \int (D^3(\rho \operatorname{div} u) - \rho D^3 \operatorname{div} u) D^3\theta \mathrm{d}x + \int D^3\big(\frac{Q}{\theta}\big) D^3\theta \mathrm{d}x \\
&\le \frac{\kappa}{8}\int \frac{1}{\theta}(D^4\theta)^2\mathrm{d}x + C(M)\|\nabla\theta\|_{L^\infty}^2\|D^3\theta\|_{L^2}^2
\end{aligned}
$$

$$
\begin{aligned}
&+\kappa C\left(\|\frac{\nabla\theta}{\theta^2}\|_{L^\infty}\|D^4\theta\|_{L^2}+\|\Delta\theta\|_{L^\infty}\|D^3\frac{1}{\theta}\|_{L^2}\right)\|D^3\theta\|_{L^2}\\
&+C\Big(\|\nabla\frac{\rho}{\theta}\|_{L^\infty}\|D^2\partial_t\theta\|_{L^2}+\|\partial_t\theta\|_{L^\infty}\|D^3\frac{\rho}{\theta}\|_{L^2}\Big)\|D^3\theta\|_{L^2}\\
&+C\|\partial_t\frac{\rho}{\theta}\|_{L^\infty}\|D^3\theta\|_{L^2}^2+C\Big(\|\nabla\frac{\rho u}{\theta}\|_{L^\infty}\|D^3\theta\|_{L^2}+\|\nabla\theta\|_{L^\infty}\|D^3\frac{\rho u}{\theta}\|_{L^2}\Big)\|D^3\theta\|_{L^2}\\
&+C\|\nabla\frac{\rho u}{\theta}\|_{L^\infty}\|D^3\theta\|_{L^2}^2+C(\|\nabla\rho\|_{L^\infty}\|D^3u\|_{L^2}+\|\operatorname{div}u\|_{L^\infty}\|D^3\rho\|_{L^2})\|D^3\theta\|_{L^2}\\
&+C\Big(\|\frac{1}{\theta}\|_{L^\infty}\|D^3Q\|_{L^2}+\|Q\|_{L^\infty}\|D^3\frac{1}{\theta}\|_{L^2}\Big)\|D^3\theta\|_{L^2}\\
&\le\frac{\kappa}{4}\int\frac{1}{\theta}(D^4\theta)^2\mathrm{d}x+C(M)+C(M)(\|D^2\partial_t\theta\|_{L^2}+\|\partial_t\theta\|_{L^\infty})\\
&+\frac{\mu}{2}\|D^4u\|_{L^2}^2+\frac{\lambda+\mu}{2}\|D^3\operatorname{div}u\|_{L^2}^2+\frac{\eta}{16}\|D^4b\|_{L^2}^2\\
&\le\frac{\kappa}{4}\int\frac{1}{\theta}(D^4\theta)^2\mathrm{d}x+\|D^2\partial_t\theta\|_{L^2}^2+C(M)\\
&+\frac{\mu}{2}\|D^4v\|_{L^2}^2+\frac{\lambda+\mu}{2}\|D^3\operatorname{div}v\|_{L^2}^2+\frac{\eta}{16}\|D^4b\|_{L^2}^2.
\end{aligned}
$$

Summing (2.14), (2.16) and the above inequality, we arrive at

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\Big(\frac{\theta}{\rho}(D^3\rho)^2+\rho|D^3v|^2+|D^3b|^2+\frac{\rho}{\theta}(D^3\theta)^2\Big)\mathrm{d}x+\frac{\mu}{2}\int|D^4v|^2\mathrm{d}x\\
&+\frac{\lambda+\mu}{2}\int(D^3\operatorname{div}v)^2\mathrm{d}x+\frac{\eta}{2}\int|D^4b|^2\mathrm{d}x+\frac{\kappa}{2}\int\frac{1}{\theta}(D^4\theta)^2\mathrm{d}x\\
&\le-\int\theta D^3\operatorname{div}uD^3\rho\mathrm{d}x-\int\rho D^3\nabla\theta\cdot D^3u\mathrm{d}x-\int\theta D^3\nabla\rho\cdot D^3u\mathrm{d}x\\
&-\int\rho D^3\operatorname{div}uD^3\theta\mathrm{d}x+\|D^2\partial_tu\|_{L^2}^2+\|D^2\partial_t\theta\|_{L^2}^2+C(M)\\
&+\int(b\times D^3\operatorname{rot}b)D^3w\mathrm{d}x+\int\rho D^3\nabla\theta\cdot D^3w\mathrm{d}x+\int\theta D^3\nabla\rho\cdot D^3w\mathrm{d}x\\
&\le\int D^3uD^3\rho\nabla\theta\mathrm{d}x+\int D^3uD^3\theta\nabla\rho\mathrm{d}x+\|D^2\partial_tu\|_{L^2}^2+\|D^2\partial_t\theta\|_{L^2}^2+C(M)\\
&+\Big|\int\operatorname{rot}(b\times D^3w)D^3b\mathrm{d}x\Big|+\Big|\int D^3\theta\operatorname{div}(\rho D^3w)\mathrm{d}x\Big|+\Big|\int D^3\rho\operatorname{div}(\theta D^3w)\mathrm{d}x\Big|\\
&\le C(M)+\|D^2\partial_tu\|_{L^2}^2+\|D^2\partial_t\theta\|_{L^2}^2.
\end{aligned}
$$

Here we have used that

$$(b\times D^3\operatorname{rot}b)\cdot D^3u+(b\times D^3u)\cdot D^3\operatorname{rot}b=0.$$

Using (2.15) and (2.17), we have

$$\|D^3(\rho,u,b,\theta)(\cdot,t)\|_{L^2}\le C_0(M_0)\exp(tC(M)).\tag{2.18}$$

On the other hand, from (1.2) it follows that

$$
\begin{aligned}
\|\partial_tv\|_{L^2}&=\Big\|\frac{1}{\rho}\Big(b\cdot\nabla b-\frac{1}{2}\nabla|b|^2+\mu\Delta u+(\lambda+\mu)\nabla\operatorname{div}u-\nabla p-\rho u\cdot\nabla u\Big)\Big\|_{L^2}\\
&\le C_0(M_0)\exp(tC(M)).
\end{aligned}\tag{2.19}
$$

Similarly, we have

$$\|\partial_t\theta\|_{L^2} \le C_0(M_0)\exp(tC(M)). \tag{2.20}$$

Combining (2.4), (2.7), (2.8), (2.9), (2.18), (2.19) and (2.20), we conclude that (1.10) holds. This completes the proof. □

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Jishan Fan
Department of Applied Mathematics, Nanjing Forestry University, Nanjing 210037, China
Email address: fanjishan@njfu.edu.cn

Yong Zhou (corresponding author)
School of Mathematics, Sun Yat-sen University, Zhuhai 519082, China.
Department of Mathematics, Zhejiang Normal University, Jinhua 321004, China
Email address: zhouyong3@mail.sysu.edu.cn