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EXISTENCE AND MULTIPLICITY OF POSITIVE SOLUTIONS FOR SINGULAR $p\&q$-LAPLACIAN PROBLEMS VIA SUB-SUPERSOLUTION METHOD

SUELLEN CRISTINA Q. ARRUDA, RÚBIA G. NASCIMENTO

Abstract. In this work we show the existence and multiplicity of positive solutions for a singular elliptic problem which the operator is non-linear and non-homogenous. We use the sub-supersolution method to study the following class of $p\&q$-singular problems

$$
\begin{gathered}
-\operatorname{div}(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u) = h(x)u^{-\gamma} + f(x,u) \quad \text{in } \Omega,\\
u > 0 \quad \text{in } \Omega,\\
u = 0 \quad \text{on } \partial\Omega,
\end{gathered}
$$

where Ω is a bounded domain in $\mathbb{R}^N$ with $N \geq 3$, $2 \leq p < N$ and $\gamma > 0$. The hypotheses on the functions a, h, and f allow us to extend this result to a large class of problems.

1. Introduction

Let us consider the semilinear problem

$$
\begin{gathered}
-\Delta u = m(x,u) \quad \text{in } \Omega,\\
u > 0 \quad \text{in } \Omega,\\
u = 0 \quad \text{on } \partial\Omega.
\end{gathered} \tag{1.1}
$$

The classical sub-supersolution method asserts that if we can find a pair of sub-supersolution $v_1, v_2 \in H_0^1(\Omega)$ with $v_1(x) \leq v_2(x)$ a.e. in Ω, then there exists a solution $v \in H_0^1(\Omega)$ such that $v_1(x) \leq v(x) \leq v_2(x)$ a.e. in Ω.

In general, a candidate to be a subsolution of (1.1) is $v_1 = \varepsilon\phi_1$, where ϕ_1 is a eigenfunction associated with λ_1, the first eigenvalue of the operator $(-\Delta, H_0^1(\Omega))$. A candidate to be a supersolution, in general, is the unique positive solution of the problem

$$
\begin{gathered}
-\Delta u = M \quad \text{in } \Omega,\\
u > 0 \quad \text{in } \Omega,\\
u = 0 \quad \text{on } \partial\Omega,
\end{gathered}
$$

where M is a constant. The sizes of ε and M together with the Comparison Principle for operator $(-\Delta, H_0^1(\Omega))$ allow us to show that the sub-supersolution are ordered.

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When the operator is non-linear and non-homogeneous, in general, we do not have eigenvalues and eigenfunctions. In this work we show that the sub-supersolution method still can be applied. More precisely, we consider a general singular elliptic problem

$$\begin{gathered} -\operatorname{div}(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u) = h(x)u^{-\gamma} + f(x,u) \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 3$, $2 \leq p < N$, $\gamma > 0$ is a fixed constant, $a : \mathbb{R}^+ \to \mathbb{R}^+$ is a function of class C^1, and $h \geq 0$ is a nontrivial measurable function. In this article we use the following assumptions:

(A1) There exists $0 < \phi_0 \in C_0^1(\overline{\Omega})$ such that $h\phi_0^{-\gamma} \in L^\infty(\Omega)$.

Remark 1.1. Note that by (A1), we have $h \in L^\infty(\Omega)$ because

$$|h| = |h\phi_0^{-\gamma}\phi_0^{\gamma}| \leq \|h\phi_0^{-\gamma}\|_\infty \phi_0^{\gamma}.$$

Here f is a Caracthéodory function defined on $\overline{\Omega} \times [0,\infty)$ and satisfying:

(A2) There exists $0 < \delta < 1/2$ such that

$$-h(x) \leq f(x,t) \leq 0 \quad \text{a.e. in } \Omega \text{ for } 0 \leq t \leq \delta.$$

The function $a : \mathbb{R}^+ \to \mathbb{R}^+$ belongs to C^1 and satisfies the following assumptions:

(A3) There exist constants k_1, k_2, k_3, $k_4 > 0$ and q with $2 \leq p \leq q < N$ such that

$$k_1 t^p + k_2 t^q \leq a(t^p)t^p \leq k_3 t^p + k_4 t^q, \quad \forall t \geq 0.$$

(A4) The function $t \mapsto a(t^p)t^{p-2}$ is increasing.

(A5) The function $t \mapsto A(t^p)$ with $A(t) = \int_0^t a(s)\,ds$ is strictly convex.

(A6) There exist positive constants μ and θ, with $\theta \in (q, q^*)$ and $q/p \leq \mu < \theta/p$, such that

$$\frac{1}{\mu} a(t)\,t \leq A(t), \quad \forall t \geq 0.$$

We point out that, since $p < q$ and Ω is bounded, it follows that $W_0^{1,p}(\Omega) \cap W_0^{1,q}(\Omega) = W_0^{1,q}(\Omega)$. Therefore, to prove the existence and multiplicity of solutions for (1.2), we consider the Sobolev space $W_0^{1,q}(\Omega)$ endowed with the norm

$$\|u\|_{1,q} = \Big(\int_\Omega |\nabla u|^q dx\Big)^{1/q}.$$

Moreover, we say that $u \in W_0^{1,q}(\Omega)$ is a weak solution of (1.2) if $u > 0$ in Ω and satisfies

$$\int_\Omega a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u \nabla\phi\,dx = \int_\Omega h(x)u^{-\gamma}\phi\,dx + \int_\Omega f(x,u)\phi\,dx, \quad \forall \phi \in W_0^{1,q}(\Omega).$$

Our first result is the existence of a weak solution for (1.2).

Theorem 1.2. *Assume that conditions* (A1)–(A5) *hold. If* $\|h\|_\infty$ *is sufficiently small, then problem* (1.2) *has a weak solution.*

Setting $F(x,t) = \int_0^t f(x,s)\,ds$, we define the conditions below for proving the existence of two solutions for problem (1.2).

(A7) There exists $1 < r < q^* = \frac{Nq}{(N-q)}$ ($q^* = \infty$ if $q \geq N$) such that

$$f(x,t) \leq h(x)(t^{r-1} + 1) \quad \text{a.e. in } \Omega \text{ for all } t \geq 0.$$

(A8) There exists $t_0 > 0$ such that
$$0 < \theta F(x,t) \leq t f(x,t) \quad \text{a.e. in } \Omega \text{ for all } t \geq t_0,$$
where θ is defined by (A6).

Theorem 1.3. *Assume that conditions* (A1)–(A8) *hold. If* $\|h\|_\infty$ *is sufficiently small, then problem* (1.2) *has two weak solutions.*

This class of problems has been extensively studied in the previous ten years. The singular term presents difficulties that make the problem very interesting. Since it is not possible to cite all, we make a brief bibliographical review in chronological order of the papers with singular term and the sub-supersolution method.

In [3, 13] the authors studied the problem (1.2) with p-Laplacian operator and suitable truncation techniques. The case with p-Laplacian operator without the Ambrosetti and Rabinowitz condition was studied in [10]. The case with p-Laplacian operator and concave and convex nonlinearities was considered in [8]. In [12] the authors studied the case with Laplacian operator and the singular term appearing in the left-hand side. In [6] was studied the case with Laplacian operator and a nonlinearity depending on the gradient. The case supercritical with Laplacian operator was studied in [17]. In [5] it was studied the existence of solutions for nonlocal systems involving the $p(x)$-Laplacian operator.

Our arguments are strongly influenced by results in [3, 5, 6, 8, 10, 12, 13, 17]. Below we list the main contributions of our article.

- (i) This work considers a large class of quasilinear operators which includes but it is not restricted to p-Laplacian operator. In general, operators $p\&q$-Laplacian type are non-linear and non-homogeneous. See below several examples of operators we can consider.
- (ii) Since we work with a general operator, some estimates are more refined then the standard ones. See for example the proof of Theorems 1.2 and 1.3.
- (iii) Unlike the works mentioned above, no truncation was necessary in this paper. Moreover, we do not use the parameter as it was used there.
- (iv) In the same way as in [13], our result is valid for every $\gamma > 0$.
- (v) The results in this paper are valid for a general function f, including when f is negative near on the origin.

We would like to indicate that our theorems can be applied for the nonlinearity
$$f(x,t) = h(x)(t^{r-1} - \delta^{r-1}).$$
To illustrate the degree of generality of the kind of problems studied here, we present some examples of functions a which are also interesting from the mathematical point of view, and have a wide range of applications in physics and related sciences.

Example 1.4. If $a \equiv 1$, our operator is the p-Laplacian and so problem (1.2) becomes
$$\begin{gathered} -\Delta_p u = h(x)u^{-\gamma} + f(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered}$$
with $q = p$, $k_1 + k_2 = 1$ and $k_3 + k_4 = 1$.

Example 1.5. If $a(t) = 1 + t^{\frac{q-p}{p}}$, we obtain
$$-\Delta_p u - \Delta_q u = h(x)u^{-\gamma} + f(x,u) \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial\Omega,$$

with $k_1 = k_2 = k_3 = k_4 = 1$.

Example 1.6. If $a(t) = 1 + \frac{1}{(1+t)^{\frac{p-2}{p}}}$, we obtain

$$\begin{gathered} -\operatorname{div}(|\nabla u|^{p-2}\nabla u + \frac{|\nabla u|^{p-2}\nabla u}{(1+|\nabla u|^p)^{\frac{p-2}{p}}}) = h(x)u^{-\gamma} + f(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

with $q = p$, $k_1 + k_2 = 1$ and $k_3 + k_4 = 2$.

Example 1.7. If $a(t) = 1 + t^{\frac{q-p}{p}} + \frac{1}{(1+t)^{\frac{p-2}{p}}}$, we obtain

$$\begin{gathered} -\Delta_p u - \Delta_q u - \operatorname{div}(\frac{|\nabla u|^{p-2}\nabla u}{(1+|\nabla u|^p)^{\frac{p-2}{p}}}) = h(x)u^{-\gamma} + f(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

where $k_1 = k_2 = k_4 = 1$ and $k_3 = 2$.

2. Proof of Theorem 1.2

We combine the sub-supersolution method with minimization arguments. For this, the lemma below establishes the existence of a subsolution and a supersolution for problem (1.2) whenever we fix the value of $\|h\|_\infty$.

We say that the pair $(\underline{u}, \overline{u})$ is a sub-supersolution for problem (1.2), if $\underline{u}, \overline{u} \in W_0^{1,q}(\Omega) \cap L^\infty(\Omega)$ with

(i) $\underline{u} \le \overline{u}$ in Ω and $\underline{u} = 0 \le \overline{u}$ on $\partial\Omega$,

(ii) for each $\phi \in W_0^{1,q}(\Omega)$, with $\phi \ge 0$, we have

$$\begin{gathered} \int_\Omega a(|\nabla \underline{u}|^p)|\nabla \underline{u}|^{p-2}\nabla \underline{u}\nabla\phi\,dx \le \int_\Omega h(x)\underline{u}^{-\gamma}\phi\,dx + \int_\Omega f(x,\underline{u})\phi\,dx, \\ \int_\Omega a(|\nabla \overline{u}|^p)|\nabla \overline{u}|^{p-2}\nabla \overline{u}\nabla\phi\,dx \ge \int_\Omega h(x)\overline{u}^{-\gamma}\phi\,dx + \int_\Omega f(x,\overline{u})\phi\,dx. \end{gathered}$$

Lemma 2.1. *Assume that* (A1)–(A4) *are satisfied. If $\|h\|_\infty$ is sufficiently small, then there exist $\underline{u}, \overline{u} \in C^1(\overline{\Omega})$ such that*

(i) *$h\underline{u}^{-\gamma} \in L^\infty(\Omega)$ and $\|\underline{u}\|_\infty \le \delta$, where δ is given by (A2).*

(ii) *$0 < \underline{u}(x) \le \overline{u}(x)$ a.e. in Ω.*

(iii) *$\underline{u}$ is a subsolution and $\overline{u}$ is a supersolution for problem* (1.2).

Proof. From [4, Lemma 2.1], Minty-Browder's Theorem [2, Theorem 5.15], and the Maximum Principle, the problem

$$\begin{gathered} -\operatorname{div}(a(|\nabla \underline{u}|^p)|\nabla \underline{u}|^{p-2}\nabla \underline{u}) = h(x) \quad \text{in } \Omega, \\ \underline{u} = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{2.1}$$

has an unique positive solution $\underline{u} \in W_0^{1,q}(\Omega)$. By Remark 1.1 and (A3), we can use the same arguments in [9] to obtain that $\underline{u} \in C^1(\overline{\Omega})$. Thus, it follows from Lemmas 4.1 and 4.2 in the Appendix that there exists $C > 0$ such that $\underline{u}/\phi_0 \ge C$. Consequently, by (A1) we obtain

$$|h\underline{u}^{-\gamma}| = \big|h\frac{\underline{u}^{-\gamma}}{\phi_0^{-\gamma}}\phi_0^{-\gamma}\big| \le C^{-\gamma}\|h\phi_0^{-\gamma}\|_\infty, \tag{2.2}$$

implying that $h\underline{u}^{-\gamma} \in L^\infty(\Omega)$. Moreover, arguing as in [16, Lemma 4.5], there exist $C_* > 0$ and $\alpha > 0$ such that $\|\underline{u}\|_\infty \le C_* \|h\|_\infty^\alpha$, where C_* is a constant that does not depend on h or $\underline{u}$. Therefore, we choose $\|h\|_\infty$ sufficiently small such that $\|\underline{u}\|_\infty \le \delta < 1/2$. This completes the proof of (i).

To prove (ii), we use [4, Lemma 2.1], Minty-Browder's Theorem [2, Theorem 5.15], and the Maximum Principle once again to obtain that the problem

$$\begin{gathered} -\operatorname{div}(a(|\nabla \overline{u}|^p)|\nabla \overline{u}|^{p-2}\nabla \overline{u}) = h(x)\underline{u}^{-\gamma} \quad \text{in } \Omega, \\ \overline{u} = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{2.3}$$

has an unique positive solution $\overline{u} \in W_0^{1,q}(\Omega)$. Since $h\underline{u}^{-\gamma} \in L^\infty(\Omega)$, we can repeat the same arguments above to obtain $\overline{u} \in C^1(\overline{\Omega})$. Furthermore, using (2.2) we can write

$$\|\overline{u}\|_\infty \le C_* \|h\underline{u}^{-\gamma}\|_\infty^\alpha \le C_* \|h\|_\infty^\alpha C^{-\gamma\alpha} \|\phi_0\|_\infty^{-\gamma\alpha}.$$

So, choosing $\|h\|_\infty$ sufficiently small we conclude that

$$\|\overline{u}\|_\infty \le \delta < \frac{1}{2}. \tag{2.4}$$

Now, since $\|\underline{u}\|_\infty$ and $\|\overline{u}\|_\infty$ are small from 2.1 and 2.3 it follows that

$$\begin{aligned} \int_\Omega a(|\nabla \overline{u}|^p)|\nabla \overline{u}|^{p-2}\nabla \overline{u}\nabla\phi \, dx &= \int_\Omega h(x)\underline{u}^{-\gamma}\phi \, dx \\ &\ge \int_\Omega h(x)\phi \, dx \\ &= \int_\Omega a(|\nabla \underline{u}|^p)|\nabla \underline{u}|^{p-2}\nabla \underline{u}\nabla\phi \, dx. \end{aligned}$$

Therefore, applying the Weak Comparison Principle for the $p\&q$-Laplacian operator, see [4, Lemma 2.2], we conclude that $0 < \underline{u}(x) \le \overline{u}(x)$ a.e. in Ω.

Finally it is necessary to verify that condition (iii) is satisfied. Indeed, we use (A2), (2.1) and (i) to obtain

$$\begin{aligned} &\int_\Omega a(|\nabla \underline{u}|^p)|\nabla \underline{u}|^{p-2}\nabla \underline{u}\nabla\phi \, dx - \int_\Omega h(x)\underline{u}^{-\gamma}\phi \, dx - \int_\Omega f(x,\underline{u})\phi \, dx \\ &\le 2\int_\Omega h(x)\phi \, dx - \int_\Omega h(x)\underline{u}^{-\gamma}\phi \, dx \le 0, \end{aligned}$$

which implies that $\underline{u}$ is a subsolution for problem (1.2). On the other hand, we use (A2), (2.3), (2.4) and (ii) to obtain

$$\begin{aligned} &\int_\Omega a(|\nabla \overline{u}|^p)|\nabla \overline{u}|^{p-2}\nabla \overline{u}\nabla\phi \, dx - \int_\Omega h(x)\overline{u}^{-\gamma}\phi \, dx - \int_\Omega f(x,\overline{u})\phi \, dx \\ &\ge \int_\Omega (\underline{u}^{-\gamma} - \overline{u}^{-\gamma})h(x)\phi \, dx \ge 0, \end{aligned}$$

which implies that $\overline{u}$ is a supersolution for problem (1.2). $\square$

Proof of Theorem 1.2. Consider the function

$$g(x,t) = \begin{cases} h(x)\overline{u}(x)^{-\gamma} + f(x,\overline{u}(x)), & t > \overline{u}(x) \\ h(x)t^{-\gamma} + f(x,t), & \underline{u}(x) \le t \le \overline{u}(x) \\ h(x)\underline{u}(x)^{-\gamma} + f(x,\underline{u}(x)), & t < \underline{u}(x) \end{cases} \tag{2.5}$$

and the auxiliary problem

$$
\begin{gathered}
-\operatorname{div}(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u) = g(x,u) \quad \text{in } \Omega, \\
u > 0 \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial\Omega.
\end{gathered} \tag{2.6}
$$

We define the energy functional $\Phi : W_0^{1,q}(\Omega) \to \mathbb{R}$ associated with (2.6) by

$$
\Phi(u) = \frac{1}{p}\int_\Omega A(|\nabla u|^p)\,dx - \int_\Omega G(x,u)\,dx,
$$

where $G(x,t) = \int_0^t g(x,s)\,ds$. It follows from Lemma 2.1(i)-(ii), (A2), (2.4) and (2.5) that

$$
|g(x,t)| \le K \quad \text{a.e. in } \Omega, \text{ for some } K > 0 \text{ and all } t \in \mathbb{R}. \tag{2.7}
$$

Note that by (A3), the functional $\Phi(u)$ is well defined. Moreover, by standard arguments, Φ is of class C^1 on $W_0^{1,q}(\Omega)$.

Next, consider the set

$$
M = \{u \in W_0^{1,q}(\Omega) : \underline{u} \le u \le \overline{u} \text{ a.e. in } \Omega\}.
$$

For all $u \in M$, we apply (A3), (2.7) and continuous embedding $W_0^{1,q}(\Omega) \hookrightarrow L^{1,q}(\Omega)$ to get that Φ is coercive in M. Moreover, since (A5) holds and $g \in L^\infty(\Omega)$ we have that Φ is weak lower semi-continuous on M. Thus, as M is closed and convex in $W_0^{1,q}(\Omega)$, we use [15, Theorem 1.2] to conclude that Φ is bounded from below in M and attains it is infimum at a point $u \in M$.

Using the same argument as in the proof of [15, Theorem 2.4], we see that this minimum point is a critical point of Φ in all space and hence, u is a weak solution of the auxiliary problem (2.6). However, since $g(x,t) = h(x)t^{-\gamma} + f(x,t)$ for all $t \in [\underline{u}, \overline{u}]$, problem (1.2) has a weak solution $u \in W_0^{1,q}(\Omega)$ such that

$$
0 < \underline{u}(x) \le u(x) \le \overline{u}(x) \text{ a.e. in } \Omega. \qquad \square
$$

3. Proof of Theorem 1.3

Let $\underline{u} \in C^1(\overline{\Omega})$ be the subsolution of problem (1.2) and let $\widehat{g}$ be a Carathéodory function defined on $\overline{\Omega} \times \mathbb{R}$ given by

$$
\widehat{g}(x,t) = \begin{cases} h(x)t^{-\gamma} + f(x,t), & t > \underline{u}(x), \\ h(x)\underline{u}(x)^{-\gamma} + f(x,\underline{u}(x)), & t \le \underline{u}(x). \end{cases} \tag{3.1}
$$

We consider the auxiliary problem

$$
\begin{gathered}
-\operatorname{div}(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u) = \widehat{g}(x,u) \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial\Omega,
\end{gathered} \tag{3.2}
$$

and define the energy functional $\widehat{\Phi} : W_0^{1,q}(\Omega) \to \mathbb{R}$ associated with (3.2) by

$$
\widehat{\Phi}(u) = \frac{1}{p}\int_\Omega A(|\nabla u|^p)\,dx - \int_\Omega \widehat{G}(x,u)\,dx, \quad \forall u \in W_0^{1,q}(\Omega),
$$

where $\widehat{G}(x,t) = \int_0^t \widehat{g}(x,s)\,ds$.

Note that by the definition of $\widehat{g}$ and (A7), there exists $c_1 > 0$ such that

$$
\widehat{G}(x,t) \le h(x)\underline{u}(x)^{-\gamma}|t| + h(x)(c_1|t|^r + |t|) \quad \text{a.e. in } \Omega \text{ and all } t \in \mathbb{R}. \tag{3.3}
$$

Again, we can prove that $\widehat{\Phi} \in C^1(W_0^{1,q}(\Omega), \mathbb{R})$ with the Fréchet derivative

$$\widehat{\Phi}'(u)\phi = \int_\Omega a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u \nabla \phi \, dx - \int_\Omega \widehat{g}(x,u)\phi \, dx, \quad \forall \phi \in W_0^{1,q}(\Omega).$$

Furthermore, a straightforward calculation shows that any critical point of $\widehat{\Phi}$ is a weak solution for the auxiliary problem (3.2).

The next result shows that $\widehat{\Phi}$ satisfies the geometries of the Mountain Pass Theorem [1].

Lemma 3.1. *Suppose* (A1)–(A8) *are satisfied. Then* $\widehat{\Phi}$ *satisfies the following conditions*

(1) *There exist* R, α, β *with* $R > \|\underline{u}\|_{1,q}$ *and* $\alpha < \beta$ *such that*

$$\widehat{\Phi}(\underline{u}) \le \alpha < \beta \le \inf_{\partial B_R(0)} \widehat{\Phi}.$$

(2) *There exists* $e \in W_0^{1,q}(\Omega) \setminus \overline{B_R(0)}$ *such that* $\widehat{\Phi}(e) < \beta$.

Proof. Since $\underline{u}$ is a subsolution of problem (1.2) it follows from (A2), Lemma 2.1(i), and (3.1) that

$$\widehat{G}(x, \underline{u}) \ge \big(h(x)\underline{u}(x)^{-\gamma} - h(x)\big)\underline{u}(x) \quad \text{a.e. in } \Omega$$

and hence, there exists $0 < \alpha \in \mathbb{R}$ such that

$$\widehat{\Phi}(\underline{u}) \le \frac{1}{p}\int_\Omega A(|\nabla \underline{u}|^p)\, dx \equiv \alpha. \tag{3.4}$$

We invoke (A3), (3.3), Lemma 2.1(i), Remark 1.1 and the Sobolev embedding to obtain $c_2, c_3, c_4 > 0$ such that

$$\widehat{\Phi}(u) \ge \frac{k_2}{q}\|u\|_{1,q}^q - c_2\|h\underline{u}^{-\gamma}\|_\infty \|u\|_{1,q} - c_3\|h\|_\infty\|u\|_{1,q} - c_4\|h\|_\infty\|u\|_{1,q}^r, \tag{3.5}$$

for all $u \in W^{1,q}(\Omega)$. Thus, taking $\|u\|_{1,q} = R$ with $R > \max\{1, \|\underline{u}\|_{1,q}\}$, we may choice $\|h\|_\infty$ sufficiently small so that there exists $0 < \beta \in \mathbb{R}$ such that $\widehat{\Phi}(u) \ge \beta > \alpha$, for all $u \in \partial B_R(0)$. Therefore, the choices of α, β, R and $\|h\|_\infty$ combined with the inequalities (3.4) and (3.5) show that the condition (1) is satisfied.

Now, by the definition of $\widehat{g}$, we have

$$\widehat{G}(x, t\underline{u}) \ge F(x, t\underline{u}) \quad \text{a.e. in } \Omega, \text{ and all } t \ge 1$$

and hence, using (A3), (A8) and Sobolev embedding, there exist $c_5, c_6 > 0$ such that

$$\widehat{\Phi}(x, t\underline{u}) \le \frac{k_3}{p}t^p\|\underline{u}\|_{1,p}^p + \frac{k_4}{q}t^q\|\underline{u}\|_{1,q}^q - c_5 t^\theta \|\underline{u}\|_{1,q}^\theta + c_6.$$

Since $2 \le p \le q < \theta < q^*$ there exists $t^* > 0$ such that $e = t^*\underline{u} \in W_0^{1,q}(\Omega)$ satisfying $\|e\|_{1,q} > R$ and $\widehat{\Phi}(e) < \beta$, which completes the proof. $\square$

Lemma 3.2. *The functional* $\widehat{\Phi}$ *satisfies the Palais-Smale condition.*

Proof. Consider $(u_n) \subset W_0^{1,p}(\Omega)$ a sequence such that

$$\widehat{\Phi}(u_n) \to c \in \mathbb{R}, \widehat{\Phi}'(u_n) \to 0. \tag{3.6}$$

Thus, for all n sufficiently large, we use (A3) and (A6) to obtain $C > 0$ such that

$$C(1 + \|u_n\|_{1,q}) \ge \big(\frac{1}{p\mu} - \frac{1}{\theta}\big)k_2\|u_n\|_{1,q}^q + \int_\Omega \big[\frac{1}{\theta}\widehat{g}(x,u_n)u_n - \widehat{G}(x,u_n)\big]\, dx. \tag{3.7}$$

For t_0 given in (A8), from (3.1) it follows that

$$
\begin{aligned}
&\int_\Omega \big[\frac{1}{\theta}\widehat{g}(x,u_n)u_n - \widehat{G}(x,u_n)\big]\,dx \\
&= \big(\frac{1}{\theta}-1\big)\int_{\{u_n\le \underline{u}\}} h(x)u_n^{1-\gamma}\,dx + \big(\frac{1}{\theta}-\frac{1}{1-\gamma}\big)\int_{\{u_n>\underline{u}\}} h(x)u_n^{1-\gamma}\,dx \\
&\quad + \int_{\Omega\cap\{|u_n|\ge t_0\}}\Big(\frac{1}{\theta}f(x,u_n)u_n - F(x,u_n)\Big)dx \\
&\quad + \int_{\Omega\cap\{|u_n|< t_0\}}\Big(\frac{1}{\theta}f(x,u_n)u_n - F(x,u_n)\Big)dx
\end{aligned}
$$

and hence, by (A8) and (3.7) there exists $\widehat{C}>0$ such that

$$
\begin{aligned}
C(1+\|u_n\|_{1,q}) \ge \big(\frac{1}{p\mu}-\frac{1}{\theta}\big)k_2\|u_n\|_{1,q}^q + \big(\frac{1}{\theta}-1\big)\int_{\{u_n\le \underline{u}\}} h(x)u_n^{1-\gamma}\,dx \\
+ \big(\frac{1}{\theta}-\frac{1}{1-\gamma}\big)\int_{\{u_n>\underline{u}\}} h(x)u_n^{1-\gamma}\,dx - \widehat{C}.
\end{aligned} \tag{3.8}
$$

Now, applying (A6), Lemma 2.1(i), and (3.8) we consider the following cases:

Case 1: $\gamma>1$. Then there exists $M>0$ such that

$$
M + C\|u_n\|_{1,q} \ge \big(\frac{1}{p\mu}-\frac{1}{\theta}\big)k_2\|u_n\|_{1,q}^q.
$$

Case 2: $0<\gamma<1$. We can apply Holder's inequality in (3.8) to obtain $M>0$ such that

$$
M + C\|u_n\|_{1,q} + \big(\frac{1}{1-\gamma}-\frac{1}{\theta}\big)\|h\|_{1,q}^{q+(\gamma-1)}\|u_n\|_{1,q}^{1-\gamma} \ge \big(\frac{1}{p\mu}-\frac{1}{\theta}\big)k_2\|u_n\|_{1,q}^q.
$$

Case 3: $\gamma=1$ in (3.1). Then there exists $M>0$ such that

$$
M + C\|u_n\|_{1,q} + \|h\|_\infty\|u_n\|_{1,q} \ge \big(\frac{1}{p\mu}-\frac{1}{\theta}\big)k_2\|u_n\|_{1,q}^q.
$$

Therefore, by analyzing the three cases above, since $\theta>p\mu$ we conclude that (u_n) is bounded in $W_0^{1,q}(\Omega)$. Thus, up to subsequence, there exists $u\in W_0^{1,q}(\Omega)$ such that

$$
\begin{gathered}
u_n \rightharpoonup u \quad \text{in } W_0^{1,q}(\Omega),\\
u_n \to u \quad \text{in } L^s(\Omega),\ 1\le s<q^*,\\
u_n(x)\to u(x) \quad \text{a.e. in } \Omega,\\
|u_n(x)| \le \varphi(x)\in L^s(\Omega), \quad 1\le s<q^*.
\end{gathered} \tag{3.9}
$$

Invoking (A4) we can argue as in [4, Lemma 2.1] to obtain

$$
C_q\|u_n-u\|_{1,q}^q \le \int_\Omega a(|\nabla u_n|^p)|\nabla u_n|^p dx - \int_\Omega a(|\nabla u_n|^p)|\nabla u_n|^{p-2}\nabla u_n\nabla u\,dx + o_n(1),
$$

where

$$
o_n(1) = \int_\Omega a(|\nabla u|^p)|\nabla u|^p dx - \int_\Omega a(|\nabla u|^p|\nabla u|^{p-2}\nabla u_n\nabla u\,dx.
$$

But, in view of (3.6) and (3.9) we have

$$
C_q\|u_n-u\|_{1,q}^q \le \int_\Omega \widehat{g}(x,u_n)(u_n-u)\,dx. \tag{3.10}
$$

Now, we use Lemma 2.1(i), Remark 1.1, (3.9) and the Lebesgue Dominated Convergence Theorem to conclude that

$$\int_\Omega \widehat{g}(x,u_n)(u_n-u)\,dx \to 0 \quad \text{as } n\to 0, \tag{3.11}$$

and hence, from (3.10) and (3.11) it follows that $u_n \to u$ in $W_0^{1,q}(\Omega)$. □

Proof of Theorem 1.3. Let $\underline{u},\overline{u}$ be the subsolution and the supersolution, respectively, of problem (1.2) given in Lemma 2.1, and w be the weak solution of (1.2) obtained in Theorem 1.2. By using Lemmas 3.1 and 3.2, from the Mountain Pass Theorem it follows that there exists $v\in W_0^{1,q}(\Omega)$ such that

$$\widehat{\Phi}'(v)=0 \quad \text{and} \quad \beta<\widehat{\Phi}(v)=c,$$

where $c=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\widehat{\Phi}(\gamma(t))$ with

$$\Gamma=\big\{\,\gamma\in C\big([0,1],W_0^{1,q}(\Omega)\big):\gamma(0)=\underline{u} \text{ and } \gamma(1)=e\big\}$$

which is the minimax value of $\widehat{\Phi}$.

Since $g(x,t)=\widehat{g}(x,t)$, for all $t\in[0,\overline{u}]$, it follows that $\Phi(u)=\widehat{\Phi}(u)$, for all $u\in[0,\overline{u}]$. Therefore, $\widehat{\Phi}(w)=\inf_M\Phi$, where $w\in[\underline{u},\overline{u}]$ and M is given in the proof of Theorem 1.2. Thus, auxiliary problem (3.2) has two positive weak solutions $w,v\in W_0^{1,q}(\Omega)$ such that

$$\widehat{\Phi}(w)\le\widehat{\Phi}(\underline{u})\le\alpha<\beta\le\widehat{\Phi}(v)=c.$$

Finally, let us show that $\underline{u}\le v$. Indeed, taking $(\underline{u}-v)^+\in C_0^\infty(\Omega)$, we have

$$\begin{aligned}
\int_\Omega a(|\nabla v|^p)|\nabla v|^{p-2}\nabla v\nabla(\underline{u}-v)^+dx &= \int_\Omega \widehat{g}(x,v)(\underline{u}-v)^+dx\\
&= \int_{\{v<\underline{u}\}}[h(x)\underline{u}^{-\gamma}+f(x,\underline{u})](\underline{u}-v)^+dx.
\end{aligned}$$

Since $\underline{u}$ is a subsolution for problem (1.2), it follows that

$$\int_\Omega a(|\nabla \underline{u}|^p)|\nabla \underline{u}|^{p-2}\nabla \underline{u}\nabla(\underline{u}-v)^+dx-\int_\Omega a(|\nabla v|^p)|\nabla v|^{p-2}\nabla v\nabla(\underline{u}-v)^+dx\le 0$$

and hence,

$$C_q\int_\Omega|\nabla(\underline{u}-v)^+|^q\le 0,$$

which implies that $(\underline{u}-v)^+=0$. So, we conclude that $0<\underline{u}\le v$ a.e. in Ω, as claimed.

It follows from (3.1) that $\widehat{g}(x,v)=h(x)v^{-\gamma}+f(x,v)$ in Ω. Therefore, $v,w\in W_0^{1,q}(\Omega)$ are two weak solutions for problem (1.2). □

4. Appendix

Lemma 4.1. *Let $\Omega\subset\mathbb{R}^N$ be a bounded domain with smooth boundary. If $u\in C^1(\overline{\Omega})\cap W_0^{1,q}(\Omega)$, with $2\le p\le q<N$, and*

$$\begin{gathered}
-\operatorname{div}(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u)\ge 0 \quad \text{in } \Omega,\\
u>0 \quad \text{in } \Omega,\\
u=0 \quad \text{on } \partial\Omega.
\end{gathered}$$

Then, $\partial u/\partial\eta<0$ on $\partial\Omega$, where η is the outward normal to $\partial\Omega$.

Proof. The proof of this lemma is the same as that of [14, Hoppf's Lemma], replacing the operator $-\Delta_p u$ by $-\operatorname{div}(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u)$, and replacing the Weak Comparison Principle for p-Laplacian operator by the Weak Comparison Principle given in [4, Lemma 2.2]. $\square$

The following result can be found in [11, Lemma 2.6], and its proof is presented for the completeness of this paper.

Lemma 4.2. *Let $\phi, \omega > 0$ be any functions on $C_0^1(\overline{\Omega})$. If $\partial\phi/\partial\nu > 0$ in $\partial\Omega$, where ν is the inward normal to $\partial\Omega$, then there exists $C > 0$ such that*
$$\frac{\phi(x)}{\omega(x)} \geq C > 0, \quad \forall x \in \Omega.$$

Proof. For $\delta > 0$ sufficiently small, we consider the set
$$\Omega_\delta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \delta\}.$$
Since $\phi, \omega > 0$ in Ω, and $\Omega \setminus \Omega_\delta$ is compact, there exists $m > 0$ such that
$$\frac{\phi(x)}{\omega(x)} \geq m, \quad \forall x \in \Omega \setminus \Omega_\delta. \tag{4.1}$$

It follows from $\partial\phi/\partial\nu > 0$ on $\partial\Omega$ that $\partial\phi/\partial\eta < 0$, where η is the outwards normal to $\partial\Omega$. Furthermore, since $\Omega \subset \mathbb{R}^n$ is bounded domain, $\partial\Omega$ is a compact set and consequently, there exists $C_1 < 0$ satisfying
$$\frac{\partial\phi(x)}{\partial\eta} \leq C_1, \quad \forall x \in \overline{\Omega}_\delta.$$
Thus, since $\omega \in C_0^1(\overline{\Omega})$, there exists $C_2 > 0$ such that $|\frac{\partial\omega(x)}{\partial\eta}| \leq C_2$ for all $x \in \overline{\Omega}_\delta$.

Consider $K_0 = \inf_{\overline{\Omega}_\delta} \frac{\partial\omega}{\partial\eta} < 0$ and define the function
$$H(x) = \alpha\omega(x) - \phi(x), \quad \forall x \in \overline{\Omega}_\delta \text{ and } \alpha \in \mathbb{R} \text{ to be chosen.}$$
Since $0 < \alpha < C_1/K_0$ we obtain
$$\frac{\partial H(x)}{\partial\eta} = \alpha\frac{\partial\omega(x)}{\partial\eta} - \frac{\partial\phi(x)}{\partial\eta} \geq \alpha K_0 - C_1 > 0, \quad \forall x \in \overline{\Omega}_\delta.$$

Now, we fix $x \in \overline{\Omega}_\delta$ and consider the function
$$f(x) = H(x + s\eta), \quad \forall s \in \mathbb{R}.$$
For each $x \in \overline{\Omega}_\delta$, we choose an unique $\widetilde{x} \in \overline{\Omega}_\delta$, so that there exists $\widehat{s} > 0$ for which $x + \widehat{s}\eta = \widetilde{x} \in \partial\Omega$. Since $H(\partial\Omega) \equiv 0$, we have $f(\widehat{s}) = H(x + \widehat{s}\eta) = H(\widetilde{x}) = 0$.

Next, applying the Medium Value Theorem, there exists $\xi \in (0, \widehat{s})$ such that
$$f(\widehat{s}) - f(0) = f'(\xi)(\widehat{s} - 0),$$
which implies
$$-H(x) = \frac{\partial H}{\partial\eta}(x + \xi\eta)\widehat{s} > 0 \quad \text{in } \overline{\Omega}_\delta.$$
Therefore, $H(x) \leq 0$ for all $x \in \overline{\Omega}_\delta$ and hence, $\alpha\omega(x) - \phi(x) \leq 0$ for all $x \in \overline{\Omega}_\delta$, which implies $\alpha\omega(x) \leq \phi(x)$ for all $x \in \overline{\Omega}_\delta$. Therefore,
$$\frac{\phi(x)}{\omega(x)} \geq \alpha > 0, \quad \forall x \in \overline{\Omega}_\delta. \tag{4.2}$$

From (4.1) and (4.2) we conclude that there exists $C > 0$ such that

$$\frac{\phi(x)}{\omega(x)} \geq C, \quad \forall x \in \Omega. \qquad \square$$

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Suellen Cristina Q. Arruda
Faculdade de Ciências Exatas e Tecnologia, Universidade Federal do Pará - UFPA, Abaetetuba, PA 68440-000, Brazil
Email address: `scqarruda@ufpa.br`

Rúbia G. Nascimento
Instituto de Ciências Exatas e Naturais, Universidade Federal do Pará - UFPA, Belém-PA, PA 66075-110, Brazil
Email address: `rubia@ufpa.br`