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CONTINUOUS IMBEDDING IN MUSIELAK SPACES WITH AN APPLICATION TO ANISOTROPIC NONLINEAR NEUMANN PROBLEMS

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Abstract. We prove a continuous embedding that allows us to obtain a boundary trace imbedding result for anisotropic Musielak-Orlicz spaces, which we then apply to obtain an existence result for Neumann problems with nonlinearities on the boundary associated to some anisotropic nonlinear elliptic equations in Musielak-Orlicz spaces constructed from Musielak-Orlicz functions on which and on their conjugates we do not assume the Δ_2-condition. The uniqueness of weak solutions is also studied.

1. Introduction

Let Ω be an open bounded subset of $\mathbb{R}^N$, $(N \geq 2)$. We denote by $\vec{\phi} : \Omega \times \mathbb{R}^+ \to \mathbb{R}^N$ the vector function $\vec{\phi} = (\phi_1, \dots, \phi_N)$ where for every $i \in \{1, \dots, N\}$, ϕ_i is a Musielak-Orlicz function differentiable with respect to its second argument whose complementary Musielak-Orlicz function is denoted by ϕ_i^* (see preliminaries). We consider the problem

$$\begin{gathered} -\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u) + b(x)\varphi_{\max}(x, |u(x)|) = f(x, u) \quad \text{in } \Omega, \\ u \geq 0 \quad \text{in } \Omega, \\ \sum_{i=1}^{N} a_i(x, \partial_{x_i} u)\nu_i = g(x, u) \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\partial_{x_i} = \frac{\partial}{\partial_{x_i}}$ and for every $i = 1, \dots, N$, we denote by ν_i the i^{th} component of the outer normal unit vector and $a_i : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that there exist a locally integrable Musielak-Orlicz function (see definition 1.1 below) $P_i : \overline{\Omega} \times \mathbb{R}^+ \to \mathbb{R}^+$ with $P_i \ll \phi_i$, a positive constant c_i and a nonnegative function $d_i \in E_{\phi_i^*}(\Omega)$ satisfying for all s, $t \in \mathbb{R}$ and for almost every $x \in \Omega$ the following assumptions

$$|a_i(x, s)| \leq c_i\big(d_i(x) + (\phi_i^*)^{-1}(x, P_i(x, s))\big), \tag{1.2}$$

$$\phi_i(x, |s|) \leq a_i(x, s)s \leq A_i(x, s), \tag{1.3}$$

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$$\big(a_i(x,s)-a_i(x,t)\big)\cdot\big(s-t\big)>0,\ \text{for all } s\neq t, \tag{1.4}$$

the function $A_i:\Omega\times\mathbb{R}\to\mathbb{R}$ is defined by

$$A_i(x,s)=\int_0^s a_i(x,t)dt.$$

Here and in what follows, we define

$$\phi_{\min}(x,s)=\min_{i=1,\dots,N}\phi_i(x,s)\quad\text{and}\quad\phi_{\max}(x,s)=\max_{i=1,\dots,N}\phi_i(x,s).$$

Let $\varphi_{\max}(x,y)=\frac{\partial\phi_{\max}}{\partial y}(x,y)$. We also assume that there exist a locally integrable Musielak-Orlicz function $R:\overline{\Omega}\times\mathbb{R}^+\to\mathbb{R}^+$ with $R\ll\phi_{\max}$ and a nonnegative function $D\in E_{\phi^*_{\max}}(\Omega)$, such that for all s, $t\in\mathbb{R}$ and for almost every $x\in\Omega$,

$$|\varphi_{\max}(x,s)|\le D(x)+(\phi^*_{\max})^{-1}(x,R(x,s)), \tag{1.5}$$

where $\phi^*_{\max}$ stands for the complementary function of $\phi_{\max}$ defined below in (2.1). As regards the data, we suppose that $f:\Omega\times\mathbb{R}^+\to\mathbb{R}^+$ and $g:\partial\Omega\times\mathbb{R}^+\to\mathbb{R}^+$ are Carathéodory functions. We define the antiderivatives $F:\Omega\times\mathbb{R}\to\mathbb{R}$ and $G:\partial\Omega\times\mathbb{R}\to\mathbb{R}$ of f and g respectively by

$$F(x,s)=\int_0^s f(x,t)dt,\quad G(x,s)=\int_0^s g(x,t)dt.$$

We say that a Musielak-Orlicz function ϕ satisfies the Δ_2-condition, if there exists a positive constant $k>0$ and a nonnegative function $h\in L^1(\Omega)$ such that

$$\phi(x,2t)\le k\phi(x,t)+h(x).$$

Remark that the condition (Δ_2) is equivalent to the following condition: for all $\alpha>1$ there exists a positive constant $k>0$ and a nonnegative function $h\in L^1(\Omega)$ such that

$$\phi(x,\alpha t)\le k\phi(x,t)+h(x).$$

We assume now that there exist two positive constants k_1 and k_2 and two locally integrable Musielak-Orlicz functions M and $H:\overline{\Omega}\times\mathbb{R}^+\to\mathbb{R}^+$ satisfying the Δ_2-condition and differentiable with respect to their second arguments with $M\ll\phi^{**}_{\min}$, $H\ll\phi^{**}_{\min}$ and $H\ll\psi_{\min}$, such that the functions f and g satisfy for all $s\in\mathbb{R}_+$ the following assumptions

$$|f(x,s)|\le k_1 m(x,s),\ \text{for a.e. } x\in\Omega, \tag{1.6}$$

$$|g(x,s)|\le k_2 h(x,s),\ \text{for a.e. } x\in\partial\Omega, \tag{1.7}$$

where

$$\begin{gathered}\psi_{\min}(x,t)=\Big((\phi^{**}_{\min})_*(x,t)\Big)^{\frac{N-1}{N}},\quad m(x,s)=\frac{\partial M(x,s)}{\partial s},\\ h(x,s)=\frac{\partial H(x,s)}{\partial s}.\end{gathered} \tag{1.8}$$

Finally, for the function b involved in (1.1), we assume that there exists a constant $b_0>0$ such that b satisfies the hypothesis

$$b\in L^\infty(\Omega)\text{ and } b(x)\ge b_0,\quad \text{or a.e. } x\in\Omega. \tag{1.9}$$

Observe that (1.4) and the relation $a_i(x,\zeta)=\nabla_\zeta A_i(x,\zeta)$ imply in particular that for any $i=1,\dots,N$, the function $\zeta\to A_i(\cdot,\zeta)$ is convex.

Let us put ourselves in the particular case of $\vec{\phi}=(\phi_i)_{i\in\{1,\dots,N\}}$ where for $i\in\{1,\dots,N\}$, $\phi_i(x,t)=|t|^{p_i(x)}$ with $p_i\in C_+(\bar{\Omega})=\{h\in C(\bar{\Omega}):\inf_{x\in\Omega}h(x)>1\}$.

Defining $p_{\max}(x) = \max_{i\in\{1,\dots,N\}} p_i(x)$ and $p_{\min}(x) = \min_{i\in\{1,\dots,N\}} p_i(x)$, one has $\phi_{\max}(x,t) = |t|^{p_M(x)}$ and then $\varphi_{\max}(x,t) = p_M(x)|t|^{p_M(x)-2}t$, where p_M is $p_{\max}$ or $p_{\min}$ according to whether $|t| \geq 1$ or $|t| \leq 1$ and then the space $W^1L_{\vec{\phi}}(\Omega)$ is nothing but the anisotropic space with variable exponent $W^{1,\vec{p}(\cdot)}(\Omega)$, where $\vec{p}(\cdot) = (p_1(\cdot),\dots,p_N(\cdot))$ (see [7] for more details on this space). Therefore, the problem (1.1) can be rewritten as

$$
\begin{gathered}
-\sum_{i=1}^{N} \partial_{x_i} a_i(x,\partial_{x_i}u) + b_1(x)|u|^{p_M(x)-2}u = f(x,u) \quad \text{in } \Omega,\\
u \geq 0 \quad \text{in } \Omega,\\
\sum_{i=1}^{N} a_i(x,\partial_{x_i}u)\nu_i = g(x,u) \quad \text{on } \partial\Omega,
\end{gathered} \tag{1.10}
$$

where $b_1(x) = p_M(x)b(x)$. Boureanu and Rădulescu [2] have proved the existence and uniqueness of the weak solution of (1.10). They prove an imbedding and a trace results which they use together with a classical minimization existence result for functional reflexive framework (see [22, Theorem 1.2]). Problem (1.10) with Dirichlet boundary condition and $b_1(x) = 0$ was treated in [15]. The authors proved that if $f(\cdot,u) = f(\cdot) \in L^\infty(\Omega)$ then (1.10) admits a unique solution by using [22, Theorem 1.2]. The problem (1.10) with for all $i = 1,\dots,N$

$$
a_i(x,s) = a(x,s) = s^{p(x)-1},
$$

with $p \in C^1(\overline{\Omega})$ and $b_1 = g = 0$ was treated in [12], where the authors proved the three nontrivial smooth solutions, two of which have constant sign (one positive, the other negative). In connection with Neumann problems, the authors [21] studied the problem

$$
\begin{gathered}
-\operatorname{div} a(\nabla u(z)) + \big(\zeta(z)+\lambda\big)u(z)^{p-1} = f(z,u(z)) \quad \text{in } \Omega,\\
\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega\\
u > 0,\ \lambda > 0,\ 1 < p < +\infty,
\end{gathered} \tag{1.11}
$$

where the function $a : \mathbb{R}^N \to \mathbb{R}^N$ is strictly monotone, continuous and satisfies certain other regularity and growth conditions. The function ζ involved in (1.11) changes its sign and is such that $\zeta \in L^\infty(\Omega)$. The reaction term $f(z,x)$ is a Carathéodory function. They proved the existence of a critical parameter value $\lambda_* > 0$ such that if $\lambda > \lambda_*$ problem (1.11) has at least two positive solutions, if $\lambda = \lambda_*$ (1.11) has at least a positive solution and if $\lambda \in (0,\lambda_*)$ problem (1.11) has no positive solution.

Let us mention some related results in the framework of Orlicz-Sobolev spaces. Le and Schmitt [17] proved an existence result for the boundary value problem

$$
\begin{gathered}
-\operatorname{div}(A(|\nabla u|^2)\nabla u) + F(x,u) = 0, \quad \text{in } \Omega,\\
u = 0 \quad \text{on } \partial\Omega,
\end{gathered}
$$

in $W_0^1L_\phi(\Omega)$ where $\phi(s) = A(|s|^2)s$ and F is a Carathéodory function satisfying some growth conditions. This result extends the one obtained in [11] with $F(x,u) = -\lambda\psi(u)$, where ψ is an odd increasing homeomorphism of $\mathbb{R}$ onto $\mathbb{R}$. In [11, 17] the authors assume that the N-function ϕ^* complementary to the N-function ϕ satisfies the Δ_2 condition, which is used to prove that the functional $u \to \int_\Omega \Phi(|\nabla u|)\,dx$ is coercive and of class $\mathcal{C}^1$, where Φ is the antiderivative of ϕ vanishing at the origin.

Here we are interested in proving the existence and uniqueness of the weak solutions for problem (1.1) without any additional condition on the Musielak-Orlicz function ϕ_i or its complementary ϕ_i^* for $i = 1, \dots, N$. Therefore, the resulting Musielak-Orlicz spaces $L_{\phi_i}(\Omega)$ are neither reflexive nor separable and thus classical existence results can not be applied.

The approach we use consists in proving first a continuous imbedding and a trace result which we then apply to solve the problem (1.1). The results we prove extend to the anisotropic Musielak-Orlicz-Sobolev spaces the continuous imbedding obtained in [6] under some extra conditions and the trace result proved in [18]. The imbedding result we obtain extends to Musielak spaces a part of the one obtained in [19] in the anisotropic case and that of Fan [9] in the isotropic case (see Remark 3.2). In the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ where $1 < p_+ = \sup_{x\in\Omega} p(x) < N$, other imbedding results can be found for instance in [3, 4, 16] while the case $1 \le p_- \le p_+ \le N$ was investigated in [13].

To the best of our knowledge, the trace result we obtain here is new and does not exist in the literature. The main difficulty we found when we deal with problem (1.1) is the coercivity of the energy functional. We overcome this by using both our continuous imbedding and trace results. Then we prove the boundedness of a minimization sequence and by a compactness argument, we are led to obtain a minimizer which is a weak solution of problem (1.1).

Definition 1.1. Let Ω be an open subset of $\mathbb{R}^N$, $(N \ge 2)$. We say that a Musielak-Orlicz function ϕ is locally-integrable, if for every compact subset K of Ω and every constant $c > 0$, we have

$$\int_K \phi(x,c)\,dx < \infty.$$

The article is organized as follows: Section 2 contains some definitions. In Section 3, we give and prove our main results, which we then apply in Section 4 to solve problem (1.1). In the last section we give an appendix which contains some important lemmas that are necessary for the accomplishment of the proofs of the results.

2. Preliminaries

2.1. **Anisotropic Musielak-Orlicz-Sobolev spaces.** Let Ω be an open subset of $\mathbb{R}^N$. A real function $\phi : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ will be called a Musielak-Orlicz function if it satisfies the following conditions

(i) $\phi(\cdot, t)$ is a measurable function on Ω.

(ii) $\phi(x, \cdot)$ is an N-function, that is a convex nondecreasing function with $\phi(x,t) = 0$ if only if $t = 0$, $\phi(x,t) > 0$ for all $t > 0$ and for almost every $x \in \Omega$,

$$\lim_{t\to 0^+} \frac{\phi(x,t)}{t} = 0 \quad \text{and} \quad \lim_{t\to+\infty} \inf_{x\in\Omega} \frac{\phi(x,t)}{t} = +\infty.$$

We will extend these Musielak-Orlicz functions into even functions on all $\Omega \times \mathbb{R}$. The complementary function ϕ^* of the Musilek-Orlicz function ϕ is defined by

$$\phi^*(x,s) = \sup_{t\ge 0}\{st - \phi(x,t)\}. \tag{2.1}$$

It can be checked that ϕ^* is also a Musielak-Orlicz function (see [20]). Moreover, for every $t,\, s \geq 0$ and a.e. $x \in \Omega$ we have the so-called Young inequality (see [20])

$$ts \leq \phi(x,t) + \phi^*(x,s).$$

For any function $h : \mathbb{R} \to \mathbb{R}$ the second complementary function $h^{**} = (h^*)^*$ (cf. (2.1)), is convex and satisfies

$$h^{**}(x) \leq h(x), \tag{2.2}$$

with equality when h is convex. Roughly speaking, h^{**} is a convex envelope of h, that is the biggest convex function smaller or equal to h.

Let ϕ and ψ be two Musielak-Orlicz functions. We say that ψ grows essentially more slowly than ϕ, denote $\psi \ll \phi$, if

$$\lim_{t\to+\infty} \sup_{x\in\Omega} \frac{\psi(x,t)}{\phi(x,ct)} = 0,$$

for every constant $c > 0$ and for almost every $x \in \Omega$. We point out that if $\psi : \overline{\Omega} \times \mathbb{R}^+ \to \mathbb{R}^+$ is locally integrable then $\psi \ll \phi$ implies that for all $c > 0$ there exists a nonnegative function $h \in L^1(\overline{\Omega})$ such that

$$\psi(x,t) \leq \phi(x,ct) + h(x), \quad \text{for all } t \in \mathbb{R} \text{ and for a.e. } x \in \Omega.$$

The Musielak-Orlicz space $L_\phi(\Omega)$ is defined by

$$L_\phi(\Omega) = \Big\{ u : \Omega \to \mathbb{R} \text{ measurable } : \int_\Omega \phi\Big(x, \frac{u(x)}{\lambda}\Big) < +\infty \text{ for some } \lambda > 0 \Big\}.$$

Endowed with the so-called Luxemborg norm

$$\|u\|_\phi = \inf \Big\{ \lambda > 0 : \int_\Omega \phi\Big(x, \frac{u(x)}{\lambda}\Big)\, dx \leq 1 \Big\},$$

$(L_\phi(\Omega), \|\cdot\|_\phi)$ is a Banach space. Observe that since $\lim_{t\to+\infty} \inf_{x\in\Omega} \frac{\phi(x,t)}{t} = +\infty$ and if Ω has finite measure then we have the following continuous imbedding

$$L_\phi(\Omega) \hookrightarrow L^1(\Omega). \tag{2.3}$$

We will also use the space

$$E_\phi(\Omega) = \Big\{ u : \Omega \to \mathbb{R} \text{ measurable } : \int_\Omega \phi\Big(x, \frac{u(x)}{\lambda}\Big) < +\infty \text{ for all } \lambda > 0 \Big\}.$$

Observe that for every $u \in L_\phi(\Omega)$ the following inequality holds

$$\|u\|_\phi \leq \int_\Omega \phi(x, u(x))\, dx + 1. \tag{2.4}$$

For two complementary Musielak-Orlicz functions ϕ and ϕ^*, Hölder's inequality (see [20])

$$\int_\Omega |u(x)v(x)|\, dx \leq 2\|u\|_\phi \|v\|_{\phi^*} \tag{2.5}$$

holds for every $u \in L_\phi(\Omega)$ and $v \in L_{\phi^*}(\Omega)$. Define ϕ^{*-1} for every $s \geq 0$ by

$$\phi^{*-1}(x,s) = \sup\{\tau \geq 0 : \phi^*(x,\tau) \leq s\}.$$

Then, for almost every $x \in \Omega$ and for every $s \in \mathbb{R}$ we have

$$\phi^*(x, \phi^{*-1}(x,s)) \leq s, \tag{2.6}$$

$$s \leq \phi^{*-1}(x,s)\phi^{-1}(x,s) \leq 2s, \tag{2.7}$$

$$\phi(x,s) \le s\frac{\partial \phi(x,s)}{\partial s} \le \phi(x,2s). \tag{2.8}$$

Definition 2.1. Let $\vec{\phi} : \Omega \times \mathbb{R}_+ \longrightarrow \mathbb{R}^N$ be the vector function $\vec{\phi} = (\phi_1, \dots, \phi_N)$ where for every $i \in \{1, \dots, N\}$, ϕ_i is a Musielak-Orlicz function. We define the anisotropic Musielak-Orlicz-Sobolev space by

$$W^1 L_{\vec{\phi}}(\Omega) = \Big\{ u \in L_{\phi_{\max}}(\Omega);\ \partial_{x_i} u \in L_{\phi_i}(\Omega) \text{ for all } i = 1, \cdots, N \Big\}.$$

By the continuous imbedding (2.3), we obtain that $W^1 L_{\vec{\phi}}(\Omega)$ is a Banach space with respect to the following norm

$$\|u\|_{W^1 L_{\vec{\phi}}(\Omega)} = \|u\|_{\phi_{\max}} + \sum_{i=1}^{N} \|\partial_{x_i} u\|_{\phi_i}.$$

Moreover, we have the continuous embedding $W^1 L_{\vec{\phi}}(\Omega) \hookrightarrow W^{1,1}(\Omega)$.

3. Main results

In this section we prove an imbedding theorem and a trace result. Let us assume the conditions

$$\int_0^1 \frac{(\phi_{\min}^{**})^{-1}(x,t)}{t^{1+\frac{1}{N}}} dt < +\infty \quad \text{and} \quad \int_1^{+\infty} \frac{(\phi_{\min}^{**})^{-1}(x,t)}{t^{1+\frac{1}{N}}} dt = +\infty, \ \ \forall x \in \overline{\Omega}. \tag{3.1}$$

Thus, we define the Sobolev conjugate $(\phi_{\min}^{**})_*$

$$(\phi_{\min}^{**})_*^{-1}(x,s) = \int_0^s \frac{(\phi_{\min}^{**})^{-1}(x,t)}{t^{1+\frac{1}{N}}} dt, \quad \text{for } x \in \overline{\Omega} \text{ and } s \in [0, +\infty). \tag{3.2}$$

It may readily be checked that $(\phi_{\min}^{**})_*$ is a Musielak-Orlicz function. We assume that there exist two positive constants $\nu < \frac{1}{N}$ and $c_0 > 0$ such that

$$\Big|\frac{\partial(\phi_{\min}^{**})_*}{\partial x_i}(x,t)\Big| \le c_0 \Big[(\phi_{\min}^{**})_*(x,t) + ((\phi_{\min}^{**})_*(x,t))^{1+\nu} \Big], \tag{3.3}$$

for all $t \in \mathbb{R}$ and for almost every $x \in \Omega$, provided that for every $i = 1, \dots, N$ the derivative $\frac{\partial(\phi_{\min}^{**})_*}{\partial x_i}(x,t)$ exists.

3.1. Imbedding theorem.

Theorem 3.1. *Let Ω be an open bounded subset of $\mathbb{R}^N$, $(N \ge 2)$, with the cone property. Assume that* (3.1) *and* (3.3) *are fulfilled, $(\phi_{\min}^{**})_*(\cdot, t)$ is Lipschitz continuous on $\overline{\Omega}$ and $\phi_{\max}$ is locally integrable. Then, there is a continuous embedding*

$$W^1 L_{\vec{\phi}}(\Omega) \hookrightarrow L_{(\phi_{\min}^{**})_*}(\Omega).$$

Some remarks about Theorem 3.1 are in order. We discuss how Theorem 3.1 include some previous results known in the literature when reducing to some particular Musielak-Orlicz functions.

Remark 3.2. (1) Let $M(x,t) = t^{p(x)}$ and $m(x,t) = \frac{\partial M(x,t)}{\partial t} = p(x)t^{p(x)-1}$, where $p(\cdot)$ is Lipschitz continuous on $\overline{\Omega}$, with $1 < p_- = \inf_{x\in\Omega} p(x) \le p(x) \le p_+ =$

$\sup_{x\in\Omega} p(x) < N$. Since $M(\cdot,t)$ and $m(\cdot,t)$ are continuous on $\overline{\Omega}$, we can use Lemma 5.8 (given in Appendix) to define the following Musielak-Orlicz function

$$\phi(x,t) = \begin{cases} \frac{t_1^{p(x)}}{t_1^{\alpha}} t^{\alpha} & \text{if } t \le t_1, \\ t^{p(x)} & \text{if } t \ge t_1, \end{cases}$$

where $t_1 > 1$ and $\alpha > 1$ are two constants mentioned in the proof of Lemma 5.8. Let us now consider the particular case where for all $i = 1,\dots,N$,

$$\phi_i(x,t) = \phi(x,t) = \begin{cases} \frac{t_1^{p(x)}}{t_1^{\alpha}} t^{\alpha} & \text{if } t \le t_1, \\ t^{p(x)} & \text{if } t \ge t_1. \end{cases} \tag{3.4}$$

It is worth pointing out that since Ω is of finite Lebesgue measure, it can be seen easily that $W^1L_{\vec{\phi}}(\Omega) = W^1L_{\phi}(\Omega) = W^{1,p(\cdot)}(\Omega)$. Thus, $\phi_{\min}^{**}(x,t) = \phi_{\min}(x,t) = \phi(x,t)$ and

$$(\phi_{\min}^{**})_*(x,t) = (\phi_{\min})_*(x,t) = \phi_*(x,t) = \begin{cases} \big(\frac{(N-\alpha)t}{N\alpha t_1}\big)^{\frac{N\alpha}{N-\alpha}} t_1^{\frac{Np(x)}{N-\alpha}} & \text{if } t \le t_1, \\ \big(\frac{1}{p_*(x)} t\big)^{p_*(x)} & \text{if } t \ge t_1, \end{cases}$$

provided that $\alpha < N$. Now we shall prove that $(\phi_{\min}^{**})_*$ satisfies (3.3) and our imbedding result include some previous result known in the literature. For every $t \in \mathbb{R}$ and for almost every $x \in \Omega$ we have

$$\frac{\partial(\phi_{\min}^{**})_*}{\partial x_i}(x,t) = \begin{cases} \frac{N}{N-\alpha}\frac{\partial p(x)}{\partial x_i} \log(t_1)(\phi_{\min}^{**})_*(x,t) & \text{if } t \le t_1, \\ \frac{\partial p_*(x)}{\partial x_i} \log\big(\frac{t}{ep_*(x)}\big)(\phi_{\min}^{**})_*(x,t) & \text{if } t \ge t_1. \end{cases}$$

- If $t \le t_1$, then

$$\big|\frac{\partial(\phi_{\min}^{**})_*}{\partial x_i}(x,t)\big| = \frac{N}{N-\alpha}\big|\frac{\partial p}{\partial x_i}(x)\big| \log(t_1)(\phi_{\min}^{**})_*(x,t).$$

Since $p(\cdot)$ is Lipschitz continuous on $\overline{\Omega}$ there exists a constant $C_1 > 0$ satisfying $\big|\frac{\partial p}{\partial x_i}(x)\big| \le C_1$ thus we obtain

$$\big|\frac{\partial(\phi_{\min}^{**})_*}{\partial x_i}(x,t)\big| \le C_1\frac{N}{N-\alpha} \log(t_1)(\phi_{\min}^{**})_*(x,t). \tag{3.5}$$

- If $t \ge t_1$, then

$$\big|\frac{\partial(\phi_{\min}^{**})_*}{\partial x_i}(x,t)\big| = \big|\frac{\partial p_*}{\partial x_i}(x)\big|\,\big|\log\Big(\frac{t}{ep_*(x)}\Big)\big|(\phi_{\min}^{**})_*(x,t).$$

Since $p(\cdot)$ is Lipschitz continuous on $\overline{\Omega}$, it can be seen easily that $p_*(\cdot)$ is also Lipschitz continuous on $\overline{\Omega}$. Then, there exists a constant $C_2 > 0$ satisfying $\Big|\frac{\partial p_*}{\partial x_i}(x)\Big| \le C_2$. So that we have

$$\big|\frac{\partial(\phi_{\min}^{**})_*}{\partial x_i}(x,t)\big| \le C_2\big|\log\Big(\frac{t}{ep_*(x)}\Big)\big|(\phi_{\min}^{**})_*(x,t).$$

Let $0 < \epsilon < 1/N$. For all $t > 0$ we can easily check that

$$\log(t) \le \frac{1}{\epsilon^2 Ne} t^{\epsilon}. \tag{3.6}$$

Now, since the Musielak-Orlicz function $(\phi_{\min}^{**})_*$ has a superlinear growth, we can choose $A>0$ for which there exists $t_0>\max\{t_1,e\}$ (not depending on x) such that $At\le(\phi_{\min}^{**})_*(x,t)$ whenever $t\ge t_0$. Therefore,

• If $t\ge t_0$ then by (3.6) we obtain

$$
\begin{aligned}
&\Big|\frac{\partial(\phi_{\min}^{**})_*}{\partial x_i}(x,t)\Big|\\
&\le C_2\Big(\log\Big(\frac{t}{e}\Big)+\log(\frac{N^2}{N-p_+})\Big)(\phi_{\min}^{**})_*(x,t)\\
&\le \frac{C_2}{\epsilon^2Ne^{1+\epsilon}}t^{\epsilon}(\phi_{\min}^{**})_*(x,t)+C_2\log(\frac{N^2}{N-p_+})(\phi_{\min}^{**})_*(x,t)\\
&\le \frac{C_2}{\epsilon^2Ne^{1+\epsilon}A^{\epsilon}}((\phi_{\min}^{**})_*(x,t))^{1+\epsilon}+C_2\log(\frac{N^2}{N-p_+})(\phi_{\min}^{**})_*(x,t).
\end{aligned}
\tag{3.7}
$$

• If $t_1<t\le t_0$, then

$$
\Big|\frac{\partial(\phi_{\min}^{**})_*}{\partial x_i}(x,t)\Big|\le C_2\Big(\log(t_0)+\log\Big(\frac{eN^2}{N-p_+}\Big)\Big)(\phi_{\min}^{**})_*(x,t). \tag{3.8}
$$

Therefore, from (3.5), (3.7) and (3.8), we obtain that for every $t\ge0$ and for almost every $x\in\Omega$, there is a constant $c_0>0$ such that

$$
\Big|\frac{\partial(\phi_{\min}^{**})_*}{\partial x_i}(x,t)\Big|\le c_0\Big((\phi_{\min}^{**})_*(x,t)+((\phi_{\min}^{**})_*(x,t))^{1+\epsilon}\Big).
$$

Before we show that our imbedding result includes some previous known results in the literature, we remark that the proof of Theorem 3.1 relies to the application of Lemma 5.4 in Appendix for the function $g(x,t)=((\phi_{\min}^{**})_*(x,t))^{\alpha}$, $\alpha\in(0,1)$, where we have used the fact that Ω is bounded to ensure that $\max_{x\in\overline{\Omega}}g(x,t)<\infty$ for some $t>0$. In the case of the variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ built upon the Musielak-Orlicz function given in (3.4), we do not need Ω to be bounded, since

$$
\phi_*(x,t)\le\max\{t_1^{\frac{N\alpha}{N-\alpha}},t^{\frac{N^2}{N-p_+}}\}<\infty,\quad\text{for some } t>0.
$$

Therefore, the embedding result in Theorem 3.1 can be seen as an extension to the Musielak-Orlicz framework of the one obtained in [9, Theorem 1.1].

(2) Let us consider the particular case where, for $i\in\{1,\dots,N\}$,

$$
\phi_i(x,t)=\begin{cases}\frac{t_1^{p_i(x)}}{t_1^{\alpha}}t^{\alpha} & \text{if } t\le t_1,\\ t^{p_i(x)} & \text{if } t\ge t_1\end{cases}
$$

where $t_1>1$, $1<\alpha<N$ and $\vec{\phi}=(\phi_i)_{i\in\{1,\dots,N\}}$ with

$$
p_i\in C_+(\overline{\Omega})=\{h\in C(\overline{\Omega}):\inf_{x\in\overline{\Omega}}h(x)>1\},
$$

$1<p_i(x)<N$, $N\ge3$. We define $p_i^-=\inf_{x\in\Omega}p_i(x)$, $p_M(x)=\max_{i\in\{1,\dots,N\}}p_i(x)$, $p_m(x)=\min_{i\in\{1,\dots,N\}}p_i(x)$. Then

$$
\phi_{\min}^{**}(x,t)=\phi_{\min}(x,t)=\begin{cases}\frac{t_1^{p_m(x)}}{t_1^{\alpha}}t^{\alpha} & \text{if } t\le t_1,\\ t^{p_m(x)} & \text{if } t\ge t_1,\end{cases}
$$

whose Sobolev conjugate function is

$$(\phi_{\min}^{**})_*(x,t)=\begin{cases}\big(\frac{(N-\alpha)t}{N\alpha t_1}\big)^{\frac{N\alpha}{N-\alpha}}t_1^{\frac{Np_m(x)}{N-\alpha}} & \text{if } t\le t_1,\\ \big(\frac{1}{(p_m)_*(x)}t\big)^{(p_m)_*(x)} & \text{if } t\ge t_1.\end{cases}$$

Let us define $p_-^*=\frac{N}{\sum_{i=1}^N\frac{1}{p_i^-}-1}$. Notice that $p_i^->p_m^-$ implies

$$p_-^*>\frac{Np_m^-}{N-p_m^-}=(p_m^-)_*. \tag{3.9}$$

Since Ω is of finite Lebesgue measure, it can be seen easily that $W^1L_{\vec{\phi}}(\Omega)=W^{1,\vec{p}(\cdot)}(\Omega)$. So, by Theorem 3.1 we have $W^{1,\vec{p}(\cdot)}(\Omega)\hookrightarrow L^{(p_m)_*(\cdot)}(\Omega)$ and since $(p_m)_*(x)\ge(p_m^-)_*$ for each $x\in\overline{\Omega}$, we deduce that $W^{1,\vec{p}(\cdot)}(\Omega)\hookrightarrow L^{(p_m^-)_*}(\Omega)$. Therefore, by (3.9) the result we obtain can be found in [19, Theorem 1].

(3) Let us now consider the case where

$$\phi_i(x,t)=\begin{cases}\frac{t_1^{p_i(x)}\log(t_1+1)}{t_1^\alpha}t^\alpha & \text{if } t\le t_1,\\ t^{p_i(x)}\log(t+1) & \text{if } t\ge t_1,\end{cases}$$

where $t_1>1$, $1<\alpha<N$ and for each $i\in\{1,\dots,N\}$ the function $p_i(\cdot)$ is Lipschitz continuous on $\overline{\Omega}$ with $1<\inf_{x\in\overline{\Omega}}p_i(x)\le p_i(x)\le\sup_{x\in\overline{\Omega}}p_i(x)<N-1$. Define $p_M(x)=\max_{i\in\{1,\dots,N\}}p_i(x)$, $p_m(x)=\min_{i\in\{1,\dots,N\}}p_i(x)$ and $\phi_{\min}(x,t)=\min_{i\in\{1,\dots,N\}}\phi_i(x,t)$. Then

$$\phi_{\min}(x,t)=\phi_{\min}^{**}(x,t)=\begin{cases}\frac{t_1^{p_m(x)}\log(t_1+1)}{t_1^\alpha}t^\alpha & \text{if } t\le t_1,\\ t^{p_m(x)}\log(t+1) & \text{if } t\ge t_1.\end{cases}$$

Set $A(x,t)=t^{p_m(x)}\log(t+1)$. By [18, Example 2] there exist $\sigma<\frac{1}{N}$, $C_0>0$ and $t_0>0$ such that

$$\big|\frac{\partial A_*}{\partial x_i}(x,t)\big|\le C_0(A_*(x,t))^{1+\sigma},$$

for $x\in\Omega$ and $t\ge t_0$. Choosing this $t_0>0$ in Lemma 5.8 given in Appendix, we can take $t_1>t_0+1$ obtaining

$$\big|\frac{\partial A_*}{\partial x_i}(x,t)\big|\le C_0(A_*(x,t))^{1+\sigma},\quad \text{for all } t\ge t_1. \tag{3.10}$$

On the other hand, for $t\le t_1$ we have

$$(\phi_{\min}^{**})_*(x,t)=\Big(\frac{(N-\alpha)t}{N\alpha t_1}\Big)^{\frac{N\alpha}{N-\alpha}}\Big(\frac{t_1^{p_m(x)}}{\log(t_1+1)}\Big)^{\frac{N}{N-\alpha}}.$$

Thus

$$\big|\frac{\partial(\phi_{\min}^{**})_*}{\partial x_i}(x,t)\big|=\frac{N\log(t_1)}{N-\alpha}\big|\frac{\partial p_m}{\partial x_i}(x)\big|(\phi_{\min}^{**})_*(x,t).$$

Since $p_m(\cdot)$ is Lipschitz continuous on $\overline{\Omega}$ there exists a constant $C_3>0$ satisfying $\Big|\frac{\partial p_m}{\partial x_i}(x)\Big|\le C_3$. So we have

$$\big|\frac{\partial(\phi_{\min}^{**})_*}{\partial x_i}(x,t)\big|\le\frac{C_3N\log(t_1)}{N-\alpha}(\phi_{\min}^{**})_*(x,t). \tag{3.11}$$

Therefore, by (3.10) and (3.11) the function $(\phi_{\min}^{**})_*$ satisfies the assertions of Theorem 3.1 and then we obtain the continuous embedding

$$W^1L_{\vec{\phi}}(\Omega) \hookrightarrow L_{(\phi_{\min}^{**})_*}(\Omega).$$

Proof ofTheorem 3.1. Let $u \in W^1L_{\vec{\phi}}(\Omega)$. Assume first that the function u is bounded and $u \neq 0$. Defining $f(s) = \int_\Omega (\phi_{\min}^{**})_*\big(x, \frac{|u(x)|}{s}\big)\,dx$, for $s > 0$, one has $\lim_{s\to 0^+} f(s) = +\infty$ and $\lim_{s\to\infty} f(s) = 0$. Since f is continuous on $(0, +\infty)$, there exists $\lambda > 0$ such that $f(\lambda) = 1$. Then by the definition of the Luxemburg norm, we obtain

$$\|u\|_{(\phi_{\min}^{**})_*} \leq \lambda. \tag{3.12}$$

On the other hand,

$$f(\|u\|_{(\phi_{\min}^{**})_*}) = \int_\Omega (\phi_{\min}^{**})_*\Big(x, \frac{u(x)}{\|u\|_{(\phi_{\min}^{**})_*}}\Big)\,dx \leq 1 = f(\lambda)$$

and since f is decreasing,

$$\lambda \leq \|u\|_{(\phi_{\min}^{**})_*}. \tag{3.13}$$

So that by (3.12) and (3.13), we obtain $\lambda = \|u\|_{(\phi_{\min}^{**})_*}$ and

$$\int_\Omega (\phi_{\min}^{**})_*\Big(x, \frac{u(x)}{\lambda}\Big)\,dx = 1. \tag{3.14}$$

From (3.2) we can easily check that $(\phi_{\min}^{**})_*$ satisfies the differential equation

$$(\phi_{\min}^{**})^{-1}(x, (\phi_{\min}^{**})_*(x,t))\frac{\partial(\phi_{\min}^{**})_*}{\partial t}(x,t) = ((\phi_{\min}^{**})_*(x,t))^{\frac{N+1}{N}}.$$

Hence, by (2.7) we obtain the inequality

$$\frac{\partial(\phi_{\min}^{**})_*}{\partial t}(x,t) \leq ((\phi_{\min}^{**})_*(x,t))^{\frac{1}{N}}(\phi_{\min}^{**})^{*-1}(x, (\phi_{\min}^{**})_*(x,t)), \tag{3.15}$$

for a.e. $x \in \Omega$. Let

$$h(x) = \Big[(\phi_{\min}^{**})_*\Big(x, \frac{u(x)}{\lambda}\Big)\Big]^{\frac{N-1}{N}}. \tag{3.16}$$

Since $(\phi_{\min}^{**})_*(\cdot, t)$ is Lipschitz continuous on $\overline{\Omega}$ and $(\phi_{\min}^{**})_*(x, \cdot)$ is locally Lipschitz continuous on $\mathbb{R}^+$, the function h is Lipschitz continuous on $\overline{\Omega}$. Hence, we can compute using Lemma 5.6 (given in Appendix) for $f = h$, obtaining for a.e. $x \in \Omega$,

$$\begin{aligned} \frac{\partial h}{\partial x_i}(x) = \frac{N-1}{N}\Big((\phi_{\min}^{**})_*\Big(x, \frac{u(x)}{\lambda}\Big)\Big)^{-\frac{1}{N}}\Big[&\frac{\partial(\phi_{\min}^{**})_*}{\partial t}\Big(x, \frac{u(x)}{\lambda}\Big)\frac{\partial_{x_i}u}{\lambda}(x) \\ &+ \frac{\partial(\phi_{\min}^{**})_*}{\partial x_i}\Big(x, \frac{u(x)}{\lambda}\Big)\Big], \end{aligned}$$

where $\partial_{x_i}u := \frac{\partial u}{\partial x_i}$. Therefore,

$$\sum_{i=1}^N \Big|\frac{\partial h}{\partial x_i}(x)\Big| \leq I_1 + I_2, \quad \text{for a.e. } x \in \Omega, \tag{3.17}$$

where we have set

$$I_1 = \frac{N-1}{N\lambda}\Big((\phi_{\min}^{**})_*\Big(x, \frac{u(x)}{\lambda}\Big)\Big)^{\frac{-1}{N}}\frac{\partial(\phi_{\min}^{**})_*}{\partial t}\Big(x, \frac{u(x)}{\lambda}\Big)\sum_{i=1}^N |\partial_{x_i}u(x)|,$$

$$I_2 = \frac{N-1}{N}\Big((\phi_{\min}^{**})_*\Big(x, \frac{u(x)}{\lambda}\Big)\Big)^{\frac{-1}{N}} \sum_{i=1}^{N} \Big|\frac{\partial(\phi_{\min}^{**})_*}{\partial x_i}\Big(x, \frac{u(x)}{\lambda}\Big)\Big|.$$

Now we estimate the two integrals $\int_\Omega I_1(x)\,dx$ and $\int_\Omega I_2(x)\,dx$. By (3.15), we can write

$$I_1(x) \le \frac{N-1}{N\lambda}(\phi_{\min}^{**})^{*-1}\Big(x, (\phi_{\min}^{**})_*\Big(x, \frac{u(x)}{\lambda}\Big)\Big) \sum_{i=1}^{N} |\partial_{x_i} u(x)|. \tag{3.18}$$

By (2.6), we have

$$\int_\Omega (\phi_{\min}^{**})^*\Big(x, (\phi_{\min}^{**})^{*-1}\Big(x, (\phi_{\min}^{**})_*\Big(x, \frac{u(x)}{\lambda}\Big)\Big)\Big)\,dx \le \int_\Omega (\phi_{\min}^{**})_*\Big(x, \frac{u(x)}{\lambda}\Big)\,dx = 1.$$

Hence

$$\big\|(\phi_{\min}^{**})^{*-1}\Big(\cdot, (\phi_{\min}^{**})_*\Big(\cdot, \frac{u(\cdot)}{\lambda}\Big)\Big)\big\|_{(\phi_{\min}^{**})^*} \le 1. \tag{3.19}$$

From (2.5), (3.18) and (3.19) it follows that

$$\begin{aligned}
&\int_\Omega I_1(x)\,dx \\
&\le \frac{2(N-1)}{N\lambda}\big\|(\phi_{\min}^{**})^{*-1}\Big(\cdot, (\phi_{\min}^{**})_*\Big(\cdot, \frac{u(\cdot)}{\lambda}\Big)\Big)\big\|_{(\phi_{\min}^{**})^*} \sum_{i=1}^{N} \big\|\partial_{x_i} u\big\|_{\phi_{\min}^{**}} \\
&\le \frac{2(N-1)}{N\lambda} \sum_{i=1}^{N} \big\|\partial_{x_i} u\big\|_{\phi_{\min}^{**}} \\
&\le \frac{2}{\lambda} \sum_{i=1}^{N} \big\|\partial_{x_i} u\big\|_{\phi_{\min}^{**}}.
\end{aligned} \tag{3.20}$$

Recalling the definition of $\phi_{\min}$ and (2.2), we obtain $\|\partial_{x_i} u(x)\|_{\phi_{\min}^{**}} \le \|\partial_{x_i} u(x)\|_{\phi_i}$, so that (3.20) implies

$$\int_\Omega I_1(x)\,dx \le \frac{2}{\lambda} \sum_{i=1}^{N} \big\|\partial_{x_i} u(x)\big\|_{\phi_i}. \tag{3.21}$$

Using (3.3) we can write

$$I_2(x) \le c_1\Big[\Big((\phi_{\min}^{**})_*\Big(x, \frac{u(x)}{\lambda}\Big)\Big)^{1-\frac{1}{N}} + \Big((\phi_{\min}^{**})_*\Big(x, \frac{u(x)}{\lambda}\Big)\Big)^{1-\frac{1}{N}+\nu}\Big],$$

with $c_1 = c_0(N-1)$. Since $(\phi_{\min}^{**})_*(\cdot, t)$ is continuous on $\overline{\Omega}$ and $\nu < \frac{1}{N}$, we can apply Lemma 5.4 (given in Appendix) with the functions $g(x,t) = \frac{((\phi_{\min}^{**})_*(x,t))^{1-\frac{1}{N}+\nu}}{t}$ and $f(x,t) = \frac{(\phi_{\min}^{**})_*(x,t)}{t}$ and $\epsilon = \frac{1}{8c_1c_*}$ obtaining for $t = \frac{|u(x)|}{\lambda}$

$$\Big[(\phi_{\min}^{**})_*\Big(x, \frac{u(x)}{\lambda}\Big)\Big]^{1-\frac{1}{N}+\nu} \le \frac{1}{8c_1c_*}(\phi_{\min}^{**})_*\Big(x, \frac{u(x)}{\lambda}\Big) + K_0\frac{|u(x)|}{\lambda}. \tag{3.22}$$

Using again Lemma 5.4 with the functions $g(x,t) = \frac{((\phi_{\min}^{**})_*(x,t))^{1-\frac{1}{N}}}{t}$ and $f(x,t) = \frac{(\phi_{\min}^{**})_*(x,t)}{t}$ and $\epsilon = \frac{1}{8c_1c_*}$, we obtain by substituting t by $\frac{|u(x)|}{\lambda}$

$$\Big[(\phi_{\min}^{**})_*\Big(x, \frac{u(x)}{\lambda}\Big)\Big]^{1-\frac{1}{N}} \le \frac{1}{8c_1c_*}(\phi_{\min}^{**})_*\Big(x, \frac{u(x)}{\lambda}\Big) + K_0\frac{|u(x)|}{\lambda}, \tag{3.23}$$

where c_* is the constant in the continuous embedding $W^{1,1}(\Omega) \hookrightarrow L^{\frac{N}{N-1}}(\Omega)$, that is

$$\|w\|_{L^{\frac{N}{N-1}}(\Omega)} \le c_* \|w\|_{W^{1,1}(\Omega)}, \text{ for all } w \in W^{1,1}(\Omega). \tag{3.24}$$

By (3.22) and (3.23), we obtain

$$\int_\Omega I_2(x)\,dx \le \frac{1}{4c_*} + \frac{2K_0c_1}{\lambda}\|u\|_{L^1(\Omega)}. \tag{3.25}$$

Putting together (3.21) and (3.25) in (3.17) we obtain

$$\begin{aligned}\sum_{i=1}^N \|\partial_{x_i} h\|_{L^1(\Omega)} &\le \frac{1}{4c_*} + \frac{2}{\lambda}\sum_{i=1}^N \|\partial_{x_i} u(x)\|_{\phi_i} + \frac{2K_0c_1}{\lambda}\|u\|_{L^1(\Omega)}\\ &\le \frac{1}{4c_*} + \frac{2}{\lambda}\sum_{i=1}^N \|\partial_{x_i} u(x)\|_{\phi_i} + \frac{2K_0c_1c_2}{\lambda}\|u\|_{\phi_{\max}},\end{aligned}$$

where c_2 is the constant in the continuous embedding (2.3). Then it follows that

$$\sum_{i=1}^N \|\partial_{x_i} h\|_{L^1(\Omega)} \le \frac{1}{4c_*} + \frac{c_3}{\lambda}\|u\|_{W^1L_{\vec{\phi}}(\Omega)}, \tag{3.26}$$

with $c_3 = \max\{2, 2K_0c_1c_2\}$. Now, using again Lemma 5.4 (in Appendix) with the functions $g(x,t) = \big[(\phi_{\min}^{**})_*(x,t)\big]^{1-\frac{1}{N}}/t$ and $f(x,t) = (\phi_{\min}^{**})_*(x,t)/t$ and $\epsilon = \frac{1}{4c_*}$, for $t = |u(x)|/\lambda$, we obtain

$$h(x) \le \frac{1}{4c_*}(\phi_{\min}^{**})_*\Big(x, \frac{u(x)}{\lambda}\Big) + K_0\frac{|u(x)|}{\lambda},$$

From (2.3), we obtain

$$\|h\|_{L^1(\Omega)} \le \frac{1}{4c_*} + \frac{K_0c_2}{\lambda}\|u\|_{L_{\phi_{\max}}(\Omega)}. \tag{3.27}$$

Thus, by (3.26) and (3.27) we obtain

$$\|h\|_{W^{1,1}(\Omega)} \le \frac{1}{2c_*} + \frac{c_4}{\lambda}\|u\|_{W^1L_{\vec{\phi}}(\Omega)},$$

where $c_4 = c_3 + K_0c_2$, which shows that $h \in W^{1,1}(\Omega)$ and which together with (3.24) yield

$$\|h\|_{L^{\frac{N}{N-1}}(\Omega)} \le \frac{1}{2} + \frac{c_4c_*}{\lambda}\|u\|_{W^1L_{\vec{\phi}}(\Omega)}.$$

Having in mind (3.14), we obtain

$$\int_\Omega [h(x)]^{\frac{N}{N-1}}\,dx = \int_\Omega (\phi_{\min}^{**})_*\Big(x, \frac{u(x)}{\lambda}\Big)\,dx = 1.$$

So that one has

$$\|u\|_{(\phi_{\min}^{**})_*} = \lambda \le 2c_4c_*\|u\|_{W^1L_{\vec{\Phi}}(\Omega)}. \tag{3.28}$$

We now extend the estimate (3.28) to an arbitrary $u \in W^1L_{\vec{\phi}}(\Omega)$. Let T_n, $n > 0$, be the truncation function at levels $\pm n$ defined on $\mathbb{R}$ by $T_n(s) = \min\{n, \max\{s, -n\}\}$. Since $\phi_{\max}$ is locally integrable, by [1, Lemma 8.34.] one has $T_n(u) \in W^1L_{\vec{\phi}}(\Omega)$. So that in view of (3.28)

$$\|T_n(u)\|_{(\phi_{\min}^{**})_*} \le 2c_4c_*\|T_n(u)\|_{W^1L_{\vec{\phi}(\Omega)}} \le 2c_4c_*\|u\|_{W^1L_{\vec{\phi}(\Omega)}}. \tag{3.29}$$

Let $k_n = \|T_n(u)\|_{(\phi^{**}_{\min})_*}$. Thanks to (3.29), the sequence $\{k_n\}_{n=1}^{\infty}$ is nondecreasing and converges. If we denote $k = \lim_{n\to\infty} k_n$, by Fatou's lemma we have

$$\int_\Omega (\phi^{**}_{\min})_*\Big(x, \frac{|u(x)|}{k}\Big)\,dx \le \liminf \int_\Omega (\phi^{**}_{\min})_*\Big(x, \frac{|T_n(u)|}{k_n}\Big)\,dx \le 1.$$

This implies that $u \in L_{(\phi^{**}_{\min})_*}(\Omega)$ and

$$\|u\|_{(\phi^{**}_{\min})_*} \le k = \lim_{n\to\infty} \|T_n(u)\|_{(\phi^{**}_{\min})_*} \le 2c_4c_*\|u\|_{W^1L_{\vec{\phi}(\Omega)}}.$$

Inequality (3.28) trivially holds if $u = 0$. Then proof is complete. □

Corollary 3.3. *Let Ω be an open bounded subset of $\mathbb{R}^N$, $N \ge 2$, with the cone property. Assume that* (3.1), (3.3) *are fulfilled, $(\phi^{**}_{\min})_*(\cdot,t)$ is Lipschitz continuous on $\overline{\Omega}$ and $\phi_{\max}$ is locally integrable. Let A be a Musielak-Orlicz function where the function $A(\cdot,t)$ is continuous on $\overline{\Omega}$ and $A \ll (\phi^{**}_{\min})_*$. Then, the embedding $W^1L_{\vec{\phi}}(\Omega) \hookrightarrow L_A(\Omega)$ is compact.*

Proof. Let $\{u_n\}$ is a bounded sequence in $W^1L_{\vec{\phi}}(\Omega)$. By Theorem 3.1, $\{u_n\}$ is bounded in $L_{(\phi^{**}_{\min})_*}(\Omega)$. Since the embedding $W^1L_{\vec{\phi}}(\Omega) \hookrightarrow W^{1,1}(\Omega)$ is continuous and the imbedding $W^{1,1}(\Omega) \hookrightarrow L^1(\Omega)$ is compact, we deduce that there exists a subsequence of $\{u_n\}$ still denoted by $\{u_n\}$ which converges in measure in Ω. Since $A \ll (\phi^{**}_{\min})_*$, by Lemma 5.5 (in Appendix) the sequence $\{u_n\}$ converges in norm in $L_A(\Omega)$. □

3.2. **Trace result.** We prove here a trace result which is a useful tool to prove the coercivity of some energy functionals. Recall that $\psi_{\min}(x,t) = [(\phi^{**}_{\min})_*(x,t)]^{\frac{N-1}{N}}$ is a Musielak-Orlicz function. Indeed, we have

$$\frac{\partial}{\partial t}(\psi_{\min})^{-1}(x,t) = \frac{\partial}{\partial t}(\phi^{**}_{\min})_*^{-1}\big(x, t^{\frac{N}{N-1}}\big).$$

By (3.2), we obtain

$$\frac{\partial}{\partial t}(\psi_{\min})^{-1}(x,t) = \frac{N}{N-1} t^{\frac{1}{N-1}} \frac{(\phi^{**}_{\min})^{-1}\big(x, t^{\frac{N}{N-1}}\big)}{t^{\frac{N}{N-1}+\frac{1}{N-1}}} = \frac{N}{N-1} \frac{(\phi^{**}_{\min})^{-1}\big(x, t^{\frac{N}{N-1}}\big)}{t^{\frac{N}{N-1}}}.$$

Being the inverse of a Musielak-Orlicz function, it is clear that $(\phi^{**}_{\min})^{-1}$ satisfies

$$\lim_{\tau\to+\infty} \frac{(\phi^{**}_{\min})^{-1}(x,\tau)}{\tau} = 0 \quad \text{and} \quad \lim_{\tau\to 0^+} \frac{(\phi^{**}_{\min})^{-1}(x,\tau)}{\tau} = +\infty.$$

Moreover, $(\phi^{**}_{\min})^{-1}(x,\cdot)$ is concave so that if $0 < \tau < \sigma$ then we obtain

$$\frac{(\phi^{**}_{\min})^{-1}(x,\tau)}{(\phi^{**}_{\min})^{-1}(x,\sigma)} \ge \frac{\tau}{\sigma}.$$

Hence, if $0 < s_1 < s_2$, then

$$\frac{\frac{\partial}{\partial t}(\psi_{\min})^{-1}(x,s_1)}{\frac{\partial}{\partial t}(\psi_{\min})^{-1}(x,s_2)} = \frac{(\phi^{**}_{\min})^{-1}\big(x, s_1^{\frac{N}{N-1}}\big)}{(\phi^{**}_{\min})^{-1}\big(x, s_2^{\frac{N}{N-1}}\big)} \frac{s_2^{\frac{N}{N-1}}}{s_1^{\frac{N}{N-1}}} \ge \frac{s_1^{\frac{N}{N-1}}}{s_2^{\frac{N}{N-1}}} \frac{s_2^{\frac{N}{N-1}}}{s_1^{\frac{N}{N-1}}} = 1.$$

It follows that $\frac{\partial}{\partial t}(\psi_{\min})^{-1}(x,t)$ is positive and decreases monotonically from $+\infty$ to 0 as t increases from 0 to $+\infty$ and thus $\psi_{\min}$ is a Musielak-Orlicz function.

Theorem 3.4. *Let Ω be an open bounded subset of $\mathbb{R}^N$, $N \geq 2$, with the cone property. Assume that* (3.1), (3.3) *are fulfilled, $(\phi_{\min}^{**})_*(\cdot,t)$ is Lipschitz continuous on $\overline{\Omega}$ and $\phi_{\max}$ is locally integrable. Let $\psi_{\min}$ the Musielak-Orlicz function defined in* (1.8)*. Then, the following boundary trace embedding $W^1L_{\vec{\phi}}(\Omega) \hookrightarrow L_{\psi_{\min}}(\partial\Omega)$ is continuous.*

Remark 3.5. In the case where for all $i = 1, \dots, N$,

$$\phi_i(x,t) = \phi(x,t) = \begin{cases} \frac{t_1^{p(x)}}{t_1^{\alpha}} t^{\alpha} & \text{if } t \leq t_1, \\ t^{p(x)} & \text{if } t \geq t_1, \end{cases}$$

for some $t_1 > 0$, with $p \in L^\infty(\Omega)$, $1 \leq \inf_{x\in\Omega} p(x) \leq \sup_{x\in\Omega} p(x) < N$, $|\nabla p| \in L^{\gamma(\cdot)}(\Omega)$, where $\gamma \in L^\infty(\Omega)$ and $\inf_{x\in\Omega} \gamma(x) > N$. It is worth pointing out that since Ω is of finite Lebesgue measure, it can be seen easily that $W^1L_{\vec{\phi}}(\Omega) = W^1L_{\phi}(\Omega) = W^{1,p(\cdot)}(\Omega)$. Then $\phi_{\min}^{**}(x,s) = \phi_{\min}(x,t) = \phi(x,t)$ and so

$$(\phi_{\min}^{**})_*(x,t) = (\phi_{\min})_*(x,t) = \phi_*(x,t) = \begin{cases} \big(\frac{(N-\alpha)t}{N\alpha t_1}\big)^{\frac{N\alpha}{N-\alpha}} t_1^{\frac{Np(x)}{N-\alpha}} & \text{if } t \leq t_1, \\ \big(\frac{1}{p_*(x)} t\big)^{p_*(x)} & \text{if } t \geq t_1. \end{cases}$$

As above we can prove that $(\phi_{\min}^{**})_*$ satisfies the conditions of Theorem 3.4 and then our trace result is an extension to Musielak-Orlicz framework of the one proved by Fan in [8].

Proof of Theorem 3.4. Let $u \in W^1L_{\vec{\phi}}(\Omega)$. Because of the continuous embedding $W^1L_{\vec{\phi}}(\Omega) \hookrightarrow L_{(\phi_{\min}^{**})_*}(\Omega)$, the function u belongs to $L_{(\phi_{\min}^{**})_*}(\Omega)$ and then u belongs to $L_{\psi_{\min}}(\Omega)$. Clearly $W^1L_{\vec{\phi}}(\Omega) \hookrightarrow W^{1,1}(\Omega)$ and by the Gagliardo trace theorem (see [10]) we have the embedding $W^{1,1}(\Omega) \hookrightarrow L^1(\partial\Omega)$. Hence, we conclude that for all $u \in W^1L_{\vec{\phi}}(\Omega)$ there holds $u|_{\partial\Omega} \in L^1(\partial\Omega)$. Therefore, for every $u \in W^1L_{\vec{\phi}}(\Omega)$ the trace $u|_{\partial\Omega}$ is well defined. Assume first that u is bounded and $u \neq 0$. Since $(\phi_{\min}^{**})_*(\cdot,t)$ is continuous on $\partial\Omega$, the function u belongs to $L_{\psi_{\min}}(\partial\Omega)$. Let

$$k = \|u\|_{L_{\psi_{\min}}(\partial\Omega)} = \inf\Big\{\lambda > 0 : \int_{\partial\Omega} \psi_{\min}\Big(x, \frac{u(x)}{\lambda}\Big)\,dx \leq 1\Big\}.$$

We distinguish the two cases: $k \geq \|u\|_{L_{(\phi_{\min}^{**})_*}(\Omega)}$ and $k < \|u\|_{L_{(\phi_{\min}^{**})_*}(\Omega)}$.

Case 1: Assume that

$$k \geq \|u\|_{L_{(\phi_{\min}^{**})_*}(\Omega)}. \tag{3.30}$$

Going back to (3.16) we can repeat exactly the same lines with $l(x) = \psi_{\min}\big(x, \frac{u(x)}{k}\big)$ instead of the function h, obtaining

$$\|l\|_{W^{1,1}(\Omega)} \leq \Big(\frac{1}{4c} + \frac{c_3}{k}\|u\|_{W^1L_{\vec{\phi}}(\Omega)} + \|l\|_{L^1(\Omega)}\Big), \tag{3.31}$$

where c is the constant in the imbedding $W^{1,1}(\Omega) \hookrightarrow L^1(\partial\Omega)$, that is

$$\|w\|_{L^1(\partial\Omega)} \leq c\|w\|_{W^{1,1}(\Omega)}, \quad \text{for all } w \in W^{1,1}(\Omega). \tag{3.32}$$

Since $(\phi_{\min}^{**})_*(\cdot,t)$ is continuous on $\overline{\Omega}$, using Lemma 5.4 (in Appendix) with the functions $f(x,t) = \frac{(\phi_{\min}^{**})_*(x,t)}{t}$ and $g(x,t) = \frac{l(x)}{t}$ and $\epsilon = \frac{1}{4c}$, we obtain for $t = \frac{|u(x)|}{k}$

$$l(x) \leq \frac{1}{4c}(\phi_{\min}^{**})_*\big(x, \frac{u(x)}{k}\big) + K_0\frac{|u(x)|}{k}. \tag{3.33}$$

By (3.30), we have

$$\int_\Omega (\phi^{**}_{\min})_*\big(x, \frac{u(x)}{k}\big)\,dx \le \int_\Omega (\phi^{**}_{\min})_*\Big(x, \frac{u(x)}{\|u\|_{(\phi^{**}_{\min})_*}}\Big)\,dx \le 1.$$

Integrating (3.33) over Ω, we obtain

$$\|l\|_{L^1(\Omega)} \le \frac{1}{4c} + \frac{K_0c_2}{k}\|u(x)\|_{L_{\phi_{\max}}(\Omega)} \le \frac{1}{4c} + \frac{K_0c_2}{k}\|u\|_{W^1L_{\vec\phi}(\Omega)}, \tag{3.34}$$

where c_2 is the constant of the imbedding (2.3). Thus, by (3.31) and (3.34) we obtain

$$\|l\|_{W^{1,1}(\Omega)} \le \frac{1}{2c} + \frac{C_4}{k}\|u\|_{W^1L_{\vec\phi}(\Omega)},$$

where $C_4 = c_2K_0 + c_3$. This implies that $l \in W^{1,1}(\Omega)$ and by (3.32) we arrive at

$$\|l\|_{L^1(\partial\Omega)} \le \frac{1}{2} + \frac{cC_4}{k}\|u\|_{W^1L_{\vec\phi}(\Omega)}.$$

As

$$\|l\|_{L^1(\partial\Omega)} = \int_{\partial\Omega} |l(x)|\,dx = \int_{\partial\Omega} \psi_{\min}\big(x, \frac{u(x)}{k}\big)\,dx = 1,$$

we obtain $\|u\|_{L_{\psi_{\min}}(\partial\Omega)} = k \le 2cC_4\|u\|_{W^1L_{\vec\phi}(\Omega)}$.

Case 2: Assume that $k < \|u\|_{(\phi^{**}_{\min})_*}$. By Theorem 3.1, there is a constant $c > 0$ such that

$$\|u\|_{L_{\psi_{\min}}(\partial\Omega)} = k < \|u\|_{(\phi^{**}_{\min})_*} \le c\|u\|_{W^1L_{\vec\phi}(\Omega)}.$$

Finally, in both cases there exists a constant $c > 0$ such that

$$\|u\|_{L_{\psi_{\min}}(\partial\Omega)} \le c\|u\|_{W^1L_{\vec\phi}(\Omega)}.$$

For an arbitrary $u \in W^1L_{\vec\phi}(\Omega)$, we proceed as in the proof of Theorem 3.1 by truncating the function u. $\square$

4. Application to anisotropic elliptic equations

In this section, we apply the above results to obtain the existence and uniqueness of the weak solution for the problem (1.1).

4.1. **Properties of the energy functional.**

Definition 4.1. Let Ω be an open bounded subset of $\mathbb{R}^N$, $N \ge 2$. By a weak solution of problem (1.1), we mean a function $u \in W^1L_{\vec\phi}(\Omega)$ satisfying for all $v \in C^\infty(\overline{\Omega})$ the identity

$$\begin{aligned} &\int_\Omega \sum_{i=1}^N a_i(x, \partial_{x_i}u)\partial_{x_i}v\,dx + \int_\Omega b(x)\varphi_{\max}(x,u)v\,dx \\ &- \int_\Omega f(x,u)v\,dx - \int_{\partial\Omega} g(x,u)v\,ds = 0. \end{aligned} \tag{4.1}$$

We note that all the terms in (4.1) make sense. Indeed, for the first term in the right hand side in (4.1), we can write by using (2.8)

$$\begin{aligned} \int_\Omega \phi_i^*(x, \phi_i^{*-1}(x, P_i(x, \partial_{x_i}u(x))))\,dx &\le \int_\Omega P_i(x, \partial_{x_i}u(x))\,dx \\ &\le \int_\Omega p_i(x, \partial_{x_i}u(x))\partial_{x_i}u(x)\,dx, \end{aligned}$$

where P_i is the Musielak-Orlicz function given in (1.2) and $p_i(x,s) = \frac{\partial P_i}{\partial s}(x,s)$. Since P_i is locally integrable and $P_i \ll \phi_i$, we can use Lemma 5.7 (see the Appendix) obtaining $p_i(\cdot, \partial_{x_i} u(\cdot)) \in L_{P_i^*}(\Omega)$. So that by Hölder's inequality (2.5), we obtain

$$\int_\Omega \phi_i^*(x, \phi_i^{*-1}(x, P_i(x, \partial_{x_i} u(x))))\, dx \leq 2\|p_i(\cdot, \partial_{x_i} u(\cdot))\|_{P_i^*}\|\partial_{x_i} u\|_{P_i} < \infty.$$

Thus, $\phi_i^{*-1}(\cdot, P_i(\cdot, \partial_{x_i} u(\cdot))) \in L_{\phi_i^*}(\Omega)$. Since $v \in C^\infty(\overline{\Omega})$ and $\phi_{\max}$ is locally integrable, then $v \in W^1 L_{\vec{\phi}}(\Omega)$. So we can use the growth condition (1.2) and again the Hölder inequality (2.5) to write

$$\begin{aligned} &\int_\Omega a_i(x, \partial_{x_i} u)\partial_{x_i} v\, dx \\ &\leq 2c_i\|d_i(\cdot)\|_{\phi_i^*}\|\partial_{x_i} v\|_{\phi_i} + 2c_i\|\phi_i^{*-1}(\cdot, P_i(\cdot, \partial_{x_i} u(\cdot)))\|_{\phi_i^*}\|\partial_{x_i} v\|_{\phi_i} < \infty. \end{aligned} \tag{4.2}$$

For the second term, the inequality (2.8) enables us to write

$$\int_\Omega \phi_{\max}^*(x, \phi_{\max}^{*-1}(x, R(x, u(x))))\, dx \leq \int_\Omega R(x, u(x))\, dx \leq \int_\Omega r(x, u(x))u(x)\, dx,$$

where R is the Musielak-Orlicz function given in (1.5) and $r(x,s) = \frac{\partial R(x,s)}{\partial s}$. Since R is locally integrable and $R \ll \phi_{\max}$, Lemma 5.7 (in Appendix) gives

$$\int_\Omega \phi_{\max}^*(x, \phi_{\max}^{*-1}(x, R(x, \partial_i u)))\, dx \leq 2\|r(\cdot, u(\cdot))\|_{R^*}\|u\|_R < \infty,$$

which shows that $\varphi_{\max}(\cdot, u(\cdot)) \in L_{\phi_{\max}^*}(\Omega)$. Thus,

$$\int_\Omega b(x)\varphi_{\max}(x,u)v\, dx \leq 2\|b\|_\infty \|\varphi_{\max}(\cdot, u(\cdot))\|_{\phi_{\max}^*}\|v\|_{\phi_{\max}} < \infty. \tag{4.3}$$

We now turn to the third term in the right hand side in (4.1). By using (1.6) and the Hölder inequality (2.5), one has

$$\Big|\int_\Omega f(x,u)v\, dx\Big| \leq k_1\|m(\cdot, u(\cdot))\|_{L_{M^*}(\Omega)}\|v\|_{L_M(\Omega)}. \tag{4.4}$$

Since M is locally integrable and $M \ll \phi_{\min}^{**}$, then $M \ll \phi_{\max}$ and Lemma 5.7 ensures that $|\int_\Omega f(x,u)v\, dx| < \infty$. For the last term in the right hand side in (4.1), using (1.7) to have

$$\Big|\int_{\partial\Omega} g(x,u)v\, ds\Big| \leq k_2 \int_{\partial\Omega} |h(x,u)v|\, ds.$$

Since the primitive H of h is a locally integrable function satisfying $H \ll \phi_{\min}^{**}$ we can use a similar way as in Lemma 5.7 to obtain $h(\cdot, u) \in L_{H^*}(\partial\Omega)$ and since $\partial\Omega$ is a bounded set, the imbedding (2.3) gives $h(x,u) \in L^1(\partial\Omega)$. On the other hand, since $v \in C^\infty(\overline{\Omega})$ one has $v \in L^\infty(\partial\Omega)$. Therefore,

$$\Big|\int_{\partial\Omega} g(x,u)v\, ds\Big| < \infty.$$

Define the functional $I : W^1 L_{\vec{\phi}}(\Omega) \to \mathbb{R}$ by

$$\begin{aligned} I(u) = &\int_\Omega \sum_{i=1}^N A_i(x, \partial_i u)\, dx + \int_\Omega b(x)\phi_{\max}(x,u)\, dx \\ &- \int_\Omega F(x,u)\, dx - \int_{\partial\Omega} G(x,u) ds. \end{aligned} \tag{4.5}$$

Some basic properties of I are established in the following lemma.

Lemma 4.2. *Let Ω be an open bounded subset of $\mathbb{R}^N$, $N \geq 2$. Then*

(i) *The functional I is well defined on $W^1 L_{\vec{\phi}}(\Omega)$.*

(ii) *The functional I has a Gâteaux derivative $I'(u)$ for every $u \in W^1 L_{\vec{\phi}}(\Omega)$. Moreover, for every $v \in W^1 L_{\vec{\phi}}(\Omega)$*

$$\begin{aligned}\langle I'(u), v\rangle &= \int_\Omega \sum_{i=1}^N a_i(x, \partial_i u)\partial_i v\, dx + \int_\Omega b(x)\varphi_{\max}(x, u)v\, dx \\ &\quad - \int_\Omega f(x, u)v\, dx - \int_{\partial\Omega} g(x, u)v\, ds.\end{aligned}$$

So that, the critical points of I are weak solutions to the problem (1.1).

Proof. (i) For almost every $x \in \Omega$ and for every $\zeta \in \mathbb{R}$, we can write

$$A_i(x, \zeta) = \int_0^1 \frac{d}{dt} A_i(x, t\zeta)dt = \int_0^1 a_i(x, t\zeta)\zeta dt.$$

Then, by (1.2) we obtain

$$|A_i(x, \zeta)| \leq c_i d_i(x)|\zeta| + \int_0^1 \phi_i^{*-1}(x, P_i(x, t\zeta))|\zeta|dt \leq c_i d_i(x)|\zeta| + \phi_i^{*-1}(x, P_i(x, \zeta))|\zeta|.$$

In a similar manner as in (4.2), we arrive at

$$\Big|\int_\Omega A_i(x, \partial_i u(x))\, dx\Big| < \infty.$$

Hence, the first term in the right hand side in (4.5) is well defined. For the second term, we using (2.8), the Hölder inequality (2.5) and (4.3) obtaining

$$\Big|\int_\Omega b(x)\phi_{\max}(x, u(x))\, dx\Big| \leq 2\|b\|_\infty \|\varphi_{\max}(\cdot, u(\cdot))\|_{\phi_{\max}^*} \|u\|_{\phi_{\max}} < \infty,$$

while for the third term we can estimate it using (1.6) and (2.8) as follows

$$\int_\Omega \big|F(x, u(x))\big|\, dx \leq k_1 \int_\Omega |m(x, u)u|\, dx.$$

So that by Hölder's inequality (2.5) we obtain

$$\int_\Omega \big|F(x, u(x))\big|\, dx \leq 2k_1 \|m(\cdot, u(\cdot))\|_{M^*} \|u\|_M < \infty.$$

Regarding the last term in the right hand side in (4.5), we can use (1.7) and (2.8) to have

$$\int_{\partial\Omega} |G(x, u)|\, ds \leq k_2 \int_{\partial\Omega} h(x, u)u ds.$$

Since the function H is locally integrable and satisfies $H \ll \phi_{\min}^{**}$, it follows that $H \ll \phi_{\max}$ and in a similar manner as in Lemma 5.7 (in Appendix) we obtain $h(\cdot, u) \in L_{H^*}(\partial\Omega)$. Applying the Hölder inequality (2.5) one has

$$\int_{\partial\Omega} |G(x, u)|\, ds < \infty.$$

(ii) For $i = 1, \dots, N$ we define the functional $\Lambda_i : W^1 L_{\vec{\phi}}(\Omega) \to \mathbb{R}$ by

$$\Lambda_i(u) = \int_\Omega A_i(x, \partial_i u(x))\, dx.$$

We denote by B, L_1, $L_2 : W^1L_{\vec{\phi}}(\Omega) \to \mathbb{R}$ the functionals

$$B(u) = \int_\Omega b(x)\phi_{\max}(x, u(x))\, dx, \quad L_1(u) = \int_\Omega F(x, u(x))\, dx,$$
$$L_2(u) = \int_{\partial\Omega} G(x, u(x)) ds.$$

Observe that for $u \in W^1L_{\vec{\phi}}(\Omega)$, $v \in C^\infty(\overline{\Omega})$, and $r > 0$, we have

$$\frac{\Lambda_i(u + rv) - \Lambda_i(u)}{r} = \int_\Omega \frac{1}{r}\Big(A_i\Big(x, \frac{\partial u}{\partial x_i}(x) + r\frac{\partial v}{\partial x_i}(x)\Big) - A_i\Big(x, \frac{\partial u}{\partial x_i}(x)\Big)\Big)\, dx,$$
$$\frac{1}{r}\Big(A_i\Big(x, \frac{\partial u}{\partial x_i}(x) + r\frac{\partial v}{\partial x_i}(x)\Big) - A_i\Big(x, \frac{\partial u}{\partial x_i}(x)\Big)\Big) \longrightarrow a_i\Big(x, \frac{\partial u}{\partial x_i}(x)\Big)\frac{\partial v}{\partial x_i}(x),$$

as $r \to 0$ for almost every $x \in \Omega$. On the other hand, by the mean value theorem there exists $\nu \in [0, 1]$ such that

$$\begin{aligned} &\frac{1}{r}\big|A_i\Big(x, \frac{\partial u}{\partial x_i}(x) + r\frac{\partial v}{\partial x_i}(x)\Big) - A_i\Big(x, \frac{\partial u}{\partial x_i}(x)\Big)\big| \\ &= \Big|a_i\Big(x, \frac{\partial u}{\partial x_i}(x) + \nu r\frac{\partial v}{\partial x_i}(x)\Big)\Big|\big|\frac{\partial v}{\partial x_i}(x)\big|. \end{aligned}$$

Hence, by using this equality and (1.2) we obtain

$$\begin{aligned} &\frac{1}{r}\Big|A_i\Big(x, \frac{\partial u}{\partial x_i}(x) + r\frac{\partial v}{\partial x_i}(x)\Big) - A_i\Big(x, \frac{\partial u}{\partial x_i}(x)\Big)\Big| \\ &\le c_i\Big(d_i(x) + \phi_i^{*-1}\Big(x, \phi_i\Big(x, \frac{\partial u}{\partial x_i}(x) + \nu r\frac{\partial v}{\partial x_i}(x)\Big)\Big)\Big)\Big|\frac{\partial v}{\partial x_i}(x)\Big|. \end{aligned}$$

Next, by Hölder's inequality (2.5) we obtain

$$c_i\Big(d_i(x) + \phi_i^{*-1}\Big(x, \phi_i\Big(x, \frac{\partial}{\partial x_i}u(x) + \nu r\frac{\partial}{\partial x_i}v(x)\Big)\Big)\Big)\Big|\frac{\partial}{\partial x_i}v(x)\Big| \in L^1(\Omega).$$

The dominated convergence theorem can be applied to obtain

$$\lim_{r\to 0} \frac{\Lambda_i(u + rv) - \Lambda_i(u)}{r} = \langle \Lambda_i'(u), v\rangle := \int_\Omega a_i\Big(x, \frac{\partial u}{\partial x_i}(x)\Big)\frac{\partial v}{\partial x_i}(x)\, dx,$$

for $i = 1, \dots, N$. By a similar calculus as in above, we can show that

$$\langle B'(u), v\rangle = \int_\Omega b(x)\varphi_{\max}(x, u)v\, dx, \quad \langle L_1'(u), v\rangle = \int_\Omega f(x, u)v\, dx,$$
$$\langle L_2'(u), v\rangle = \int_\Omega g(x, u)v\, dx.$$

□

4.2. **Existence of solutions.** Our main existence result reads as follows.

Theorem 4.3. *Let Ω be an open bounded subset of $\mathbb{R}^N$, $N \ge 2$, with the cone property. Assume that* (1.2), (1.3), (1.4), (1.6), (1.7), (1.9), (3.1) *and* (3.3) *are fulfilled and suppose that $\phi_{\max}$ and $\phi_{\min}^*$ are locally integrable and $(\phi_{\min}^{**})_*(\cdot, t)$ is Lipschitz continuous on $\overline{\Omega}$. Then, problem* (1.1) *admits at least a weak solution in $W^1L_{\vec{\phi}}(\Omega)$.*

Proof. We divide the proof into three steps.
Step 1: Weak* lower semicontinuity property of I. We define the functional $J : W^1L_{\vec{\phi}}(\Omega) \to \mathbb{R}$ by

$$J(u) = \int_\Omega \sum_{i=1}^N A_i(x, \partial_i u)\,dx + \int_\Omega b(x)\phi_{\max}(x,u)\,dx,$$

so that $I(u) = J(u) - L_1(u) - L_2(u)$, where

$$L_1(u) = \int_\Omega F(x,u)\,dx, \quad L_2(u) = \int_{\partial\Omega} G(x,u)ds.$$

First, we claim that J is sequentially weakly lower semicontinuous. Indeed, since $u \mapsto \phi_{\max}(x,u)$ is continuous, it is sufficient to show that the functional

$$u \mapsto K(u) = \int_\Omega \sum_{i=1}^N A_i(x, \partial_i u)\,dx,$$

is sequentially weakly$-^*$ lower semicontinuous. To do this, let $u_n \overset{*}{\rightharpoonup} u$ in $W^1L_{\vec{\phi}}(\Omega)$ in the sense that

$$\int_\Omega u_n\varphi\,dx \to \int_\Omega u\varphi\,dx, \quad \text{for all } \varphi \in E_{\phi^*_{\max}}, \tag{4.6}$$

$$\int_\Omega \partial_i u_n\varphi\,dx \to \int_\Omega \partial_i u\varphi\,dx, \quad \text{for all } \varphi \in E_{\phi^*_i}. \tag{4.7}$$

By the definitions of $\phi_{\min}$ and $\phi_{\max}$, (4.6) and (4.7) hold for every $\varphi \in E_{\phi^*_{\min}}(\Omega)$. Being $\phi^*_{\min}$ locally integrable, one has $L^\infty(\Omega) \subset E_{\phi^*_{\min}}(\Omega)$. Therefore, for every $i \in \{1,\dots,N\}$

$$\partial_i u_n \rightharpoonup \partial_i u \quad \text{and} \quad u_n \rightharpoonup u \quad \text{in } L^1(\Omega), \tag{4.8}$$

for the weak topology $\sigma(L^1, L^\infty)$. Since the embedding $W^1L_{\vec{\phi}}(\Omega) \hookrightarrow W^{1,1}(\Omega)$ is continuous and the embedding $W^{1,1}(\Omega) \hookrightarrow L^1(\Omega)$ is compact, we conclude that the sequence $\{u_n\}$ is relatively compact in $L^1(\Omega)$. Therefore, there exist a subsequence still indexed by n and a function $v \in L^1(\Omega)$ such that $u_n \to v$ strongly in $L^1(\Omega)$. In view of (4.8), we have $v = u$ almost everywhere on Ω and $u_n \to u$ in $L^1(\Omega)$. Passing once more to a subsequence, we can have $u_n \to u$ almost everywhere on Ω. Recall that $\zeta \to A_i(x,\zeta)$ is a convex function, so by (1.3) we can use [5, Theorem 2.1, Chapter 8] obtaining

$$K(u) = \int_\Omega \sum_{i=1}^N A_i(x, \partial_i u)\,dx \le \liminf \int_\Omega \sum_{i=1}^N A_i(x, \partial_i u_n)\,dx = \liminf K(u_n).$$

We shall now prove that L_1 and L_2 are continuous. Since $M \ll \phi^{**}_{\min}$, it follows that $M \ll (\phi^{**}_{\min})_*$, then by Corollary 3.3 we have $u_n \to u$ in $L_M(\Omega)$. Thus, there exists n_0 such that for every $n \ge n_0$, $\|u_n - u\|_M < \frac{1}{2}$. By (1.6), we obtain

$$\int_\Omega |F(x, u_n(x))|\,dx \le k_1 \int_\Omega M(x, u_n(x))\,dx.$$

Let $\theta_n = \|u_n - u\|_M$. By the convexity of M, we can write

$$M(x, u_n(x)) = M\Big(x, \theta_n\Big(\frac{u_n(x) - u(x)}{\theta_n}\Big) + (1-\theta_n)\frac{u(x)}{1-\theta_n}\Big)$$

$$\leq \theta_n M\Big(x, \frac{u_n(x)-u(x)}{\theta_n}\Big) + (1-\theta_n) M\Big(x, \frac{u(x)}{1-\theta_n}\Big).$$

Hence,

$$\int_\Omega M(x, u_n(x))\, dx \leq \theta_n + (1-\theta_n) \int_\Omega M\Big(x, \frac{u(x)}{1-\theta_n}\Big)\, dx. \tag{4.9}$$

Moreover,

$$M\Big(x, \frac{u(x)}{1-\theta_n}\Big) \leq M(x, 2u(x)).$$

Since M is locally integrable and $M \ll \phi_{\max}$, there exists a nonnegative function $h \in L^1(\Omega)$ such that

$$\int_\Omega M(x, 2|u(x)|)\, dx \leq \int_\Omega \phi_{\max}\Big(x, \frac{|u(x)|}{\|u\|_{\phi_{\max}}}\Big)\, dx + \int_\Omega h(x)\, dx < \infty.$$

Thus, the Lebesgue dominated convergence theorem yields

$$\lim_{n\to\infty} \int_\Omega M\Big(x, \frac{u(x)}{1-\theta_n}\Big)\, dx = \int_\Omega M(x, u(x))\, dx$$

and therefore by (4.9) we have

$$\limsup_{n\to\infty} \int_\Omega M(x, u_n(x))\, dx \leq \int_\Omega M(x, u(x))\, dx.$$

In addition, by Fatou's Lemma we obtain

$$\int_\Omega M(x, u(x))\, dx \leq \liminf_{n\to\infty} \int_\Omega M(x, u_n(x))\, dx.$$

Gathering the two inequalities above, we obtain

$$\lim_{n\to+\infty} \int_\Omega M(x, u_n(x))\, dx = \int_\Omega M(x, u(x))\, dx.$$

Applying [14, Theorem 13.47] we obtain $M(x, u_n(x)) \to M(x, u(x))$ strongly in $L^1(\Omega)$ which in turn implies that $M(x, u_n(x))$ is equi-integrable and then so is $F(x, u_n(x))$. Since $F(x, u_n) \to F(x, u)$ almost everywhere on Ω by Vitali's theorem we have $L_1(u_n) \to L_1(u)$. Similarly, we can show that $L_2(u_n) \to L_2(u)$. That is to say that L_1 and L_2 are continuous. Since J is weakly$-^*$ lower semicontinuous, we conclude that I is weakly-* lower semicontinuous.

Step 2: Coercivity of the functional I. By (1.3), (1.9) and (2.4) we can write

$$\begin{aligned}
I(u) &\geq \int_\Omega \sum_{i=1}^N \phi_i(x, \partial_i u)\, dx + b_0 \int_\Omega \phi_{\max}(x, u)\, dx - \int_\Omega F(x, u)\, dx - \int_{\partial\Omega} G(x, u) ds \\
&\geq \sum_{i=1}^N \|\partial_i u\|_{\phi_i} + b_0 \|u\|_{\phi_{\max}} - N - b_0 - \int_\Omega F(x, u)\, dx - \int_{\partial\Omega} G(x, u) ds \\
&\geq \min\{1, b_0\} \|u\|_{W^1 L_{\vec{\phi}}(\Omega)} - \int_\Omega F(x, u)\, dx - \int_{\partial\Omega} G(x, u) ds - N - b_0.
\end{aligned}$$

By (1.6) and (1.7) we obtain

$$I(u) \geq \min\{1, b_0\} \|u\|_{W^1 L_{\vec{\phi}}(\Omega)} - k_1 \int_\Omega M(x, u)\, dx - k_2 \int_{\partial\Omega} H(x, u) ds - N - b_0.$$

As $M \ll (\phi_{\min}^{**})_*$ and $H \ll \psi_{\min}$, by Theorems 3.1 and 3.4 there exist two positive constant $C_1 > 0$ and $C_2 > 0$ such that $\|u\|_{L_M(\Omega)} \leq C_1 \|u\|_{W^1 L_{\vec{\phi}}(\Omega)}$ and $\|u\|_{L_H(\partial\Omega)} \leq$

$C_2\|u\|_{W^1L_{\vec{\phi}}(\Omega)}$. Since M and H satisfy the Δ_2-condition, there exist two positive constants $r_1 > 0$ and $r_2 > 0$ and two nonnegative functions $h_1 \in L^1(\Omega)$ and $h_2 \in L^1(\partial\Omega)$ such that

$$\begin{aligned}
I(u) &\geq \min\{1, b_0\}\|u\|_{W^1L_{\vec{\phi}}(\Omega)} - k_1 r_1 \int_\Omega M\Big(x, \frac{|u(x)|}{C_1\|u\|_{W^1L_{\vec{\phi}}(\Omega)}}\Big)\,dx \\
&\quad - k_2 r_2 \int_{\partial\Omega} H\Big(x, \frac{|u(x)|}{C_2\|u\|_{W^1L_{\vec{\phi}}(\Omega)}}\Big)ds - \int_\Omega h_1(x)\,dx - \int_{\partial\Omega} h_2(x)ds - N - b_0 \\
&\geq \min\{1, b_0\}\|u\|_{W^1L_{\vec{\phi}}(\Omega)} - k_1 r_1 \int_\Omega M\Big(x, \frac{|u(x)|}{\|u\|_{L_M(\Omega)}}\Big)\,dx \\
&\quad - k_2 r_2 \int_{\partial\Omega} H\Big(x, \frac{|u(x)|}{\|u\|_{L_H(\partial\Omega)}}\Big)ds - \int_\Omega h_1(x)\,dx - \int_{\partial\Omega} h_2(x)ds - N - b_0 \\
&\geq \min\{1, b_0\}\|u\|_{W^1L_{\vec{\phi}}(\Omega)} - \int_\Omega h_1(x)\,dx - \int_{\partial\Omega} h_2(x)ds - k_1 r_1 - k_2 r_2 - N - b_0,
\end{aligned}$$

which implies $I(u) \to \infty$ as $\|u\|_{W^1L_{\vec{\phi}}(\Omega)} \to \infty$.

Step 3: Existence of a weak solution. Let $\lambda > 0$ be arbitrary. Since I is coercive there exists $R > 0$ such that

$$\|u\|_{W^1L_{\vec{\phi}}(\Omega)} > R \Rightarrow I(u) > \lambda.$$

We define $E_\lambda = \{u \in W^1L_{\vec{\phi}}(\Omega) : I(u) \leq \lambda\}$ and denote by $B_R(0)$ the closed ball in $W^1L_{\vec{\phi}}(\Omega)$ of radius R centered at the origin. We claim that $\alpha = \inf_{v\in W^1L_{\vec{\phi}}(\Omega)} I(v) > -\infty$. If not, for all $n > 0$ there is a sequence $u_n \in E_\lambda$ such that $I(u_n) < -n$. As $E_\lambda \subset B_R(0)$, by the Banach-Alaoglu-Bourbaki theorem there exists $u \in B_R(0)$ such that, passing to a subsequence if necessary, we can assume that $u_n \rightharpoonup u$ weak* in $W^1L_{\vec{\phi}}(\Omega)$. So that the weak-* lower semicontinuity of I implies $I(u) = -\infty$ which contradicts the fact that I is well defined on $W^1L_{\vec{\phi}}(\Omega)$. Therefore, for every $n > 0$ there exists a sequence $u_n \in E_\lambda$ such that $I(u_n) \leq \alpha + \frac{1}{n}$. Thus, there exists $u \in B_R(0)$ such that for a subsequence still indexed by n, $u_n \rightharpoonup u$ weak$-^*$ in $W^1L_{\vec{\phi}}(\Omega)$. Since I is weakly$-^*$ lower semicontinuous we obtain

$$I(u) = J(u) - L_1(u) - L_2(u) \leq \liminf_{n\to\infty}\Big(J(u_n) - L_1(u_n) - L_2(u_n)\Big) = \liminf_{n\to\infty} I(u_n) \leq \alpha.$$

Note that u belongs also to E_λ, which yields $I(u) = \alpha \leq \lambda$. This shows that $I(u) = \min\{I(v) : v \in W^1L_{\vec{\phi}}(\Omega)\}$. Moreover, inserting $v = -u^-$ as test function in (4.1) and then using (2.8), we obtain $u \geq 0$. This ends the proof of Theorem 4.3. ◻

4.3. **Uniqueness.** To prove the uniqueness of the weak solution we need the following monotonicity assumptions:

$$\big(f(x,s) - f(x,t)\big)\big(s - t\big) < 0 \quad \text{for a.e. } x \in \Omega,\ \forall s,t \in \mathbb{R} \text{ with } s \neq t, \tag{4.10}$$

$$\big(g(x,s) - g(x,t)\big)\big(s - t\big) < 0 \quad \text{for a.e. } x \in \Omega,\ \forall s,t \in \mathbb{R} \text{ with } s \neq t, \tag{4.11}$$

$$\big(\varphi_{\max}(x,s) - \varphi_{\max}(x,t)\big)\big(s - t\big) > 0 \quad \text{for a.e. } x \in \Omega,\ \forall s,t \in \mathbb{R} \text{ with } s \neq t. \tag{4.12}$$

Theorem 4.4. *Let u be the weak solution of* (1.1) *given by Theorem 4.3. If in addition* (1.4), (4.10), (4.11) *and* (4.12) *are fulfilled, then the weak solution u is unique.*

Proof. Suppose that there exists another weak solution w of problem (1.1). We choose $v = u - w$ as a test function in (4.1) obtaining

$$\begin{aligned}&\int_\Omega \sum_{i=1}^N a_i(x, \partial_{x_i} u)\partial_{x_i}(u-w)\,dx + \int_\Omega b(x)\varphi_{\max}(x,u)(u-w)\,dx\\&- \int_\Omega f(x,u)(u-w)\,dx - \int_{\partial\Omega} g(x,u)(u-w)\,ds = 0.\end{aligned}$$

Then choosing $v = w - u$ as a test function in the weak formulation of solution (4.1) solved by w, we obtain

$$\begin{aligned}&\int_\Omega \sum_{i=1}^N a_i(x, \partial_{x_i} w)\partial_{x_i}(w-u)\,dx + \int_\Omega b(x)\varphi_{\max}(x,w)(w-u)\,dx\\&- \int_\Omega f(x,w)(w-u)\,dx - \int_{\partial\Omega} g(x,w)(w-u)\,ds = 0.\end{aligned}$$

Combining the previous two equalities, we obtain

$$\begin{aligned}&\int_\Omega \sum_{i=1}^N \Big(a_i(x, \partial_{x_i} u) - a_i(x, \partial_{x_i} w)\Big)(\partial_{x_i} u - \partial_{x_i} w)\,dx\\&+ \int_\Omega b(x)\Big(\varphi_{\max}(x,u) - \varphi_{\max}(x,w)\Big)(u-w)\,dx\\&- \int_\Omega \Big(f(x,u) - f(x,w)\Big)(u-w)\,dx - \int_{\partial\Omega} \Big(g(x,u) - g(x,w)\Big)(u-w)ds = 0.\end{aligned}$$

In view of (1.4), (4.10), (4.11) and (4.12) we obtain $u = w$ a.e. in Ω. □

5. Appendix

We present some important results that are necessary for the accomplishment of the proofs of the above results.

Lemma 5.1 ([23, Theorem B.1]). *Let ϕ be a locally integrable Musielak-Orlicz function. The space E_ϕ is separable.*

Lemma 5.2 ([23, Lemma B.4]). *Let ϕ and ϕ^* be two complementary Musielak-Orlicz functions. For every $\eta \in L_{\phi^*}(\Omega)$, the linear functional F_η defined for every $\zeta \in E_\phi(\Omega)$ by*

$$F_\eta(\zeta) = \int_\Omega \zeta(x)\eta(x)\,dx \tag{5.1}$$

belongs to the dual space of $E_\phi(\Omega)$, denoted $E_\phi(\Omega)^*$, and its norm $\|F_\eta\|$ satisfies

$$\|F_\eta\| \le 2\|\eta\|_{\phi^*}, \tag{5.2}$$

where $\|F_\eta\| = \sup\{|F_\eta(u)| : \|u\|_{L_M(\Omega)} \le 1\}$.

Lemma 5.3. *Let ϕ be a locally integrable Musielak-Orlicz function. The dual space of $E_\phi(\Omega)$ can be identified to the Musielak-Orlicz space $L_{\phi^*}(\Omega)$.*

Proof. According to Lemma 5.2 any element $\eta \in L_{\phi^*}(\Omega)$ defines a bounded linear functional F_η on $L_\phi(\Omega)$ and also on $E_\phi(\Omega)$ which is given by (5.1). It remains to show that every bounded linear functional on $E_\phi(\Omega)$ is of the form F_η for some $\eta \in L_{\phi^*}(\Omega)$. The proof of this claim is done in [23]. For the convenience of the

reader we give it here. Given $F \in E_\phi(\Omega)^*$ we define the complex measure λ by setting
$$\lambda(E) = F(\chi_E),$$
where E is a measurable subset of Ω having a finite measure and χ_E stands for the characteristic function of E. Due to the fact that ϕ is locally integrable, the measurable function $\phi\big(\cdot, \phi^{-1}(x_0, \frac{1}{2|E|})\chi_E(\cdot)\big)$ belongs to $L^1(\Omega)$ for any $x_0 \in \Omega$. Hence, there is an $\epsilon > 0$ such that for any measurable subset Ω' of Ω, one has
$$|\Omega'| < \epsilon \Rightarrow \int_{\Omega'} \phi\Big(x, \phi^{-1}\big(x_0, \frac{1}{2|E|}\big)\chi_E(x)\Big)\, dx \le \frac{1}{2}.$$
As $\phi(\cdot, s)$ is measurable on E, Luzin's theorem implies that for $\epsilon > 0$ there exists a closed set $K_\epsilon \subset E$, with $|E \setminus K_\epsilon| < \epsilon$, such that the restriction of $\phi(\cdot, s)$ to K_ϵ is continuous. Let k be the point where the maximum of $\phi(\cdot, s)$ is reached in the set K_ϵ.
$$\begin{aligned}
&\int_E \phi\Big(x, \phi^{-1}\big(k, \frac{1}{2|E|}\big)\Big)\, dx \\
&= \int_{K_\epsilon} \phi\Big(x, \phi^{-1}\big(k, \frac{1}{2|E|}\big)\Big)\, dx + \int_{E\setminus K_\epsilon} \phi\Big(x, \phi^{-1}\big(k, \frac{1}{2|E|}\big)\Big)\, dx.
\end{aligned}$$
For the first term in the right-hand side of the equality, we can write
$$\int_{K_\epsilon} \phi\Big(x, \phi^{-1}\big(k, \frac{1}{2|E|}\big)\Big)\, dx \le \int_E \phi\Big(k, \phi^{-1}\big(k, \frac{1}{2|E|}\big)\Big)\, dx \le \frac{1}{2}.$$
Since $|E \setminus K_\epsilon| < \epsilon$, the second term can be estimated as
$$\int_{E\setminus K_\epsilon} \phi\Big(x, \phi^{-1}\big(k, \frac{1}{2|E|}\big)\Big) \le \frac{1}{2}.$$
Thus, we obtain
$$\int_\Omega \phi\Big(x, \phi^{-1}\big(k, \frac{1}{2|E|}\big)\chi_E(x)\Big)\, dx \le 1.$$
It follows that
$$|\lambda(E)| \le \|F\| \|\chi_E\|_\phi \le \frac{\|F\|}{\phi^{-1}\big(k, \frac{1}{2|E|}\big)}.$$
As the right-hand side tends to zero when $|E|$ converges to zero, the measure λ is absolutely continuous with respect to the Lebesgue measure and so by Radon-Nikodym's Theorem (see for instance [1, Theorem 1.52]), it can be expressed in the form
$$\lambda(E) = \int_E \eta(x)\, dx,$$
for some nonnegative function $\eta \in L^1(\Omega)$ unique up to sets of Lebesgue measure zero. Thus,
$$F(\zeta) = \int_\Omega \zeta(x)\eta(x)\, dx$$
holds for every measurable simple function ζ. Note first that since ϕ is locally integrable any measurable simple function lies in $E_\phi(\Omega)$ and the set of measurable simple functions is dense in $(E_\phi(\Omega), \|\cdot\|_\phi)$. Indeed, for nonnegative $\zeta \in E_\phi(\Omega)$, there exists a sequence of increasing measurable simple functions ζ_j converging almost everywhere to ζ and $|\zeta_j(x)| \le |\zeta(x)|$ on Ω. By the theorem of dominated

convergence one has $\zeta_j \to \zeta$ in $E_\phi(\Omega)$. For an arbitrary $\zeta \in E_\phi(\Omega)$, we obtain the same result splitting ζ into positive and negative parts.

Let $\zeta \in E_\phi(\Omega)$ and let ζ_j be a sequence of measurable simple functions converging to ζ in $E_\phi(\Omega)$. By Fatou's Lemma and the inequality (5.2) we can write

$$\begin{aligned}
\Big| \int_\Omega \zeta(x)\eta(x)\,dx \Big| &\le \liminf_{j\to+\infty} \int_\Omega |\zeta_j(x)\eta(x)|\,dx \\
&= \liminf_{j\to+\infty} F(|\zeta_j| \operatorname{sgn}\eta) \\
&\le \|F\| \liminf_{j\to+\infty} \|\zeta_j\|_\phi \le \|F\|\|\zeta\|_\phi,
\end{aligned}$$

which implies that $\eta \in L_{\phi^*}(\Omega)$. Thus, the linear functional $F_\eta(\zeta) = \int_\Omega \zeta(x)\eta(x)\,dx$ and F are both defined on $E_\phi(\Omega)$ and have the same values on the set of measurable simple functions, so by a density argument they agree on $E_\phi(\Omega)$. $\square$

Lemma 5.4. *Let Ω be an open bounded subset of $\mathbb{R}^N$. Let $f, g : \Omega \times (0,+\infty) \to (0,+\infty)$ be continuous nondecreasing functions with respect to there second argument and $g(\cdot,t)$ is continuous on $\overline{\Omega}$ with $\lim_{t\to\infty} \frac{f(x,t)}{g(x,t)} = +\infty$, then for all $\epsilon > 0$, there exists a positive constant K_0 such that*

$$g(x,t) \le \epsilon f(x,t) + K_0, \quad \text{for all } t > 0.$$

Proof. Let $\epsilon > 0$ be arbitrary. There exists $t_0 > 0$ such that $t \ge t_0$ implies $g(x,t) \le \epsilon f(x,t)$. Then, for all $t \ge 0$,

$$g(x,t) \le \epsilon f(x,t) + K(x),$$

where $K(x) = \sup_{t\in(0,t_0)} g(x,t)$. Being $g(\cdot,t)$ continuous on $\overline{\Omega}$, one has $g(x,t) \le \epsilon f(x,t) + K_0$ with $K_0 = \max_{x\in\overline{\Omega}} K(x)$. $\square$

Lemma 5.5. *Let Ω be an open bounded subset of $\mathbb{R}^N$. Let A, B be two Musielak-Orlicz functions such that $B \ll A$, with $B(\cdot,t)$ is continuous on $\overline{\Omega}$. If a sequence $\{u_n\}$ is bounded in $L_A(\Omega)$ and converges in measure in Ω then it converges in norm in $L_B(\Omega)$.*

Proof. Let us fix $\epsilon > 0$. Defining $v_{j,k}(x) = \frac{u_j(x)-u_k(x)}{\epsilon}$, we shall prove that $\{u_j\}$ is a Cauchy sequence in the Banach space $L_B(\Omega)$. Clearly $\{v_{j,k}\}$ is bounded in $L_A(\Omega)$, say $\|v_{j,k}\|_A \le K$ for all j and k and for some positive constant K. Since $B \ll A$ there exists a positive number t_0 such that for $t \ge t_0$ one has

$$B(x,t) \le \frac{1}{4} A\Big(x, \frac{t}{K}\Big).$$

On the other hand, since $B(\cdot,t)$ is continuous on $\overline{\Omega}$ we denote x_0 the point where the maximum of $B(\cdot,t)$ is reached in $\overline{\Omega}$. Let $\delta = \frac{1}{4B(x_0,t_0)}$ and set

$$\Omega_{j,k} = \Big\{ x \in \Omega : |v_{j,k}| \ge B^{-1}\Big(x_0, \frac{1}{2|\Omega|}\Big) \Big\}.$$

Since $\{u_j\}$ converges in measure, there exists an integer N_0 such that $|\Omega_{j,k}| \le \delta$ whenever $j, k \ge N_0$. Defining

$$\Omega'_{j,k} = \{x \in \Omega_{j,k} : |v_{j,k}| \ge t_0\} \text{ and } \Omega''_{j,k} = \Omega_{j,k} \setminus \Omega'_{j,k},$$

one has

$$\begin{aligned}\int_\Omega B(x,|v_{j,k}(x)|)\,dx &= \int_{\Omega\setminus\Omega_{j,k}} B(x,|v_{j,k}(x)|)\,dx + \int_{\Omega'_{j,k}} B(x,|v_{j,k}(x)|)\,dx \\ &\quad + \int_{\Omega''_{j,k}} B(x,|v_{j,k}(x)|)\,dx.\end{aligned}\tag{5.3}$$

For the first term in the right-hand side of (5.3), we can write

$$\begin{aligned}\int_{\Omega\setminus\Omega_{j,k}} B(x,|v_{j,k}(x)|)\,dx &\le \int_{\Omega\setminus\Omega_{j,k}} B\Big(x, B^{-1}\Big(x_0,\frac{1}{2|\Omega|}\Big)\Big)\,dx \\ &\le \int_\Omega B\Big(x_0, B^{-1}\Big(x_0,\frac{1}{2|\Omega|}\Big)\Big)\,dx \le \frac12.\end{aligned}$$

Since $B \ll A$, the second term in the right hand side of (5.3) can be estimated as follows

$$\int_{\Omega'_{j,k}} B(x,|v_{j,k}(x)|)\,dx \le \frac14\int_\Omega A\Big(x,\frac{|v_{j,k}|}{K}\Big)\,dx \le \frac14,$$

while for the third term in the right hand side of (5.3), we obtain

$$\int_{\Omega''_{j,k}} B(x,|v_{j,k}(x)|)\,dx \le \int_{\Omega_{j,k}} B(x,t_0)\,dx \le \delta B(x_0,t_0) = \frac14.$$

Finally, putting all the above estimates in (5.3), we arrive at

$$\int_\Omega B(x,|v_{j,k}(x)|)\,dx \le 1, \text{ for every } j,k \ge N_0,$$

which yields $\|u_j - u_k\|_B \le \epsilon$. Thus, $\{u_j\}$ converges in the Banach space $L_B(\Omega)$. $\square$

Lemma 5.6. *Let $u \in W^{1.1}_{loc}(\Omega)$ and let $F : \overline{\Omega}\times\mathbb{R}^+ \to \mathbb{R}^+$ be a Lipschitz continuous function. If $f(x) = F(x,u(x))$ then $f \in W^{1.1}_{loc}(\Omega)$. Moreover, for every $j = 1,\dots,N$, the weak derivative $\partial_{x_j} f$ of f is such that*

$$\partial_{x_j} f(x) = \frac{\partial F(x,u(x))}{\partial x_j} + \frac{\partial F(x,u(x))}{\partial t}\partial_{x_j} u(x), \quad \textit{for a.e. } x\in\Omega.$$

Proof. Let $\varphi \in D(\Omega)$ and let $\{e_j\}_{j=1}^N$ be the standard basis in $\mathbb{R}^N$. We can write

$$\begin{aligned}&-\int_\Omega F(x,u(x))\partial_{x_j}\varphi(x)\,dx \\ &= -\lim_{h\to0}\int_\Omega F(x,u(x))\frac{\varphi(x)-\varphi(x-he_j)}{h}\,dx \\ &= \lim_{h\to0}\int_\Omega \frac{F(x+he_j,u(x+he_j))-F(x,u(x))}{h}\varphi(x)\,dx \\ &= \lim_{h\to0}\int_\Omega \frac{F(x+he_j,u(x+he_j))-F(x,u(x+he_j))}{h}\varphi(x)\,dx \\ &\quad + \lim_{h\to0}\int_\Omega \frac{F(x,u(x+he_j))-F(x,u(x))}{h}\varphi(x)\,dx \\ &= \lim_{h\to0}\int_\Omega Q_1(x,h)\varphi(x)\,dx + \lim_{h\to0}\int_\Omega Q_2(x,h)\frac{u(x+he_j)-u(x)}{h}\varphi(x)\,dx,\end{aligned}$$

where

$$Q_1(x,h)=\begin{cases}\frac{F(x+he_j,u(x+he_j))-F(x,u(x+he_j))}{h} & \text{if } h\neq 0,\\ \frac{\partial F(x,u(x))}{\partial x_j} & \text{if } h=0\end{cases}$$

and

$$Q_2(x,h)=\begin{cases}\frac{F(x,u(x+he_j))-F(x,u(x))}{u(x+he_j)-u(x)} & \text{if } u(x+he_j)\neq u(x),\\ \frac{\partial F(x,u(x))}{\partial t} & \text{otherwise.}\end{cases}$$

Since $F(\cdot,\cdot)$ is Lipschitz continuous, there exist two positive constants k_1 and k_2 not depending on h, such that $\|Q_1(\cdot,h)\|_\infty \le k_1$ and $\|Q_2(\cdot,h)\|_\infty \le k_2$. Thus, for some sequence of values of h tending to zero, $Q_1(\cdot,h)$ converges to $\frac{\partial F(x,u(x))}{\partial x_j}$ and $Q_2(\cdot,h)$ converges to $\frac{\partial F(x,u(x))}{\partial t}$ both in $L^\infty(\Omega)$ for the weak-star topology $\sigma^*(L^\infty(\Omega),L^1(\Omega))$. On the other hand, since $u\in W^{1,1}(\operatorname{supp}(\varphi))$ we have

$$\lim_{h\to 0}\int_{\operatorname{supp}(\varphi)}\frac{u(x+he_j)-u(x)}{h}\varphi(x)\,dx=\int_{\operatorname{supp}(\varphi)}\partial_j u(x)\varphi(x)\,dx.$$

It follows that

$$\begin{aligned}&-\int_\Omega F(x,u(x))\partial_{x_j}\varphi(x)\,dx\\ &=\int_\Omega \frac{\partial F(x,u(x))}{\partial x_j}\varphi(x)\,dx+\int_\Omega \frac{\partial F(x,u(x))}{\partial t}\partial_{x_j}u(x)\varphi(x)\,dx,\end{aligned}$$

which completes the proof. □

Lemma 5.7. *Let Ω be an open bounded subset of $\mathbb{R}^N$. Let A and ϕ be two Musielak-Orlicz functions. We assume that $\phi:\overline{\Omega}\times\mathbb{R}^+\to\mathbb{R}^+$ is locally integrable, differentiable with respect to its second argument and $\phi\ll A$. Then $\varphi(\cdot,s)\in L_{\phi^*}(\Omega)$ for every $s\in L_A(\Omega)$ where $\varphi(x,s)=\frac{\partial\phi(x,s)}{\partial s}$.*

Proof. Let $s\in L_A(\Omega)$. By (2.8) we can write

$$\begin{aligned}\int_\Omega \phi^*(x,\varphi(x,s))\,dx&=\int_\Omega\int_0^{\varphi(x,s)}\varphi^{-1}(x,\tau)d\tau\,dx\\ &\le \int_\Omega |s|\varphi(x,|s|)\,dx\le\int_\Omega \phi(x,2|s|)\,dx.\end{aligned}$$

It is obvious that if $s=0$ then $\varphi(\cdot,s)\in L_{\phi^*}(\Omega)$. Assume that $s\neq 0$. Since ϕ is locally integrable and $\phi\ll A$, there exists a nonnegative function $h\in L^1(\overline{\Omega})$ such that $\phi(x,2|s|)\le A\Big(x,\frac{|s|}{\|s\|_A}\Big)+h(x)$ for a.e. $x\in\Omega$. Thus,

$$\int_\Omega\phi(x,2|s|)\,dx\le\int_\Omega A\Big(x,\frac{|s|}{\|s\|_A}\Big)\,dx+\int_\Omega h(x)\,dx<\infty.$$

Hence, $\varphi(\cdot,s)\in L_{\phi^*}(\Omega)$. □

Let $\phi:\overline{\Omega}\times\mathbb{R}^+\to\mathbb{R}^+$ be a real function such that the partial function $\phi(x,\cdot)$ is convex. The function ϕ is called the principal part of the Musielak-Orlicz function M if $M(x,t)=\phi(x,t)$ for large values of the argument t.

Lemma 5.8. *Let $t_0>0$ be arbitrary and let $\phi:\overline{\Omega}\times[t_0,+\infty[\to\mathbb{R}^+$ be a real function where the partial function $\phi(x,\cdot)$ is convex. We define the function $\varphi(x,t)=\frac{\partial\phi(x,t)}{\partial t}$. If $\phi(\cdot,t)$ and $\varphi(\cdot,t)$ are continuous on $\overline{\Omega}$ and $\lim_{t\to+\infty}\inf_{x\in\Omega}\varphi(x,t)=+\infty$. Then $\phi(x,t)$ is the principal part of a Musielak-Orlicz function $M(x,t)$.*

Proof. Since $\lim_{t\to+\infty}\inf_{x\in\Omega}\varphi(x,t) = +\infty$, then there exists $t_1 > t_0+1$ (not depending on x) such that $\sup_{x\in\overline{\Omega}}\varphi(x,t_0+1)+\sup_{x\in\overline{\Omega}}\phi(x,t_0)\le\varphi(x,t_1)$. Thus, we have

$$\begin{aligned}
\inf_{x\in\overline{\Omega}}\phi(x,t_1)\le\phi(x,t_1) &= \int_{t_0}^{t_0+1}\varphi(x,\tau)d\tau+\int_{t_0+1}^{t_1}\varphi(x,\tau)d\tau+\phi(x,t_0)\\
&\le \sup_{x\in\overline{\Omega}}\varphi(x,t_0+1)+\sup_{x\in\overline{\Omega}}\phi(x,t_0)+(t_1-t_0-1)\varphi(x,t_1)\\
&\le (t_1-t_0)\varphi(x,t_1)\\
&< t_1\varphi(x,t_1)\\
&\le t_1\sup_{x\in\overline{\Omega}}\varphi(x,t_1),
\end{aligned}$$

from which it follows that

$$\alpha=\frac{t_1\sup_{x\in\overline{\Omega}}\varphi(x,t_1)}{\inf_{x\in\overline{\Omega}}\phi(x,t_1)}>1.$$

We define the function

$$M(x,t)=\begin{cases}\frac{\phi(x,t_1)}{t_1^\alpha}t^\alpha & \text{if } t\le t_1,\\ \phi(x,t) & \text{if } t\ge t_1.\end{cases}$$

The function $M(x,t)$ is a Musielak-Olicz function inasmuch as its derivative,

$$\frac{\partial M(x,t)}{\partial t}=\begin{cases}\frac{\alpha\phi(x,t_1)}{t_1^\alpha}t^{\alpha-1} & \text{if } t\le t_1,\\ \varphi(x,t) & \text{if } t\ge t_1,\end{cases}$$

is a function which is positive for $t>0$, right-continuous for $t\ge 0$ non-decreasing, and $\lim_{t\to+\infty}\frac{\partial M(x,t)}{\partial t}=+\infty$. □

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