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WEAKLY MONOTONE DECREASING SOLUTIONS TO ELLIPTIC SCHRÖDINGER INTEGRAL SYSTEMS

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Abstract. In this article, we study positive solutions to an elliptic Schrödinger system in $\mathbb{R}^n$ for $n \geq 2$. We give general conditions guaranteeing the non-existence of positive solutions and introduce weakly monotone decreasing functions. We also establish lower-bounds on the decay rates of positive solutions and obtain upper-bounds when these are weakly monotone decreasing.

1. Introduction and main results

In this article, we investigate positive solutions to the elliptic Schrödinger integral system

$$
\begin{aligned}
u(x) &= \int_{\mathbb{R}^n} \frac{\phi(y)u(y)^r v(y)^q + \Gamma_1(y,u,v)}{|x-y|^{n-\alpha}|y|^{\sigma_1}}\,\mathrm{d}y, \quad x \in \mathbb{R}^n,\\
v(x) &= \int_{\mathbb{R}^n} \frac{\psi(y)u(y)^p v(y)^s + \Gamma_2(y,u,v)}{|x-y|^{n-\alpha}|y|^{\sigma_2}}\,\mathrm{d}y, \quad x \in \mathbb{R}^n,
\end{aligned} \tag{1.1}
$$

where

$$
n \geq 2, \quad \alpha \in (0,n), \quad p,q,r,s \geq 0, \quad r,s \in [0,1], \quad \sigma_1,\sigma_2 \in (-\infty,\alpha). \tag{1.2}
$$

We assume that ϕ, ψ, Γ_1 and Γ_2 are non-negative in their arguments and that

$$
\liminf_{|x|\to\infty} \phi(x) > 0 \quad \text{and} \quad \liminf_{|x|\to\infty} \psi(x) > 0. \tag{1.3}
$$

These integral systems are closely related, and equivalent under the appropriate regularity and decay assumptions (see Vétois [3] and Villavert [4, 5] for results regarding this relationship), to differential equations of the form

$$
\begin{aligned}
(-\Delta)^{\alpha/2}u(x) &\equiv (\phi(x)v(x)^q u(x)^r + \Gamma_1(x,u,v))|x|^{-\sigma_1},\\
(-\Delta)^{\alpha/2}v(x) &\equiv (\psi(x)u(x)^p v(x)^s + \Gamma_2(x,u,v))|x|^{-\sigma_2}
\end{aligned}
$$

with $x \in \mathbb{R}^n \setminus \{0\}$. Systems of the form in (1.1) arise in nonlinear optics and in the modelling of Bose-Einstein double condensates (consult Vétois [3] and the references therein). It is also worth noting that Schrödinger equations in the whole $\mathbb{R}^n$ with $\Gamma_1, \Gamma_2 \equiv 0$ and $\phi \equiv \psi \equiv 1$ are central in the blow-up analysis of solutions to more general equations on manifolds and domains in $\mathbb{R}^n$. Furthermore, a priori

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decay estimates for solutions of (1.1) are useful in establishing the symmetry of solutions (see, for instance, Liu-Ma [2] and Vétois [3]).

When obtaining a priori estimates, it is common to consider *decay solutions*, i.e. solution pairs (u, v) such that $u(x) \simeq |x|^{-\theta_1}$ and $v(x) \simeq |x|^{-\theta_2}$, for some $\theta_1, \theta_2 > 0$. Here, $u(x) \simeq |x|^{-\theta}$ means that there exists a constant $C > 0$ such that

$$\frac{1}{C}|x|^{-\theta} \leq u(x) \leq C|x|^{-\theta}, \quad \text{as } |x| \to \infty.$$

This decay assumption was made in Villavert [4] when considering positive bounded solutions to the Hardy-Sobolev type system

$$\begin{aligned} u(x) &= \int_{\mathbb{R}} \frac{v(y)^q}{|x-y|^{n-\alpha}|y|^{\sigma_1}}\, \mathrm{d}y, \\ v(x) &= \int_{\mathbb{R}} \frac{u(y)^p}{|x-y|^{n-\alpha}|y|^{\sigma_2}}\, \mathrm{d}y \end{aligned} \tag{1.4}$$

with $\sigma_1, \sigma_2 \in [0, \alpha)$. We now introduce the notion of a weakly monotone decreasing function, which extends the concept of a decay solution.

Definition 1.1. A measurable function $f : \mathbb{R}^n \to (0, \infty]$ is said to be *weakly monotone decreasing* provided f is finite almost everywhere and there exist constants $C, R > 0$ such that $f(x) \leq Cf(y)$ whenever $|x| \geq |y| \geq R$.

Remark 1.2. If f is weakly monotone decreasing, then $\{f = \infty\}$ must also be bounded.

The set of all weakly monotone decreasing functions shall henceforth be denoted by $\mathcal{W}_m(\mathbb{R}^n)$. It is not difficult to see that all decay functions are weakly monotone decreasing. Thus, it is natural to view weakly monotone decreasing functions as a generalization of decay solutions. This notion of weak monotonicity will play a crucial role when deducing upper-bounds on the decay rates of positive solutions to (1.1).

Let us now define two positive constants that play a fundamental role in our asymptotic analysis:

$$\begin{aligned} r_0 &:= \frac{p(\alpha - \sigma_1) + (\alpha - \sigma_2)(1 - r)}{pq - (1 - s)(1 - r)}, \\ s_0 &:= \frac{q(\alpha - \sigma_2) + (\alpha - \sigma_1)(1 - s)}{pq - (1 - s)(1 - r)}. \end{aligned}$$

Recall that we use the notation $f(x) \lesssim g(x)$ to state that there exists $C, R > 0$ such that $f(x) \leq Cg(x)$ for all x satisfying $|x| \geq R$.

Theorem 1.3. *Suppose that* (1.2)-(1.3) *hold and let* (u, v) *be a positive solution pair to* (1.1). *Then*

$$u(x) \gtrsim \begin{cases} (1 + |x|)^{-\min\{n-\alpha, (q+r)(n-\alpha)-(\alpha-\sigma_1)\}}, & (q + r)(n - \alpha) \neq n - \sigma_1, \\ (1 + |x|)^{-(n-\alpha)} \ln(1 + |x|), & (q + r)(n - \alpha) = n - \sigma_1 \end{cases} \tag{1.5}$$

and

$$v(x) \gtrsim \begin{cases} (1 + |x|)^{-\min\{n-\alpha, (p+s)(n-\alpha)-(\alpha-\sigma_2)\}}, & (p + s)(n - \alpha) \neq n - \sigma_2, \\ (1 + |x|)^{-(n-\alpha)} \ln(1 + |x|), & (p + s)(n - \alpha) = n - \sigma_2. \end{cases} \tag{1.6}$$

Suppose, in addition, that u and v are weakly monotone decreasing. If

$$pq > (1-r)(1-s),$$

then

$$u(x) \lesssim |x|^{-s_0} \text{ and } v(x) \lesssim |x|^{-r_0}. \tag{1.7}$$

In several cases, the lower and upper estimates obtained in Theorem 1.3 are known to be sharp. Villavert [4] showed that all integrable solutions (u, v) to (1.4) decay precisely with the rates in (1.5)-(1.6). The lower bounds are also known to be optimal in the case $r = s = \sigma_{1,2} = 0$, $\Gamma_1 \equiv \Gamma_2 \equiv 0$ and $\phi \equiv \psi \equiv 1$ (see Vétois [3]). The bounds in (1.5)-(1.6) were also found to be sharp for positive $C^2(\mathbb{R}^n)$ radially symmetric solutions of the equation $\Delta u + K(x)u^p \equiv 0$, under suitable conditions for K and p (the reader may consult Li [1] for more details). In fact, Li [1] also showed that these radial $C^2(\mathbb{R}^n)$ solutions to $\Delta u+K(x)u^p \equiv 0$ decay with the rates in (1.7) when $u \not\simeq |x|^{2-\alpha}$. We also point out that the upper-bound estimates in (1.7) were obtained in Villavert [4] for bounded decay solutions to (1.4). Moreover, in Villavert [4] it was also established that the estimates in (1.7) are sharp for all non-integrable decay solutions to (1.4).

The first section is devoted to the proof of Theorem 1.3. In the second section, we shall instead give conditions under which no positive or weakly monotone decreasing solution pairs to (1.1) can exist. We also provide bounds on the weighting terms σ_1 and σ_2 required for the existence of solutions. These are contained within the following theorem.

Theorem 1.4. *Assume* (1.2)-(1.3) *hold. System* (1.1) *admits no positive solutions if either $pq = 0$,*

$$\sigma_1 \leq \alpha - (q + r)(n - \alpha), \quad \text{or} \quad \sigma_2 \leq \alpha - (p + s)(n - \alpha).$$

Furthermore, no weakly monotone decreasing solutions exist if $pq \leq (1 - r)(1 - s)$.

2. Decay estimates

For the entirety of this section, we assume that u and v are positive functions defined on $\mathbb{R}^n$ and that (1.2)-(1.3) hold. We begin by deriving a priori upper-bound estimates for weakly monotone decreasing solution pairs. For the remainder of this paper, we denote by $\text{meas}(\cdot)$ the Lebesgue measure on $\mathbb{R}^n$.

Proposition 2.1. *Let (u, v) be a positive weakly monotone decreasing solution pair to* (1.1)*. If $pq > (1-r)(1-s)$, then*

$$u(x) \lesssim |x|^{-s_0} \quad \text{and} \quad v(x) \lesssim |x|^{-r_0}.$$

Proof. We shall follow the strategy illustrated in Villavert [4]. Since u and v are both weakly monotone decreasing, we are free to choose positive constants R and C such that u and v satisfy

$$Cu(x) \leq u(y) \quad \text{and} \quad Cv(x) \leq v(y)$$

whenever $|x| \geq |y| \geq R$. Moreover, by invoking (1.3), we are free to assume that

$$\min\{\phi(x), \psi(x)\} \geq \gamma_0 > 0, \quad \forall |x| \geq R,$$

where γ_0 is a constant. For $|x| \geq 2R$ we define an annulus in space

$$A_x := \{y \in \mathbb{R}^n : \frac{|x|}{2} < |y| < |x|\}$$

and deduce from the non-negativity of u and v that, for all such x,

$$u(x) \geq \int_{\mathbb{R}^n} \frac{\phi(y)v(y)^q u(y)^r}{|x-y|^{n-\alpha}|y|^{\sigma_1}}\,\mathrm{d}y \geq \gamma_0 \int_{A_x} \frac{v(y)^q u(y)^r}{|x-y|^{n-\alpha}|y|^{\sigma_1}}\,\mathrm{d}y.$$

Now, using that both u and v are weakly monotone decreasing, we find (after a correction of the constant C)

$$\begin{aligned} u(x) &\geq C \int_{A_x} \frac{v(x)^q u(x)^r}{|x-y|^{n-\alpha}|y|^{\sigma_1}}\,\mathrm{d}y \\ &\geq C u(x)^r v(x)^q |x|^{\alpha-n} \int_{A_x} \frac{1}{|y|^{\sigma_1}}\,\mathrm{d}y \\ &\geq C u(x)^r v(x)^q |x|^{\alpha-n-\sigma_1}\,\mathrm{meas}(A_x), \end{aligned}$$

where we have used that $|x-y| \leq 2|x|$ and $|y| \leq |x|$. Since

$$\mathrm{meas}(A_x) = c\big(|x|^n - \frac{|x|^n}{2^n}\big)$$

for a constant $c > 0$, it follows that

$$u(x) \geq C u(x)^r v(x)^q |x|^{\alpha-\sigma_1}, \quad \text{as } |x| \to \infty. \tag{2.1}$$

By symmetry of the system, a verbatim argument yields

$$v(x) \geq C u(x)^p v(x)^s |x|^{\alpha-\sigma_2}, \quad \text{as } |x| \to \infty. \tag{2.2}$$

We now distinguish two possible cases.

Case 1: $r, s \in [0,1)$. Using (2.1) and (2.2) we have, as $|x| \to \infty$,

$$u(x) \geq C v(x)^{\frac{q}{1-r}} |x|^{\frac{\alpha-\sigma_1}{1-r}} \quad \text{and} \quad v(x) \geq C u(x)^{\frac{p}{1-s}} |x|^{\frac{\alpha-\sigma_2}{1-s}}.$$

Combining these inequalities yields, for $|x|$ large,

$$u(x) \geq C u(x)^{\frac{pq}{(1-s)(1-r)}} |x|^{\frac{q(\alpha-\sigma_2)}{(1-s)(1-r)} + \frac{\alpha-\sigma_1}{1-r}}.$$

The above implies that, as $|x| \to \infty$,

$$u(x)^{\frac{pq-(1-s)(1-r)}{(1-s)(1-r)}} \leq C|x|^{-\frac{q(\alpha-\sigma_2)+(\alpha-\sigma_1)(1-s)}{(1-s)(1-r)}}.$$

Consequently, as $|x| \to \infty$

$$u(x) \leq C|x|^{-\frac{q(\alpha-\sigma_2)+(\alpha-\sigma_1)(1-s)}{pq-(1-s)(1-r)}} = C|x|^{-s_0}.$$

A symmetric argument shows that $v(x) \lesssim |x|^{-r_0}$ as well.

Case 2: $r = 1$ or $s = 1$. We may assume without loss of generality that $r = 1$. We invoke equation (2.1) to find that, after a correction of C,

$$v(x) \leq C|x|^{-\frac{\alpha-\sigma_1}{q}} = C|x|^{-r_0}, \quad \text{as } |x| \to \infty. \tag{2.3}$$

Similarly, if $s = 1$ we use (2.2) and take roots to obtain

$$u(x) \leq C|x|^{-\frac{\alpha-\sigma_2}{p}} = C|x|^{-s_0}, \quad \text{as } |x| \to \infty.$$

On the other hand, if $0 \leq s < 1$, it follows from (2.3) that for all suitably large x,

$$v(x)^{1-s} \leq C|x|^{-\frac{(\alpha-\sigma_1)(1-s)}{q}}.$$

Combining the above estimate with (2.2) grants us the following, which is valid for all $|x|$ large,

$$C u(x)^p |x|^{\alpha-\sigma_2} \leq v(x)^{1-s} \leq C'|x|^{-\frac{(\alpha-\sigma_1)(1-s)}{q}}$$

whence we have
$$u(x)^p \le C|x|^{-\frac{q(\alpha-\sigma_2)+(\alpha-\sigma_1)(1-s)}{q}}, \quad \text{as } |x| \to \infty.$$
Taking roots we obtain
$$u(x) \le C|x|^{-\frac{q(\alpha-\sigma_2)+(\alpha-\sigma_1)(1-s)}{pq}} = C|x|^{-s_0}, \quad \text{as } |x| \to \infty.$$
A verbatim argument applies to the case of $s = 1$ and $0 \le r < 1$. This completes the proof. □

Lemma 2.2. *Let* (u, v) *be a positive solution pair to* (1.1)*. Then*
$$\min\{u(x), v(x)\} \gtrsim \frac{1}{(1+|x|)^{n-\alpha}}. \tag{2.4}$$

Proof. By (1.3), we may choose $R > 0$ such that
$$\min\{\phi(x), \psi(x)\} \ge \gamma_0 > 0$$
whenever $|x| \ge R - 1$. Once again, we define an annulus in $\mathbb{R}^n$
$$A := \{y \in \mathbb{R}^n : R - 1 < |y| < R\}.$$
Let $x \in \mathbb{R}^n$ be such that $|x| \ge R$ and let $y \in A$. Then $|x - y| \le |x| + R$, whence
$$u(x) \ge \gamma_0 \int_A \frac{v(y)^q u(y)^r}{|x-y|^{n-\alpha}|y|^{\sigma_1}}\, \mathrm{d}y \ge \frac{C}{(R+|x|)^{n-\alpha}} \int_A \frac{v(y)^q u(y)^r}{|y|^{\sigma_1}}\, \mathrm{d}y.$$
By taking x such that $u(x) < \infty$, it follows that $\int_A \frac{v(y)^q u(y)^r}{|y|^{\sigma_1}}\, \mathrm{d}y$ is a finite positive constant independent of x, thereby yielding the desired inequality for u. By a symmetric argument, the same inequality holds true for v. □

We are now capable of proving our generalized version of Villavert [4, THM-1].

Proof of Theorem 1.3. We shall prove this result in two steps. The first establishes lower bounds for all positive solutions and the second step gives a sharper estimate on positive solutions in the cases
$$(q + r)(n - \alpha) = n - \sigma_1 \quad \text{and} \quad (p + s)(n - \alpha) = n - \sigma_2.$$

Step 1. *Suppose* u *and* v *are positive solutions to* (1.1)*. Then*
$$u(x) \gtrsim (1 + |x|)^{-\min\{n-\alpha,(q+r)(n-\alpha)-(\alpha-\sigma_1)\}},$$
$$v(x) \gtrsim (1 + |x|)^{-\min\{n-\alpha,(p+s)(n-\alpha)-(\alpha-\sigma_2)\}}.$$

Proof of Step 1. For $|x| > 0$ we define an open ball
$$B_x := \{y \in \mathbb{R}^n : |x - y| < \frac{|x|}{2}\},$$
and observe that by letting $|x| \to \infty$, we can make $y \in B_x$ arbitrarily large. Thus, by Lemma 2.2, as $|x| \to \infty$ we have (letting γ_0 be the same as in the previous lemma)
$$\begin{aligned}
u(x) &\ge \gamma_0 \int_{B_x} \frac{v(y)^q u(y)^r}{|x-y|^{n-\alpha}|y|^{\sigma_1}}\, \mathrm{d}y \\
&\ge C \int_{B_x} \frac{1}{(1+|y|)^{(n-\alpha)(q+r)}|x-y|^{n-\alpha}|y|^{\sigma_1}}\, \mathrm{d}y \\
&\ge \frac{C}{(1+|x|)^{(n-\alpha)(q+r)}} \int_{B_x} \frac{1}{|x-y|^{n-\alpha}|y|^{\sigma_1}}\, \mathrm{d}y
\end{aligned}$$

where, in this last step, we used that

$$(1+|y|)^{(n-\alpha)(q+r)} \le \Big(1+\frac{3}{2}|x|\Big)^{(n-\alpha)(q+r)} \le \big(\frac{3}{2}\big)^{(n-\alpha)(q+r)}(1+|x|)^{(n-\alpha)(q+r)}.$$

Thus, for $|x|$ sufficiently large, we obtain the lower-bound estimate

$$u(x) \ge \frac{C}{(1+|x|)^{(q+r)(n-\alpha)+\sigma_1}} \int_{B_x} \frac{1}{|x-y|^{n-\alpha}}\,\mathrm{d}y.$$

The estimate for u follows from the above once we observe that

$$\begin{aligned}
\int_{B_x} \frac{1}{|x-y|^{n-\alpha}}\,\mathrm{d}y &= \widetilde{C}\int_0^{|x|/2} \frac{1}{\rho^{n-\alpha}}\cdot \rho^{n-1}\,\mathrm{d}\rho \\
&= \widetilde{C}\int_0^{|x|/2} \rho^{\alpha-1}\,\mathrm{d}\rho \\
&= \widetilde{C}|x|^{\alpha} \sim \widetilde{C}(1+|x|)^{\alpha}.
\end{aligned}$$

This concludes the first step since a similar argument will yield the symmetric inequality for v.

Step 2. *Let (u,v) be a positive solution pair to* (1.1)*. Then*

$$\begin{gathered}
u(x) \gtrsim (1+|x|)^{-(n-\alpha)}\ln(1+|x|), \quad \text{if } (q+r)(n-\alpha) = n-\sigma_1, \\
v(x) \gtrsim (1+|x|)^{-(n-\alpha)}\ln(1+|x|), \quad \text{if } (p+s)(n-\alpha) = n-\sigma_2.
\end{gathered}$$

Proof of Step 2. We shall make use of an argument from Vétois [3] (see Theorem 1.1–Step 3.4 in this paper). An application of Lemma 2.2 shows that one shall always have the estimates

$$u(x) \gtrsim |x|^{\alpha-n}, \quad v(x) \gtrsim |x|^{\alpha-n}. \tag{2.5}$$

For fixed $k\in\mathbb{N}$, we define

$$A_0 := \inf_{|x|<1} v(x), \quad A_k := \inf_{2^{k-1}<|x|<2^k} v(x)$$

as well as

$$I_{j,k} := \inf_{2^{k-1}<|x|<2^k} \int_{B(0,2^j)\setminus B(0,2^{j-1})} |x-y|^{\alpha-n}\,\mathrm{d}y.$$

Let $k\in\mathbb{N}$ be large and fix $x\in\mathbb{R}^n$ such that $2^{k-1}<|x|<2^k$. Using that $\liminf_{|x|\to\infty}\psi(x)>0$ we obtain for $R>0$ and $N\in\mathbb{N}$ sufficiently large,

$$\begin{aligned}
v(x) &\ge c\int_{|y|\ge R} u(y)^p v(y)^s |x-y|^{\alpha-n}|y|^{-\sigma_2}\,\mathrm{d}y \\
&\ge c\sum_{j\ge N}\int_{B(0,2^j)\setminus B(0,2^{j-1})} u(y)^p v(y)^s |x-y|^{\alpha-n}|y|^{-\sigma_2}\,\mathrm{d}y.
\end{aligned}$$

Thus, by the estimates in (2.5),

$$\begin{aligned}
v(x) &\ge c\sum_{j\ge N}\int_{B(0,2^j)\setminus B(0,2^{j-1})} 2^{-jp(n-\alpha)-j\sigma_2} v(y)^s |x-y|^{\alpha-n}\,\mathrm{d}y \\
&\ge c\sum_{j\ge N}\int_{B(0,2^j)\setminus B(0,2^{j-1})} 2^{-jp(n-\alpha)-j\sigma_2} A_j^s |x-y|^{\alpha-n}\,\mathrm{d}y \\
&= c\sum_{j\ge N} 2^{-jp(n-\alpha)-j\sigma_2} A_j^s \int_{B(0,2^j)\setminus B(0,2^{j-1})} |x-y|^{\alpha-n}\,\mathrm{d}y
\end{aligned}$$

$$\geq c \sum_{j\geq N} 2^{-jp(n-\alpha)-j\sigma_2} A_j^s I_{j,k}.$$

This implies that there exists an $N \in \mathbb{N}$ and $c > 0$ such that for all positive integers k sufficiently large

$$A_k \geq c \sum_{j\geq N} 2^{-j(p(n-\alpha)+\sigma_2)} A_j^s I_{j,k}. \tag{2.6}$$

Now, let k be large and $j \in \{N, N+1, \dots, k\}$; if $2^{k-1} < |x| < 2^k$ we have

$$\begin{aligned} \int_{B(0,2^j)\setminus B(0,2^{j-1})} |x-y|^{\alpha-n}\,\mathrm{d}y &\geq c2^{-k(n-\alpha)} \int_{B(0,2^j)\setminus B(0,2^{j-1})} \mathrm{d}y \\ &= c2^{-k(n-\alpha)} \cdot (2^{nj} - 2^{n(j-1)}) \end{aligned}$$

which implies that for all k large,

$$I_{j,k} \geq c2^{nj-k(n-\alpha)}, \quad \forall j \in \{N, N+1, \dots, k\}. \tag{2.7}$$

We now carry all we need in order to complete the proof. By (2.6)-(2.7), if k is an integer much larger than N,

$$\begin{aligned} A_k &\geq c \sum_{j\geq N} 2^{-j(p(n-\alpha)+\sigma_2)} A_j^s I_{j,k} \\ &\geq c \sum_{j=N}^{k} 2^{-j(p(n-\alpha)+\sigma_2)} A_j^s I_{j,k} \\ &\geq c \sum_{j=N}^{k} 2^{-j(p(n-\alpha)+\sigma_2)} \cdot 2^{nj-k(n-\alpha)} A_j^s \\ &\geq c2^{-k(n-\alpha)} \sum_{j=N}^{k} 2^{-j(p(n-\alpha)+\sigma_2-n)} \cdot 2^{-sj(n-\alpha)} \\ &= c2^{-k(n-\alpha)} \sum_{j=N}^{k} 2^{-j((p+s)(n-\alpha)+\sigma_2-n)} \\ &= c2^{-k(n-\alpha)}(k-N). \end{aligned}$$

Since $k \sim (k-N)$ as $k \to \infty$, it follows that

$$v(x) \gtrsim |x|^{\alpha-n} \ln|x|.$$

An identical argument applies to u in the case $(q+r)(n-\alpha) = n - \sigma_1$. This concludes the proof of step 2.

The lower-bounds from the statement of the theorem follow immediately from these previous two steps combined with Lemma 2.2. If u and v are assumed to be weakly monotone decreasing, the upper-bounds follow from Proposition 2.1. $\square$

3. Non-existence results

In this section we prove Theorem 1.4, which gives the non-existence results justifying our assumptions on the constants appearing in system (1.1). Throughout this section, we assume that (1.2)-(1.3) hold and that both u and v are non-trivial.

Lemma 3.1. *Let $f : \mathbb{R}^n \to (0, \infty]$ be a weakly monotone decreasing function. Then*

$$\limsup_{|x|\to\infty} f(x) < \infty.$$

Proof. Since f is weakly monotone decreasing we may take $y \in \mathbb{R}$ so large in norm that $f(x) \le Cf(y)$ whenever $|x| \ge |y|$, where C is some positive constant independent of x. Without loss of generality suppose that $f(y) < \infty$. This implies that $\limsup_{|x|\to\infty} f(x) \le Cf(y) < \infty$, as was asserted. □

Proposition 3.2. *System* (1.1) *does not admit any non-trivial weakly monotone decreasing solution pairs when $0 < pq \le (1-r)(1-s)$.*

Proof. For this proof we borrow ideas from Villavert [4, PROP-8] and Villavert [5, THM-6]. Since we are handling the case $pq > 0$ we are assuming, especially, that $r, s \in [0, 1)$. We may also assume without loss of generality that $\sigma_{1,2} \ge 0$. Suppose, by way of contradiction, that $(u, v) \in \mathcal{W}_m(\mathbb{R}^n) \times \mathcal{W}_m(\mathbb{R}^n)$ is a positive solution pair to system (1.1) when $pq \le (1-r)(1-s)$. Using Lemma 2.2 it follows that

$$u(x) \gtrsim |x|^{-b_0}$$

where we set $b_0 = n - \alpha$. Combining this with (1.3) shows that we may choose $R > 0$ so large that $u(x) \ge c|x|^{-b_0}$, $\phi(x) \ge \gamma_0 > 0$,

$$cu(x) \le u(y) \quad \text{and} \quad cv(x) \le v(y)$$

whenever $|x| \ge |y| \ge R$. For $|x|$ sufficiently large we consider the annulus

$$A_x := \{y \in \mathbb{R}^n : R < |y| < |x|\}.$$

Then

$$\begin{aligned} v(x) &\ge Cv(x)^s u(x)^p |x|^{-\sigma_2} \int_{A_x} \frac{1}{|x-y|^{n-\alpha}}\,\mathrm{d}y \quad \ge Cv(x)^s u(x)^p |x|^{\alpha-\sigma_2-n}\,\mathrm{meas}(A_x) \\ &\ge Cv(x)^s u(x)^p |x|^{\alpha-\sigma_2} \\ &\ge Cv(x)^s |x|^{-pb_0+\alpha-\sigma_2}, \quad \text{as } |x| \to \infty. \end{aligned}$$

Hence,

$$v(x) \ge C|x|^{-a_1} \quad \text{where } a_1 := \frac{pb_0 - \alpha + \sigma_2}{1-s}$$

as $|x| \to \infty$. Repeating this procedure and taking R sufficiently large in each step, one can find by induction that

$$u(x) \gtrsim |x|^{-b_k} \quad \text{and} \quad v(x) \gtrsim |x|^{-a_k}$$

where

$$a_{k+1} := \frac{pb_k - \alpha + \sigma_2}{1-s} \quad \text{and} \quad b_k := \frac{qa_k - \alpha + \sigma_1}{1-r}.$$

The idea is to rewrite the induced recurrence relation in simpler terms to estimate b_k. Let us now define

$$\begin{aligned} P &:= \frac{p}{1-s}, \quad A := \frac{\alpha}{1-s}, \quad \Sigma_1 := \frac{\sigma_1}{1-r}, \\ Q &:= \frac{q}{1-r}, \quad B := \frac{\alpha}{1-r}, \quad \Sigma_2 := \frac{\sigma_2}{1-s}. \end{aligned}$$

Using the above notation, we rewrite the recurrence relation of interest as

$$a_{k+1} := Pb_k + \Sigma_2 - A, \quad b_k := Qa_k + \Sigma_1 - B.$$

By way of determining a closed form, let $k \in \mathbb{N}$ be large and $1 \le j \le k$ an integer. The reader may verify by direct substitution that

$$\begin{aligned} b_k &= Q^j P^j b_{k-j} + (Q + Q^2 P + Q^3 P^2 + \dots + Q^j P^{j-1})(\Sigma_2 - A) \\ &\quad + (1 + QP + \dots + Q^{j-1} P^{j-1})(\Sigma_1 - B). \end{aligned}$$

Now, taking $j = k$ we find

$$b_k = (PQ)^k b_0 + Q(\Sigma_2 - A)\sum_{\ell=0}^{k-1}(PQ)^\ell + (\Sigma_1 - B)\sum_{\ell=0}^{k-1}(PQ)^\ell. \tag{3.1}$$

Which yields the following simple expression for b_k,

$$b_k = (PQ)^k b_0 + [Q(\Sigma_2 - A) + (\Sigma_1 - B)]\sum_{\ell=0}^{k-1}(PQ)^\ell.$$

There are now two cases to distinguish.

Case 1: Assume $pq = (1-s)(1-r)$. Then $PQ = 1$ so that $b_k \to -\infty$ as $k \to \infty$.

Case 2: Suppose $pq < (1-s)(1-r)$. We then have

$$0 < PQ = \frac{pq}{(1-s)(1-r)} < 1,$$

whence

$$b_k = (PQ)^k b_0 + [Q(\Sigma_2 - A) + (\Sigma_1 - B)]\frac{(PQ)^k - 1}{PQ - 1}.$$

Now, we calculate

$$\begin{aligned} Q(\Sigma_2 - A) + (\Sigma_1 - B) &= \frac{q}{1-r}\Big(\frac{\sigma_2 - \alpha}{1-s}\Big) + \frac{\sigma_1 - \alpha}{1-r} \\ &= \frac{q(\sigma_2 - \alpha) + (\sigma_1 - \alpha)(1-s)}{(1-r)(1-s)}. \end{aligned}$$

Finally,

$$PQ - 1 = \frac{pq}{(1-s)(1-r)} - 1 = \frac{pq - (1-s)(1-r)}{(1-s)(1-r)},$$

whence

$$(Q(\Sigma_2 - A) + (\Sigma_1 - B))\frac{1}{PQ - 1} = \frac{q(\sigma_2 - \alpha) + (\sigma_1 - \alpha)(1-s)}{pq - (1-s)(1-r)} = -s_0.$$

Under our conditions we have $-s_0 > 0$ implying that $b_k < 0$ for large enough k.

In either case we may make $b_k < 0$ for all $k \in \mathbb{N}$ sufficiently large. Hence, for suitable k it holds

$$u(x) \gtrsim |x|^{-b_k} \quad \text{where } b_k < 0$$

which implies $\lim_{|x|\to\infty} u(x) = \infty$. However, this contradicts Lemma 3.1. □

Proposition 3.3. *If $p = 0$ there is no positive solution pair to* (1.1). *Similarly, there is no positive solution if $q = 0$.*

Proof. Without loss of generality, we may assume that $\sigma_{1,2} \ge 0$. We handle only the case $q = 0$; a similar argument applies when $p = 0$. From Lemma 3.4 it follows that $u(x) \ge c|x|^{-(n-\alpha)}$ as $|x| \to \infty$, for some constant $c > 0$. Fix $R > 0$ so large that $u(x) \geq c|x|^{-(n-\alpha)}$ and $\phi(x) \geq \gamma_0 > 0$ whenever $|x| \geq R$ (this can be done by (1.3)). Given $|x| \geq 2R$, we define as in the proof of Proposition 2.1

$$A_x := \{y \in \mathbb{R}^n : \frac{|x|}{2} < |y| < |x|\}$$

so that

$$\begin{aligned} u(x) &\geq \int_{A_x} \frac{\phi(y)u(y)^r}{|x-y|^{n-\alpha}|y|^{\sigma_1}} \,\mathrm{d}y \\ &\geq c|x|^{-r(n-\alpha)-\sigma_1} \int_{A_x} \frac{1}{|x-y|^{n-\alpha}} \,\mathrm{d}y \\ &\geq c|x|^{-r(n-\alpha)-\sigma_1+\alpha-n} \operatorname{meas}(A_x) \\ &\sim c|x|^{-r(n-\alpha)+\alpha-\sigma_1}, \quad \text{as } |x| \to \infty. \end{aligned}$$

Or, rather,

$$u(x) \gtrsim |x|^{-(rb_0+\sigma_1-\alpha)}, \quad \text{where } b_0 := n-\alpha.$$

Of course, we may repeat this argument inductively on $k \in \mathbb{N}$ to find that

$$u(x) \gtrsim |x|^{-b_k}, \quad \text{where } b_k := rb_{k-1} + \sigma_1 - \alpha \tag{3.2}$$

for all $k \in \mathbb{N}$. By properties of a geometric sum, it is easy to verify that for each $k \in \mathbb{N}$

$$b_k = \begin{cases} r^k b_0 + (\sigma_1 - \alpha)\frac{1-r^k}{1-r}, & \text{if } r < 1, \\ b_0 + k(\sigma_1 - \alpha), & \text{if } r = 1. \end{cases}$$

Since $\sigma_1 < \alpha$, by taking $k \to \infty$, we can make $b_k < 0$ for some $k \in \mathbb{N}$. Fix $R > 0$ large and assume that $|x| < R$; it then holds

$$\begin{aligned} u(x) &\geq c \int_{B_R(0)^{\complement}} \frac{u(y)^r}{|x-y|^{n-\alpha}|y|^{\sigma_1}} \,\mathrm{d}y \\ &\geq c \int_{B_R(0)^{\complement}} |y|^{-rb_k+\alpha-n-\sigma_1} \,\mathrm{d}y \\ &\geq c \int_R^\infty \rho^{-rb_k+\alpha-\sigma_1-1} \,\mathrm{d}\rho \end{aligned}$$

where this last integral is convergent if and only if $-rb_k + \alpha - \sigma_1 < 0$. Hence, we obtain that $u(x) = \infty$ in $|x| < R$: a contradiction. $\square$

Having established these results, we must only show that the following holds.

Lemma 3.4. *System* (1.1) *admits no positive solutions if either*

$$-\sigma_1 \geq (q+r)(n-\alpha) - \alpha \quad \text{or} \quad -\sigma_2 \geq (p+s)(n-\alpha) - \alpha.$$

Proof. We proceed by way of contradiction; without loss of generality assume that

$$-\sigma_1 \geq (q+r)(n-\alpha).$$

By invoking Lemma 2.2, we may choose a constant $C > 0$ such that

$$u(x) \geq C|x|^{-(n-\alpha)} \quad \text{and} \quad v(x) \geq C|x|^{-(n-\alpha)}$$

for all $|x|$ sufficiently large. Also, by (1.3), there exists $\gamma_0 > 0$ such that $\phi(x) \geq \gamma_0$ for all such x. Hence, for a sufficiently large $R > 0$ it holds

$$u(x) \geq \gamma_0 \int_{|y|\geq R} |y|^{-\sigma_1} \frac{u(y)^r v(y)^q}{|x-y|^{n-\alpha}} \,\mathrm{d}y$$

$$
\begin{aligned}
&\geq \gamma_0 \int_{|y|\geq R} |y|^{(q+r)(n-\alpha)-\alpha} \frac{u(y)^r v(y)^q}{|x-y|^{n-\alpha}} \,\mathrm{d}y \\
&\geq C\gamma_0 \int_{|y|\geq R} \frac{|y|^{(q+r)(n-\alpha)-(q+r)(n-\alpha)-\alpha}}{|x-y|^{n-\alpha}} \,\mathrm{d}y \\
&\geq C \int_{|y|\geq R} |x-y|^{-n} \,\mathrm{d}y.
\end{aligned}
$$

Since $\int_{|y|\geq R} |x-y|^{-n}\,\mathrm{d}y = \infty$, it follows that $u \equiv \infty$. This completes the proof of the lemma. □

Proof of Theorem 1.4. Proposition 3.3 clearly implies that there does not exist a positive solution if either $q = 0$ or $p = 0$. Likewise, it is a consequence of Proposition 3.2 that there does not exist any weakly monotone decreasing solutions whenever $pq \leq (1-r)(1-s)$. The theorem then follows at once from Lemma 3.4. □

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