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EXISTENCE AND MULTIPLICITY FOR RADIALLY SYMMETRIC SOLUTIONS TO HAMILTON-JACOBI-BELLMAN EQUATIONS

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Abstract. This article concerns the existence and multiplicity of radially symmetric nodal solutions to the nonlinear equation

$$\begin{gathered} -\mathcal{M}_{\mathcal{C}}^{\pm}(D^2u) = \mu f(u) \quad \text{in } \mathcal{B}, \\ u = 0 \quad \text{on } \partial\mathcal{B}, \end{gathered}$$

where $\mathcal{M}_{\mathcal{C}}^{\pm}$ are general Hamilton-Jacobi-Bellman operators, μ is a real parameter and $\mathcal{B}$ is the unit ball. By using bifurcation theory, we determine the range of parameter μ in which the above problem has one or multiple nodal solutions according to the behavior of f at 0 and ∞, and whether f satisfies the signum condition $f(s)s > 0$ for $s \neq 0$ or not.

1. Introduction and main results

In scientific fields such as engineering, economical business, and mechanics, one encounters the problem of how to control systems in an optimal way [2, 6, 10]. For such systems, states are governed by the stochastic differential equation

$$\frac{dX_t}{dt} = \tau(X_t, \alpha_t)\xi_t + b(X_t, \alpha_t) \quad \text{for } t \geq 0,\ X_0 = x \in \mathbb{R}^N,$$

where ξ_t is the typical 'white noise', τ and b are matrix-valued and vector-valued functions defined on $\mathbb{R}^N \times A$ respectively, A is a separable metric space, and α_t as the control process is an stochastic process taking its values in A.

Then one defines a cost function

$$J(x, \alpha_t) = E\Big(\int_0^\infty f(X_t, \alpha_t) \exp\Big(-\int_0^t c(X_s, \alpha_s) ds\Big) dt\Big),$$

where E denotes the expectation, $f(x, \alpha)$ and $c(x, \alpha)$ are real valued functions on $\mathbb{R}^N \times A$, and c is a function often called the discount factor.

The purpose of optimal stochastic control theory is to determine the optimal cost function (also called the value function, or the criterion)

$$u(x) = \inf\Big\{\frac{J(x, \alpha_t)}{\alpha_t} \text{ stochastic process with values in } A\Big\}. \tag{1.1}$$

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A fundamental tool for finding u is given by the dynamic programming principle introduced by Bellman [1]. This principle indicates that u should, in some way, be the solution of the partial differential equation

$$\sup_{\alpha\in A}\{A_\alpha u(x) - f_\alpha(x) = 0 \text{ in } \mathbb{R}^N\}, \tag{1.2}$$

where $f_\alpha(\cdot) = f(\cdot,\alpha)$, $A_\alpha = -\sum_{i,j} a_{ij}(x,\alpha)\partial_{ij} - \sum_i b_i(x,\alpha)\partial_i + c(x,\alpha)$ and $a = \frac{1}{2}\tau\tau^T$. Equation (1.2) is called the Hamilton-Jacobi-Bellman (HJB in short) equation associated with the control problem (1.1). In some sense it is an extension of the classical first-order Hamilton-Jacobi equations occurring in Calculus of variations, see P. L. Lions [6]. We refer to the book of Bensoussan and J. L. Lions [2] or the papers of P. L. Lions [7, 8, 9] for further relation between a general HJB and stochastic control.

1.1. **Existing results.** Quass and Allendes [14] considered the radially symmetric fully nonlinear equation involving extremal operators of Pucci type,

$$\begin{gathered} -\mathcal{M}^{\pm}_{\mathcal{C}}(D^2u) = f(u) \quad \text{in } \mathcal{B},\\ u = 0 \quad \text{on } \partial\mathcal{B}, \end{gathered} \tag{1.3}$$

where $\mathcal{B}$ is the unit ball in $\mathbb{R}^N$ with $N \geq 1$, $\mathcal{M}^{\pm}_{\mathcal{C}}$ are general HJB operators. Specifically these operators are defined as

$$\mathcal{M}^{+}_{\mathcal{C}}(M) = \sup_{\sigma(A)\in\mathcal{C}} \operatorname{tr}(AM), \quad \mathcal{M}^{-}_{\mathcal{C}}(M) = \inf_{\sigma(A)\in\mathcal{C}} \operatorname{tr}(AM),$$

where $\mathcal{C}$ is any subset of the cube $[\lambda,\Lambda]^N$ which is invariant with respect to permutations of coordinates, $\sigma(A)$ is the set of eigenvalues of A, and the parameters λ,Λ satisfy $0<\lambda\leq\Lambda$. These operators reduce to classical Pucci type operators when $\mathcal{C} = [\lambda,\Lambda]^N$, and to the Laplacian when $\lambda=\Lambda=1$. When $\lambda \leq \frac{1}{N}, \Lambda = 1-\lambda(N-1)$ and $\mathcal{C} = \{a \in [\lambda,\Lambda]^N \mid \sum_{i=1}^N a_i = 1\}$, the operator corresponds to Pucci's operators, see [12, 13]. Clearly, problem (1.3) is a special case of (1.2).

Based on the bifurcation theory, Quass and Allendes established a multiplicity result for (1.3) and showed that the eigenvalue problem

$$\begin{gathered} -\mathcal{M}^{+}_{\mathcal{C}}(D^2u) = \mu u \quad \text{in } \mathcal{B},\\ u = 0 \quad \text{on } \partial\mathcal{B} \end{gathered} \tag{1.4}$$

has two unbounded increasing sequences μ_k^+ and μ_k^-, such that

$$\begin{gathered} 0<\mu_1^+<\mu_2^+<\dots<\mu_k^+<\dots,\\ 0<\mu_1^-<\mu_2^-<\dots<\mu_k^-<\dots. \end{gathered}$$

Moreover, the set of radial solutions of (1.4) for $\mu=\mu_k^+$ is positively spanned by a function φ_k^+, which is positive at the origin and has exactly $k-1$ zeros in $(0,1)$, all these zeros being simple. The same holds for $\mu=\mu_k^-$, but considering φ_k^- is negative at the origin. Then they studied the global bifurcation phenomenon of the problem

$$\begin{gathered} -\mathcal{M}^{+}_{\mathcal{C}}(D^2u) = \mu u + f(u,\mu) \quad \text{in } \mathcal{B},\\ u = 0 \quad \text{on } \partial\mathcal{B}, \end{gathered} \tag{1.5}$$

where f is continuous, $f(s,\mu) = o(|s|)$ near $s=0$, uniformly for $\mu\in\mathbb{R}$. They showed that, for each $k\in\mathbb{N}, k\geq 1$, there are two connected components $H_k^{\pm}\subset S_k^{\pm}$ of nontrivial solutions to (1.5), whose closures contain $(\mu_k^{\pm},0)$. Moreover, $H_k^{\pm}$ are unbounded and $(\mu, u) \in S_k^+(S_k^-)$ implies that u possesses exactly $k-1$ zeros in $(0,1)$, u is positive (negative) near 0.

For notational simplicity, we write $\mathcal{M}_{\mathcal{C}}^{\pm}$ in (1.3) to mean the two problems, one with the operator $\mathcal{M}_{\mathcal{C}}^{+}$ and the other with $\mathcal{M}_{\mathcal{C}}^{-}$. In the remaining, the situation is similar.

Dai [5], by using bifurcation approach with the generalized limit theorem, studied the existence and multiplicity of nodal solutions for the special problem

$$\begin{gathered} -\mathcal{M}_{\lambda,\Lambda}^{\pm}(D^2u) = \mu f(u) \quad \text{in } \mathcal{B}, \\ u = 0 \quad \text{on } \partial\mathcal{B}, \end{gathered} \tag{1.6}$$

where $\mathcal{M}_{\lambda,\Lambda}^{\pm}$ denote Pucci's extremal operator.

Motivated by works mentioned above, our aim is to extend the results from $\mathcal{C} = [\lambda, \Lambda]^N$ in [5, Theorems 1.8-1.9] to the general HJB operators. To be specific, in consideration of the Rabinowitz global bifurcation theory, and according to the asymptotic behavior of the nonlinear term f at 0 and ∞, with signum condition, we focus on the existence, multiplicity and nonexistence of nodal solutions for the radially symmetric non proper equation of the type

$$\begin{gathered} -\mathcal{M}_{\mathcal{C}}^{\pm}(D^2u) = \mu f(u) \quad \text{in } \mathcal{B}, \\ u = 0 \quad \text{on } \partial\mathcal{B}. \end{gathered} \tag{1.7}$$

Moreover, we consider the global behavior of nodal solutions for (1.7) without signum condition. We shall show that the branches bifurcating from infinity and the trivial solution line are disjoint. Hence the essential role is played by the fact whether f possesses zeros in $\mathbb{R} \setminus \{0\}$ or not.

1.2. **Statement of main results.** To obtain our main results, we shall give a bifurcation theorem from infinity for problem (1.5) under the assumption that

$$\lim_{|s|\to+\infty} \frac{f(\mu, s)}{s} = 0 \quad \text{uniformly for } \mu \in \mathbb{R}. \tag{1.8}$$

Let

$$E = \{u \in C[0,1] : u'(0) = u(1) = 0\}$$

with the usual norm $\|\cdot\|_\infty$. Let S_k^+ denote the set of functions in E which have exactly $k-1$ interior nodal (i.e. non-degenerate) zeros in $(0,1)$ and are positive at 0. Set $S_k^- = -S_k^+$ and $S_k = S_k^+ \cup S_k^-$. It is clear that S_k^+ and S_k^- are disjoint and open in E.

Theorem 1.1. *Let condition* (1.8) *hold. There exists an unbounded component* $\mathcal{D}_k^\nu \subset (\{(\mu_k^\nu, \infty)\} \cup (\mathbb{R} \times S_k^\nu))$ *of solutions to problem* (1.5). *Moreover, either*

1. $\mathcal{D}_k^\nu$ *meets* $\mathcal{R} = \{(\mu, 0) : \mu \in \mathbb{R}\}$, *or*
2. $\mathcal{D}_k^\nu$ *has an unbounded projection on* $\mathbb{R}$.

On account of Theorem 1.1 and [14, Theorem 1.4], we shall investigate the existence and multiplicity of nodal solutions for problem (1.7). We use the following assumptions:

(A1) $f(s)s > 0$ for any $s \neq 0$;

(A2) there exist $f_0, f_\infty \in [0, +\infty]$ such that

$$\lim_{|s|\to 0} \frac{f(s)}{s} = f_0, \quad \lim_{|s|\to+\infty} \frac{f(s)}{s} = f_\infty;$$

(A3) there exist two constants $s_2 < 0 < s_1$ such that $f(s_2) = f(s_1) = 0$ and $f(s)s > 0$ for $s \in \mathbb{R} \setminus \{s_2, 0, s_1\}$;

(A4) there exist two constants $\gamma_1 > 0$ and $\gamma_2 < 0$ such that

$$\lim_{s \to s_1^-} \frac{f(s)}{s_1 - s} = \gamma_1, \quad \lim_{s \to s_2^+} \frac{f(s)}{s - s_2} = \gamma_2;$$

According to the asymptotic behavior of f at 0 and ∞, we have the results in Theorem 1.2.

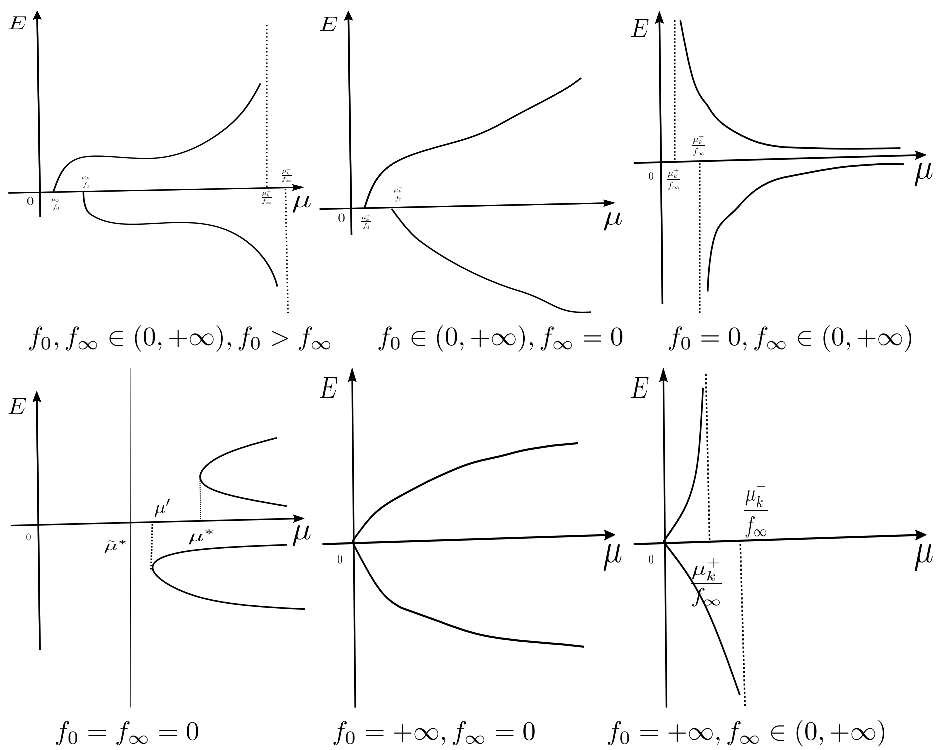


FIGURE 1. Bifurcation diagrams for Theorem 1.2

Theorem 1.2. *Suppose that f satisfies* (A1) *and* (A2).

(a) If $f_0, f_\infty \in (0,+\infty)$ with $f_0 \neq f_\infty$, then for $k \in \mathbb{N}$,

$$\mu \in \Big(\min\Big\{\frac{\mu_k^\nu}{f_0}, \frac{\mu_k^\nu}{f_\infty}\Big\}, \max\Big\{\frac{\mu_k^\nu}{f_0}, \frac{\mu_k^\nu}{f_\infty}\Big\}\Big),$$

problem (1.7) *has at least one nodal solution u_k^ν, such that νu_k^ν has exactly $k-1$ simple zeros in $(0,1)$ and is positive near 0, where $\nu \in \{+,-\}$.*

(b) If $f_0 \in (0,+\infty)$ and $f_\infty = 0$, then for any $k \in \mathbb{N}, \mu \in (\frac{\mu_k^\nu}{f_0}, +\infty)$, problem (1.7) *has at least one nodal solution u_k^ν, such that νu_k^ν has exactly $k-1$ simple zeros in $(0,1)$ and is positive near 0.*

(c) If $f_0 = 0$ and $f_\infty \in (0,+\infty)$, then for any $k \in \mathbb{N}, \mu \in (\frac{\mu_k^\nu}{f_\infty}, +\infty)$, problem (1.7) *has at least one nodal solution u_k^ν, such that νu_k^ν has exactly $k-1$ simple zeros in $(0,1)$ and is positive near 0.*

(d) If $f_0 = f_\infty = 0$, then there exists $\mu^* > 0$, such that for $k \in \mathbb{N}, \mu \in (\mu^*, +\infty)$, problem (1.7) has at least two nodal solutions u_k^{1+}, u_k^{2+}. Moreover, u_k^{1+}, u_k^{2+} have exactly $k-1$ simple zeros in $(0,1)$ and are positive near 0; there exists $\mu' > 0$, such that problem (1.7) has at least two nodal solutions u_k^{1-}, u_k^{2-} for all $\mu \in (\mu', +\infty)$, and u_k^{1-}, u_k^{2-} have exactly $k-1$ simple zeros in $(0,1)$ and are negative near 0. Furthermore, there exists $\tilde{\mu}^* > 0$, such that the problem (1.7) has no nodal solution for $\mu \in (0, \tilde{\mu}^*)$.

(e) If $f_0 = +\infty$ and $f_\infty = 0$, then for any $\mu \in (0, +\infty)$, $k \in \mathbb{N}$, problem (1.7) has at least one nodal solution u_k^ν, such that νu_k^ν has exactly $k-1$ simple zeros in $(0,1)$ and is positive near 0.

(f) If $f_0 = +\infty$ and $f_\infty \in (0, +\infty)$, then for any $\mu \in (0, \frac{\mu_k^\nu}{f_\infty})$, $k \in \mathbb{N}$, problem (1.7) has at least one nodal solution u_k^ν, such that νu_k^ν has exactly $k-1$ simple zeros in $(0,1)$ and is positive near 0.

See illustrations in Figure 1.

It is worth mentioning that the signum condition $f(s)s > 0$ for $s \neq 0$ plays an important role in the Theorem 1.2. In the following, one considers the global behavior of nodal solutions for (1.7) without signum condition. We shall show that the branches bifurcating from infinity and the trivial solution line are disjoint. Concretely, it has the following interesting results.

Theorem 1.3. *Let (A2)–(A4) hold. If $f_0, f_\infty \in (0, +\infty)$ with $f_0 \neq f_\infty$, then problem (1.7) has at least one nodal solution u_k^ν for*

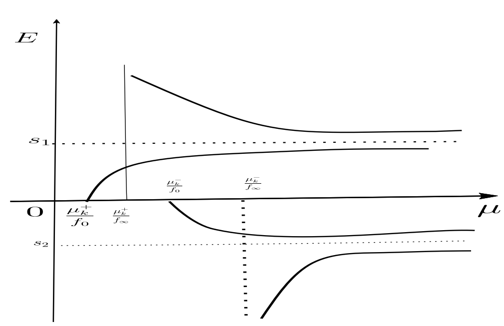
$$\mu \in \Big(\min\{\frac{\mu_k^\nu}{f_0}, \frac{\mu_k^\nu}{f_\infty}\}, \max\{\frac{\mu_k^\nu}{f_0}, \frac{\mu_k^\nu}{f_\infty}\}\Big),$$

$k \in \mathbb{N}$, such that νu_k^ν has exactly $k-1$ simple zeros in $(0,1)$ and is positive near 0; (1.7) has at least four nodal solutions $u_{k,0}^+, u_{k,\infty}^+, u_{k,0}^-$ and $u_{k,\infty}^-$ for

$$\mu \in \big(\max\{\frac{\mu_k^+}{f_0}, \frac{\mu_k^-}{f_0}, \frac{\mu_k^+}{f_\infty}, \frac{\mu_k^-}{f_\infty}\}, +\infty\big),$$

such that they have exactly $k-1$ simple zeros in $(0,1)$, $u_{k,0}^+$ and $u_{k,\infty}^+$ are positive near 0, $u_{k,0}^-$ and $u_{k,\infty}^-$ are negative near 0. Moreover, it derives $\|u_{k,0}^+\| \to s_1^-$ and $\|u_{k,0}^-\| \to (-s_2)^-$ as $\mu \to +\infty$.

See illustrations in Figure 2.



$f_0, f_\infty \in (0, +\infty), f_0 > f_\infty$

FIGURE 2. Bifurcation diagrams of Theorem 1.3.

The results obtained above are also valid for problem (1.7) if we replace $\mathcal{M}_{\mathcal{C}}^+$ by $\mathcal{M}_{\mathcal{C}}^-$, so to simplify our presentation, we only consider the operator $\mathcal{M}_{\mathcal{C}}^+$. For simplicity, we only consider the case when f does not depend on x, and f is asymptotically linear at 0 and ∞. In fact, Theorem 1.3 is still valid for the case of f depending on x or f satisfying other asymptotic behaviors with obvious changes.

The conclusions of Theorem 1.3 are not only significate in theory, but also meaningful in economics. For example, the conclusion of Theorem 1.3 with $\nu = +$ means that if the reaction function f which can denote the investment strategy is linear near 0 and ∞, and the diffusion coefficient $d := 1/\mu$ which can denote the rate of investment belongs to some interval (α, β) with $0 < \alpha < \beta < +\infty$, then there at least exists one optimal cost function.

This article is arranged as follows. In Section 2, we recall some preliminary results and give the proof of Theorem 1.1. In Section 3, according to the different asymptotic behaviors of f at 0 and ∞, we prove Theorem 1.2 and derive the existence, nonexistence and multiplicity of nodal solutions for problem (1.7) with signum condition. In Section 4, we give the proof of Theorem 1.3, which considers the global behavior of nodal solutions for (1.7) without signum condition, and we shall show that the branches bifurcating from infinity and the trivial solution line are disjoint.

2. Preliminary results and Proof of Theorem 1.1

We start this section by studying the operator acting on radial functions, details can be seen in [14, Section 3]. We define the operator $\mathcal{M}_{\mathcal{C}}^+$ acting on C^2 radially symmetric functions as

$$\mathcal{M}_{\mathcal{C}}^+(D^2u) = \sup_{(a_1,a_2)\in\tilde{\mathcal{C}}} \Big(a_1 u'' + \frac{(N-1)a_2 u'}{r}\Big),$$

where $\tilde{\mathcal{C}} := \{(a_1, \frac{1}{N-1}\sum_{i=2}^N a_i) \in \mathbb{R}^2 : (a_1, a_2, \dots, a_n) \in \mathcal{C}\}$. In the rest of this article we write $\mathcal{C}$ for $\tilde{\mathcal{C}}$ to simplify the notation. To describe the set $\mathcal{C}$ in a more convenient way. To avoid trivialization, we make an additional assumption.

(A5) The set $\mathcal{C} \subset \mathbb{R}_+^2$ is compact, convex and its projection onto the $y-$axis is not a singleton.

Assuming (A5) we exclude the case when the projection of $\mathcal{C}$ onto the y-axis is a singleton, which is equivalent to $\mathcal{C} = \{(a_1, a_2)\}$. This particular case can be analyzed as the radial Laplacian. Observe that $\mathcal{C}$ is a symmetric set.

Under assumption (A5), we can describe $\partial\mathcal{C}$ by means of two functions. Let $0 < \theta_{\min} < \theta_{\max}$ be defined as $\theta_{\min} = \min\{\theta : (x,\theta) \in \mathcal{C}\}$ and $\theta_{\max} = \max\{\theta : (x,\theta) \in \mathcal{C}\}$, and define the functions $S, \tilde{S} : [\theta_{\min}, \theta_{\max}] \to \mathbb{R}^+$ as

$$S(\theta) = \min\{x : (x,\theta) \in \mathcal{C}\}, \quad \tilde{S}(\theta) = \max\{x : (x,\theta) \in \mathcal{C}\}.$$

With these definitions we see that S is convex, $\tilde{S}$ is concave and

$$\mathcal{C} = \{(x,\theta) : \theta \in [\theta_{\min}, \theta_{\max}], S(\theta) \le x \le \tilde{S}(\theta)\}.$$

Being S convex, it has one-sided derivatives $S'_-(\theta)$ and $S'_+(\theta)$, consequently it is locally Lipschitz continuous in $(\theta_{\min}, \theta_{\max})$. The sub-differential of S is then defined as $\partial S(\theta) = [S'_-(\theta), S'_+(\theta)]$ for $\theta \in (\theta_{\min}, \theta_{\max})$. The cases $\theta = \theta_{\min}$ and $\theta = \theta_{\max}$

are special. At $\theta_{\max}$ we have two possibilities, either $S'_{-}(\theta_{\max})$ exists, and then we define $\partial S(\theta_{\max}) = [S'_{-}(\theta_{\max}), +\infty)$, or

$$\lim_{t\to 0^-} \frac{S(\theta_{\max}+t) - S(\theta_{\max})}{t} = +\infty.$$

An analogous situation occurs at $\theta_{\min}$. We observe that with these definitions, for every $Q \in \mathbb{R}$ there is at least one solution $\theta \in [\theta_{\min}, \theta_{\max}]$, such that

$$\partial S(\theta)\theta - S(\theta) \ni Q.$$

In each of this maximal intervals $[\theta_i^-, \theta_i^+]$, with $\theta_i^- < \theta_i^+$ where the function S is affine, we may write

$$S(\theta) = d_i\theta - Q_i, \quad \forall \theta \in [\theta_i^-, \theta_i^+],$$

for numbers d_i and Q_i. We define the function $d : \mathbb{R} \to \mathbb{R}$ as $d(Q) \in \partial S(\theta)$ such that

$$d(Q)\theta - S(\theta) = Q.$$

All the above hold for $\tilde{S}$ with natural modification since $\tilde{S}$ is concave and $\partial\tilde{S}$ is the super-differential of $\tilde{S}$. We consider $\Theta : \mathbb{R} \to \mathbb{R}$ as $\Theta(Q_i) = \theta_i^+$ in each interval where S or $\tilde{S}$ are affine functions.

Now, we can easily show that the radially symmetric solutions of problem (1.7) are also solutions to

$$\begin{gathered} u''(r) + (N_d - 1)\frac{u'(r)}{r} + \frac{\mu f(u)}{\theta} = 0 \quad \text{in } (0,1), \\ u'(0) = 0, \quad u(1) = 0, \end{gathered} \tag{2.1}$$

where $\theta = \Theta(\frac{u}{(N-1)u'})$ when $u' \neq 0$ and

$$N_d = \begin{cases} \frac{S(\theta)}{\theta}(N-1), & \text{if } u' < 0, \\ \frac{\tilde{S}(\theta)}{\theta}(N-1), & \text{if } u' > 0. \end{cases}$$

When $u' = 0$, then $\theta := \theta_{\min}$ if $u > 0$ and $\theta := \theta_{\max}$ if $u < 0$. Notice that the functions $\theta(r)$ and $N_d(r)$ are measurable functions, having discontinuities whenever r is so that $\frac{u(r)r}{(N-1)u'(r)} = Q_i$ and $S, \tilde{S}$ are affine functions. Moreover, both $\theta(r)$ and $N_d(r)$ are bounded and bounded away from 0.

Similar to [14], we obtain that problem (2.1) is equivalent to the problem

$$\begin{gathered} -\big(\rho_u(r)u'(r)\big)' = \mu\tilde{\rho}_u f(u(r)) \quad \text{in } (0,1), \\ u'(0) = 0, \quad u(1) = 0, \end{gathered} \tag{2.2}$$

where $\rho_u(r) := \exp\big(\int_0^r \frac{N_d(\tau)-1}{\tau} d\tau\big)$ denotes the integral factor of the equation, $\tilde{\rho}_u(r) := \frac{\rho_u(r)}{\theta}$, ρ_u and θ are characterized by the optimal condition.

For arriving to the results in Theorem 1.2, we need the following topological lemma, see[11].

Lemma 2.1. *Let X be a Banach space and let $\mathcal{C}_n$ be a family of closed connected subsets of X. Assume that*

(i) *there exist $z_n \in \mathcal{C}_n, n = 1, 2, \dots,$ and $z^* \in X$, such that $z_n \to z^*$;*
(ii) *$r_n = \sup\{\|x\|_X : x \in \mathcal{C}_n\} = +\infty$;*
(iii) *for every $R > 0, (\cup_{n=1}^{+\infty}\mathcal{C}_n) \cap B_R$ is a relatively compact set of X, where $B_R = \{x \in X : \|x\|_X \leq R\}$.*

Then there exists an unbounded component $\mathcal{C}$ of $\mathcal{D} = \limsup_{n\to+\infty} \mathcal{C}_n$ and $z^* \in \mathcal{C}$.

Now we recall the following compactness results for the Pucci's extremal operator, see [4, Proposition 2.1].

Lemma 2.2. *Let $\{F_n\}_{n>0}$ be a sequence of uniformly elliptic concave (or convex) operators with ellipticity constants λ and Λ, such that $F_n \to F$ is uniformly in compact sets of $S_n \times \Omega$ (S_n is the set of symmetric matrices). In addition, suppose that $u_n \in C(\bar{\Omega}) \cap W^{2,N}_{\rm loc}(\Omega)$ satisfies*

$$F_n(D^2 u_n, x) = 0 in\ \Omega, \quad u_n = 0 \ \text{on}\ \partial\Omega$$

and that u_n converges uniformly to u. Then, $u \in C(\bar{\Omega})$ is a solution to

$$F(D^2 u, x) = 0 \ \text{in}\ \Omega, \quad u = 0 \ \text{on}\ \partial\Omega.$$

Before giving the strong maximum principle and a version of the Hopf's boundary lemma, let us recall the notion of viscosity sub-solution and super-solution for extremal operators of Pucci type.

Lemma 2.3. *Given $\gamma \geq 0$, a radially symmetric continuous function $u : \mathcal{B} \to \mathbb{R}$ is a viscosity super-solution (sub-solution) of*

$$-\mathcal{M}^+_{\mathcal{C}}(D^2 u) + \gamma u = 0 \quad \text{in}\ \mathcal{B}, \tag{2.3}$$

when the following condition holds: If $x_0 \in (0,1], \phi \in C^2(0,1)$, such that $u - \phi$ has a local minimum (maximum) at x_0, and $\varphi'(x_0) \neq 0$, then

$$-\sup_{(a_1,a_2)\in\mathcal{C}} \Big(a_1\varphi''(x_0) + \frac{(N-1)a_2\varphi'(x_0)}{|x_0|}\Big) \leq \gamma u(x_0),$$
$$\Big(-\sup_{(a_1,a_2)\in\mathcal{C}} \Big(a_1\varphi''(x_0) + \frac{(N-1)a_2\varphi'(x_0)}{|x_0|}\Big) \geq \gamma u(x_0)\Big).$$

We say that u is a viscosity super-solution (sub-solution), if u satisfies

$$-\sup_{(a_1,a_2)\in\mathcal{C}} \Big(a_1 u'' + \frac{(N-1)a_2 u'}{r}\Big) + \gamma u \geq (\leq) 0$$

in the viscosity sense. We say that u is a viscosity solution of (2.3) when it is simultaneously a viscosity sub-solution and a super-solution.

Now, we give the strong maximum principle and a version of the Hopf's boundary lemma.

Lemma 2.4. *For $\gamma \geq 0$, if $u \in C^2(\mathcal{B}) \cap C(\bar{\mathcal{B}})$ satisfies*

$$\begin{gathered} -\mathcal{M}^+_{\mathcal{C}}(D^2 u) + \gamma u \geq 0 \quad \text{in}\ \mathcal{B}, \\ u \geq 0 \quad \text{on}\ \partial\mathcal{B}, \end{gathered}$$

then either $u \equiv 0$ or $u > 0$ in $\mathcal{B}$. Moreover,

$$\limsup_{x\to x_0} \frac{u(x_0) - u(x)}{|x - x_0|} < 0,$$

where $x_0 \in \partial\mathcal{B}$ and the limit is non-tangential; that is, taken over the set of x for which the angle between $x - x_0$ and the outer normal at x_0 is less than $\frac{\pi}{2} - \delta$ for some fixed $\delta > 0$.

Proof. We start by claiming that

$$u \geq 0 \quad \text{in } \mathcal{B}. \tag{2.4}$$

Otherwise, suppose there exists $x_0 \in \mathcal{B}$, such that $u(x_0) < 0$. Let $\epsilon \in (0, -u(x_0))$, then for $x \in B(x_0, \epsilon)$, the function $u_\epsilon(x) = u(x) - \frac{\epsilon}{2}|x - x_0|^2$ also has a strictly negative minimum which is achieved at $x_\epsilon \in \mathcal{B}$. Indeed if it is achieved on the boundary $x_\epsilon \in \partial\mathcal{B}$, then

$$\begin{aligned} u_\epsilon(x_\epsilon) &= u(x_\epsilon) - \frac{\epsilon}{2}|x_\epsilon - x_0|^2 \\ &\geq u(x_\epsilon) - \frac{\epsilon}{2} \\ &\geq u(x_\epsilon) + \frac{1}{2}u(x_0) \\ &= \frac{1}{2}u(x_0) \\ &> u(x_0) = u_\epsilon(x_0). \end{aligned}$$

This is impossible. At the point x_ϵ, one has $D^2u(x_\epsilon) \geq \epsilon I$, where I is the identity matrix. Therefore,

$$0 \geq \gamma u(x_\epsilon) \geq \mathcal{M}^+_{\mathcal{C}}(D^2u(x_\epsilon)) \geq \frac{1}{2}\lambda\epsilon N > 0,$$

which is a contradiction. So (2.4) holds.

On the other hand, let M and S be symmetric matrices such that $S \geq 0$ (i.e. nonnegative definite), and let $\bar{a} \in \mathcal{C}$ such that

$$\mathcal{M}^+_{\mathcal{C}}(M + S) = \sup_{a \in \mathcal{C}} \sum_{i=1}^N a_i\lambda_i(M + S) = \sum_{i=1}^N \bar{a}_i\lambda_i(M + S).$$

Then we have

$$\mathcal{M}^+_{\mathcal{C}}(M + S) - \sum_{i=1}^N \bar{a}_i\lambda_i(M) = \sum_{i=1}^N \bar{a}_i(\lambda_i(M + S) - \lambda_i(M)) \leq \Lambda \operatorname{tr}(S),$$

from which it follows that

$$\mathcal{M}^+_{\mathcal{C}}(M + S) - \mathcal{M}^+_{\mathcal{C}}(M) \leq \Lambda \operatorname{tr}(S).$$

Proceeding in a similar form we obtain

$$\mathcal{M}^+_{\mathcal{C}}(M + S) - \mathcal{M}^+_{\mathcal{C}}(M) \geq \lambda \operatorname{tr}(S).$$

So $\mathcal{M}^+_{\mathcal{C}}(M) \geq \lambda \operatorname{tr}(M^+) - \Lambda \operatorname{tr}(M^-) := H(M)$ with $S = M^-$, where $M = M^+ - M^-$ is a minimal decomposition of M into the difference of two nonnegative matrices. Hence it is sufficient to prove the conclusions when u is a super solution of

$$H(D^2u) - \gamma u = 0.$$

On the contrary, Suppose that u is not identically equal to zero and that there exists x_0 inside $\mathcal{B}$ on which $u(x_0) = 0$, we can find $x_1 \in \mathcal{B}$ and $R > 0$, such that $B(x_1, 3R/2) \subset \mathcal{B}$, and $u > 0$ in $B(x_1, R)$ with $|x_1 - x_0| = R$. So $u_1 = \inf_{|x - x_1| = R/2} u > 0$.

Let us recall that if $\varphi(\rho) = e^{-k\rho}$, the eigenvalues of $D^2\varphi$ are $\varphi''(\rho)$ with multiplicity 1 and φ'/ρ with multiplicity $N - 1$.

Then we take $k > 0$ such that

$$k^2 > \frac{2(N-1)\Lambda}{R\lambda} k + \gamma.$$

If k is as above, let m be chosen such that

$$m(e^{-kR/2} - e^{-kR}) = u_1$$

and define $v(x) = m(e^{-k\rho} - e^{-kR})$ with $\rho = |x - x_1|$. As we discussed above, u is a non negative super solution of the operator $\mathcal{M}^+_{\mathcal{C}}(D^2u) - \gamma u$, so u is a super solution of $H(D^2u) - \gamma u = 0$ since $\mathcal{M}^+_{\mathcal{C}}(D^2u) \geq H(D^2u)$. It is not difficult to check $H(D^2v) - \gamma v > 0$ in the annulus with the above choice k, which means that v is a strict subsolution of $H(D^2v) - \gamma v = 0$ in the annulus. Furthermore

$$\begin{gathered} v = u_1 \leq u \quad \text{on } |x - x_1| = \frac{R}{2}, \\ v < 0 \leq u \quad \text{on } |x - x_1| = \frac{3R}{2}, \end{gathered}$$

that is $u \geq v$ everywhere on the boundary of the annulus. In fact $u \geq v$ everywhere in the annulus, since we can use the comparison principle [3, Theorem 2.9] with $F := H(D^2u) - \gamma u$. Therefore, $u(x_0) \geq v(x_0)$. Actually, $u(x_0) = v(x_0) = 0$ since $|x_1 - x_0| = R$. On the other hand, based on the fact that u is a super solution of $H(D^2u) - \gamma u = 0$, one has

$$H(D^2v(x_0)) - \gamma v(x_0) = H(D^2u(x_0)) - \gamma u(x_0) \leq 0,$$

which clearly contradicts the definition of v. So u cannot be zero inside $\mathcal{B}$. One derives either $u \equiv 0$ or $u > 0$ in $\mathcal{B}$.

Now, we shall give the Hopf's property by using the same construction. Supposing that there is $x_0 \in \partial\mathcal{B}$ on which $u(x_0) = 0$, we can find $x_1 \in \mathcal{B}$ and $R > 0$, such that $B(x_1, R) \subset \mathcal{B}$ with $|x_1 - x_0| = R$. So $u_1 = \inf_{|x-x_1|=\frac{R}{2}} u > 0$, and we replace the previous annulus with $R/2 \leq |x - x_1| = \rho \leq R$, with the comparison principle again. Since $v = 0$ on $|x - x_1| = R, Dv \neq 0$ in $\mathcal{B}$ and $v \leq u$ on the other boundary of the annulus, one arrives at $u(x) \geq m(e^{-k\rho} - e^{-kR})$ everywhere in the annulus. Then taking $x = x_0 - h\omega$ and letting $h > 0$ go to zero, where ω is the outward pointing normal to $\partial\Omega$, it arrives

$$\frac{u(x_0) - u(x)}{h} \leq m\frac{e^{-kR} - e^{-kR+kh}}{h} \to -mke^{-kR}. \qquad \square$$

Remark 2.5. Similar to the discussion in Lemma 2.4, we have: For $\gamma \geq 0$, if $u \in C^2(\mathcal{B}) \cap C(\bar{\mathcal{B}})$ satisfies

$$\begin{gathered} \mathcal{M}^-_{\mathcal{C}}(D^2u) - \gamma u \leq 0 \quad \text{in } \mathcal{B}, \\ u \geq 0 \quad \text{on } \partial\mathcal{B}, \end{gathered}$$

then either $u \equiv 0$ or $u > 0$ in $\mathcal{B}$ and

$$\limsup_{x \to x_0} \frac{u(x_0) - u(x)}{|x - x_0|} < 0,$$

where $x_0 \in \partial\mathcal{B}$ and the limit is non-tangential.

After giving the following important result, we study the nonlinear bifurcation problem and give the proof of Theorem 1.1.

Proposition 2.6. *If* $(\bar{\mu}, 0)$ *is a bifurcation point of problem* (1.5)*, then* $\bar{\mu}$ *is an eigenvalue of* (1.4).

Proof. Since $(\bar{\mu}, 0)$ is a nonlinear bifurcation point, there is a sequence $\{(\mu_m, u_m)\}_{m\in\mathbb{N}}$ of nontrivial solutions of problem (1.5), such that $\mu_m \to \bar{\mu}$ and $u_m \to 0$ uniformly in $\mathcal{B}$. Let $v_m = u_m/\|u_m\|$, then v_m satisfies

$$\begin{gathered} -\mathcal{M}_{\mathcal{C}}^{+}(D^2 v_m) = \mu_m v_m + \frac{f(\mu_m, u_m)}{\|u_m\|} \quad \text{in } \mathcal{B}, \\ v_m = 0 \quad \text{on } \partial\mathcal{B}. \end{gathered} \tag{2.5}$$

So, the right-hand side of the equation is bounded owing to $f(s, \mu) = o(|s|)$ near $s = 0$. By the compactness of $(-\mathcal{M}_{\mathcal{C}}^{+})^{-1}$, see [14, Page 5], we can extract a subsequence, such that $v_m \to \bar{v}$ as $m \to +\infty$ and $\|\bar{v}\| = 1$. Clearly, $\bar{v}$ satisfies

$$\begin{gathered} -\mathcal{M}_{\mathcal{C}}^{+}(D^2 \bar{v}) = \bar{\mu}\bar{v} \quad \text{in } \mathcal{B}, \\ \bar{v} = 0 \quad \text{on } \partial\mathcal{B}, \end{gathered}$$

namely, $\bar{\mu}$ is an eigenvalue of problem (1.4). □

Proof of Theorem 1.1. If (μ, u) with $u \not\equiv 0$ is a solution pair of problem (1.5), dividing equation (1.5) by $\|u\|^2$ and setting $w = \frac{u}{\|u\|^2}$, it yields

$$\begin{gathered} -\mathcal{M}_{\mathcal{C}}^{+}(D^2 w) = \mu w + \frac{f(\mu, u)}{\|u\|^2} \quad \text{in } \mathcal{B}, \\ w = 0 \quad \text{on } \partial\mathcal{B}. \end{gathered} \tag{2.6}$$

We define

$$\tilde{f}(\mu, w) = \begin{cases} \|w\|^2 f(\mu, \frac{w}{\|w\|^2}), & \text{if } w \neq 0, \\ 0, & \text{if } w = 0. \end{cases}$$

Clearly, (2.6) is equivalent to

$$\begin{gathered} -\mathcal{M}_{\mathcal{C}}^{+}(D^2 w) = \mu w + \tilde{f}(\mu, w) \quad \text{in } \mathcal{B}, \\ w = 0 \quad \text{on } \partial\mathcal{B}. \end{gathered} \tag{2.7}$$

It is easy to see that $(\mu, 0)$ is always the solution of the problem (2.7).

Let $\hat{f}(\mu, u) = \max_{0\le|s|\le u} |f(\mu, s)|$ for any $\mu \in \mathbb{R}$. Then $\hat{f}$ is nondecreasing with respect to u. We define

$$\bar{f}(\mu, u) = \max_{u/2\le|s|\le u} |f(\mu, s)|$$

for any $\mu \in \mathbb{R}$. Then we arrive at

$$\hat{f}(\mu, u) \le \hat{f}(\mu, \frac{u}{2}) + \bar{f}(\mu, u), \tag{2.8}$$

$$\lim_{u\to+\infty} \frac{\bar{f}(\mu, u)}{u} = 0 \text{ uniformly for } \mu \in \mathbb{R}, \tag{2.9}$$

since $\lim_{|s|\to+\infty} \frac{f(\mu,s)}{s} = 0$ uniformly for $\mu \in \mathbb{R}$. It is not difficult to verify that, for any given $\rho > 0$, $\frac{\hat{f}(\mu,s)}{s}$ is positive and bounded for $s \in [\rho, +\infty)$. This fact and (2.8), (2.9) imply

$$\limsup_{u\to+\infty} \frac{\hat{f}(\mu, u)}{u} \le \limsup_{u\to+\infty} \frac{\hat{f}(\mu, 2u)}{u} = \limsup_{t\to+\infty} 2\frac{\hat{f}(\mu, t)}{t}$$

uniformly for $\mu \in \mathbb{R}$, where $t = 2u$. So it has

$$\lim_{u\to+\infty} \frac{\hat{f}(\mu,u)}{u} = 0 \tag{2.10}$$

uniformly for $\mu \in \mathbb{R}$. Furthermore, it follows from (2.10) that

$$\frac{f(\mu,u)}{\|u\|} \le \frac{\hat{f}(\mu,|u|)}{\|u\|} \le \frac{\hat{f}(\mu,\|u\|)}{\|u\|} \to 0 \quad \text{as } \|u\| \to +\infty \tag{2.11}$$

uniformly for $\mu \in \mathbb{R}$. By a direct computation, one can see that (2.11) implies

$$\lim_{\|w\|\to 0} \frac{\tilde{f}(\mu,w)}{\|w\|} = 0$$

uniformly for $\mu \in \mathbb{R}$.

Applying [14, Theorem 1.4] to problem (2.7), we derive that the component $\mathcal{C}_k^\nu$ of problem (2.7) containing $(\mu_k^\nu, 0)$ is unbounded and lies in $(\mathbb{R} \times \mathcal{S}_k^\nu) \cup (\mu_k^\nu, 0)$, under the inversion $w \to \frac{w}{\|w\|^2} = u$ and $\mathcal{C}_k^\nu \to \mathcal{D}_k^\nu$. It is not difficult to check that $\mathcal{D}_k^\nu$ emanates from $((\mathbb{R} \times \mathcal{S}_k^\nu) \cup (\mu_k^\nu \times \{\infty\}))$.

Next, we show that there exists a neighborhood $\mathcal{N} \subset \mathcal{U}$ of $(\mu_k^\nu \times \{\infty\})$ such that

$$\mathcal{D}_k^\nu \cap \mathcal{N} \subset ((\mathbb{R} \times \mathcal{S}_k^\nu) \cup (\mu_k^\nu \times \{\infty\})) \quad \text{for } \nu = + \text{ and } \; - .$$

We only prove the case of $\nu = +$, since the proof of the other case is similar.

It is easy to see that the inversion $w \to \frac{w}{\|w\|^2} = u$ turns $(\mu_k^+ \times \{0\})$ into $(\mu_k^+ \times \{\infty\})$. Let $\mathcal{M}$ be a bounded neighborhood of $(\mu_k^+ \times \{0\})$. Then $\mathcal{C}_k^+ \cap (\mathcal{M} \setminus (\mu_k^+ \times \{0\})) \subset \mathbb{R} \times \mathcal{S}_k^+$. By the inversion $w \to \frac{w}{\|w\|^2} = u$, $\mathcal{C}_k^+ \cap (\mathcal{M} \setminus (\mu_k^+ \times \{0\}))$ is translated to a deleted neighborhood $\mathcal{N}^0$ of $(\mu_k^+ \times \{\infty\})$. Clearly, $(\mu, w) \in \big(\mathcal{C}_k^+ \cap (\mathcal{M} \setminus (\mu_k^+ \times \{0\}))\big)$ implies that there exists a constant C_0 such that $0 < \|w\| \le C_0$. It follows that $(\mu, u) \in \mathcal{N}^0$, which implies $\frac{1}{C_0} \le \|u\| < \infty$. Consequently, we obtain $\mathcal{D}_k^+ \cap \mathcal{N} \subset \big((\mathbb{R} \times \mathcal{S}_k^+) \cup (\mu_k^+ \times \{\infty\})\big)$ by taking $\mathcal{N} := \mathcal{N}^0 \cup (\mu_k^+ \times \{\infty\})$. $\square$

At last, we give a Sturm type comparison theorem.

Lemma 2.7 ([14, Lemma 3.5])**.** *Let $a, b \in L^\infty(0,1)$ with $a \ge b$ in $(0,1)$. Assume that $u, v \in C^2[0,1] \setminus \{0\}, u'(0) = v'(0) = 0$, and respectively satisfy*

$$\begin{aligned} -\left(\rho_u(r)u'(r)\right)' &= \tilde{\rho}_u(r)a(r)u(r) \quad \text{a.e. } (0,1), \\ -\left(\rho_v(r)v'(r)\right)' &= \tilde{\rho}_v(r)b(r)v(r) \quad \text{a.e. } (0,1), \end{aligned}$$

where $\rho_u(r)$ denote the integral factor of the equation, $\tilde{\rho}_u(r) := \frac{\rho_u(r)}{\theta}, \rho_u$ and θ are characterized by the optimal condition. Then

- (i) *If v has a zero in $(0,1)$, then u also has a zero. The first zero of u is less than or equal to the first zero of v.*
- (ii) *If $(r_0, r_1) \subseteq [0,1]$, $v(r_0) = v(r_1) = 0$, $u(r) \not\equiv 0$, for $r \in (r_0, r_1)$, and $a \ge b$ in some subset of (r_0, r_1), then u has at least one zero in (r_0, r_1).*

3. Proof of Theorem 1.2

Based on [14, Theorem 1.4] and Theorem 1.1, we give the proof of Theorem 1.2.

Proof of Theorem 1.2. (a) Let $\xi \in C(\mathbb{R}, \mathbb{R})$ be such that $f(s) = f_0 s + \xi(s)$ with

$$\lim_{|s|\to 0} \frac{\xi(s)}{s} = 0 \quad \text{and} \quad \lim_{|s|\to +\infty} \frac{\xi(s)}{s} = f_\infty - f_0.$$

By [14, Theorem 1.4], we have that there is an unbounded continua $\mathcal{C}_k^\nu$, emanating from $(\frac{\mu_k^\nu}{f_0}, 0)$, such that

$$\mathcal{C}_k^\nu \subset (\{(\frac{\mu_k^\nu}{f_0}, 0)\} \cup (\mathbb{R} \times \mathcal{S}_k^\nu)),$$

where $\nu \in \{+, -\}$. To complete the proof, it will be sufficient to show that $\mathcal{C}_k^\nu$ connects $(\frac{\mu_k^\nu}{f_0}, 0)$ to $(\frac{\mu_k^\nu}{f_\infty}, +\infty)$. Let $(\mu_n, u_n) \in \mathcal{C}_k^\nu$ with $u_n \not\equiv 0$ satisfying

$$\mu_n + \|u_n\| \to +\infty.$$

We note that $\mu_n > 0$ for all $n \in \mathbb{N}$, since 0 is the only solution of the problem (1.7) for $\mu = 0$ and $\mathcal{C}_k^\nu \cap (\{0\} \times E) = \emptyset$. We divide the remainder of the proof into two steps.

Step 1: One shows if there exists a constant $M > 0$ such that $\mu_n \subset (0, M]$ for sufficiently large $n \in \mathbb{N}$, then $\mathcal{C}_k^\nu$ connects $(\frac{\mu_k^\nu}{f_0}, 0)$ to $(\frac{\mu_k^\nu}{f_\infty}, +\infty)$. In this case it follows that $\|u_n\| \to +\infty$.

Let $\zeta \in C(\mathbb{R}, \mathbb{R})$ be such that $f(s) = f_\infty s + \zeta(s)$ with

$$\lim_{|s|\to +\infty} \frac{\zeta(s)}{s} = 0 \quad \text{and} \quad \lim_{|s|\to 0} \frac{\zeta(s)}{s} = f_0 - f_\infty. \tag{3.1}$$

We divide both sides of the equation

$$\begin{gathered} -\mathcal{M}_\mathcal{C}^+(D^2 u_n) = \mu_n f_\infty u_n(x) + \mu_n \zeta(u_n(x)) \quad \text{in } \mathcal{B}, \\ u_n = 0 \quad \text{on } \partial\mathcal{B} \end{gathered}$$

by $\|u_n\|$ and set $\overline{u}_n = u_n/\|u_n\|$. Similar to the argument for (2.11), we obtain $\lim_{n\to+\infty} \zeta(u_n)/\|u_n\| = 0$ as $n \to +\infty$. Then one derives

$$\begin{gathered} -\mathcal{M}_\mathcal{C}^+(D^2 \overline{u}_n) = \mu_n f_\infty \overline{u}_n(x) \quad \text{in } \mathcal{B}, \\ \overline{u}_n = 0 \quad \text{on } \partial\mathcal{B}. \end{gathered}$$

By the compactness of $(-\mathcal{M}_\mathcal{C}^+)^{-1}$, see [14, Page 5], we can extract a subsequence such that $\bar{u}_m \to \bar{u}$ as $m \to +\infty$ and $\|\bar{u}\| = 1$. Clearly, $\bar{u}$ satisfies

$$\begin{gathered} -\mathcal{M}_\mathcal{C}^+(D^2 \bar{u}) = \bar{\mu} f_\infty \bar{u} \quad \text{in } \mathcal{B}, \\ \bar{u} = 0 \quad \text{on } \partial\mathcal{B}, \end{gathered}$$

where $\overline{\mu} = \lim_{n\to+\infty} \mu_n$.

Clearly, $\bar{u} \in \mathcal{S}_k^\nu$ since $\bar{u}_m \in \mathcal{S}_k^\nu$. Thus, $\bar{\mu} f_\infty = \mu_k^\nu$, i.e., $\bar{\mu} = \frac{\mu_k^\nu}{f_\infty}$. Therefore, $\mathcal{C}_k^\nu$ connects $(\frac{\mu_k^\nu}{f_0}, 0)$ to $(\frac{\mu_k^\nu}{f_\infty}, +\infty)$.

Step 2: We show that there exists a constant M such that $\mu_n \in (0, M]$ for sufficiently large $n \in \mathbb{N}$. On the contrary, suppose that $\lim_{n\to+\infty} \mu_n = +\infty$. One notes that

$$-\mathcal{M}_\mathcal{C}^+(D^2 u_n(x)) = \mu_n \widetilde{f_n}(x) u_n(x), \quad x \in \mathcal{B},$$

where

$$\widetilde{f_n}(x) = \begin{cases} \frac{f(u_n(x))}{u_n(x)}, & \text{if } u_n(x) \neq 0, \\ f_0, & \text{if } u_n(x) = 0. \end{cases}$$

As for (2.2), one derives that (μ_n, u_n) satisfies

$$
\begin{gathered}
-\big(\rho_{u_n}(r)u_n'(r)\big)' = \mu_n \tilde{\rho}_{u_n} \widetilde{f_n}(r) u_n(r) \quad \text{in } (0,1), \\
u_n'(0) = 0, \quad u_n(1) = 0,
\end{gathered}
$$

where $\rho_{u_n}(r) := \exp\big(\int_0^r \frac{N_d(\tau)-1}{\tau} d\tau\big)$ denote the integral factor of the equation $\tilde{\rho}_{u_n}(r) := \frac{\rho_{u_n}(r)}{\theta}$, ρ_{u_n} and θ are characterized by the optimal condition.

The signum condition (A1) implies that there exists a positive constant ϱ such that $\widetilde{f_n} \geq \varrho$ for $r \in [0,1]$. Thus, one has that

$$
\mu_n \widetilde{f_n} > \mu_k
$$

where μ_k is the $k-$th eigenvalue of the problem

$$
\begin{gathered}
-\mathcal{M}_{\mathcal{C}}^+(D^2 v) = \mu v(x) \quad \text{in } \mathcal{B}, \\
v = 0 \quad \text{on } \partial\mathcal{B}.
\end{gathered}
$$

i.e. (μ_k, v) satisfies

$$
\begin{gathered}
-\left(\rho_v(r)v'(r)\right)' = \mu_k \tilde{\rho}_v(r) v(r) \quad \text{in } (0,1), \\
v'(0) = 0, \quad v(1) = 0.
\end{gathered}
$$

By [14, Theorem 1.2], we know that μ_k is positive, simple and the corresponding eigenfunction v has exactly $k-1$ simple zeros in $(0,1)$. By Lemma 2.7, we obtain that u_n has at least k zeros in $(0,1)$ for n large enough, and this contradicts the fact that u_n has exactly $k-1$ zeros in $(0,1)$. Consequently, $\mu_n \leq M$ for some constant $M > 0$ and sufficiently large $n \in \mathbb{N}$.

(b) In view of (a), we only need to show that $\mathcal{C}_k^\nu$ connects $(\frac{\mu_k^\nu}{f_0}, 0)$ to $(+\infty, +\infty)$. At the beginning, we prove that $\mathcal{C}_k^\nu$ is unbounded in the direction of μ. On the contrary, suppose that there exists μ_M be a blow up point of parament μ and $\mu_M < +\infty$. Then there exists a sequence nodal solutions $\{(\mu_n, u_n)\} \in \mathcal{C}_k^\nu$, such that $\lim_{n\to+\infty} \mu_n = \lambda_M$ and $\lim_{n\to+\infty} \|u_n\| = +\infty$. Let $v_n = \frac{u_n}{\|u_n\|}$. Then v_n should be the solutions of problem

$$
-\mathcal{M}_{\mathcal{C}}^+(D^2 v_n) = \mu_n \frac{f(u_n)}{\|u_n\|} \quad \text{in } \mathcal{B}.
$$

Similar to the argument for (2.11), one obtains $\lim_{n\to+\infty} f(u_n)/\|u_n\| = 0$. By the compactness of $(-\mathcal{M}_{\mathcal{C}}^+)^{-1}$, see [14, Page 5], we have that for a subsequence $v_n \to v_0$ as $n \to +\infty$ and $v_0 \equiv 0$. This contradicts $\|v_0\| = 1$. Thus $\mathcal{C}_k^\nu$ is unbounded in the direction of μ.

Next, we show that $\mathcal{C}_k^\nu$ is unbounded in the direction of E. Suppose that $\mathcal{C}_k^\nu$ is bounded in the direction of E. Thus there exist $(\mu_n, u_n) \in \mathcal{C}_k^\nu$ and a positive constant M, such that $\mu_n \to +\infty$ as $n \to +\infty$ and $\|u_n\| \leq M$ for any $n \in \mathbb{N}$. Therefore, we can conclude that there exists a constant $\delta > 0$, such that

$$
\frac{f(u_n)}{u_n} \geq \delta.
$$

Similar to part (a) and by using Strum comparison lemma, Lemma 2.7, we can arrive that u_n has at least k simple zeros, which contradicts $(\mu_n, u_n) \in \mathcal{C}_k^\nu$. So $\mathcal{C}_k^\nu$ is unbounded in the direction of E. Therefore, conclusion (b) follows.

(c) If (μ,u) is a nontrivial solution of problem (1.7), dividing problem (1.7) by $\|u\|^2$ and setting $v=u/\|u\|^2$, we obtain

$$\begin{gathered} -\mathcal{M}_{\mathcal{C}}^{+}(D^2v)=\mu\frac{f(u(x))}{\|u\|^2},\quad \text{in } \mathcal{B},\\ v=0,\quad \text{on } \partial\mathcal{B}. \end{gathered} \tag{3.2}$$

We define

$$\tilde{f}(v)=\begin{cases} \|v\|^2f(v/\|v\|^2), & \text{if } v\neq 0,\\ 0, & \text{if } v=0. \end{cases}$$

The problem (3.2) is equivalent to

$$\begin{gathered} -\mathcal{M}_{\mathcal{C}}^{+}(D^2v)=\mu\tilde{f}(v),\quad \text{in } \mathcal{B},\\ v=0,\quad \text{on } \partial\mathcal{B}. \end{gathered} \tag{3.3}$$

Clearly, $(\mu,0)$ is always the solution of problem (3.3). By simple computation, we can show that $(\tilde{f})_0=f_\infty$ and $(\tilde{f})_\infty=f_0$. Now, applying (b) and the inversion $v\to v/\|v\|^2=u$, we achieve the conclusions.

(d) We define

$$f^n(x)=\begin{cases} x/n, & x\in[-1/n,1/n],\\ \big(f(\frac{2}{n})-1/n^2\big)(nx-2)+f(\frac{2}{n}), & x\in(1/n,2/n),\\ -\big(f(-\frac{2}{n})+1/n^2\big)(nx+2)+f(-\frac{2}{n}), & x\in(-2/n,-1/n),\\ f(x), & x\in(-\infty,-2/n]\cup[2/n,+\infty). \end{cases}$$

Then $f^n(x)\in C(\mathbb{R},\mathbb{R})$. One considers the auxiliary problem

$$\begin{gathered} -\mathcal{M}_{\mathcal{C}}^{+}(D^2u)=\mu f^n(u(x)),\quad \text{in } \Omega,\\ u=0,\quad \text{on } \partial\Omega. \end{gathered} \tag{3.4}$$

It is not difficult to check that $\lim_{n\to+\infty}f^n(x)=f(x),(f^n)_0=\frac{1}{n}$ and $(f^n)_\infty=f_\infty=0$. For any fixed $n\in\mathbb{N}$, it follows from (b) that there exists a sequence unbounded continua $\mathcal{C}_{k,n}^{\nu}$ of solutions to problem (3.4) emanating from $(n\mu_k^\nu,0)$ and connecting to $(+\infty,+\infty)$.

Let $\widetilde{\mathcal{C}}_k^\nu=\limsup_{n\to+\infty}\mathcal{C}_{k,n}^{\nu}$. For any $(\mu,u)\in\widetilde{\mathcal{C}}_k^\nu$, the definition of limit superior shows that there exists a sequence $(\mu_n,u_n)\in\mathcal{C}_{k,n}^{\nu}$, such that $(\mu_n,u_n)\to(\mu,u)$ as $n\to+\infty$. Clearly, one has $-\mathcal{M}_{\mathcal{C}}^{+}(D^2u_n)=\mu_nf^n(u_n(x))$. One applies Lemma 2.2, then it arrives that u satisfies $-\mathcal{M}_{\mathcal{C}}^{+}(D^2u)=\mu f(u(x))$, i.e., u is a solution of (1.7).

By Lemma 2.1, there exists an unbounded component $\mathcal{C}_k^\nu$ of $\widetilde{\mathcal{C}}_k^\nu$ of solutions to problem (1.7), such that $(+\infty,0)\in\mathcal{C}_k^\nu$ and $(+\infty,+\infty)\in\mathcal{C}_k^\nu$. So there exists $\mu^*>0$ such that for $\mu\in(\mu^*,+\infty)$, problem (1.7) has at least two nodal solutions $u_k^{1,+}$ and $u_k^{2,+}$. Moreover, $u_k^{1,+}$ and $u_k^{2,+}$ have exactly $k-1$ simple zeros in $(0,1)$ and are positive near 0; and there exists $\mu'>0$ such that for $\mu\in(\mu',+\infty)$, problem (1.7) has at least two nodal solutions $u_k^{1,-}$ and $u_k^{2,-}$; moreover, $u_k^{1,-}$ and $u_k^{2,-}$ have exactly $k-1$ simple zeros in $(0,1)$ and are negative near 0.

Next, one shows that there exists $\tilde{\mu}^*>0$ such that problem (1.7) has no nodal solution for any $\mu\in(0,\tilde{\mu}^*)$. On the contrary, suppose that there exists a sequence $\{(\mu_n,u_n)\}\in\mathcal{C}_k^\nu$ such that $\lim_{n\to+\infty}\mu_n=0$. On the other hand, $f_0=f_\infty=0$

implies that there exists a positive constant M such that

$$\frac{f(s)}{s} \leq M \quad \text{for any } s \neq 0.$$

Let $v_n = \frac{u_n}{\|u_n\|}$. Then, one has

$$v_n = (-\mathcal{M}_{\mathcal{C}}^{+})^{-1}\Big(\frac{\mu_n f(u_n(x))}{\|u_n\|}\Big).$$

Let $\hat{f}(u) = \max_{0\leq |s| \leq u} |f(s)|$, then $\hat{f}$ is nondecreasing with respect to u. Then we arrive at

$$\lim_{u\to 0} \frac{\hat{f}(u)}{u} = 0. \tag{3.5}$$

Furthermore, it follows from (3.5) that

$$\frac{f(u)}{\|u\|} \leq \frac{\hat{f}(|u|)}{\|u\|} \leq \frac{\hat{f}(\|u\|)}{\|u\|} \to 0 \quad \text{as } \|u\| \to 0. \tag{3.6}$$

So $\lim_{n\to+\infty} \mu_n f(u_n)/\|u_n\| = 0$. By the compactness of $(-\mathcal{M}_{\mathcal{C}}^{+})^{-1}$ again, see [14, Page 5], it receives that for some convenient subsequence $v_n \to v_0$ as $n \to +\infty$. Letting $n \to +\infty$, one has $v_0 \equiv 0$. This contradicts $\|v_0\| = 1$. So the conclusions follow.

(e) We define the cut-off function of f as

$$f^n(x) = \begin{cases} nx, & x \in [-1/n, 1/n],\\ n(f(\frac{2}{n}) - 1)(x - \frac{1}{n}) + 1, & x \in (1/n, 2/n),\\ -n(f(-\frac{2}{n}) + 1)(x + \frac{1}{n}) - 1, & x \in (-2/n, -1/n),\\ f(x), & x \in (-\infty, -2/n] \cup [2/n, +\infty). \end{cases} \tag{3.7}$$

Then $f^n \in C(\mathbb{R}, \mathbb{R})$. One considers the auxiliary problem

$$\begin{gathered} -\mathcal{M}_{\mathcal{C}}^{+}(D^2u) = \mu f^n(u(x)), \quad \text{in } \mathcal{B},\\ u = 0, \quad \text{on } \partial\mathcal{B}. \end{gathered} \tag{3.8}$$

It is easy to see that $\lim_{n\to+\infty} f^n(x) = f(x), (f^n)_0 = n$ and $(f_n)_\infty = f_\infty = 0$. (b) implies that there exists a sequence unbounded continua $\mathcal{C}_{k,n}^{\nu}$ of solutions to problem (3.8) emanating from $(\frac{\mu_k^\nu}{n}, 0)$ and connecting to $(+\infty, +\infty)$.

With the help of Lemma 2.1, we shall finish the proof. Taking $z_n = (\frac{\mu_k^\nu}{n}, +\infty)$ and $z^* = (0, 0)$, it receives that $z_n \to z^*$. So Lemma 2.1 (i) is satisfied. (ii) and (iii) are obvious. So there exists an unbounded component $\mathcal{C}_k^\nu$ of $\limsup_{n\to+\infty} \mathcal{C}_{k,n}^\nu$, such that $(0, 0) \in \mathcal{C}_k^\nu$ and $(+\infty, +\infty) \in \mathcal{C}_k^\nu$. This completes the proof.

(f) We define the function $f^n(x)$ as in (3.7) and consider the auxiliary problem (3.8) again, but in this case $(f^n)_0 = n$ and $(f_n)_\infty = f_\infty \in (0, +\infty)$. In view of (a), one derives that there exists a sequence unbounded continua $\mathcal{C}_{k,n}^\nu$ of solutions to problem (3.8) emanating from $(\frac{\mu_k^\nu}{n}, 0)$ and connecting to $(\frac{\mu_k^\nu}{f_\infty}, +\infty)$.

On account of Lemma 2.1 again, we can easily obtain that there exists an unbounded component $\mathcal{C}_k^\nu$ of $\limsup_{n\to+\infty} \mathcal{C}_{k,n}^\nu$, such that $(0, 0) \in \mathcal{C}_k^\nu$ and $(\frac{\mu_k^\nu}{f_\infty}, +\infty)$. This completes the proof. $\square$

4. Proof of Theorem 1.3

Based on the Theorem 1.2, we give the proof of Theorem 1.3.

Proof of Theorem 1.3. The argument of Theorem 1.2(a) implies that there is an unbounded continua $\mathcal{C}_k^\nu$, emanating from $(\frac{\mu_k^\nu}{f_0},0)$, such that it satisfies

$$\mathcal{C}_k^\nu \subset (\{(\frac{\mu_k^\nu}{f_0},0)\}\cup(\mathbb{R}^+\times\mathcal{S}_k^\nu)),\quad \text{where } \nu\in\{+,-\}.$$

Let $\eta\in C(\mathbb{R},\mathbb{R})$ be such that

$$f(u)=f_\infty u+\eta(u)\quad\text{with}\ \lim_{|s|\to+\infty}\frac{\eta(s)}{s}=0.$$

Let us consider

$$\begin{gathered}-\mathcal{M}_{\mathcal{C}}^+(D^2u)=\mu f_\infty u+\mu\eta(u),\quad\text{in }\mathcal{B},\\ u=0,\quad\text{on }\partial\mathcal{B}\end{gathered}\tag{4.1}$$

as a bifurcation problem from infinity. Applying Theorem 1.1 to (4.1), it shows that there exists an unbounded continua $\mathcal{D}_k^\nu$ of solutions of (4.1), emanating from $(\frac{\mu_k^\nu}{f_\infty},+\infty)$.

Next, we show that the components $\mathcal{C}_k^\nu$ and $\mathcal{D}_k^\nu$ are disjoint under the assumption (A3), that is, one wants to obtain for any $(\mu,u)\in\mathcal{C}_k^+\cup\mathcal{C}_k^-, s_2<u(r)<s_1$ for all $r\in[0,1]$; and for any $(\mu,u)\in\mathcal{D}_k^+\cup\mathcal{D}_k^-$, $\max\{u(r)|r\in[0,1]\}>s_1$ or $\min\{u(r)|r\in[0,1]\}<s_2$.

On the contrary, suppose that there exists $(\mu,u)\in\mathcal{C}_k^+\cup\mathcal{C}_k^-\cup\mathcal{D}_k^+\cup\mathcal{D}_k^-$, such that either $\max\{u(r)|r\in[0,1]\}=s_1$ or $\min\{u(r)|r\in[0,1]\}=s_2$. We only discuss the case of $\max\{u(r)|r\in[0,1]\}=s_1$. The discussion for the other case $\min\{u(r)|r\in[0,1]\}=s_2$ is closely similar, so we omit it here. In this case, there exists $j\in\{0,1,\dots,k-1\}$ such that $\max\{u(r):r\in[0,1]\}=s_1$ and $0\le u(r)\le s_1$ for all $r\in[\tau_j,\tau_{j+1}]$, where $[\tau_j,\tau_{j+1}]\subset[0,1]$.

We claim that there exists $0<m<+\infty$ such that $f(s)\le m(s_1-s)$ for any $s\in[0,s_1]$. With the aid of (A3), it is easy to see that the claim is true for the cases $s=0$ and $s=s_1$. For any $\epsilon\in(0,\gamma_1)$, it follows from (A4) that there exists $\delta>0$ such that

$$f(s)<(\gamma_1+\epsilon)(s_1-s)$$

for any $s\in(s_1-\delta,s_1)$. From (A3), one arrives at

$$\max_{s\in[0,s_1-\delta]}\frac{f(s)}{s_1-s}:=\rho>0.$$

So the claim is verified by choosing $m=\max\{\rho,\gamma_1+\epsilon\}$.

Now, we consider an equivalent problem of (1.7) as follows,

$$\begin{gathered}-\mathcal{M}_{\mathcal{C}}^+(D^2(s_1-u))+\mu m(s_1-u)=\mu m(s_1-u)-\mu f(u),\quad |x|\in[\tau_j,\tau_{j+1}],\\ s_1-u>0,\quad |x|=\tau_j,\tau_{j+1}.\end{gathered}$$

It is obvious that $f(s)\le m(s_1-s)$ for any $s\in[0,s_1]$ implies

$$\begin{gathered}-\mathcal{M}_{\mathcal{C}}^+(D^2(s_1-u))+\mu m(s_1-u)\ge 0,\quad |x|\in[\tau_j,\tau_{j+1}],\\ s_1-u>0,\quad |x|=\tau_j,\tau_{j+1}.\end{gathered}\tag{4.2}$$

Let $v = s_1 - u$, then (4.2) is equivalent to

$$\begin{gathered}\mathcal{M}_{\mathcal{C}}^{-}(D^2 v) - \mu m v \le 0, \quad |x| \in [\tau_j, \tau_{j+1}],\\ v = s_1, \quad |x| = \tau_j, \tau_{j+1}.\end{gathered} \tag{4.3}$$

The strong maximum principle (Remark 2.5) implies that, if v satisfies (4.3), then it has $v > 0$ in $[\tau_j, \tau_{j+1}]$; that is $s_1 > u(r)$ on $[\tau_j, \tau_{j+1}]$. This is a contradiction.

So for any $(\mu, u) \in \mathcal{C}_k^+ \cup \mathcal{C}_k^-$, $s_2 < u(r) < s_1$, $r \in [0,1]$; for $(\mu, u) \in \mathcal{D}_k^+ \cup \mathcal{D}_k^-$, $\max\{u(r) | r \in [0,1]\} > s_1$ or $\min\{u(r) | r \in [0,1]\} < s_2$. Therefore, $(\frac{\mu_k^\nu}{f_0}, +\infty) \subseteq \mathrm{Proj}(\mathcal{C}_k^\nu)$ and $\mathcal{D}_k^\nu$ has an unbounded projection on $\mathbb{R}$. Immediately, from the global structures of $\mathcal{C}_k^\nu$ and $\mathcal{D}_k^\nu$, one obtain that problem (1.7) has at least one nodal solution u_k^ν for any $\mu \in (\min\{\frac{\mu_k^\nu}{f_0}, \frac{\mu_k^\nu}{f_\infty}\}, \max\{\frac{\mu_k^\nu}{f_0}, \frac{\mu_k^\nu}{f_\infty}\})$, $k \in \mathbb{N}$, such that νu_k^ν has exactly $k-1$ simple zeros in $(0,1)$ and is positive near 0; (1.7) at least has four nodal solutions $u_{k,0}^+, u_{k,\infty}^+, u_{k,0}^-$ and $u_{k,\infty}^-$ for any $\mu \in (\max\{\frac{\mu_k^+}{f_0}, \frac{\mu_k^-}{f_0}, \frac{\mu_k^+}{f_\infty}, \frac{\mu_k^-}{f_\infty}\}, +\infty)$, such that they have exactly $k-1$ simple zeros in $(0,1)$, $u_{k,0}^+$ and $u_{k,\infty}^+$ are positive near 0, $u_{k,0}^-$ and $u_{k,\infty}^-$ are negative near 0.

Finally, we show $\|u_{k,0}^+\| \to s_1^-$ and $\|u_{k,0}^-\| \to (-s_2)^-$ as $\mu \to +\infty$. Here, we only prove the case of $\nu = +$. Because the proof of $\nu = -$ is similar. Suppose, by contradiction, that there exists $\eta \in (0, s_1)$, such that $\|u_{k,0}^+\| \le \eta$. The assumption (A3) implies that there exists a positive constant $\delta > 0$, such that

$$\frac{f(u_{k,0}^+)}{u_{k,0}^+} \ge \delta.$$

Similar to Theorem 1.2 (a), the Sturm type comparison theorem, Lemma 2.7, implies that $u_{k,0}^+$ has at least k zeros for μ large enough, which is a contradiction; so the conclusion follows. $\square$

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