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AN ELEMENTARY METHOD FOR OBTAINING GENERAL SOLUTIONS TO SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. An analytical method is proposed for finding the general solution of a system of ordinary differential equations (ODEs). The general solution is expressed as a series which in some cases can be summed to give an expression in closed form. A sufficient condition for the series to converge is derived. Illustrative examples are given for scalar first-order ODEs (Riccati, Abel, homogeneous, Bernoulli, linear, separable) and for higher order ODEs (Airy, linear oscillator, Liénard, van der Pol). The method relies only on a calculus background.

1. Introduction

Consider a system of ordinary differential equations (ODEs)

$$\frac{\mathrm{d}y_j}{\mathrm{d}x} = f_j(x, y_1, y_2, \dots, y_d), \quad j = 1, 2, \dots, d,$$

where $y_j = y_j(x)$ for $j = 1, 2, \dots, d$. In matrix form this system can be expressed as

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}x} = \mathbf{f}(x, \mathbf{y}), \tag{1.1}$$

where $\mathbf{y} = (y_1, y_2, \dots, y_d)$ and $\mathbf{f} = (f_1, f_2, \dots, f_d)$. An important special case of (1.1) is the scalar first-order ODE (i.e. $d = 1$)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y). \tag{1.2}$$

It is well known that a scalar mth-order ODE can always be rewritten in the form (1.1).

There are a number of techniques for finding particular solutions of (1.1), most notably using symmetry methods [2, 3, 4, 7]. A good compilation of techniques for obtaining exact and approximate solutions of ODEs is given in [9]. The reader is also referred to the encyclopedic treatises [5, 8] on exact solutions of ODEs. It is to be noted that in most cases the solution techniques tend to be *ad hoc*, i.e. specific to the type of ODE being considered. However, Lie group analysis can be quite general but is arguably non-elementary.

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The main objective of this article is to propose an elementary analytical method to find the general solution of (1.1). The term “elementary” here means a calculus background is sufficient. By “general solution” we mean a solution of (1.1) with d arbitrary constants. The general solution is expressed as a series which in some cases can be summed to obtain a closed-form expression. A convergence condition for the series is derived and yields an approximate solution when a closed form is difficult to obtain.

A secondary objective of this article is pedagogical in nature. In a first course on differential equations, students are taught “recipes” for solving different types of ODEs. A solution technique may be valid for one ODE but not for another. This is of course to be expected. However, it would be useful for the students to realize that many of these techniques can be considered under a unified framework.

The outline of this paper is as follows. The derivation of the proposed solution method is given in Section 2. Illustrative examples for the scalar ODE (1.2) are provided in Section 3, namely (i) separable equations, (ii) Bernoulli and first-order linear equations, (iii) homogeneous equations, (iv) Riccati equations and (v) Abel equations of the first and second kind. In Section 4 we apply the solution method to (i) second-order linear equations and (ii) Liénard equations. Brief concluding remarks are given in Section 5.

2. Derivation of the solution method

Before giving the derivation of the method for the system of ODEs (1.1), it is worthwhile to look at the scalar ODE (1.2) to elucidate the underlying idea. The general solution of (1.2) can be expressed implicitly by

$$z(x, y) = c,$$

where $c \in \mathbb{R}$. This can be viewed as a level curve of the function $z = z(x, y)$. We wish to define sequences $(z_n)_{n=0}^{\infty}$ and $(S_n)_{n=0}^{\infty}$ of functions $z_n = z_n(x, y)$ and $S_n = z_n(x, y)$, respectively, such that

$$S_n(x, y) = z_0(x, y) + z_1(x, y) + \dots + z_n(x, y)$$

and

$$\lim_{n\to\infty} S_n(x, y) = z(x, y) = c.$$

Thus $S_n(x, y)$ is an approximation to $z(x, y)$. This is reminiscent of the idea in [6] but a different convergence criterion will be derived here that is amenable to a generalization to systems of ODEs.

Returning now to (1.1), we can express the general solution as

$$z^{(j)}(x, \mathbf{y}) = c_j, \quad j = 1, 2, \dots, d,$$

where $c_1, c_2, \dots, c_d \in \mathbb{R}$. These can be thought of as level surfaces of the functions $z^{(j)} = z^{(j)}(x, \mathbf{y})$ for $j = 1, 2, \dots, d$. Define the operator

$$\nabla_y = \Big(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \dots, \frac{\partial}{\partial y_d}\Big).$$

Using (1.1), we see that

$$\frac{\partial z^{(j)}}{\partial x} + \nabla_y z^{(j)} \cdot \mathbf{f} = 0, \quad j = 1, 2, \dots, d. \tag{2.1}$$

Define the sequences $(z_n^{(j)})_{n=0}^{\infty}$, where $z_n^{(j)} = z_n^{(j)}(x,\mathbf{y})$ for each $j = 1,2,\dots,d$, such that

$$\frac{\partial z_n^{(j)}}{\partial x} = -\nabla_y z_{n-1}^{(j)} \cdot \mathbf{f}, \quad n \geq 1. \tag{2.2}$$

Assume that $z_0^{(j)}$ for $j = 1,2,\dots,d$ are given. Define also the sequences $(S_n^{(j)})_{n=0}^{\infty}$, where $S_n^{(j)} = S_n^{(j)}(x,\mathbf{y})$ for each $j = 1,2,\dots,d$, such that

$$\begin{aligned} S_n^{(j)}(x,\mathbf{y}) &= z_0^{(j)}(x,\mathbf{y}) + z_1^{(j)}(x,\mathbf{y}) + \dots + z_n^{(j)}(x,\mathbf{y}) \\ &= \sum_{k=0}^{n} z_k^{(j)}(x,\mathbf{y}), \quad j = 1,2,\dots,d. \end{aligned} \tag{2.3}$$

Since $S_n^{(j)}(x,\mathbf{y})$ is desired to be an approximation to $z^{(j)}(x,\mathbf{y})$, it must be true that

$$\lim_{n\to\infty} S_n^{(j)}(x,\mathbf{y}) = z^{(j)}(x,\mathbf{y}), \quad j = 1,2,\dots,d.$$

Therefore the exact general solution of (1.1) is given implicitly by

$$\sum_{k=0}^{\infty} z_k^{(j)}(x,\mathbf{y}) = c_j, \quad j = 1,2,\dots,d. \tag{2.4}$$

To evaluate how "close" $S_n^{(j)}(x,\mathbf{y})$ is to $z^{(j)}(x,\mathbf{y})$, let us compare the gradient vectors associated with $S_n^{(j)}(x,\mathbf{y})$ and $z^{(j)}(x,\mathbf{y})$ at each $(x,\mathbf{y})$. Observing (2.1), the following conditions must hold:

$$\lim_{n\to\infty} \Big[\frac{\partial S_n^{(j)}}{\partial x}(x,\mathbf{y}) + \nabla_y S_n^{(j)}(x,\mathbf{y}) \cdot \mathbf{f}(x,\mathbf{y})\Big] = 0, \quad j = 1,2,\dots,d. \tag{2.5}$$

It can be seen from (2.3) and (2.2) that

$$\begin{aligned} \frac{\partial S_n^{(j)}}{\partial x} + \nabla_y S_n^{(j)} \cdot \mathbf{f} &= \frac{\partial z_0^{(j)}}{\partial x} + \sum_{k=1}^{n} \frac{\partial z_k^{(j)}}{\partial x} + \nabla_y z_0^{(j)} \cdot \mathbf{f} + \sum_{k=1}^{n} (\nabla_y z_k^{(j)} \cdot \mathbf{f}) \\ &= \frac{\partial z_0^{(j)}}{\partial x} + \nabla_y z_0^{(j)} \cdot \mathbf{f} + \sum_{k=1}^{n} (\nabla_y z_k^{(j)} \cdot \mathbf{f} - \nabla_y z_{k-1}^{(j)} \cdot \mathbf{f}). \end{aligned}$$

But the last term on the right-hand side is a telescoping sum that collapses to

$$\sum_{k=1}^{n} (\nabla_y z_k^{(j)} \cdot \mathbf{f} - \nabla_y z_{k-1}^{(j)} \cdot \mathbf{f}) = \nabla_y z_n^{(j)} \cdot \mathbf{f} - \nabla_y z_0^{(j)} \cdot \mathbf{f}.$$

Thus

$$\frac{\partial S_n^{(j)}}{\partial x} + \nabla_y S_n^{(j)} \cdot \mathbf{f} = \frac{\partial z_0^{(j)}}{\partial x} + \nabla_y z_n^{(j)} \cdot \mathbf{f}.$$

Therefore the convergence condition (2.5) is equivalent to

$$\lim_{n\to\infty} \Big[\frac{\partial z_0^{(j)}}{\partial x}(x,\mathbf{y}) + \nabla_y z_n^{(j)}(x,\mathbf{y}) \cdot \mathbf{f}(x,\mathbf{y})\Big] = 0, \quad j = 1,2,\dots,d. \tag{2.6}$$

In general, (2.6) may be valid only for all $(x,\mathbf{y})$ belonging to some region in $\mathbb{R}^{d+1}$.

Remark 2.1. If $z_0^{(j)}$ does not depend on x for $j = 1,2,\dots,d$, then (2.6) becomes

$$\lim_{n\to\infty} \nabla_y z_n^{(j)}(x,\mathbf{y}) \cdot \mathbf{f}(x,\mathbf{y}) = 0, \quad j = 1,2,\dots,d. \tag{2.7}$$

In particular, when $d = 1$, (2.7) simplifies further to

$$\lim_{n\to\infty} \frac{\partial z_n}{\partial y}(x, y) = 0.$$

This is different from the convergence condition obtained in [6], where

$$\lim_{n\to\infty} \frac{(\partial z_n/\partial y)(x, y)}{(\partial S_n/\partial y)(x, y)} = 0.$$

The reason is that the tangent vector was considered in [6], while here the gradient vector is used. The latter allows the solution method to be generalizable to systems of ODEs.

3. Examples: scalar case

In this section several illustrative examples are given for some classes of scalar first-order ODEs (i.e. $d = 1$).

3.1. **Separable equations.** Consider the separable equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) = \frac{g(x)}{h(y)}. \tag{3.1}$$

Take

$$z_0(x, y) = \int h(y)\,\mathrm{d}y$$

in (2.2), so that

$$\frac{\partial z_1}{\partial x} = -\frac{g(x)}{h(y)}\frac{\partial z_0}{\partial y} \quad \text{or} \quad z_1(x, y) = -\int g(x)\,\mathrm{d}x.$$

Then (2.2) yields $z_n(x, y) = 0$ for $n \geq 2$ and therefore

$$S_n(x, y) = z_0(x, y) + z_1(x, y), \quad n \geq 1.$$

Equation (2.4) gives the general solution of the separable equation (3.1) as

$$\int h(y)\,\mathrm{d}y - \int g(x)\,\mathrm{d}x = c,$$

as is well known. It is obvious that the convergence condition in (2.7) is always satisfied.

3.2. **Bernoulli and first-order linear equations.** Let us take a look at the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) = g(x)y + h(x)y^p. \tag{3.2}$$

Without loss of generality, we may assume that $p \neq 1$, otherwise (3.2) reduces to a separable equation. Equatiion (3.2) is a first-order linear equation when $p = 0$ and a Bernoulli equation when $p \neq 0$. Choose $z_0(x, y) = y^{1-p}$. We claim that

$$z_n(x, y) = F_n(x)y^{1-p} + G_n(x), \quad n \geq 1, \quad F_0(x) = 1, \quad G_0(x) = 0, \tag{3.3}$$

where $F_n = F_n(x)$ and $G_n = G_n(x)$ for $n \geq 1$ are to be determined. Substituting (3.3) into (2.2), we have

$$F_n'(x)y^{1-p} + G_n'(x) = (p-1)g(x)F_{n-1}(x)y^{1-p} + (p-1)h(x)F_{n-1}(x),$$

which implies that

$$F_n'(x) = (p-1)g(x)F_{n-1}(x), \quad G_n'(x) = (p-1)h(x)F_{n-1}(x), \quad n \geq 1.$$

By induction on n, we can show that

$$F_n(x) = \frac{(p-1)^n}{n!}\Big[\int g(x)\,\mathrm{d}x\Big]^n, \quad n \geq 1$$

and therefore

$$G_n(x) = \int \frac{(p-1)^n}{(n-1)!}\Big[\int g(x)\,\mathrm{d}x\Big]^{n-1} h(x)\,\mathrm{d}x.$$

Equation (3.3) becomes

$$\begin{aligned} z_n(x,y) &= \frac{(p-1)^n}{n!}\Big[\int g(x)\,\mathrm{d}x\Big]^n y^{1-p} \\ &\quad + \int \frac{(p-1)^n}{(n-1)!}\Big[\int g(x)\,\mathrm{d}x\Big]^{n-1} h(x)\,\mathrm{d}x, \quad n \geq 1 \end{aligned}$$

and so (2.3) yields

$$\begin{aligned} S_n(x,y) &= y^{1-p} + y^{1-p}\sum_{k=1}^{n} \frac{(p-1)^k}{k!}\Big[\int g(x)\,\mathrm{d}x\Big]^k \\ &\quad + \sum_{k=1}^{n} \int \frac{(p-1)^k}{(k-1)!}\Big[\int g(x)\,\mathrm{d}x\Big]^{k-1} h(x)\,\mathrm{d}x \\ &= y^{1-p}\sum_{k=0}^{n} \frac{(p-1)^k}{k!}\Big[\int g(x)\,\mathrm{d}x\Big]^k + \sum_{k=0}^{n-1} \frac{(p-1)^{k+1}}{k!}\Big[\int g(x)\,\mathrm{d}x\Big]^k h(x)\,\mathrm{d}x. \end{aligned}$$

As $n \to \infty$, we see that

$$\exp\Big((p-1)\int g(x)\,\mathrm{d}x\Big) y^{1-p} + (p-1)\int \exp\Big((p-1)\int g(x)\,\mathrm{d}x\Big) h(x)\,\mathrm{d}x = c.$$

Solving for y above recovers the well-known solutions to the Bernoulli and first-order linear equations in one go. By comparison, the standard technique is to first solve a first-order linear ODE using an integrating factor, and then transform a Bernoulli equation to a first-order linear equation. Note that the convergence condition (2.7) holds since

$$\lim_{n\to\infty} \frac{\partial z_n}{\partial y}(x,y) = (1-p)y^{-p} \lim_{n\to\infty} \frac{1}{n!}\Big[(p-1)\int g(x)\,\mathrm{d}x\Big]^n = 0.$$

3.3. **Homogeneous equations.** A homogeneous equation has the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y) = g\Big(\frac{y}{x}\Big). \tag{3.4}$$

Take $z_0(x,y) = -\log(|y|)$. Define

$$F_n(u) = -\int u^{-n-1} g(u)^n\,\mathrm{d}u, \quad F_{n-1}(u) = -\int u^{-n} g(u)^{n-1}\,\mathrm{d}u,$$
$$G(u) = \int \frac{1}{g(u)-u}\,\mathrm{d}u.$$

Then it is straightforward to show that

$$F_n'(u) = -u^{-n-1}g(u)^n,\ F_{n-1}'(u) = -u^{-n}g(u)^{n-1},\ F_n'(u) = \frac{g(u)}{u}F_{n-1}'(u). \tag{3.5}$$

For later use we note that

$$\begin{aligned} -\log(|u|)+\sum_{k=1}^{\infty}F_k(u) &= -\sum_{k=0}^{\infty}\int \frac{1}{u}\Big[\frac{g(u)}{u}\Big]^k \,\mathrm{d}u \\ &= -\int \frac{1}{u}\frac{1}{1-g(u)/u}\,\mathrm{d}u = G(u), \end{aligned} \tag{3.6}$$

where we used the geometric series expansion and assumed that $|g(u)| < |u|$. We claim that

$$z_n(x,y) = F_n\Big(\frac{y}{x}\Big), \quad n \ge 1.$$

Substituting this into (2.2) and recalling (3.5) gives

$$\frac{\partial z_n}{\partial x} + g\Big(\frac{y}{x}\Big)\frac{\partial z_{n-1}}{\partial y} = -\frac{y}{x^2}F_n'\Big(\frac{y}{x}\Big) + \frac{1}{x}g\Big(\frac{y}{x}\Big)F_{n-1}'\Big(\frac{y}{x}\Big) = 0.$$

This proves the claim. The left-hand side of (2.4) can be written with the aid of (3.6) as

$$\begin{aligned} -\log(|y|)+\sum_{k=1}^{\infty}F_k\Big(\frac{y}{x}\Big) &= -\log\Big(|\frac{y}{x}|\Big)+\sum_{k=1}^{\infty}F_k\Big(\frac{y}{x}\Big)-\log(|x|) \\ &= G\Big(\frac{y}{x}\Big)-\log(|x|). \end{aligned}$$

Hence the general solution of the homogeneous equation (3.4) from (2.4) is

$$G\Big(\frac{y}{x}\Big)-\log(|x|) = c, \quad G(u) = \int \frac{1}{g(u)-u}\,\mathrm{d}u. \tag{3.7}$$

By comparison, (3.7) can be also be obtained by a dependent variable transformation and solving the resulting separable equation. The convergence condition (2.7) is

$$\lim_{n\to\infty}\frac{\partial z_n}{\partial y}(x,y) = -\frac{1}{y}\lim_{n\to\infty}\Big[\frac{g(y/x)}{y/x}\Big]^n = 0,$$

provided $|g(y/x)| < |y/x|$, which is what we assumed for g. Note, however, that (3.7) is valid even without this assumption.

3.4. **Riccati equations.** A general Riccati equation has the form (see [8, Section 1.2.1])

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g_2(x)y^2 + g_1(x)y + g_0(x),$$

which can be transformed to a second-order linear ODE with variable coefficients. A general Riccati equation can also be transformed to the canonical form [8]

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y) = y^2 + g(x). \tag{3.8}$$

Unlike the previous examples, the solution to (3.8) for an arbitrary g is not known.

Take $z_0(x,y) = y$. We claim that

$$\begin{aligned} z_{2n-1}(x,y) &= \sum_{j=0}^{n}F_{n,j}(x)y^{2n-2j}, \\ z_{2n}(x,y) &= \sum_{j=0}^{n}G_{n,j}(x)y^{2n-2j+1}, \quad n\ge 1, \end{aligned} \tag{3.9}$$

where the functions $F_{n,j} = F_{n,j}(x)$ and $G_{n,j} = G_{n,j}(x)$ for $n \geq 1$ and $0 \leq j \leq n$ are to be determined. Define $G_{0,0}(x,y) = 1$. We infer from the second equation of (3.9) that

$$z_{2n-2}(x,y) = \sum_{j=0}^{n-1} G_{n-1,j}(x) y^{2n-2j-1}, \quad n \geq 2. \tag{3.10}$$

Equation (2.2) is

$$\frac{\partial z_m}{\partial x} = -[y^2 + g(x)] \frac{\partial z_{m-1}}{\partial y}, \quad m \geq 1. \tag{3.11}$$

From (3.11) we see that

$$z_1(x,y) = -xy^2 - \int g(x)\,\mathrm{d}x.$$

Therefore we deduce from (3.9) that

$$F_{1,0}(x) = -x, \quad F_{1,1}(x) = -\int g(x)\,\mathrm{d}x.$$

Moreover, (3.11) gives

$$z_2(x,y) = x^2 y^3 + 2y \int x g(x)\,\mathrm{d}x.$$

Thus (3.9) yields

$$G_{1,0}(x) = x^2, \quad G_{1,1}(x) = 2 \int x g(x)\,\mathrm{d}x.$$

Suppose that $m = 2n$ in (3.11), i.e.

$$\frac{\partial z_{2n}}{\partial x} = -[y^2 + g(x)] \frac{\partial z_{2n-1}}{\partial y}. \tag{3.12}$$

Substituting (3.9) into (3.12), we have

$$\begin{aligned}
\sum_{j=0}^{n} G'_{n,j}(x) y^{2n-2j+1} &= -[y^2 + g(x)] \sum_{j=0}^{n} (2n-2j) F_{n,j}(x) y^{2n-2j-1} \\
&= \sum_{j=0}^{n-1} (-2n+2j) F_{n,j}(x) y^{2n-2j+1} \\
&\quad + \sum_{j=0}^{n-1} (-2n+2j) g(x) F_{n,j}(x) y^{2n-2j-1}.
\end{aligned}$$

However,

$$\sum_{j=0}^{n-1} (-2n+2j) g(x) F_{n,j}(x) y^{2n-2j-1} = \sum_{j=1}^{n} (-2n+2j-2) g(x) F_{n,j-1}(x) y^{2n-2j+1},$$

so that

$$\begin{aligned}
&\sum_{j=1}^{n-1} G'_{n,j}(x) y^{2n-2j+1} + G'_{n,0}(x) y^{2n+1} + G'_{n,n}(x) y \\
&= \sum_{j=1}^{n-1} (-2n+2j) F_{n,j}(x) y^{2n-2j+1} - 2n F_{n,0}(x) y^{2n+1}
\end{aligned}$$

$$+\sum_{j=1}^{n-1}(-2n+2j-2)g(x)F_{n,j-1}(x)y^{2n-2j+1}-2g(x)F_{n,n-1}(x)y.$$

Equating coefficients of terms of like powers gives

$$\begin{gathered}G'_{n,0}(x)=-2nF_{n,0}(x),\quad n\geq 2,\\ G'_{n,j}(x)=(-2n+2j)F_{n,j}(x)+(-2n+2j-2)g(x)F_{n,j-1}(x),\\ n\geq 2,\quad 1\leq j\leq n-1,\\ G'_{n,n}(x)=-2g(x)F_{n,n-1}(x),\quad n\geq 2.\end{gathered}\tag{3.13}$$

Now suppose that $m=2n-1$ in (3.11), i.e.

$$\frac{\partial z_{2n-1}}{\partial x}=-[y^2+g(x)]\frac{\partial z_{2n-2}}{\partial y}.\tag{3.14}$$

Substituting (3.9) and (3.10) into (3.14) gives

$$\begin{aligned}\sum_{j=0}^{n}F'_{n,j}(x)y^{2n-2j}&=-[y^2+g(x)]\sum_{j=0}^{n-1}(2n-2j-1)G_{n-1,j}(x)y^{2n-2j-2}\\ &=\sum_{j=0}^{n-1}(-2n+2j+1)G_{n-1,j}(x)y^{2n-2j}\\ &\quad+\sum_{j=0}^{n-1}(-2n+2j+1)g(x)G_{n-1,j}(x)y^{2n-2j-2}.\end{aligned}$$

However,

$$\sum_{j=0}^{n-1}(-2n+2j+1)g(x)G_{n-1,j}(x)y^{2n-2j-2}=\sum_{j=1}^{n}(-2n+2j-1)g(x)G_{n-1,j-1}(x)y^{2n-2j},$$

which implies that

$$\begin{aligned}&\sum_{j=1}^{n-1}F'_{n,j}(x)y^{2n-2j}+F'_{n,0}(x)y^{2n}+F'_{n,n}(x)\\ &=\sum_{j=1}^{n-1}(-2n+2j+1)G_{n-1,j}(x)y^{2n-2j}+(-2n+1)G_{n-1,0}(x)y^{2n}\\ &\quad+\sum_{j=1}^{n-1}(-2n+2j-1)g(x)G_{n-1,j-1}(x)y^{2n-2j}-g(x)G_{n-1,n-1}(x).\end{aligned}$$

Equating coefficients of terms of like powers gives

$$\begin{gathered}F'_{n,0}(x)=(-2n+1)G_{n-1,0}(x),\quad n\geq 2,\\ F'_{n,j}(x)=(-2n+2j+1)G_{n-1,j}(x)+(-2n+2j-1)g(x)G_{n-1,j-1}(x),\\ n\geq 2,\quad 1\leq j\leq n-1,\\ F'_{n,n}(x)=-g(x)G_{n-1,n-1}(x),\quad n\geq 2.\end{gathered}\tag{3.15}$$

The recursive equations in (3.13) and (3.15) together provide determining equations for $F_{n,j}(x)$ and $G_{n,j}(x)$ for $n\geq 2$ and $0\leq j\leq n$. If we set

$$F_{n,0}(x)=-x^{2n-1},\quad G_{n,0}(x)=x^{2n},\quad n\geq 2,$$

then

$$G'_{n,0}(x) + 2nF_{n,0} = 0, \quad F'_{n,0}(x) + (2n-1)G_{n-1,0}(x) = 0$$

and the first equations in (3.13) and (3.15) hold. The other equations in (3.13) and (3.15) can be solved recursively once g has been specified. In summary, we have

$$\begin{gathered}
G_{0,0}(x,y) = 1, \quad G_{1,0}(x) = x^2, \quad G_{1,1}(x) = 2\int xg(x)\,\mathrm{d}x, \\
F_{1,0}(x) = -x, \quad F_{1,1}(x) = -\int g(x)\,\mathrm{d}x, \\
F_{n,0}(x) = -x^{2n-1}, \quad G_{n,0}(x) = x^{2n}, \quad n \geq 2, \\
F'_{n,j}(x) = (-2n+2j+1)G_{n-1,j}(x) + (-2n+2j-1)g(x)G_{n-1,j-1}(x), \\
n \geq 2, \quad 1 \leq j \leq n-1, \\
G'_{n,j}(x) = (-2n+2j)F_{n,j}(x) + (-2n+2j-2)g(x)F_{n,j-1}(x), \\
n \geq 2, \quad 1 \leq j \leq n-1, \\
F'_{n,n}(x) = -g(x)G_{n-1,n-1}(x), \quad G'_{n,n}(x) = -2g(x)F_{n,n-1}(x), \quad n \geq 2.
\end{gathered} \tag{3.16}$$

The general solution of the canonical Riccati equation (3.8) from (2.4) is

$$\begin{aligned}
&y + \sum_{k=1}^{\infty}\sum_{j=0}^{k} F_{k,j}(x)y^{2k-2j} + \sum_{k=1}^{\infty}\sum_{j=0}^{k} G_{k,j}(x)y^{2k-2j+1} \\
&= y + \sum_{k=1}^{\infty}\sum_{j=0}^{k}[F_{k,j}(x) + yG_{k,j}(x)]y^{2k-2j} = c.
\end{aligned} \tag{3.17}$$

The convergence condition (2.7) for the canonical Riccati equation (3.8) is

$$\lim_{n\to\infty}\sum_{j=0}^{n}[(2n-2j)F_{n,j}(x)y^{2n-2j-1} + (2n-2j+1)G_{n,j}(x)y^{2n-2j}] = 0. \tag{3.18}$$

3.4.1. *Particular example: canonical Riccati equation with $g(x) = 0$.* In this case (3.16) simplifies to

$$\begin{gathered}
G_{0,0}(x,y) = 1, \quad G_{1,0}(x) = x^2, \quad G_{1,1}(x) = 0, \quad F_{1,0}(x) = -x, \quad F_{1,1}(x) = 0, \\
F_{n,0}(x) = -x^{2n-1}, \quad G_{n,0}(x) = x^{2n}, \quad n \geq 2, \\
F'_{n,j}(x) = (-2n+2j+1)G_{n-1,j}(x), \quad n \geq 2, \quad 1 \leq j \leq n-1, \\
G'_{n,j}(x) = (-2n+2j)F_{n,j}(x), \quad n \geq 2, \quad 1 \leq j \leq n-1, \\
F'_{n,n}(x) = 0, \quad G'_{n,n}(x) = 0, \quad n \geq 2.
\end{gathered}$$

It follows that $F_{n,0}(x) = -x^{2n-1}$, $G_{n,0}(x) = x^{2n}$ and $F_{n,j}(x) = G_{n,j}(x) = 0$ for all $n \geq 1$ and $1 \leq j \leq n$. The general solution (3.17) of $\mathrm{d}y/\mathrm{d}x = y^2$ simplifies to

$$y - \sum_{k=1}^{\infty} x^{2k-1}y^{2k} + \sum_{k=1}^{\infty} x^{2k}y^{2k+1} = y + \Big(y - \frac{1}{x}\Big)\sum_{k=1}^{\infty}(x^2y^2)^k = \frac{y}{1+xy} = c$$

provided $|xy| < 1$. Solving for y gives the explicit general solution

$$y = \frac{c}{1-cx}.$$

Note that this remains a solution even if the the condition $|xy| < 1$ is removed, as a direct substitution into the ODE shows.

3.4.2. *Particular example: canonical Riccati equation with $g(x) = x$.* In this case (3.16) becomes

$$\begin{gathered}
G_{0,0}(x,y) = 1, \quad G_{1,0}(x) = x^2, \quad G_{1,1}(x) = \frac{2}{3}x^3\,\mathrm{d}x, \\
F_{1,0}(x) = -x, \quad F_{1,1}(x) = -\frac{1}{2}x^2, \\
F_{n,0}(x) = -x^{2n-1}, \quad G_{n,0}(x) = x^{2n}, \quad n \geq 2, \\
F'_{n,j}(x) = (-2n+2j+1)G_{n-1,j}(x) + (-2n+2j-1)xG_{n-1,j-1}(x), \\
n \geq 2, \quad 1 \leq j \leq n-1, \\
G'_{n,j}(x) = (-2n+2j)F_{n,j}(x) + (-2n+2j-2)xF_{n,j-1}(x), \\
n \geq 2, \quad 1 \leq j \leq n-1, \\
F'_{n,n}(x) = -xG_{n-1,n-1}(x), \quad G'_{n,n}(x) = -2xF_{n,n-1}(x), \quad n \geq 2.
\end{gathered}$$

Then

$$F_{n,j}(x) = a_{n,j}x^{2n+j-1}, \quad G_{n,j}(x) = b_{n,j}x^{2n+j}, \quad n \geq 1, \quad 0 \leq j \leq n,$$

where $a_{n,j}$ and $b_{n,j}$ are defined recursively by

$$\begin{gathered}
a_{1,0} = -1, \quad a_{1,1} = -\frac{1}{2}, \quad b_{0,0} = 1, \quad b_{1,0} = 1, \quad b_{1,1} = \frac{2}{3}, \\
a_{n,0} = -1, \quad b_{n,0} = 1, \quad n \geq 2, \\
a_{n,j} = \frac{-2n+2j+1}{2n+j-1}b_{n-1,j} + \frac{-2n+2j-1}{2n+j-1}b_{n-1,j-1}, \quad n \geq 2, \quad 1 \leq j \leq n-1, \\
b_{n,j} = \frac{-2n+2j}{2n+j}a_{n,j} + \frac{-2n+2j-2}{2n+j}a_{n,j-1}, \quad n \geq 2, \quad 1 \leq j \leq n-1, \\
a_{n,n} = -\frac{1}{3n-1}b_{n-1,n-1}, \quad b_{n,n} = -\frac{2}{3n}a_{n,n-1}, \quad n \geq 2.
\end{gathered}$$

From (3.17) the general solution of $\mathrm{d}y/\mathrm{d}x = y^2 + x$ is

$$y + \sum_{k=1}^{\infty}\sum_{j=0}^{k}(a_{k,j}x^{2k+j-1}y^{2k-2j} + b_{k,j}x^{2k+j}y^{2k-2j+1}) = c.$$

The convergence condition (3.18) is

$$\lim_{n\to\infty}\sum_{j=0}^{n}[(2n-2j)a_{n,j}x^{2n+j-1}y^{2n-2j-1} + (2n-2j+1)b_{n,j}x^{2n+j}y^{2n-2j}] = 0.$$

3.5. **Abel equations.** An Abel equation of the first kind has the form (see [8, Section 1.4.1])

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g_3(x)y^3 + g_2(x)y^2 + g_1(x)y + g_0(x),$$

which can be transformed to the canonical form [8]

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y) = y^3 + g(x). \tag{3.19}$$

Similarly, a canonical Abel equation of the second kind has the form (see [8, Section 1.3.1])

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y) = 1 + g(x)y^{-1}. \tag{3.20}$$

As with the Riccati equation (3.8), the solution to an Abel equation for an arbitrary g is not known. Here we will just consider (3.19) since (3.20) can be similarly handled.

Take $z_0(x,y)=y$. Equation (2.2) here is

$$\frac{\partial z_n}{\partial x}=-[y^3+g(x)]\frac{\partial z_{n-1}}{\partial y},\quad n\ge 1. \tag{3.21}$$

It is easy to see that

$$\frac{\partial z_1}{\partial x}=-[y^3+g(x)]\quad\text{or}\quad z_1(x,y)=-xy^3-\int g(x)\,\mathrm{d}x$$

and

$$\frac{\partial z_2}{\partial x}=-[y^3+g(x)](-3xy^2)\quad\text{or}\quad z_2(x,y)=\frac{3}{2}x^2y^5+3y^2\int xg(x)\,\mathrm{d}x.$$

We claim that

$$z_n(x,y)=\sum_{j=0}^{n-1}F_{n,j}(x)y^{2n-3j+1}\quad n\ge 2, \tag{3.22}$$

where $F_{n,j}=F_{n,j}(x)$ for $n\ge 2$ and $0\le j\le n-1$ are to be determined. We see that

$$F_{2,0}(x)=\frac{3}{2}x^2,\quad F_{2,1}(x)=3\int xg(x)\,\mathrm{d}x.$$

Furthermore, (3.22) implies that

$$z_{n-1}(x,y)=\sum_{j=0}^{n-2}F_{n-1,j}(x)y^{2n-3j-1},\quad n\ge 3.$$

Thus, taking $n\ge 3$ and substituting (3.22) into (3.21), we obtain

$$\begin{aligned}\sum_{j=0}^{n-1}F'_{n,j}(x)y^{2n-3j+1}&=-[y^3+g(x)]\sum_{j=0}^{n-2}(2n-3j-1)F_{n-1,j}(x)y^{2n-3j-2}\\&=\sum_{j=0}^{n-2}(-2n+3j+1)F_{n-1,j}(x)y^{2n-3j+1}\\&\quad+\sum_{j=0}^{n-2}(-2n+3j+1)g(x)F_{n-1,j}(x)y^{2n-3j-2}.\end{aligned}$$

However,

$$\begin{aligned}&\sum_{j=0}^{n-2}(-2n+3j+1)g(x)F_{n-1,j}(x)y^{2n-3j-2}\\&=\sum_{j=1}^{n-1}(-2n+3j-2)g(x)F_{n-1,j-1}(x)y^{2n-3j+1}.\end{aligned}$$

Hence

$$\sum_{j=1}^{n-2}F'_{n,j}(x)y^{2n-3j+1}+F'_{n,0}(x)y^{2n+1}+F'_{n,n-1}(x)y^{-n+4}$$

$$
\begin{aligned}
&= \sum_{j=1}^{n-2}(-2n+3j+1)F_{n-1,j}(x)y^{2n-3j+1} + (-2n+1)F_{n-1,0}(x)y^{2n+1} \\
&\quad + \sum_{j=1}^{n-2}(-2n+3j-2)g(x)F_{n-1,j-1}(x)y^{2n-3j+1} \\
&\quad + (n-5)g(x)F_{n-1,n-2}(x)y^{-n+4}.
\end{aligned}
$$

Equating coefficients of terms of like powers gives

$$
\begin{gathered}
F'_{n,0}(x) = (-2n+1)F_{n-1,0}(x), \quad n \geq 3, \\
F'_{n,j}(x) = (-2n+3j+1)F_{n-1,j}(x) + (-2n+3j-2)g(x)F_{n-1,j-1}(x), \\
n \geq 3, \quad 1 \leq j \leq n-2, \\
F'_{n,n-1}(x) = (n-5)g(x)F_{n-1,n-2}(x), \quad n \geq 3.
\end{gathered} \tag{3.23}
$$

Recall that $F_{2,0}(x) = 3x^2/2$. We claim that

$$
F_{n,0}(x) = a_{n,0}x^n, \quad n \geq 3, \quad a_{2,0} = \frac{3}{2},
$$

where $(a_{n,0})_{n=3}^{\infty}$ is to be determined. Substituting into the first equation in (3.23), we see that

$$
na_{n,0}x^{n-1} = (-2n+1)a_{n-1,0}x^{n-1} \quad \text{or} \quad a_{n,0} = \Big(-2+\frac{1}{n}\Big)a_{n-1,0}, \quad n \geq 3.
$$

It follows that

$$
a_{n,0} = \frac{3}{2}\prod_{k=3}^{n}\Big(-2+\frac{1}{k}\Big), \quad n \geq 3.
$$

The expressions $F_{n,j}(x)$ for $n \geq 3$ and $1 \leq j \leq n-1$ can be computed recursively from the second and third equations in (3.23). From (2.4) we deduce that the general solution of the canonical Abel equation of the first kind (3.19) is

$$
\begin{aligned}
&y - xy^3 - \int g(x)\,\mathrm{d}x + \frac{3}{2}x^2y^5 + 3y^2\int xg(x)\,\mathrm{d}x \\
&+ \sum_{n=3}^{\infty}\Big[\frac{3}{2}x^n y^{2n+1}\prod_{k=3}^{n}\Big(-2+\frac{1}{k}\Big) + \sum_{j=1}^{n-1}F_{n,j}(x)y^{2n-3j+1}\Big] = c.
\end{aligned} \tag{3.24}
$$

The convergence condition (2.7) for the canonical Abel equation of the first kind given in (3.19) is

$$
\lim_{n\to\infty}\sum_{j=0}^{n-1}(2n-3j+1)F_{n,j}(x)y^{2n-3j} = 0. \tag{3.25}
$$

3.5.1. *Particular example: canonical Abel equation of the first kind with $g(x) = x$.* First we claim that

$$
F_{n,j}(x) = a_{n,j}x^{n+j}, \quad n \geq 3, \quad 1 \leq j \leq n-1.
$$

Substitution of this ansatz into the second equation of (3.23) yields

$$
(n+j)a_{n,j}x^{n+j-1} = (-2n+3j+1)a_{n-1,j}x^{n-1+j} + (-2n+3j-2)xa_{n-1,j-1}x^{n+j-2},
$$

so that

$$a_{n,j} = \frac{-2n+3j+1}{n+j} a_{n-1,j} + \frac{-2n+3j-2}{n+j} a_{n-1,j-1}\,, \\ n \geq 3,\ 1 \leq j \leq n-2. \tag{3.26}$$

Moreover, substituting the ansatz into the third equation of (3.23) gives

$$(2n-1)a_{n,n-1}x^{2n-2} = (n-5)xa_{n-1,n-2}x^{2n-3} \quad \text{or} \quad a_{n,n-1} = \frac{n-5}{2n-1} a_{n-1,n-2}.$$

Since $F_{2,1}(x) = x^3$, we see that $a_{2,1} = 1$. Define $b_n = a_{n-1,n-2}$ for $n \geq 3$. Hence

$$b_{n+1} = \frac{n-5}{2n-1} b_n, \quad n \geq 3, \quad b_3 = 1$$

and we deduce that

$$b_n = \prod_{k=3}^{n-1} \frac{k-5}{2k-1}, \quad n \geq 4.$$

Thus

$$a_{n,n-1} = b_{n+1} = \prod_{k=3}^{n} \frac{k-5}{2k-1}, \quad n \geq 3.$$

We see from (3.24) that the general solution of $\mathrm{d}y/\mathrm{d}x = y^3 + x$ is

$$y - xy^3 - \frac{1}{2}x^2 + \frac{3}{2}x^2y^5 + x^3y^2 + \sum_{n=3}^{\infty} \sum_{j=0}^{n-1} a_{n,j} x^{n+j} y^{2n-3j+1} = c,$$

where

$$a_{n,0} = \frac{3}{2} \prod_{k=3}^{n} \Big(-2 + \frac{1}{k}\Big), \quad a_{n,n-1} = \prod_{k=3}^{n} \frac{k-5}{2k-1}, \quad n \geq 3$$

and $a_{n,j}$ for $n \geq 3$ and $1 \leq j \leq n-2$ are calculated recursively from (3.26), while the convergence condition (3.25) is

$$\lim_{n\to\infty} \sum_{j=0}^{n-1} (2n-3j+1) a_{n,j} x^{n+j} y^{2n-3j} = 0.$$

4. Examples: system case

Let us now turn to some examples of systems of ODEs with $d = 2$.

4.1. **Second-order linear ODEs.**

Consider the second-order linear ODE

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + g(x)y = 0. \tag{4.1}$$

The general solution of (4.1) is not known for an arbitrary g. If one nontrivial solution can be found, then another linearly independent solution can be obtained through the method of variation of parameters and hence the general solution can be determined. However, there is no general procedure for finding a particular solution when g is arbitrary.

To express (4.1) as a system of the form (1.1), define $y_1 = y$ and $y_2 = \mathrm{d}y/\mathrm{d}x$, yielding

$$\begin{aligned} \frac{\mathrm{d}y_1}{\mathrm{d}x} &= f_1(x, y_1, y_2) = y_2, \\ \frac{\mathrm{d}y_2}{\mathrm{d}x} &= f_2(x, y_1, y_2) = -g(x)y_1. \end{aligned} \tag{4.2}$$

Take

$$z_0^{(1)}(x,y_1,y_2)=y_1,\quad z_0^{(2)}(x,y_1,y_2)=y_2.$$

We claim that

$$z_{2n-1}^{(1)}(x,y_1,y_2)=F_n^{(1)}(x)y_2,\quad z_{2n}^{(1)}(x,y_1,y_2)=G_n^{(1)}(x)y_1,\quad n\ge 1$$

and

$$z_{2n-1}^{(2)}(x,y_1,y_2)=F_n^{(2)}(x)y_1,\quad z_{2n}^{(2)}(x,y_1,y_2)=G_n^{(2)}(x)y_2,\quad n\ge 1,$$

where $F_n^{(j)}=F_n^{(j)}(x)$ and $G_n^{(j)}=G_n^{(j)}(x)$ for $j=1,2$ and $n\ge 1$ are to be determined such that $G_0^{(1)}(x)=1$ and $G_0^{(2)}(x)=1$. Equation (2.2) becomes

$$\begin{gathered}
\frac{\partial z_{2n-1}^{(1)}}{\partial x}=-y_2\frac{\partial z_{2n-2}^{(1)}}{\partial y_1}+g(x)y_1\frac{\partial z_{2n-2}^{(1)}}{\partial y_2},\\
\frac{\partial z_{2n}^{(1)}}{\partial x}=-y_2\frac{\partial z_{2n-1}^{(1)}}{\partial y_1}+g(x)y_1\frac{\partial z_{2n-1}^{(1)}}{\partial y_2},\\
\frac{\partial z_{2n-1}^{(2)}}{\partial x}=-y_2\frac{\partial z_{2n-2}^{(2)}}{\partial y_1}+g(x)y_1\frac{\partial z_{2n-2}^{(2)}}{\partial y_2},\\
\frac{\partial z_{2n}^{(2)}}{\partial x}=-y_2\frac{\partial z_{2n-1}^{(2)}}{\partial y_1}+g(x)y_1\frac{\partial z_{2n-1}^{(2)}}{\partial y_2}.
\end{gathered}$$

It follows that

$$\frac{\partial z_1^{(1)}}{\partial x}=-y_2\quad\text{or}\quad z_1^{(1)}(x,y_1,y_2)=-xy_2$$

and

$$\frac{\partial z_2^{(1)}}{\partial x}=-xg(x)y_1\quad\text{or}\quad z_2^{(1)}(x,y_1,y_2)=-y_1\int xg(x)\,\mathrm{d}x.$$

This allows us to identify

$$F_1^{(1)}(x)=-x,\quad G_1^{(1)}(x)=-\int xg(x)\,\mathrm{d}x.$$

Similarly, we see that

$$\frac{\partial z_1^{(2)}}{\partial x}=g(x)y_1\quad\text{or}\quad z_1^{(2)}(x,y_1,y_2)=y_1\int g(x)\,\mathrm{d}x$$

and

$$\frac{\partial z_2^{(2)}}{\partial x}=-y_2\int g(x)\,\mathrm{d}x\quad\text{or}\quad z_2^{(2)}(x,y_1,y_2)=-y_2\int\Big[\int g(x)\,\mathrm{d}x\Big]\,\mathrm{d}x,$$

from which we identify

$$F_1^{(2)}(x)=\int g(x)\,\mathrm{d}x,\quad G_1^{(2)}(x)=-\int\Big[\int g(x)\,\mathrm{d}x\Big]\,\mathrm{d}x.$$

More generally, for $n\ge 2$, we obtain

$$\begin{gathered}
\frac{\mathrm{d}}{\mathrm{d}x}(F_n^{(1)})y_2=-y_2G_{n-1}^{(1)},\quad \frac{\mathrm{d}}{\mathrm{d}x}(G_n^{(1)})y_1=g(x)y_1F_n^{(1)},\\
\frac{\mathrm{d}}{\mathrm{d}x}(F_n^{(2)})y_1=g(x)y_1G_{n-1}^{(2)},\quad \frac{\mathrm{d}}{\mathrm{d}x}(G_n^{(2)})y_2=-y_2F_n^{(2)}.
\end{gathered}$$

These are equivalent to

$$\begin{gathered}\frac{\mathrm{d}^2}{\mathrm{d}x^2}F_n^{(1)}+g(x)F_{n-1}^{(1)}=0,\quad n\ge 2,\quad F_1^{(1)}(x)=-x,\\ \frac{\mathrm{d}^2}{\mathrm{d}x^2}G_n^{(2)}+g(x)G_{n-1}^{(2)}=0,\quad n\ge 2,\quad G_1^{(2)}(x)=-\int\Big[\int g(x)\,\mathrm{d}x\Big]\,\mathrm{d}x,\\ G_n^{(1)}(x)=\int g(x)F_n^{(1)}(x)\,\mathrm{d}x,\quad F_n^{(2)}(x)=\int g(x)G_{n-1}^{(2)}(x)\,\mathrm{d}x,\quad n\ge 1.\end{gathered}\tag{4.3}$$

Therefore (2.4) in this case is

$$\begin{gathered}y_1+y_2\sum_{n=1}^{\infty}F_n^{(1)}(x)+y_1\sum_{n=1}^{\infty}G_n^{(1)}(x)=y_1\sum_{n=0}^{\infty}G_n^{(1)}(x)+y_2\sum_{n=1}^{\infty}F_n^{(1)}(x)=c_1\,,\\ y_2+y_1\sum_{n=1}^{\infty}F_n^{(2)}(x)+y_2\sum_{n=1}^{\infty}G_n^{(2)}(x)=y_1\sum_{n=1}^{\infty}F_n^{(2)}(x)+y_2\sum_{n=0}^{\infty}G_n^{(2)}(x)=c_2.\end{gathered}\tag{4.4}$$

But (4.4) is a linear system in y_1 and y_2, and whose solution is

$$\begin{gathered}y_1=\frac{c_1\sum_{n=0}^{\infty}G_n^{(2)}(x)-c_2\sum_{n=1}^{\infty}F_n^{(1)}(x)}{\sum_{n=0}^{\infty}G_n^{(1)}(x)\sum_{n=0}^{\infty}G_n^{(2)}(x)-\sum_{n=1}^{\infty}F_n^{(1)}(x)\sum_{n=1}^{\infty}F_n^{(2)}(x)}\,,\\ y_2=\frac{-c_1\sum_{n=1}^{\infty}F_n^{(2)}(x)+c_2\sum_{n=0}^{\infty}G_n^{(1)}(x)}{\sum_{n=0}^{\infty}G_n^{(1)}(x)\sum_{n=0}^{\infty}G_n^{(2)}(x)-\sum_{n=1}^{\infty}F_n^{(1)}(x)\sum_{n=1}^{\infty}F_n^{(2)}(x)}.\end{gathered}$$

Recalling that $y_1=y$, the general solution of the second-order linear equation (4.1) is therefore

$$y=\frac{c_1\sum_{n=0}^{\infty}G_n^{(2)}(x)-c_2\sum_{n=1}^{\infty}F_n^{(1)}(x)}{\sum_{n=0}^{\infty}G_n^{(1)}(x)\sum_{n=0}^{\infty}G_n^{(2)}(x)-\sum_{n=1}^{\infty}F_n^{(1)}(x)\sum_{n=1}^{\infty}F_n^{(2)}(x)},\tag{4.5}$$

where $F_n^{(j)}(x)$ and $G_n^{(j)}(x)$ for $j=1,2$ and $n\ge 1$ are as given recursively in (4.3). Note that $G_0^{(1)}(x)=G_0^{(2)}(x)=1$ by assumption.

It is easy to see that

$$\begin{gathered}\frac{\partial z_{2n}^{(1)}}{\partial y_1}(x,y_1,y_2)=G_n^{(1)}(x),\quad \frac{\partial z_{2n-1}^{(1)}}{\partial y_1}(x,y_1,y_2)=0,\\ \frac{\partial z_{2n}^{(2)}}{\partial y_2}(x,y_1,y_2)=G_n^{(2)}(x),\quad \frac{\partial z_{2n-1}^{(2)}}{\partial y_2}(x,y_1,y_2)=0.\end{gathered}$$

Therefore sufficient conditions for the convergence condition (2.7) to be valid are

$$\lim_{n\to\infty}G_n^{(1)}(x)=0,\quad \lim_{n\to\infty}G_n^{(2)}(x)=0.\tag{4.6}$$

4.1.1. *Particular example: second-order linear equation with* $g(x)=1$. Suppose that we take $g(x)=1$ in (4.1), giving the classical oscillator equation [9]. It is straightforward to show that the solution of (4.3) is

$$F_n^{(1)}(x)=(-1)^n\frac{x^{2n-1}}{(2n-1)!},\quad G_n^{(2)}(x)=(-1)^n\frac{x^{2n}}{(2n)!},\quad n\ge 1$$

and

$$G_n^{(1)}(x)=(-1)^n\frac{x^{2n}}{(2n)!}=G_n^{(2)}(x),$$

$$F_n^{(2)}(x) = (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} = -F_n^{(1)}(x), \quad n \geq 1.$$

Moreover, we obtain

$$\sum_{n=0}^{\infty} G_n^{(1)}(x) = \sum_{n=0}^{\infty} G_n^{(2)}(x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} = \cos(x)$$

and

$$\begin{aligned}\sum_{n=1}^{\infty} F_n^{(1)}(x) = -\sum_{n=1}^{\infty} F_n^{(2)}(x) &= \sum_{n=1}^{\infty}(-1)^n\frac{x^{2n-1}}{(2n-1)!}\\ &= -\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} = -\sin(x).\end{aligned}$$

Using (4.5), we deduce that the general solution of $\mathrm{d}^2y/\mathrm{d}x^2 + y = 0$ is

$$y = c_1\cos(x) + c_2\sin(x)$$

as to be expected. Note that (4.6) is satisfied since

$$\lim_{n\to\infty}(-1)^n\frac{x^{2n}}{(2n)!} = 0.$$

4.1.2. *Particular example: second-order linear equation with $g(x) = -x$.* Assume that $g(x) = -x$ in (4.1), which yields the Airy ODE [1, 9] whose general solution is

$$y = c_1\operatorname{Ai}(x) + c_2\operatorname{Bi}(x),$$

where Ai and Bi are the Airy functions. Define the sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ by

$$\begin{gathered}a_n = \frac{1}{(3n-2)(3n-3)}a_{n-1}, \quad n \geq 2, \quad a_1 = -1,\\ b_n = \frac{1}{3n(3n-1)}b_{n-1}, \quad n \geq 2, \quad b_1 = \frac{1}{6}.\end{gathered}$$

We can verify by direct substitution that the respective solutions of these difference equations are

$$a_n = -\prod_{j=0}^{n-2}\frac{1}{(3j+3)(3j+4)}, \quad b_n = \frac{1}{6}\prod_{j=0}^{n-2}\frac{1}{(3j+5)(3j+6)}, \quad n \geq 2.$$

Again by direct substitution we can show that the solution of (4.3) is

$$F_n^{(1)}(x) = a_n x^{3n-2}, \quad G_n^{(2)}(x) = b_n x^{3n}, \quad n \geq 1$$

and

$$G_n^{(1)}(x) = -\frac{1}{3n}a_n x^{3n}, \quad F_n^{(2)}(x) = -\frac{1}{3n-1}b_{n-1}x^{3n-1}, \quad n \geq 1.$$

Thus from (4.5) the general solution of the Airy ODE $\mathrm{d}^2y/\mathrm{d}x^2 - xy = 0$ can also be expressed as

$$y = c_1R_1(x) + c_2R_2(x),$$

where

$$R_1(x) = \frac{\sum_{n=0}^{\infty} G_n^{(2)}(x)}{\sum_{n=0}^{\infty} G_n^{(1)}(x)\sum_{n=0}^{\infty} G_n^{(2)}(x) - \sum_{n=1}^{\infty} F_n^{(1)}(x)\sum_{n=1}^{\infty} F_n^{(2)}(x)},$$

$$R_2(x) = -\frac{\sum_{n=1}^{\infty} F_n^{(1)}(x)}{\sum_{n=0}^{\infty} G_n^{(1)}(x) \sum_{n=0}^{\infty} G_n^{(2)}(x) - \sum_{n=1}^{\infty} F_n^{(1)}(x) \sum_{n=1}^{\infty} F_n^{(2)}(x)}.$$

In principle, it should be possible to express Ai and Bi as linear combinations of R_1 and R_2, and vice versa. Equation (4.6) is satisfied if

$$\lim_{n\to\infty} \Big[\frac{1}{3n} \prod_{j=0}^{n-2} \frac{1}{(3j+3)(3j+4)}\Big] x^{3n} = 0, \quad \lim_{n\to\infty} \Big[\frac{1}{6} \prod_{j=0}^{n-2} \frac{1}{(3j+5)(3j+6)}\Big] x^{3n} = 0.$$

These are achievable, for instance, if $-1 < x < 1$.

4.2. **Liénard equations.** A Liénard equation [8, 9] has the form

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + g(y)\frac{\mathrm{d}y}{\mathrm{d}x} + h(y) = 0. \tag{4.7}$$

A particular case is the van der Pol oscillator [9], where $g(y) = -\alpha(1 - y^2)$ for $\alpha \in \mathbb{R}$ and $h(y) = y$. Exact analytical solutions to such nonlinear ODEs are not known in general.

Let $y_1 = y$ and $y_2 = \mathrm{d}y/\mathrm{d}x$, so that a Liénard equation has the equivalent form

$$\begin{gathered} \frac{\mathrm{d}y_1}{\mathrm{d}x} = f_1(x, y_1, y_2) = y_2, \\ \frac{\mathrm{d}y_2}{\mathrm{d}x} = f_2(x, y_1, y_2) = -g(y_1)y_2 - h(y_1). \end{gathered} \tag{4.8}$$

Take

$$z_0^{(1)}(x, y_1, y_2) = y_1, \quad z_0^{(2)}(x, y_1, y_2) = y_2.$$

For $n \geq 1$, (2.2) becomes

$$\begin{aligned} \frac{\partial z_n^{(1)}}{\partial x} &= -y_2 \frac{\partial z_{n-1}^{(1)}}{\partial y_1} + [g(y_1)y_2 + h(y_1)] \frac{\partial z_{n-1}^{(1)}}{\partial y_2}, \\ \frac{\partial z_n^{(2)}}{\partial x} &= -y_2 \frac{\partial z_{n-1}^{(2)}}{\partial y_1} + [g(y_1)y_2 + h(y_1)] \frac{\partial z_{n-1}^{(2)}}{\partial y_2}. \end{aligned} \tag{4.9}$$

In particular,

$$\frac{\partial z_1^{(1)}}{\partial x} = -y_2, \quad \frac{\partial z_1^{(2)}}{\partial x} = g(y_1)y_2 + h(y_1)$$

imply that

$$z_1^{(1)}(x, y_1, y_2) = -xy_2, \quad z_1^{(2)}(x, y) = x[g(y_1)y_2 + h(y_1)],$$

while

$$\begin{gathered} \frac{\partial z_2^{(1)}}{\partial x} = -x[g(y_1)y_2 + h(y_1)], \\ \frac{\partial z_2^{(2)}}{\partial x} = -xy_2[g'(y_1)y_2 + h'(y_1)] + xg(y_1)[g(y_1)y_2 + h(y_1)] \end{gathered}$$

imply that

$$\begin{gathered} z_2^{(1)}(x, y_1, y_2) = -\frac{1}{2}x^2[g(y_1)y_2 + h(y_1)], \\ z_2^{(2)}(x, y_1, y_2) = \frac{1}{2}x^2[-g'(y_1)y_2^2 - h'(y_1)y_2 + g(y_1)^2 y_2 + g(y_1)h(y_1)]. \end{gathered}$$

Define $F_0(y_1,y_2)=y_1$ and $G_0(y_1,y_2)=y_2$. We claim that

$$\begin{aligned} z_n^{(1)}(x,y_1,y_2)&=\frac{1}{n!}F_n(y_1,y_2)x^n,\\ z_n^{(2)}(x,y_1,y_2)&=\frac{1}{n!}G_n(y_1,y_2)x^n,\quad n\ge 1, \end{aligned}\tag{4.10}$$

where $F_n=F_n(y_1,y_2)$ and $G_n=G_n(y_1,y_2)$ for $n\ge 1$ are to be determined. It is not difficult to see that

$$\begin{gathered} F_1(y_1,y_2)=-y_2,\quad G_1(y_1,y_2)=g(y_1)y_2+h(y_1),\\ F_2(y_1,y_2)=-g(y_1)y_2-h(y_1),\\ G_2(y_1,y_2)=-g'(y_1)y_2^2-h'(y_1)y_2+g(y_1)^2y_2+g(y_1)h(y_1). \end{gathered}$$

Substituting (4.10) into (4.9) and equating coefficients of like powers, we obtain

$$\begin{aligned} F_n(y_1,y_2)&=-y_2\frac{\partial F_{n-1}}{\partial y_1}(y_1,y_2)+[g(y_1)y_2+h(y_1)]\frac{\partial F_{n-1}}{\partial y_2}(y_1,y_2),\\ G_n(y_1,y_2)&=-y_2\frac{\partial G_{n-1}}{\partial y_1}(y_1,y_2)+[g(y_1)y_2+h(y_1)]\frac{\partial G_{n-1}}{\partial y_2}(y_1,y_2). \end{aligned}\tag{4.11}$$

Suppose that

$$F_n(y_1,y_2)=\sum_{j=0}^{n}f_{n,j}(y_1)y_2^j,\quad G_n(y_1,y_2)=\sum_{j=0}^{n}g_{n,j}(y_1)y_2^j,\quad n\ge 1,\tag{4.12}$$

where $f_{n,j}=f_{n,j}(y_1)$ and $g_{n,j}=g_{n,j}(y_1)$ for $n\ge 1$ and $0\le j\le n$ are to be determined. Then

$$\begin{gathered} f_{1,0}(y_1)=0,\quad f_{1,1}(y_1)=-1,\quad g_{1,0}(y_1)=h(y_1),\quad g_{1,1}(y_1)=g(y_1),\\ f_{2,0}(y_1)=-h(y_1),\quad f_{2,1}(y_1)=-g(y_1),\quad f_{2,2}(y_1)=0,\\ g_{2,0}(y_1)=g(y_1)h(y_1),\quad g_{2,1}(y_1)=-h'(y_1)+g(y_1)^2,\quad g_{2,2}(y_1)=-g'(y_1). \end{gathered}$$

Substitution of the first ansatz in (4.12) into (4.11) gives

$$\begin{aligned} \sum_{j=0}^{n}f_{n,j}(y_1)y_2^j&=-y_2\sum_{j=0}^{n-1}f'_{n-1,j}(y_1)y_2^j+[g(y_1)y_2+h(y_1)]\sum_{j=0}^{n-1}jf_{n-1,j}(y_1)y_2^{j-1}\\ &=-\sum_{j=0}^{n-1}f'_{n-1,j}(y_1)y_2^{j+1}+\sum_{j=1}^{n-1}jg(y_1)f_{n-1,j}(y_1)y_2^j\\ &\quad+\sum_{j=1}^{n-1}jh(y_1)f_{n-1,j}(y_1)y_2^{j-1}. \end{aligned}$$

However,

$$\sum_{j=0}^{n-1}f'_{n-1,j}(y_1)y_2^{j+1}=\sum_{j=1}^{n}f'_{n-1,j-1}(y_1)y_2^j$$

and

$$\sum_{j=1}^{n-1}jh(y_1)f_{n-1,j}(y_1)y_2^{j-1}=\sum_{j=0}^{n-2}(j+1)h(y_1)f_{n-1,j+1}(y_1)y_2^j,$$

so that

$$
\begin{aligned}
&\sum_{j=1}^{n-2} f_{n,j}(y_1)y_2^j + f_{n,0}(y_1) + f_{n,n-1}(y_1)y_2^{n-1} + f_{n,n}(y_1)y_2^n \\
&= -\sum_{j=1}^{n-2} f'_{n-1,j-1}(y_1)y_2^j - f'_{n-1,n-2}(y_1)y_2^{n-1} - f'_{n-1,n-1}(y_1)y_2^n \\
&\quad + \sum_{j=1}^{n-2} jg(y_1)f_{n-1,j}(y_1)y_2^j + (n-1)g(y_1)f_{n-1,n-1}(y_1)y_2^{n-1} \\
&\quad + \sum_{j=1}^{n-2}(j+1)h(y_1)f_{n-1,j+1}(y_1)y_2^j + h(y_1)f_{n-1,1}(y_1).
\end{aligned}
$$

Equating coefficients of like powers, we obtain

$$
\begin{gathered}
f_{n,0}(y_1) = h(y_1)f_{n-1,1}(y_1), \quad n \geq 3, \\
f_{n,j}(y_1) = -f'_{n-1,j-1}(y_1) + jg(y_1)f_{n-1,j}(y_1) + (j+1)h(y_1)f_{n-1,j+1}(y_1), \\
n \geq 3, \quad 1 \leq j \leq n-2, \\
f_{n,n-1}(y_1) = -f'_{n-1,n-2}(y_1) + (n-1)g(y_1)f_{n-1,n-1}(y_1), \quad n \geq 3, \\
f_{n,n}(y_1) = -f'_{n-1,n-1}(y_1), \quad n \geq 3.
\end{gathered}
$$

In a similar manner, substitution of the second ansatz in (4.12) into (4.11) and replacing all occurrences of $f_{n,j}$ by $g_{n,j}$ gives

$$
\begin{gathered}
g_{n,0}(y_1) = h(y_1)g_{n-1,1}(y_1), \\
g_{n,j}(y_1) = -g'_{n-1,j-1}(y_1) + jg(y_1)g_{n-1,j}(y_1) + (j+1)h(y_1)g_{n-1,j+1}(y_1), \\
n \geq 3, \quad 1 \leq j \leq n-2, \\
g_{n,n-1}(y_1) = -g'_{n-1,n-2}(y_1) + (n-1)g(y_1)g_{n-1,n-1}(y_1), \quad n \geq 3, \\
g_{n,n}(y_1) = -g'_{n-1,n-1}(y_1), \quad n \geq 3.
\end{gathered}
$$

Hence from (2.4) we get the general solution of (4.8), namely

$$
\begin{gathered}
y_1 - xy_2 + \sum_{k=2}^{\infty} \frac{x^k}{k!} \sum_{j=0}^{k} y_2^j f_{k,j}(y_1) = c_1, \\
y_2 + x[g(y_1)y_2 + h(y_1)] + \sum_{k=2}^{\infty} \frac{x^k}{k!} \sum_{j=0}^{k} y_2^j g_{k,j}(y_1) = c_2.
\end{gathered}
$$

Identifying $y_1 = y$ therefore gives the general solution of the Liénard equation (4.7) from the above pair of equations. Since

$$
\frac{\partial z_n^{(1)}}{\partial y_1}(x, y_1, y_2) = \frac{x^n}{n!} \sum_{j=0}^{n} y_2^j f'_{n,j}(y_1), \quad \frac{\partial z_n^{(2)}}{\partial y_2}(x, y_1, y_2) = \frac{x^n}{n!} \sum_{j=0}^{n} j y_2^{j-1} g_{n,j}(y_1),
$$

the convergence condition (2.7) is satisfied if

$$
\lim_{n\to\infty} \frac{x^n}{n!} \sum_{j=0}^{n} y_2^j f'_{n,j}(y_1) = 0, \quad \lim_{n\to\infty} \frac{x^n}{n!} \sum_{j=1}^{n} j y_2^{j-1} g_{n,j}(y_1) = 0.
$$

5. Concluding remarks

In this article we proposed an elementary analytical method for finding the general solution of a system of ODEs given by (1.1). The exact analytical solution (2.4) is expressed as a series and a convergence criterion (2.6) was derived for the series solution. The criterion determines a region in $(x, \mathbf{y})$-space for which the series will converge. By choosing different expressions for the "initial seeds" $z_0^{(j)}$ for $j = 1, 2, \dots, d$, other regions of convergence can be obtained. In some cases, the series can be summed and a closed-form expression for the general solution can be deduced. When this is not possible, approximate analytical formulas for the general solutions can nevertheless be written down. A careful error analysis of such analytical approximations is outside the scope of this article but will be investigated in future work.

We gave several illustrative examples in one and two dimensions. In particular, for Riccati, Abel and Liénard equations, analytical solutions are not known in the general case (cf. [8] for a compendium of solvable special cases). The proposed method here is an alternative to Lie symmetry analysis and is a useful addition to the applied mathematician's toolbox. It was also shown that many known solvable ODEs (e.g. separable, first-order linear, homogeneous, Bernoulli, Airy and other second-order linear equations) can be handled with the same methodology and can thus be introduced in a differential equations class at a more elementary level. Another future research direction will be to extend the proposed method here to fractional-order ODEs and certain classes of partial differential equations.

References

[1] Abramowitz, M.; Stegun, I.A.; *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, 9th edn. New York: Dover, 1972
[2] Bluman, G. W.; Anco, S.C.; *Symmetry and Integration Methods for Differential Equations*. Applied Mathematical Sciences vol. 154. New York: Springer, 2002
[3] Cantwell, B. J.; *Introduction to Symmetry Analysis*. Cambridge: Cambridge University Press, 2002
[4] Hill, J .M.; *Differential Equations and Group Methods for Scientists and Engineers*. Boca Raton: CRC Press, 1992.
[5] Kamke, E.; *Differentialgleichungen. Lösungsmethoden und Lösungen I. Gewöhnliche Differentialgleichungen*. Leipzig: B.G. Teubner, 1971.
[6] McGarvey, J. F.; Approximating the general solution of a differential equation. *SIAM Review*, 24 (1982) (3), pp. 333–337.
[7] Olver, P. J.; *Applications of Lie Groups to Differential Equations*. Graduate Texts in Mathematics vol. 107. New York: Springer, 1998.
[8] Polyanin, A .D.; Zaitsev, V. F.; *Handbook of Exact Solutions for Ordinary Differential Equations*, 2nd ed. Boca Raton: CRC Press, 2003.
[9] Zwillinger, D.; *Handbook of Differential Equations*. New York: Academic Press, 1998.

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