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SOLUTION TO NAVIER-STOKES EQUATIONS FOR TURBULENT CHANNEL FLOWS

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Abstract. In this article, we continue the work done in [12] for turbulent channel flows described by the Navier-Stokes and the Navier-Stokes-α equations. We study non-stationary solutions in special function spaces. In particular, we show the term representing the sum of pressure and potential is harmonic in the space variable. We find an optimal choice for the function class.

1. Introduction

Turbulence is a fluid regime with the characteristics of being unsteady, irregular, seemingly random and chaotic [14]. It can be used for modeling the weather, ocean currents, water flow in a pipe and air flow around aircraft wings. Studying turbulent fluid flows involves some of the most difficult and fundamental problems in classical physics, and is also of tremendous practical importance. The Navier-Stokes equations (NSE) have been widely used to describe the motion of turbulent fluid flows [9, 19]. However, solving NSE using the direct numerical simulation method for turbulent flows is difficult, since accurate simulation of turbulent flows should account for the interactions of a wide range of scales which leads to high computational costs.

Turbulence modeling could provide qualitative and in some cases quantitative measures for many applications [18]. There are several types of turbulence modeling methods, for example Reynolds Average Navier-Stokes (RANS) and Large Eddy Simulation (LES). As done in [6, 7], we accept that the Navier-Stokes-α (NS-α) (also called viscous Camassa-Holm equations or Lagrangian averaged Navier-Stokes equations) is a well-suited mathematical model for the dynamics of appropriately averaged turbulent fluid flows. The possibility that the NS-α is an averaged version of the NSE, first considered in [3, 4], was entailed by several auspicious facts. Namely, the NS-α analogue of the Poiseuille, resp, Hagen, solution in a channel, resp, a pipe, displays both the classical Von Kármán and the recent Barenblatt-Chorin laws [16]. In addition, the NS-α analogue of the Hagen solution, when suitably calibrated, yields good approximations to many experimental data [4]. Moreover, the NS-α have been proved to have regular solutions [1]. Therefore, continuing the study of NS-α is extremely useful and important in the aspects of both mathematical theories and down-to-earth applications.

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To understand the connection for fluid flows described by NSE and by NS-α, in [12], we used a simple Reynolds type averaging. We restricted our consideration to channel flows having special function forms prescribed as a function class called $\mathcal{P}$. This function class $\mathcal{P}$ was inspired by the concept of regular part of the weak attractor of the 3D NSE ([10, 11]) as well as by that of the sigma weak attractor introduced in [2]. This led us to consider the solution for the channel flows whose averaged form has both the second and third velocity component to be zero. This will be our assumption for the discussion done in current work. Starting from there, a physical model for the wall roughness of the channel was subsequently provided to show that the NS-α model occurs naturally as the fluid flows. Moreover, by restricting to consider functions from $\mathcal{P}$, a rational explanation was given to facilitate the understanding of why, as the Reynolds number increases, the fluid becomes in favor of the NS-α model instead of the NSE. The class $\mathcal{P}$ was composed by five assumptions, each assumption plays an unique and important role.

In this article, we study the properties of solutions in class $\mathcal{P}$. We first try to find the explicit formula of the non-stationary solutions for NSE and NS-α. This particular solution has the form which only the first velocity component is nonzero. From there, we can recover the classic Poiseuille flow. Moreover, we prove the symmetric property of the integration form of this Poiseuille flow. Explicit and detailed energy estimate of the velocity field in class $\mathcal{P}$ is presented and is of use to show the connection between $\mathcal{P}$ and the weak global attractor of the equations. Moreover, for the sum of the pressure and potential, we also prove that it is actually harmonic in the space variable. Studying the properties of the sum of the pressure and potential, we find an alternative weaker condition of the last assumption in class $\mathcal{P}$. Therefore, we have found an optimal choice of the class $\mathcal{P}$.

The article is organized as follows. Section 2 gives elementary results on the Navier-Stokes equations and the Navier-Stokes-α model as well as the definition of the class $\mathcal{P}$. In section 3, we solve the channel flows whose velocity fields have a special form. Section 4 contains the energy estimate for solutions in class $\mathcal{P}$. In section 5, we discuss the harmonicity of the sum of the pressure and potential in the space variable and its consequences. The last section contains some basic inequalities together with their proofs.

2. Preliminaries

2.1. **Mathematical background.** Throughout, we consider an incompressible viscous fluid in an immobile region $\mathcal{O} \subset \mathbb{R}^3$ subjected to a potential body force $F = -\nabla\Phi$, with a time independent potential $\Phi = \Phi(x) \in C^\infty(\mathcal{O})$. The velocity field of such flows,

$$u = u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t)), \quad x = (x_1, x_2, x_3) \in \mathcal{O} \tag{2.1}$$

satisfies the NSE equation

$$\frac{\partial}{\partial t}u + (u \cdot \nabla)u = \nu\Delta u - \nabla P, \quad \nabla \cdot u = 0, \tag{2.2}$$

where $P := p + \Phi$, t denotes the time, $\nu > 0$ the kinematic viscosity, and $p = p(x,t)$ the pressure.

The NS-α equation is

$$\frac{\partial}{\partial t}v + (u\cdot\nabla)v + \sum_{j=1}^{3} v_j \nabla u_j = \nu\Delta v - \nabla Q, \quad \nabla\cdot u = 0, \tag{2.3}$$

where

$$v = (v_1, v_2, v_3) = (1-\alpha^2\Delta)u = ((1-\alpha^2\Delta)u_1, (1-\alpha^2\Delta)u_2, (1-\alpha^2\Delta)u_3), \tag{2.4}$$

and Q in (2.3) (like P in (2.2)) may depend on the time t.

We impose the following boundary conditions for both (2.2) and (2.3),

$$u(x,t) = 0, \text{ for } x \in \partial\mathcal{O} := \text{boundary of } \mathcal{O}\,. \tag{2.5}$$

One can observe that if $\alpha = 0$, the NS-α (2.3) reduce to NSE (2.2), so that (2.3) is also referred as an α-model of (2.2).

In the case of a channel flow, that is, $\mathcal{O} = \mathbb{R}\times\mathbb{R}\times[x_3^{(l)}, x_3^{(u)}]$, where $h := x_3^{(u)} - x_3^{(l)} > 0$ is the "height" of the channel, we recall that a vector of the form

$$(U(x_3), 0, 0) \tag{2.6}$$

is a stationary solution (i.e., time independent) of the NSE (2.2) if and only if

$$U(x_3) = b\Big(1 - \frac{(x_3 - \frac{x_3^{(u)}+x_3^{(l)}}{2})^2}{(h/2)^2}\Big), \quad x_3 \in [x_3^{(l)}, x_3^{(u)}], \tag{2.7}$$

where b is a constant velocity.

Moreover, $(U(x_3), 0, 0)$ is a stationary (i.e., time independent) solution of the NS-α (2.3) if and only if,

$$U(x_3) = a_1\Big(1 - \frac{\cosh\big((x_3 - \frac{x_3^{(u)}+x_3^{(l)}}{2})/\alpha\big)}{\cosh h/(2\alpha)}\Big) + a_2\Big(1 - \frac{(x_3 - \frac{x_3^{(u)}+x_3^{(l)}}{2})^2}{(h/2)^2}\Big), \tag{2.8}$$

for $x_3 \in [x_3^{(l)}, x_3^{(u)}]$, where a_1, a_2 are constant velocities (cf. [4, formula (9.6)]). Above, $\cosh(x) = \frac{e^x+e^{-x}}{2}$ is the hyperbolic cosine function.

To simplify our notation, we will assume that $x_3^{(l)} = 0$ and $x_3^{(u)} = h$.

2.2. **Class $\mathcal{P}$.** As in [12], a function $u(x,t)$ belongs to class $\mathcal{P}$ if it satisfies conditions (A1)–(A5) below.

(A1) $u(x,t) \in C^\infty(\mathcal{O}\times\mathbb{R})$.

(A2) $u(x,t)$ is periodic in x_1 and x_2, with periods Π_1 and Π_2, respectively, i.e.,

$$u(x_1+\Pi_1, x_2, x_3, t) = u(x_1, x_2, x_3, t), \quad u(x_1, x_2+\Pi_2, x_3, t) = u(x_1, x_2, x_3, t). \tag{2.9}$$

(A3) $u(x,t)$ exists for all $t\in\mathbb{R}$, and has bounded energy per mass, i.e.,

$$\int_0^{\Pi_1}\int_0^{\Pi_2}\int_0^h u(x,t)\cdot u(x,t)dx < \infty, \quad \forall t\in\mathbb{R}. \tag{2.10}$$

(A4) there exists a constant $\bar{p} < \infty$ for which

$$0 < -p_1(t) \le \bar{p}, \quad |p_2(t)| \le \bar{p},$$

for all $t\in\mathbb{R}$, where $p_1(t)$ and $p_2(t)$ are defined in (2.11).

(A5) $P = P(x,t)$ is bounded in the x_2 direction, i.e.,

$$\sup_{x_2\in\mathbb{R}} P(x_1, x_2, x_3, t) < \infty, \quad \forall x_1, x_3,\ t\in\mathbb{R}.$$

Remark 2.1. From (2.2) and (A2), it follows that

$$\begin{aligned} P(x_1+\Pi_1,x_2,x_3,t)-P(x_1,x_2,x_3,t) &=: p_1(t),\\ P(x_1,x_2+\Pi_2,x_3,t)-P(x_1,x_2,x_3,t) &=: p_2(t). \end{aligned} \tag{2.11}$$

Note that, as done in [12], one can show $p_2(t)=0$.

As mentioned in the introduction, we additionally assume that $u_3(x,t)=0$. The reality condition on u becomes, when viewed in the Fourier space,

$$\hat{u}^*(t;k_1,k_2,k)=\hat{u}(t;-k_1,-k_2,k), \tag{2.12}$$

and because $u_3=0$, the divergence free condition in(2.2) reduces to

$$\frac{2\pi k_1}{\Pi_1}\hat{u}_1+\frac{2\pi k_2}{\Pi_2}\hat{u}_2=0. \tag{2.13}$$

Using $u_3=0$, we see that (2.2) becomes

$$\begin{gathered} \frac{\partial}{\partial t}u_1+u_1\frac{\partial}{\partial x_1}u_1+u_2\frac{\partial}{\partial x_2}u_1-\nu(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2})u_1=-\frac{\partial}{\partial x_1}P,\\ \frac{\partial}{\partial t}u_2+u_1\frac{\partial}{\partial x_1}u_2+u_2\frac{\partial}{\partial x_2}u_2-\nu(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2})u_2=-\frac{\partial}{\partial x_2}P,\\ -\frac{\partial}{\partial x_3}P=0,\\ \frac{\partial}{\partial x_1}u_1+\frac{\partial}{\partial x_2}u_2=0. \end{gathered} \tag{2.14}$$

We define the Reynolds type average of a scalar function $\phi=\phi(x)$ as

$$\langle\phi\rangle(x_3):=\frac{1}{\Pi_1\Pi_2}\int_0^{\Pi_1}\int_0^{\Pi_2}\phi\,dx_2dx_1. \tag{2.15}$$

Proposition 2.2 ([12])**.** *For all $u(x,t)\in\mathcal{P}$, we have*

$$\langle u_2(t)\rangle(x_3)=0, \tag{2.16}$$

for all $x_3\in[0,h],t\in\mathbb{R}$.

Then the averaged velocity field takes the form

$$\langle u(t)\rangle(x_3)=\begin{pmatrix}\langle u_1(t)\rangle(x_3)\\ \langle u_2(t)\rangle(x_3)\\ \langle u_3(t)\rangle(x_3)\end{pmatrix}=\begin{pmatrix}\langle u_1(t)\rangle(x_3)\\ 0\\ 0\end{pmatrix}. \tag{2.17}$$

The following kernel representation of the averaged velocity component $\langle u_1(t)\rangle(x_3)$ is also given in [12].

Proposition 2.3 ([12])**.** *For $u(x,t)\in\mathcal{P}$ we have*

$$\langle u_1(t)\rangle(x_3)=\int_{-\infty}^t K(x_3,t-\tau)p_1(\tau)d\tau, \tag{2.18}$$

where, the kernel $K(x,t)$ is defined by the series,

$$K(x,t)=\sum_{k=1}^{\infty}\frac{2((-1)^k-1)}{\Pi_1 k\pi}e^{-\nu(\frac{\pi k}{h})^2t}\sin\frac{\pi kx}{h}. \tag{2.19}$$

3. Channel flows with velocity field of a particular form

As shown in Proposition 2.2, the averaged velocity field in the solution of (2.2) has a special form, namely, both the second and the third components vanish (see (2.17)). Thus, it is worth to consider the solutions with this form of the NSE and NS-α in a more general setting. See also (2.7) and (2.8) for time independent solutions of this form.

3.1. **The NSE case.**

Consider (2.2) with solution of the form

$$u = (U(x,t), 0, 0), \tag{3.1}$$

for $0 \le x_3 \le h$, satisfying assumptions (A1)–(A3).

Simple form of the NSE. Using the form (3.1) of u, NSE (2.2) becomes

$$\begin{gathered} \frac{\partial U}{\partial t} - \nu\Delta U + \frac{\partial P}{\partial x_1} = 0, \\ \frac{\partial P}{\partial x_2} = 0, \\ \frac{\partial P}{\partial x_3} = 0, \\ \frac{\partial U}{\partial x_1} = 0. \end{gathered} \tag{3.2}$$

Using the first and fourth equations in (3.2), we obtain that U is independent of x_1 and $\frac{\partial^2 P}{\partial x_1^2} = 0$, which combined with the second and third equations in (3.2) for P, imply that P must be of the form

$$P = P(x_1, t) = \tilde{p}_0(t) + x_1 \tilde{p}_1(t). \tag{3.3}$$

Therefore, $U = U(x_2, x_3, t)$ satisfies

$$\frac{\partial U}{\partial t} - \nu\Big(\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\Big)U = -\tilde{p}_1(t). \tag{3.4}$$

Solving (3.2)*.* Based on the periodicity (2.9) in (A2), we can expand U in Fourier series

$$U(x_2, x_3, t) = \sum_{n=-\infty}^{\infty} \hat{U}(n, x_3, t) e^{\frac{i2\pi n}{\Pi_2} x_2}, \tag{3.5}$$

where

$$\hat{U}(n, x_3, t) := \frac{1}{\Pi_2} \int_0^{\Pi_2} U(x_2, x_3, t) e^{-\frac{i2\pi n}{\Pi_2} x_2} dx_2,$$

in particular

$$\hat{U}(0, x_3, t) = \frac{1}{\Pi_2} \int_0^{\Pi_2} U(x_2, x_3, t) dx_2.$$

So equation (3.4) can be written as

$$\begin{aligned} &\sum_{n=-\infty}^{\infty} e^{\frac{i2\pi n}{\Pi_2} x_2} \Big[\frac{\partial}{\partial t}\hat{U}(n, x_3, t) - \nu - \hat{U}(n, x_3, t)(\frac{2\pi n}{\Pi_2})^2 + \frac{\partial^2}{\partial x_3^2}\hat{U}(n, x_3, t)\Big] \\ &= -\tilde{p}_1(t), \end{aligned} \tag{3.6}$$

where (2.5) implies the boundary condition

$$\hat{U}(n, x_3, t)|_{x_3=0,h} = 0. \tag{3.7}$$

We first consider the case when $n \neq 0$: Equation (3.6) implies

$$\frac{\partial}{\partial t}\hat{U}(n, x_3, t) + \nu(\frac{2\pi n}{\Pi_2})^2\hat{U}(n, x_3, t) - \nu\frac{\partial^2}{\partial x_3^2}\hat{U}(n, x_3, t) = 0. \tag{3.8}$$

Taking the dot product with $\hat{U}(n, x_3, t)$ in (3.8), and then integrating with respect to x_3 from 0 to h, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^h |\hat{U}(n, x_3, t)|^2 dx_3 + \nu(\frac{2\pi n}{\Pi_2})^2\int_0^h |\hat{U}(n, x_3, t)|^2 dx_3\\ &- \nu\int_0^h \frac{\partial^2}{\partial x_3^2}\hat{U}(n, x_3, t)\cdot\hat{U}(n, x_3, t)dx_3 = 0.\end{aligned}$$

Integrating by parts and applying the boundary condition (3.7), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^h |\hat{U}(n, x_3, t)|^2 dx_3 + \nu(\frac{2\pi n}{\Pi_2})^2\int_0^h |\hat{U}(n, x_3, t)|^2 dx_3\\ &+ \nu\int_0^h (\frac{\partial}{\partial x_3}\hat{U}(n, x_3, t))^2 dx_3 = 0.\end{aligned} \tag{3.9}$$

Using Poincaré inequality (6.1), Equation (3.9) becomes

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^h |\hat{U}(n, x_3, t)|^2 dx_3 + \nu(\frac{2\pi n}{\Pi_2})^2\int_0^h |\hat{U}(n, x_3, t)|^2 dx_3\\ &+ \nu\frac{1}{h^2}\int_0^h |\hat{U}(n, x_3, t)|^2 dx_3 \leq 0.\end{aligned}$$

Denoting $W(n, t) := \int_0^h |\hat{U}(n, x_3, t)|^2 dx_3$, we have

$$\frac{1}{2}\frac{d}{dt}W(n, t) + \nu(\frac{2\pi n}{\Pi_2})^2 W(n, t) + \nu\frac{1}{h^2}W(n, t) \leq 0,$$

which can be integrated to obtain

$$W(n, t) \leq e^{-2(t-t_0)(\nu(\frac{2\pi n}{\Pi_2})^2 + \frac{\nu}{h^2})}W(n, t_0), \quad \forall t > t_0. \tag{3.10}$$

By assumption (A3), $W(n, t_0)$ is bounded for all $t_0 \in \mathbb{R}$. Hence, by taking $t_0 \to -\infty$ in (3.10), we obtain $W(n, t) = 0$, for all $t \in \mathbb{R}$. Therefore, $\hat{U}(n, x_3, t) = 0$, for all $n \neq 0$.

So, we only need to consider the case when $n = 0$. In this case (3.6) implies

$$\frac{\partial}{\partial t}\hat{U}(0, x_3, t) - \nu\frac{\partial^2}{\partial x_3^2}\hat{U}(0, x_3, t) = -\tilde{p}_1(t), \tag{3.11}$$

and, the expansion of $U(x_2, x_3, t)$ in (3.5) becomes

$$U(x_2, x_3, t) = \hat{U}(0, x_3, t) = \frac{1}{\Pi_2}\int_0^{\Pi_2} U(x_2, x_3, t)dx_2. \tag{3.12}$$

that is, $U(x_2, x_3, t) = U(x_3, t)$ is independent of x_2.

The equation satisfied by $U(x_3, t)$ follows from equation (3.4) (or (3.11)):

$$\frac{\partial}{\partial t}U(x_3, t) - \nu\frac{\partial^2}{\partial x_3^2}U(x_3, t) = -\tilde{p}_1(t). \tag{3.13}$$

Since $U(0,t)=U(h,t)=0$, we can take the Fourier sine expansion for $U(x_3,t)$:

$$U(x_3,t)=\sum_{k=1}^{\infty}\hat{U}(k,t)\sin(\frac{\pi k x_3}{h}).$$

Equation (3.13) gives the following equation for the coefficients $\hat{U}(k,t)$:

$$\frac{\partial}{\partial t}\hat{U}(k,t)+\nu(\frac{\pi k}{h})^2\hat{U}(k,t)=-\frac{2}{h}\int_0^h \sin(\frac{\pi k x_3}{h})\cdot\tilde{p}_1(t)dx_3. \tag{3.14}$$

Thus, we obtain the explicit form $U(x_3,t)$ in the following theorem.

Theorem 3.1. *Let $u=(U(x,t),0,0)$ be a solution of the* (2.2) *with $P=P(x,t)$ satisfying* (A1)–(A3). *Then, $U=U(x,t)=U(x_3,t)$ and*

$$U(x_3,t)=\int_{-\infty}^{t}\sum_{k=1}^{\infty}\frac{2((-1)^k-1)}{k\pi}e^{-\nu(\frac{\pi k}{h})^2(t-\tau)}\sin(\frac{\pi k x_3}{h})\tilde{p}_1(\tau)d\tau. \tag{3.15}$$

Proof. Simplifying (3.14), we have

$$\frac{\partial}{\partial t}\hat{U}(k,t)+\nu(\frac{\pi k}{h})^2\hat{U}(k,t)=\frac{2}{k\pi}\tilde{p}_1(\tau)((-1)^k-1),$$

from which, upon integration, we obtain

$$\hat{U}(k,t)=e^{-\nu(\frac{\pi k}{h})^2(t-t_0)}\hat{U}(k,t_0)+\frac{2}{k\pi}((-1)^k-1)\int_{t_0}^{t}e^{-\nu(\frac{\pi k}{h})^2(t-\tau)}\tilde{p}_1(\tau)d\tau,$$

which, implies (3.15), by taking $t_0\to-\infty$. □

The above theorem shows that non-stationary solutions for (2.2) exist, and are given by (2.6) and (2.7). In particular, if $\tilde{p}_1(t)$ is a constant, the Poiseuille flow (time independent) is recovered. This matches the form given in (2.7).

Corollary 3.2. *If we assume that $\tilde{p}_1(t)=\tilde{p}_{10}$, where $\tilde{p}_{10}\in\mathbb{R}$ is a constant, then*

$$\begin{aligned}U(x_3,t)&=\sum_{k=1}^{\infty}\frac{2\tilde{p}_{10}h^2}{\nu(k\pi)^3}((-1)^k-1)\sin(\frac{\pi k x_3}{h})\\&=-\frac{\tilde{p}_{10}}{2\nu}x_3(h-x_3).\end{aligned} \tag{3.16}$$

3.1.1. *A symmetry property.* Consider the averaged quantity

$$\langle U\rangle_2(x_3,t)=\frac{1}{\Pi_2}\int_0^{\Pi_2}U(x_2,x_3,t)dx_2, \tag{3.17}$$

which, from (2.5), satisfies

$$\langle U\rangle_2(x_3,t)|_{x_3=0,h}=0. \tag{3.18}$$

We can prove that $\langle U\rangle_2(x_3,t)$ satisfies the following symmetry property, which is a priori assumption in [3]-[5] for the study of the steady solutions.

Theorem 3.3. *$\langle U\rangle_2(x_3,t)$ in* (3.17) *satisfies*

$$\langle U\rangle_2(h-x_3,t)=\langle U\rangle_2(x_3,t), \tag{3.19}$$

for all $x_3\in[0,h]$ and $t\in\mathbb{R}$.

Proof. By taking average in x_2 as defined in(3.17) of the equation (3.4), and invoking the periodicity condition (2.9), we see that $\langle U\rangle_2(x_3,t)$ must satisfy

$$\frac{\partial \langle U\rangle_2}{\partial t} - \nu \frac{\partial^2 \langle U\rangle_2}{\partial x_3^2} = -\tilde{p}_1(t). \tag{3.20}$$

Denoting

$$\widetilde{\langle U\rangle}(x_3,t) := \langle U\rangle_2(h-x_3,t) - \langle U\rangle_2(x_3,t), \tag{3.21}$$

we have

$$\frac{\partial \widetilde{\langle U\rangle}}{\partial t} - \nu \frac{\partial^2 \widetilde{\langle U\rangle}}{\partial x_3^2} = 0, \tag{3.22}$$

with boundary conditions from (3.18):

$$\widetilde{\langle U\rangle}(x_3,t)|_{x_3=0,h} = 0. \tag{3.23}$$

Taking the dot product of (3.22) with $\widetilde{\langle U\rangle}$ and then integrating with respect to x_3 from 0 to h, together with the boundary conditions (3.23), we have

$$\begin{aligned}
&\int_0^h \Big(\frac{\partial \widetilde{\langle U\rangle}}{\partial t} - \nu \frac{\partial^2 \widetilde{\langle U\rangle}}{\partial x_3^2}\Big)\widetilde{\langle U\rangle}(x_3,t)dx_3\\
&= \frac{1}{2}\frac{d}{dt}\int_0^h \Big(\widetilde{\langle U\rangle}(x_3,t)\Big)^2 dx_3 - \nu \int_0^h \frac{\partial^2 \widetilde{\langle U\rangle}}{\partial x_3^2}\widetilde{\langle U\rangle}(x_3,t)dx_3\\
&= \frac{1}{2}\frac{d}{dt}\int_0^h \Big(\widetilde{\langle U\rangle}(x_3,t)\Big)^2 dx_3 + \nu \int_0^h \Big(\frac{\partial}{\partial x_3}\widetilde{\langle U\rangle}(x_3,t)\Big)^2 dx_3 = 0.
\end{aligned}$$

Invoking the Poincaré inequality (6.1), we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^h (\widetilde{\langle U\rangle}(x_3,t))^2 dx_3 + \frac{\nu}{h^2}\int_0^h (\widetilde{\langle U\rangle}(x_3,t))^2 dx_3 \le 0. \tag{3.24}$$

Therefore, from (3.24) we obtain

$$\psi(t) \le \psi(t_0)e^{-(t-t_0)2\nu/h^2}, \tag{3.25}$$

for $-\infty < t_0 < t < \infty$, where $\psi(t) := \int_0^h (\widetilde{\langle U\rangle}(x_3,t))^2 dx_3$ is nonnegative for all $t \in \mathbb{R}$. Under the assumption (A3), we could let $t_0 \to -\infty$ in (3.25) to obtain $\psi(t) \equiv 0$ for all $t \in \mathbb{R}$. Hence, $\widetilde{\langle U\rangle}(x_3,t) \equiv 0$ for all $x_3 \in [0,h]$ and all $t \in \mathbb{R}$. Consequently, the symmetry property (3.19) on $\langle U\rangle_2(x_3,t)$ can be obtained. □

3.2. **The NS-α case.** We assume the following form of solution for (2.3),

$$u = (U(x,t),0,0) \tag{3.26}$$

and $V = (1-\alpha^2\Delta)U$, for $0 \le x_3 \le h$, with $u(x,t)$ satisfying (A1)–(A3).

Simple form of the NS-α. Using (3.26), equation (2.3) becomes

$$\begin{gathered}
\frac{\partial V}{\partial t} - \nu\Delta V + \frac{\partial Q}{\partial x_1} = 0,\\
V\frac{\partial U}{\partial x_2} = -\frac{\partial Q}{\partial x_2},\\
V\frac{\partial U}{\partial x_3} = -\frac{\partial Q}{\partial x_3},\\
\frac{\partial U}{\partial x_1} = 0.
\end{gathered} \tag{3.27}$$

Using the first and fourth equations in (3.27), we obtain

$$\frac{\partial^2 Q}{\partial x_1^2} = 0.$$

By taking partial derivative in the second and third equations in (3.27) with respect to x_1, we have

$$\frac{\partial^2 Q}{\partial x_1 \partial x_2} = \frac{\partial^2 Q}{\partial x_3 \partial x_1} = 0.$$

Thus, Q must be of the form

$$Q = Q(x_1, x_2, x_3, t) = \tilde{q}_0(x_2, x_3, t) + x_1 \tilde{q}_1(t). \tag{3.28}$$

Hence, the first equation in (3.27) can be further simplified to be

$$\frac{\partial V}{\partial t} - \nu\big(\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\big)V = -\tilde{q}_1(t), \tag{3.29}$$

which is strikingly similar to (3.4).

The no-slip boundary condition (2.5) implies

$$U(x_2, x_3, t)|_{x_3=0,h} = 0. \tag{3.30}$$

Solving (3.27). By no-slip boundary condition (3.30), one can write

$$U(x_2, x_3, t) = \sum_{n=-\infty}^{\infty} \hat{U}(n, x_3, t) e^{\frac{i2\pi n}{\Pi_2} x_2}.$$

Hence,

$$V = (1 - \alpha^2 \Delta)U = \Big(1 + \alpha^2 \big(\frac{2\pi n}{\Pi_2}\big)^2 - \alpha^2 \frac{\partial^2}{\partial x_3^2}\Big)U.$$

Similarly, we have the equations for $\hat{U}(n, x_3, t)$, which follow from (3.29): when $n = 0$

$$\Big(1 - \alpha^2 \frac{\partial^2}{\partial x_3^2}\Big)\big(\frac{\partial}{\partial t} - \nu \frac{\partial^2}{\partial x_3^2}\big)\hat{U}(n, x_3, t) = -\tilde{q}_1(t).$$

When $n \neq 0$

$$\Big(1 + \alpha^2 \big(\frac{2\pi n}{\Pi_2}\big)^2 - \alpha^2 \frac{\partial^2}{\partial x_3^2}\Big)\Big(\frac{\partial}{\partial t} + \nu \big(\frac{2\pi n}{\Pi_2}\big)^2 - \nu \frac{\partial^2}{\partial x_3^2}\Big)\hat{U}(n, x_3, t) = 0. \tag{3.31}$$

Using arguments similar to those in the previous section, we obtain $\hat{U}(n, x_3, t) = 0$, when $n \neq 0$, so $U = \hat{U}(0, x_3, t) = U(x_3, t)$ is also independent of x_2, and satisfies,

$$\Big(1 - \alpha^2 \frac{\partial^2}{\partial x_3^2}\Big)\Big(\frac{\partial}{\partial t} - \nu \frac{\partial^2}{\partial x_3^2}\Big)\hat{U}(0, x_3, t) = -\tilde{q}_1(t). \tag{3.32}$$

From (3.32) and (3.3), we obtain the following theorem.

Theorem 3.4. *Let* $u = (U(x,t), 0, 0)$ *be a solution of the NSE*(2.2) *with* $P = P(x,t)$ *given, satisfying* (A1)–(A3). *Then,* u *is also a solution of the NS-*α (2.3) *with*

$$Q = Q(x,t) = \tilde{p}_1(t)x_1 - \frac{1}{2}\Big(U^2 - \alpha^2 \big(\frac{\partial U}{\partial x_3}\big)^2\Big).$$

We apply

$$U = U(x_3, t) = \sum_{k=1}^{\infty} \hat{U}(k,t) \sin\big(\frac{\pi k x_3}{h}\big) = \sum_{k=1}^{\infty} \hat{U}(k,t) \sin\big(\frac{\pi k x_3}{h}\big).$$

From (3.32), it follows that the Fourier coefficient $\hat{U}(k,t)$ satisfies:

$$\Big(1 + \alpha^2 (\frac{\pi k}{h})^2\Big)\Big(\frac{\partial}{\partial t}\hat{U}(k,t) + \nu(\frac{\pi k}{h})^2 \hat{U}(k,t)\Big) = -\frac{2}{h}\int_0^h \sin\big(\frac{\pi k x_3}{h}\big)\tilde{q}_1(t)dx_3. \quad (3.33)$$

Hence, using the same procedure as in previous section, we obtain the following form of the solution $U = U(x_3, t)$.

Theorem 3.5. *The solution of the NSE(2.2) has the form*

$$U(x_3,t) = \int_{-\infty}^{t} \sum_{k=1}^{\infty} \frac{2((-1)^k - 1)}{\pi k(1 + \alpha^2(\pi k/h)^2)} e^{-\nu(\frac{\pi k}{h})^2(t-\tau)} \sin\big(\frac{\pi k x_3}{h}\big)\tilde{q}_1(\tau)d\tau. \quad (3.34)$$

The above theorem shows that non-stationary solutions for NS-α (2.3) exist, besides those stationary solutions given by (2.6) and (2.8). In the particular case when $\tilde{q}_1(t)$ is a constant, we obtain the steady state solution mentioned in [3]-[5], which is basically of the form (2.8).

Note that the coefficients in (3.35) satisfy [4, condition (9.7)] with $c = 0$ and $d_0 = h/2$, but there are typos in (9.7) there, namely, the left hand sides of the second and the third relations should be, respectively, π_0/ν and π_2/ν.

Corollary 3.6. *If we assume $\tilde{q}_1(t) = \tilde{q}_{10}$, where $\tilde{q}_{10} \in \mathbb{R}$ is a constant, then*

$$\begin{aligned} U(x_3,t) &= \sum_{k=1}^{\infty} \frac{2h^2}{\nu(\pi k)^3} \frac{1}{(1+\alpha^2(\pi k/h)^2)} \tilde{q}_{10}((-1)^k - 1) \sin\big(\frac{\pi k x_3}{h}\big) \\ &= \frac{\alpha^2 \tilde{q}_{10}}{\nu}\Big(1 - \frac{\cosh(\frac{x_3 - h/2}{\alpha})}{\cosh(\frac{h}{2\alpha})}\Big) - \frac{\tilde{q}_{10}}{2\nu} x_3(h - x_3). \end{aligned} \quad (3.35)$$

Proof. Equality (3.35) follows from applying

$$x(h-x) = \sum_{k=1}^{\infty} \frac{4h^2(1-(-1)^k)}{(\pi k)^3} \sin\big(\frac{\pi k x}{h}\big),$$

and

$$\cosh(\frac{h}{2\alpha}) - \cosh(\frac{x - h/2}{\alpha}) = \sum_{k=1}^{\infty} \frac{2\cosh(h/2\alpha)}{\pi k(1+\alpha^2(\pi k/h)^2)}(1-(-1)^k)\sin(\frac{\pi k x}{h}).$$

□

Remark 3.7. From Corollary 3.6, we observe that all solutions having the form (3.26) of the NS-α(2.3) with time independent $\partial Q/\partial x_1$ and satisfying (A1)–(A3) are actually stationary solutions.

4. Energy estimates

In this Section, we will use inequality (6.6) in Lemma 6.2 to obtain an inequality estimating the energy of the velocity field $u(x,t) \in \mathcal{P}$.

Taking the dot product of (2.2) with u and integrating over $\Omega := [0,\Pi_1] \times [0,\Pi_2] \times [0,h]$, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 dx + \nu \int_\Omega |\nabla u|^2 dx = -\sum_{j=1}^{3}\int_\Omega \frac{\partial P}{\partial x_j} u_j dx. \tag{4.1}$$

Note that here, the nonlinear term $\int_\Omega (u\cdot\nabla)u\cdot u dx$ vanishes. Indeed, using integration by parts and the periodicity conditions (A2), one gets, for $j = 1,2$,

$$\int_0^{\Pi_j} u_k \frac{\partial}{\partial x_j}(u_j u_k) dx_j = -\int_0^{\Pi_j} u_j u_k \frac{\partial}{\partial x_j} u_k dx_j, \quad \text{for } k = 1,2,3,$$

similarly, using the boundary condition (2.5) and integration by parts, we obtain

$$\int_0^{h} u_k \frac{\partial}{\partial x_3}(u_3 u_k) dx_3 = -\int_0^{h} u_3 u_k \frac{\partial}{\partial x_3} u_k dx_3, \text{for } k = 1,2,3.$$

Therefore,

$$\begin{aligned}
\int_\Omega (u\cdot\nabla)u\cdot u\,dx &= \int_\Omega \sum_{k=1}^{3}\sum_{j=1}^{3} u_j(\frac{\partial}{\partial x_j}u_k)u_k dx \\
&= \int_\Omega \sum_{k=1}^{3}\sum_{j=1}^{3} u_k \frac{\partial}{\partial x_j}(u_j u_k) dx \\
&= -\sum_{k=1}^{3}\int_\Omega \Big(u_1 u_k \frac{\partial}{\partial x_1}u_k + u_2 u_k \frac{\partial}{\partial x_2}u_k + u_3 u_k \frac{\partial}{\partial x_3}u_k\Big) dx \\
&= -\sum_{k=1}^{3}\int_\Omega u_k \sum_{j=1}^{3} u_j \frac{\partial}{\partial x_j} u_k dx \\
&= -\int_\Omega (u\cdot\nabla)u\cdot u dx.
\end{aligned}$$

Hence

$$\int_\Omega (u\cdot\nabla)u\cdot u dx = 0.$$

Remark 4.1. Observe that the above proof can be applied to show

$$\int_\Omega (u\cdot\nabla)v\cdot v dx = 0, \tag{4.2}$$

for $u,v \in \mathcal{P}$.

We have the following energy estimates.

Proposition 4.2. *For* $u(x,t) \in \mathcal{P}$,

$$\begin{aligned} &\frac{d}{dt}\int_\Omega |u|^2 dx + 2\nu \int_\Omega \sum_{k,l=1}^{3} (\frac{\partial u_k}{\partial x_l})^2 dx - \nu \int_\Omega \Big((\frac{\partial u_1}{\partial x_1})^2 + (\frac{\partial u_2}{\partial x_2})^2 \Big) dx \\ &\le \frac{\bar{p}^2 \Pi_1 \Pi_2 h}{6\nu} + \bar{p}^{3/2}(\Pi_1 + \Pi_2) h + \bar{p}^{1/2} \sum_{j=1}^{2} \int_0^{\Pi_{j'}} \int_0^h \langle u_j \rangle_j^2 dx_3 dx_{j'}. \end{aligned} \tag{4.3}$$

Proof. For the term on the right hand side of (4.1), by (2.11), we have

$$\begin{aligned} \int_\Omega \frac{\partial P}{\partial x_1} u_1 dx &= \int_0^{\Pi_2} \int_0^h \Big((Pu_1)|_{x_1=\Pi_1} - (Pu_1)|_{x_1=0} - \int_0^{\Pi_1} P \frac{\partial u_1}{\partial x_1} dx_1 \Big) dx_3 dx_2 \\ &= p_1(t) \int_0^{\Pi_2} \int_0^h u_1|_{x_1=0} dx_3 dx_2 - \int_\Omega P \frac{\partial u_1}{\partial x_1} dx. \end{aligned}$$

Similarly,

$$\int_\Omega \frac{\partial P}{\partial x_2} u_2 dx = p_2(t) \int_0^{\Pi_1} \int_0^h u_2|_{x_2=0} dx_3 dx_1 - \int_\Omega P \frac{\partial u_2}{\partial x_2} dx.$$

From the no-slip boundary condition (2.5),

$$\int_\Omega \frac{\partial P}{\partial x_3} u_3 dx = - \int_\Omega P \frac{\partial u_3}{\partial x_3} dx.$$

Therefore,

$$\begin{aligned} &- \int_\Omega \nabla P \cdot u dx \\ &= \int_\Omega P \nabla \cdot u dx - p_1(t) \int_0^{\Pi_2} \int_0^h u_1|_{x_1=0} dx_3 dx_2 - p_2(t) \int_0^{\Pi_1} \int_0^h u_2|_{x_2=0} dx_3 dx_1 \\ &= -p_1(t) \int_0^{\Pi_2} \int_0^h u_1|_{x_1=0} dx_3 dx_2 - p_2(t) \int_0^{\Pi_1} \int_0^h u_2|_{x_2=0} dx_3 dx_1, \end{aligned}$$

where in the last line, the incompressibility condition (i.e., the second equation in (2.2)) is used.

Therefore, using (6.6) in Lemma 6.2, relations (2.11), and denoting $j' = 3 - j$ for $j = 1, 2$, (4.1) becomes

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 dx + \nu \int_\Omega |\nabla u|^2 dx \\ &= -p_1(t) \int_0^{\Pi_2} \int_0^h u_1|_{x_1=0} dx_3 dx_2 - p_2(t) \int_0^{\Pi_1} \int_0^h u_2|_{x_2=0} dx_3 dx_1 \\ &\le \sum_{j=1}^{2} |p_j(t)| \int_0^{\Pi_{j'}} \int_0^h \Big(\langle u_j \rangle_j + \frac{\Pi_j}{2\sqrt{3}} \langle (\frac{\partial u_j}{\partial x_j})^2 \rangle_j^{1/2} \Big) dx_3 dx_{j'} \\ &= \sum_{j=1}^{2} |p_j(t)| \int_0^{\Pi_{j'}} \int_0^h \Big(\langle u_j \rangle_j + \frac{{\Pi_j}^{1/2}}{2\sqrt{3}} \Big(\int_0^{\Pi_j} (\frac{\partial u_j}{\partial x_j})^2 dx_j \Big)^{1/2} \Big) dx_3 dx_{j'}, \end{aligned} \tag{4.4}$$

where $\langle \cdot \rangle_j$ denotes the average in the x_j direction, i.e.,

$$\langle \cdot \rangle_j := \frac{1}{\Pi_j} \int_0^{\Pi_j} \cdot \, dx_j. \tag{4.5}$$

We then use Young's inequality and (A4) to obtain

$$\begin{aligned}
&|p_j(t)| \int_0^{\Pi_{j'}} \int_0^h \frac{\Pi_j{}^{1/2}}{2\sqrt{3}} \Big(\int_0^{\Pi_j} (\frac{\partial u_j}{\partial x_j})^2 dx_j \Big)^{1/2} dx_3 dx_{j'} \\
&\le \frac{\bar p^2 \Pi_1 \Pi_2 h}{24\nu} + \frac{\nu}{2} \int_\Omega (\frac{\partial u_j}{\partial x_j})^2 dx, \quad j = 1, 2.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{d}{dt} \int_\Omega |u|^2 dx + 2\nu \int_\Omega \sum_{k,l=1}^3 (\frac{\partial u_k}{\partial x_l})^2 dx - \nu \int_\Omega \Big((\frac{\partial u_1}{\partial x_1})^2 + (\frac{\partial u_2}{\partial x_2})^2 \Big) dx \\
&\le \frac{\bar p^2 \Pi_1 \Pi_2 h}{6\nu} + 2\bar p \sum_{j=1}^2 \int_0^{\Pi_{j'}} \int_0^h \langle u_j \rangle_j dx_3 dx_{j'} \\
&\le \frac{\bar p^2 \Pi_1 \Pi_2 h}{6\nu} + 2\bar p \sum_{j=1}^2 (\Pi_{j'} h)^{1/2} \Big(\int_0^{\Pi_{j'}} \int_0^h \langle u_j \rangle_j^2 dx_3 dx_{j'} \Big)^{1/2} \\
&\le \frac{\bar p^2 \Pi_1 \Pi_2 h}{6\nu} + \bar p^{3/2} (\Pi_1 + \Pi_2) h + \bar p^{1/2} \sum_{j=1}^2 \int_0^{\Pi_{j'}} \int_0^h \langle u_j \rangle_j^2 dx_3 dx_{j'}.
\end{aligned}$$

□

Remark 4.3. Following the general procedure in [8], one can start from (4.4) and show the existence of the weak global attractor of the NSE (2.2).

5. Harmonicity of $P = P(x, t)$ in the space variables

5.1. **Harmonicity of P.** The following property of P, namely, harmonicty in the space variable, could be deduced. Recall that $P = p + \Phi$, where p is the pressure and Φ is the potential. Notice that from (2.14), P is independent of x_3.

Lemma 5.1. *Let $u(x, t) \in \mathcal{P}$, then $P = P(x_1, x_2, t)$ is harmonic in the space variables x_1 and x_2.*

Proof. From (2.14), by taking $\partial/\partial x_1$ in the first equation and $\partial/\partial x_2$ in the second equation and then summing the two resulting equations, we can obtain

$$\Big(\frac{\partial u_1}{\partial x_1}\Big)^2 + 2\frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1} + \Big(\frac{\partial u_2}{\partial x_2}\Big)^2 = -\Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\Big) P, \tag{5.1}$$

where $P = P(x_1, x_2, t)$ is independent of x_3 (see the third equation in (2.14)).

In (5.1), the left-hand side (LHS) takes values zero at $x_3 = 0$ and $x_3 = h$, while the right hand side (RHS) is independent of x_3, hence

$$-\Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\Big) P = 0, \tag{5.2}$$

for all $x_1, x_2 \in \mathbb{R}$. □

Now we have an intriguing corollary for the harmonicity of P.

Corollary 5.2.

$$\frac{\partial(u_1, u_2)}{\partial(x_1, x_2)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \end{vmatrix} = 0. \tag{5.3}$$

Proof. Equation (5.3) is obtained by using the fourth relation in (2.14), and the fact that the LHS in (5.1) equals zero. □

Remark 5.3. For $u = (u_1, u_2, u_3)$ being the velocity field for channel flows, after we assume $u_3 = 0$. Corollary 5.2 tells us that the two nonzero components, u_1 and u_2, are not totally independent, one of them is, at least locally, a function of the other component.

5.2. **An estimate of P.**

Lemma 5.4. *For the term $P = P(x_1, x_2, t)$, we have*

$$\sup_{x_1,x_2} \big|\frac{\partial}{\partial x_1} P(x_1, x_2, t)\big| < \infty, \tag{5.4}$$

for all $t \in \mathbb{R}$.

Proof. According to Poisson's formula [15, 17] we have, for any $a > 0$,

$$P(z,t) = \int_{|y|=a} H(y,z)P(y,t)dy, \quad \text{for } |z| < a,$$

where $z = (x_1, x_2)$, and

$$H(y,z) = \frac{1}{2\pi a}\frac{a^2 - |z|^2}{|z-y|^2}.$$

Therefore,

$$\begin{aligned}
P(z,t) &= P(x_1,x_2,t)\\
&= \frac{1}{2\pi}\int_{|y|=1} \frac{1-|\frac{z}{a}|^2}{|\frac{z}{a}-y|^2} P(ay_1, ay_2, t)dy\\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{1-\frac{x_1^2+x_2^2}{a^2}}{1-2(\frac{x_1^2+x_2^2}{a^2})^{1/2}\cos(\theta-\omega)+\frac{x_1^2+x_2^2}{a^2}} P(a\cos\theta, a\sin\theta, t)d\theta,
\end{aligned} \tag{5.5}$$

where $x_1 + ix_2 = |z|e^{i\omega}$ and $y_1 + iy_2 = |y|e^{i\theta}$.

However, we will work with the following form which equivalent to (5.5),

$$P(x_1,x_2,t) = \frac{1}{2\pi}\int_0^{2\pi} H(z/a,\theta)P(a\cos\theta, a\sin\theta, t)d\theta, \tag{5.6}$$

where, for $|z| < 1$,

$$H(z,\theta) = \frac{1-x_1^2-x_2^2}{(x_1-\cos\theta)^2+(x_2-\sin\theta)^2} = \frac{1-x_1^2-x_2^2}{1+x_1^2+x_2^2-2(x_1\cos\theta+x_2\sin\theta)}.$$

A direct calculation yields

$$\frac{\partial}{\partial x_1}H(x_1,x_2,\theta) = \frac{-4x_1+2\cos\theta+2x_1^2\cos\theta-2x_2x_2^2\cos\theta+4x_1x_2\sin\theta}{(1+x_1^2+x_2^2-2(x_1\cos\theta+x_2\sin\theta))^2}.$$

For $a > 4|z| + 1$, this implies

$$\begin{aligned}
\frac{\partial}{\partial x_1}H(z/a,\theta) &= \frac{1}{a}\frac{\partial H}{\partial x_1}(z,\theta)\big|_{z=z/a}\\
&\le \frac{4(\frac{|x_1|}{a}+2\frac{|z|^2}{a^2}+1)}{a(1+\frac{|z|^2}{a^2}-2\frac{|z|}{a})^2} \le 32.
\end{aligned}$$

Therefore, from (5.6), recalling the bound (5.7) given in Lemma 5.5, we have

$$\Big|\frac{\partial}{\partial x_1}P(x_1,x_2,t)\Big| \le \frac{32\bar{p}}{\Pi_1} + 32a\sup\{|P(x_1,x_2,t)| : 0\le x_1<\Pi_1\},$$

where $\sup\{|P(x_1,x_2,t)| : 0\le x_1<\Pi_1\}$ is a periodic function in x_2 with period Π_2, and hence

$$\max_{x_2\in\mathbb{R}}\sup\{|P(x_1,x_2,t)| : 0\le x_1<\Pi_1\}<\infty.$$

Consequently,

$$\sup_{x_1,x_2}\Big|\frac{\partial}{\partial x_1}P(x_1,x_2,t)\Big|<\infty,$$

for all $t\in\mathbb{R}$. □

5.3. **Simple form of P.** From (2.11), we observe that

$$P(x_1+n\Pi_1,x_2,t)-P(x_1,x_2,t)=np_1(t),\quad \forall n\in\mathbb{N}^+.$$

Now, for any $y\in\mathbb{R}^+$, choose $n\in\mathbb{N}$, such that $n\Pi_1\le y<(n+1)\Pi_1$, then

$$\begin{aligned}P(y,x_2,t)&=P(y-n\Pi_1,x_2,t)+np_1(t)\\&\le \sup\{|P(x_1,x_2,t)| : 0\le x_1<\Pi_1\}+np_1(t)\\&\le \sup\{|P(x_1,x_2,t)| : 0\le x_1<\Pi_1\}+\frac{y}{\Pi_1}\bar{p}.\end{aligned}$$

Similar arguments apply to the case when $y\le 0$, and we obtain the next result.

Lemma 5.5. *Let $u(x,t)\in\mathcal{P}$ be a solution of* (2.2). *Then*

$$P(y,x_2,t)\le \sup\{|P(x_1,x_2,t)| : 0\le x_1<\Pi_1\}+\frac{|y|}{\Pi_1}\bar{p} \tag{5.7}$$

for all $y,x_2,t\in\mathbb{R}$, where $\bar{p}$ is as given in (A4).

Next, we explore the harmonicity of $P=P(x,t)=P(x_1,x_2,t)$ in x_1 and x_2, and get the following explicit formula for $P(x,t)$, namely, $P(x,t)$ is linear in the variable x_1.

Lemma 5.6. *The term $P=P(x_1,x_2,t)$ in* (2.14) *is of the form*

$$P(x_1,x_2,t)=\tilde{p}_0(t)+x_1\tilde{p}_1(t)=\tilde{p}_0(t)+x_1p_1(t)/\Pi_1. \tag{5.8}$$

Proof. Using (5.4) in Lemma 5.4 given in Appendix C, and Liouville's theorem for harmonic function $\frac{\partial}{\partial x_1}P(x_1,x_2,t)$, we conclude that

$$\frac{\partial}{\partial x_1}P(x_1,x_2,t)=\tilde{p}_1(t),$$

for some function $\tilde{p}_1(t)$ of time t, so that

$$P=P(x_1,x_2,t)=\tilde{p}_0(x_2,t)+x_1\tilde{p}_1(t), \tag{5.9}$$

and, by the harmonicity of $P(x_1,x_2,t)$ in x_1 and x_2, we have

$$\frac{\partial^2}{\partial x_2^2}\tilde{p}_0(x_2,t)=0.$$

Therefore, $\tilde{p}_0(x_2,t)$ is a linear function in x_2, but then periodicity of $P(x_1,x_2,t)$ in x_2 would imply that $\tilde{p}_0(x_2,t)$ is only a function of time t, i.e.,

$$\tilde{p}_0(x_2,t)=\tilde{p}_0(t).$$

Finally, the second equality in (5.8), namely, follows from (5.9) and relation (2.11). ☐

It follows from the harmonicity of P and Lemma 5.6 that we can replace (A5) by a weaker condition.

Corollary 5.7. *Condition* (A5) *can be replaced by the f weaker condition*

$$\limsup_{x_2\to\pm\infty} P(x_1,x_2,x_3,t)<\infty, \tag{5.10}$$

for any given x_1, x_3 and $t\in\mathbb{R}$.

We have found an optimal choice of the function class.

6. Appendix

6.1. **Classic inequalities.** In this appendix, we include several classic inequalities that are used in our discussion, together with their proofs.

Lemma 6.1 (Poincaré's inequality)**.** *For any C^1 function $\phi(y)$ defined on $[0,h]$, with $\phi(0)=\phi(h)=0$, we have*

$$\int_0^h (\phi'(y))^2dy \ge \frac{1}{h^2}\int_0^h (\phi(y))^2dy. \tag{6.1}$$

Proof. By the fundamental theorem of calculus, we have, for any $x\in\mathbb{R}$, and $x\in[0,h]$,

$$\phi^2(x)=2\int_0^x \phi(y)\phi'(y)dy, \tag{6.2}$$

and

$$\phi^2(x)=-2\int_x^h \phi(y)\phi'(y)dy. \tag{6.3}$$

Using Cauchy inequality in (6.2), we obtain

$$\phi^2(x)\le 2(\int_0^x \phi^2(y)dy)^{1/2}(\int_0^x (\phi'(y))^2dy)^{1/2}.$$

Hence,

$$\begin{aligned}
\int_0^{h/2}\phi^2(x)dx &\le 2\int_0^{h/2}\Big(\int_0^x \phi^2(y)dy\Big)^{1/2}\Big(\int_0^x (\phi'(y))^2dy\Big)^{1/2}dx\\
&\le 2\int_0^{h/2}\Big(\int_0^{h/2} \phi^2(y)dy\Big)^{1/2}\Big(\int_0^{h/2} (\phi'(y))^2dy\Big)^{1/2}dx\\
&\le h\Big(\int_0^{h/2} \phi^2(y)dy\Big)^{1/2}\Big(\int_0^{h/2} (\phi'(y))^2dy\Big)^{1/2};
\end{aligned}$$

that is,

$$\int_0^{h/2}\phi^2(x)dx\le h^2\int_0^{h/2}(\phi'(y))^2dy. \tag{6.4}$$

Similarly, using (6.3), and repeating the above steps, we have

$$\int_{h/2}^{h}\phi^2(x)dx\le h^2\int_{h/2}^{h}(\phi'(y))^2dy. \tag{6.5}$$

Combined (6.4) and (6.5), we obtain (6.1). ☐

L^∞ inequality. Recall that, in our paper, for a given function $\phi = \phi(y)$ with periodicity $\Pi > 0$, we denote its average by $\langle\phi\rangle$, i.e.,

$$\langle\phi\rangle := \frac{1}{\Pi}\int_0^\Pi \phi(y)dy.$$

Lemma 6.2. *For any continuous function $\phi = \phi(y)$ with period $\Pi > 0$, it holds that*

$$|\phi|_{L^\infty} \le \langle\phi\rangle + \frac{\Pi}{2\sqrt{3}}\langle(\phi')^2\rangle^{1/2}. \tag{6.6}$$

Consequently, if

$$\langle(\phi')^2\rangle < \infty,$$

then ϕ is continuous in $\mathbb{R}$, and thus $|\phi(y)| \le |\phi|_{L^\infty}, \forall y \in \mathbb{R}$.

Proof. Without loss of generality, we assume $\phi(0) = |\phi|_{L^\infty}$, then

$$\phi(0) \le \begin{cases} \phi(y) + \int_0^y |\phi'(z)|dz, & y \ge 0,\\ \phi(y) + \int_y^0 |\phi'(z)|dz, & y \le 0. \end{cases}$$

Thus,

$$\begin{aligned} \frac{\Pi}{2}\phi(0) &\le \begin{cases} \int_0^{\Pi/2}\phi(y)dy + \int_0^{\Pi/2}(\int_z^{\Pi/2} dy)|\phi'(z)|dz,\\ \int_{-\Pi/2}^0 \phi(y)dy + \int_{-\Pi/2}^0(\int_{-\Pi/2}^z dy)|\phi'(z)|dz, \end{cases}\\ &= \begin{cases} \int_0^{\Pi/2}\phi(y)dy + \int_0^{\Pi/2}(\Pi/2 - z)|\phi'(z)|dz,\\ \int_{-\Pi/2}^0 \phi(y)dy + \int_{-\Pi/2}^0(\Pi/2 + z)|\phi'(z)|dz. \end{cases} \end{aligned}$$

Hence,

$$\begin{aligned} \Pi|\phi|_{L^\infty} &= \Pi\phi(0)\\ &\le \Pi\langle\phi\rangle + \Big(\int_0^{\Pi/2}(\Pi/2 - z)^2dz\Big)^{1/2}\Big(\int_0^{\Pi/2}(\phi'(z))^2dz\Big)^{1/2}\\ &\quad + \Big(\int_{-\Pi/2}^0(\Pi/2 + z)^2dz\Big)^{1/2}\Big(\int_{-\Pi/2}^0(\phi'(z))^2dz\Big)^{1/2}\\ &\le \Pi\langle\phi\rangle + \frac{\Pi^2}{2\sqrt{3}}\langle(\phi')^2\rangle^{1/2}. \end{aligned}$$

Then, (6.6) follows. □

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