Electronic Journal of Differential Equations, Vol. 2020 (2020), No. 21, pp. 1–17.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

MAXIMUM AND ANTIMAXIMUM PRINCIPLES FOR THE p-LAPLACIAN WITH WEIGHTED STEKLOV BOUNDARY CONDITIONS

MABEL CUESTA, LIAMIDI LEADI, PASCALINE NSHIMIRIMANA

Abstract. We study the maximum and antimaximum principles for the p-Laplacian operator under Steklov boundary conditions with an indefinite weight

$$
\begin{gathered}
-\Delta_p u + |u|^{p-2}u = 0 \quad \text{in } \Omega, \\
|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda m(x)|u|^{p-2}u + h(x) \quad \text{on } \partial\Omega,
\end{gathered}
$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$, $N > 1$. After reviewing some elementary properties of the principal eigenvalues of the p-Laplacian under Steklov boundary conditions with an indefinite weight, we investigate the maximum and antimaximum principles for this problem. Also we give a characterization for the interval of the validity of the uniform antimaximum principle.

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^N$ of class $C^{2,\alpha}$ for some $0 < \alpha < 1$, $N \geq 1$. We consider the quasilinear problem

$$
\begin{gathered}
-\Delta_p u + |u|^{p-2}u = 0 \quad \text{in } \Omega, \\
|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda m(x)|u|^{p-2}u + h(x) \quad \text{on } \partial\Omega.
\end{gathered} \tag{1.1}
$$

Here $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the well known p-Laplacian operator, $1 < p < \infty$; m and h are given functions in $C^r(\partial\Omega)$ for some $0 < r < 1$. The weight m can change sign, and $h \geq 0$, $h \not\equiv 0$. We denote by $\nu = \nu(x)$ the outer normal at x, defined for all $x \in \partial\Omega$ and by σ the restriction to $\partial\Omega$ of the $(N-1)$-Hausdorff measure, which coincides with the usual Lebesgue surface measure as $\partial\Omega$ is regular enough. All the integral along $\partial\Omega$ will be understood with respect to the measure σ.

Problems of the form (1.1) appears in several branches of pure and applied mathematics, such as the theory of quasiregular and quasiconformal mappings in Riemannian manifolds with boundary, non-Newtonian fluids, reaction diffusion problems, flow through porous media, nonlinear elasticity, glaciology, etc.

The maximum and antimaximum principles for problem (1.1) with $m \equiv 1$, have been studied in [3]. The authors proved that every solution of (1.1) is positive if

2010 *Mathematics Subject Classification.* 35J70.
Key words and phrases. p-Laplacian; Steklov boundary conditions: indefinite weight; maximum and antimaximum principles.

Submitted November 18, 2019. Published March 2, 2020.

$\lambda \in (0, \lambda_1)$ (maximum principle) and that there exists $\delta = \delta(h) > 0$ such that, if $\lambda \in (\lambda_1, \lambda_1 + \delta)$, then every weak solution is negative (antimaximum principle). Here λ_1 denotes the smallest eigenvalue of the eigenvalue problem associated with (1.1) with $m \equiv 1$. The authors in [3] also characterize the interval of validity of the uniform antimaximum principle. A uniform antimaximum principle has also been proved in [4, 10, 11] for the p-Laplacian operator with Neumann boundary conditions.

Our main aim in this work is to extend the results proved in [3] to the problem (1.1) when the weight m is indefinite. Let us define the number

$$\hat{\lambda}_1(m) := \inf \Big\{ \|u\|_{1,p}^p ; \int_{\partial\Omega} m|u|^p = 1 \text{ and } u \in Q \Big\}$$

where

$$Q := \big\{ u \in W^{1,p}(\Omega); \exists B(x_0, \delta) \text{ s.t } u_{|_{B(x_0,\delta)\cap\overline{\Omega}}} \equiv 0 \text{ a.e.} \big\}.$$

The norm $\|\cdot\|_{1,p}$ stand here for the natural norm of $W^{1,p}(\Omega)$. We prove in Theorem 4.4 that the real number $\hat{\lambda}_1(m)$ provides an interval of validity of the uniform antimaximum principle for (1.1) to the right of $\lambda_1(m)$, where $\lambda_1(m)$ is the first positive eigenvalue of the eigenvalue problem associated with (1.1). We point out here that $\bar{\lambda}_1(m) \le \hat{\lambda}_1(m)$, where $\bar{\lambda}_1(m)$ is the real number found in [3] for the validity of the uniform antimaximum principle in the case $m \equiv 1$, given by

$$\bar{\lambda}_1(m) := \inf \Big\{ \|u\|_{1,p}^p ; \int_{\partial\Omega} |u|^p = 1 \text{ and } u \text{ vanishes in a ball of } \overline{\Omega} \Big\}.$$

We will also prove that $\hat{\lambda}_1(m) = \lambda_1(m)$ if $1 < p \le N$ and $\hat{\lambda}_1(m) > \lambda_1(m)$ if $p > N$. Furthermore, we prove in Theorem 4.5 that the value $\hat{\lambda}_1(m)$ is the *greater number* $\sigma > \lambda_1(m)$ *such that the uniform antimaximun principle holds* for any $\lambda \in (\lambda_1(m), \sigma)$.

This article is organized as follows. In Section 2, we recall some basic definitions and we review some properties of the principal eigenvalues of the p-Laplacian under Steklov boundary conditions with an indefinite weight. We prove in Section 3 some results concern in maximum principle, existence of solutions and nonexistence of positive solutions for (1.1). We conclude this paper in Section 4 with some results on the antimaximum principle and on the uniformity for this principle. Our mean results of this section are Theorem 4.4 and Theorem 4.5. We finish with some example in dimension 1.

2. Preliminaries

Throughout this work, m and h are given functions in $C^r(\partial\Omega)$, for some $0 < r < 1$; $m^{\pm} = \max\{\pm m(x), 0\}$ and $h \ge 0$, $h \not\equiv 0$ a.e.

We denote by $W^{1,p}(\Omega)$ the classical Sobolev space endowed with its natural norm

$$\|u\|_{1,p} := \Big(\int_{\Omega} (|\nabla u|^p + |u|^p) \Big)^{1/p}.$$

The Lebesgue norm of $L^p(\Omega)$ will be denoted by $\|\cdot\|_p$, and the one of $L^p(\partial\Omega)$ by $\|\cdot\|_{p,\partial\Omega}$, for any $p \in [1, +\infty]$. If $S \subset \mathbb{R}^N$ is measurable set, $|S|$ denotes the Lebesgue measure of S and for $S \subset \partial\Omega$ we will also denote by $|S|$ its σ-mesure. The weak convergence will be denoted by $\rightharpoonup$ and the strong one by $\to$. Here we will denote by $p^* := Np/(N-p)^+$ the classical critical Sobolev's exponent, and by $p_* := (N-1)p/(N-p)^+$ the critical Sobolev's exponent for the trace inclusion.

We are interested in the weak solutions of (1.1), i.e., functions $u \in W^{1,p}(\Omega)$ such that

$$\int_\Omega (|\nabla u|^{p-2}\nabla u \nabla v + |u|^{p-2}uv) = \lambda \int_{\partial\Omega} m|u|^p uv + \int_{\partial\Omega} hv,$$

holds for all $v \in W^{1,p}(\Omega)$.

Remark 2.1. The standard regularity results for quasilinear elliptic pde ensure that if $m, h \in C^r(\partial\Omega)$ for some $0 < r < 1$ then every weak solution of (1.1) lies in $C^{1,\alpha}(\overline{\Omega})$. Furthermore, observing that the $W^{1,p}(\Omega)-$norm of a solution u of (1.1) can be bounded in terms of $\|u\|_{\infty,\partial\Omega}, \|m\|_{\infty,\partial\Omega}, \|h\|_{\infty.\partial\Omega}$ and $|\lambda|$, it follows that if

$$\|u\|_{\infty,\partial\Omega}, \|m\|_{\infty,\partial\Omega}, \|h\|_{\infty.\partial\Omega}, |\lambda| \le M$$

for a constant $M > 0$, then there exists a constant $\kappa > 0$, depending on M, p, Ω, such that

$$\|u\|_{C^{1,\alpha}(\overline{\Omega})} \le \kappa,$$

see [2, 6] for the details.

Let us summarize some properties of the principal eigenvalues of the eigenvalue problem associated with problem (1.1),

$$\begin{gathered} -\Delta_p u + |u|^{p-2}u = 0 \quad \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda m(x)|u|^{p-2}u \quad \text{on } \partial\Omega. \end{gathered} \tag{2.1}$$

A real number λ is said to be an eigenvalue of (2.1) if and only if there exists $u \in W^{1,p}(\Omega) \setminus \{0\}$, called eigenfunction associated with λ, satisfying

$$\int_\Omega (|\nabla u|^{p-2}\nabla u \nabla v + |u|^{p-2}uv) = \lambda \int_{\partial\Omega} m|u|^{p-2}uv, \tag{2.2}$$

for all $v \in W^{1,p}(\Omega)$. It is proved in [5] (see also [7] and [12] for a more general problem) that (2.1) admits two principal eigenvalues which are characterized by

$$\lambda_1(m) := \min\{\|u\|_{1,p}^p; u \in W^{1,p}(\Omega), I(u) = 1\} > 0; \tag{2.3}$$

$$\lambda_{-1}(m) := -\min\{\|u\|_{1,p}^p; u \in W^{1,p}(\Omega), I(u) = -1\} < 0. \tag{2.4}$$

where $I : W^{1.p}(\Omega) \to \mathbb{R}$ is the C^1-functional

$$I(u) := \int_{\partial\Omega} m|u|^p. \tag{2.5}$$

$\lambda_1(m)$ and $\lambda_{-1}(m)$ are simple, isolated. Moreover since in the case $N \ge 2$ there exists actually two sequences of eigenvalues going one to $+\infty$ and the other to $-\infty$, we can define the second eigenvalue from the right $\lambda_2(m)$ (resp. the negative eigenvalue from the left $\lambda_{-2}(m)$) of (2.1) as follows:

$$\lambda_2(m) := \min\{\lambda \in \mathbb{R}; \lambda \text{ eigenvalue and } \lambda > \lambda_1(m)\}; \tag{2.6}$$

$$\lambda_{-2}(m) := \max\{\lambda \in \mathbb{R}; \lambda \text{ eigenvalue and } \lambda < \lambda_{-1}(m)\}. \tag{2.7}$$

See section 5, where we discuss the case $N = 1$, and let us agree to write $\lambda_2(m) = +\infty$ (resp. $\lambda_{-2}(m) = -\infty$ if no eigenvalues greater than $\lambda_1(m)$ (resp. $\lambda_{-1}(m)$) exist. Every eigenfunction u associated with a positive (resp. negative) eigenvalue

$\lambda \neq \lambda_1(m)$ (resp. $\lambda \neq \lambda_{-1}(m)$) changes sign. Furthermore, if $\mathcal{N}$ is a nodal domain of u then

$$|\mathcal{N} \cap \partial\Omega| \geq \kappa_1 > 0, \tag{2.8}$$

for some constant $\kappa_1 > 0$ independent of u, see [5]. The following result is a simple consequence of the characterizations (2.3) and (2.4).

Lemma 2.2. *Assume that $h \geq 0$, $h \not\equiv 0$. If $\lambda \in (\lambda_{-1}(m), \lambda_1(m))$ then there exists a constant $\kappa > 0$ such that*

$$\|u\|_{1,p}^p - \lambda I(u) \geq \kappa \|u\|_{1,p}^p \quad \forall u \in W^{1,p}(\Omega).$$

Proof. Assume, by contradiction, that there exists a sequence $(u_n)_{n\in\mathbb{N}^*} \subset W^{1,p}(\Omega)$ with $\|u_n\|_{1,p} = 1$ such that

$$\|u_n\|_{1,p}^p - \lambda I(u_n) < \frac{1}{n}. \tag{2.9}$$

Since $\|u_n\|_{1,p} = 1$, then $(u_n)_{n\in\mathbb{N}^*}$ is bounded in $W^{1,p}(\Omega)$ and there exists a function u such that $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$ and strongly in $L^p(\partial\Omega)$. Then we obtain

$$1 - \lambda I(u) = \lim_{n\to\infty} (1 - \lambda I(u_n)) \leq 0$$

and then $\lambda I(u) \geq 1$. In particular $I(u) \neq 0$.

It follows from (2.9) and the weak lower semicontinuity of the norm that

$$\|u\|_{1,p}^p \leq \liminf_{n\to\infty} \|u_n\|_{1,p}^p \leq \lambda I(u) \tag{2.10}$$

Consequently, if $I(u) > 0$, it follows from (2.3) and (2.10) that

$$\lambda \geq \frac{\|u\|_{1,p}^p}{I(u)} \geq \lambda_1(m),$$

which is a contradiction. If $I(u) < 0$, it follows from (2.4) and (2.10) that

$$\lambda \leq \frac{\|u\|_{1,p}^p}{I(u)} \leq \lambda_{-1}(m),$$

which is also a contradiction. □

3. Maximum principle and existence of positive solutions

In this section we prove some results on maximum principle for problem (1.1), existence and uniqueness of solutions, and nonexistence of positive solutions.

Remark 3.1. Let $u \in W^{1,p}(\Omega)$ be a nonnegative weak solution of (1.1) with $h \geq 0$. Using Harnack's inequality (see [13, Theorems 5,6 and 9 pages 264-270]) and Hopf maximum principle (see [15]), its follows that $u > 0$ a.e. in $\overline{\Omega}$.

Next, let us recall Picone's identity, see [1]. Let $v > 0$ and $u \geq 0$ be two differentiable functions a.e. in Ω and denote

$$L(u,v) = |\nabla u|^p + (p-1)\frac{u^p}{v^p}|\nabla v|^p - p\frac{u^{p-1}}{v^{p-1}}|\nabla v|^{p-2}\nabla v \cdot \nabla u;$$

$$R(u,v) = |\nabla u|^p - |\nabla v|^{p-2}\nabla v \cdot \nabla\Big(\frac{u^p}{v^{p-1}}\Big).$$

Then Picone's identity states that

(i) $L(u,v) = R(u,v)$;
(ii) $L(u,v) \geq 0$;

(iii) $L(u,v)=0$ in Ω if and only if $u=kv$ for some constant k.

As a consequence of these identities we have the following result.

Lemma 3.2. *Let $u\in C^1(\overline{\Omega})$ be a weak solution for* (1.1) *such that $u>0$ in $\overline{\Omega}$. Then*

$$\lambda I(\phi)+\int_{\partial\Omega} h\frac{|\phi|^p}{u^{p-1}}\le \|\phi\|_{1,p}^p \tag{3.1}$$

for all bounded $\phi\in W^{1,p}(\Omega)$. Moreover the equality holds if and only if ϕ is scalar multiplier of u.

Proof. By Picone's identity we obtain

$$\begin{aligned}
0&\le\int_\Omega L(|\phi|,u)=\int_\Omega R(|\phi|,u)\\
&=\int_\Omega|\nabla\phi|^p-\int_\Omega|\nabla u|^{p-2}\nabla u\cdot\nabla\Big(\frac{|\phi|^p}{u^{p-1}}\Big)\\
&=\int_\Omega|\nabla\phi|^p+\int_\Omega u^{p-1}\frac{|\phi|^p}{u^{p-1}}-\lambda\int_{\partial\Omega}mu^{p-1}\frac{|\phi|^p}{u^{p-1}}-\int_{\partial\Omega}h\frac{|\phi|^p}{u^{p-1}}\\
&=\int_\Omega|\nabla\phi|^p+\int_\Omega|\phi|^p-\lambda\int_{\partial\Omega}m|\phi|^p-\int_{\partial\Omega}h\frac{|\phi|^p}{u^{p-1}}\\
&=\|\phi\|_{1,p}^p-\lambda I(\phi)-\int_{\partial\Omega}h\frac{|\phi|^p}{u^{p-1}};
\end{aligned}\tag{3.2}$$

and (3.1) holds. Moreover, from assertion (iii) of Picone's identity we have the equality in (3.2) if and only $|\phi|=cu$, for some constant c. In particular ϕ is of constant sign in Ω. □

The following result states the maximum principle for problem (1.1) for the usual range of λ.

Theorem 3.3. *Assume $h\ge 0$, $h\not\equiv 0$. Then the maximum principle for* (1.1) *holds if $\lambda\in(\lambda_{-1}(m),\lambda_1(m))$, i.e. if u is a weak solution for* (1.1) *with $\lambda\in(\lambda_{-1}(m),\lambda_1(m))$, then $u>0$ in $\overline{\Omega}$.*

Proof. Assume by contradiction that $u^-\not\equiv 0$ and take $v=u^-$ as test function in (1.1). We have

$$\begin{aligned}
0<\int_\Omega\big(|\nabla u^-|^p+(u^-)^p\big)&=\lambda\int_{\partial\Omega}m(u^-)^p-\int_{\partial\Omega}hu^-\\
&\le\lambda\int_{\partial\Omega}m(u^-)^p.
\end{aligned}\tag{3.3}$$

If $\int_{\partial\Omega}m(u^-)^p>0$, we deduce from the variational characterization of $\lambda_1(m)$ that

$$(\lambda_1(m)-\lambda)\int_{\partial\Omega}m(u^-)^p\le 0$$

which implies that $\lambda_1(m)\le\lambda$, we have a contradiction. Similarly, if $\int_{\partial\Omega}m(u^-)^p<0$, we deduce from the variational characterization of $\lambda_{-1}(m)$ that

$$(\lambda_{-1}(m)-\lambda)\int_{\partial\Omega}m(u^-)^p\le 0$$

which implies that $\lambda_{-1}(m)\ge\lambda$, and we have a contradiction. Hence, in all cases, we obtain that $u\ge 0$ and the conclusion follows from Remark 3.1. □

Let us now prove the following uniqueness result. We stress here that the existence result is well known *for any* $h \in L^q(\partial\Omega)$ *if* $q > (p_*)'$. We give here the proof for the sake of completeness.

Proposition 3.4. *Let* $h \geq 0$, $h \not\equiv 0$ *on* $\partial\Omega$. *Then* (1.1) *has a unique solution if* $\lambda \in (\lambda_{-1}(m), \lambda_1(m))$.

Proof. Let us prove that the energy functional K associated with (1.1)

$$K(u) = \frac{1}{p}\|u\|_{1,p}^p - \frac{\lambda}{p} I(u) - \int_{\partial\Omega} hu.$$

is coercive and weakly lower semicontinuous. Since $\lambda \in (\lambda_{-1}(m), \lambda_1(m))$, it follows from Lemma 2.2 that

$$\begin{aligned} K(u) &= \frac{1}{p}\|u\|_{1,p}^p - \frac{\lambda}{p} I(u) - \int_{\partial\Omega} hu \\ &\geq \kappa \frac{1}{p}\|u\|_{1,p}^p - \|h\|_\infty \|u\|_{1,\partial\Omega} \\ &\geq \frac{\kappa}{p}\|u\|_{1,p}^p - \kappa_1 \|u\|_{1,p} \to \infty \quad \text{as } \|u\|_{1,p} \to \infty \end{aligned}$$

where $\kappa_1 = c\|h\|_\infty$ with $c > 0$ the constant from the embedding of $W^{1,p}(\Omega)$ in $L^1(\partial\Omega)$. We conclude that K is coercive. Now assume that u_n is a sequence in $W^{1,p}(\Omega)$ such that $u_n \rightharpoonup u$ for some u in $W^{1,p}(\Omega)$. Then, from the compact embedding of $W^{1,p}(\Omega)$ into $L^{q_2}(\partial\Omega)$, for all $q_2 \in [1, p_*)$ we can assume that $u_n \to u$ in $L^p(\partial\Omega)$ and in $L^1(\partial\Omega)$. Then from the lower semicontinuity of the norm we obtain

$$K(u) \leq \liminf_{n\to\infty} K(u_n)$$

and the result follows. Since K is coercive and weakly lower semicontinuous, then the $\inf\{K(u), u \in W^{1,p}(\Omega)\}$ is achieved, providing us with a weak solution of (1.1) (i.e. a critical point of K).

To prove the uniqueness of the solution, assume that $u, v \in W^{1,p}(\Omega)$ are two solutions of (1.1) for a fixed $\lambda \in (\lambda_{-1}(m), \lambda_1(m))$. From Theorem 3.3 we have that u and v are positive and from Lemma 3.2 that

$$\lambda I(v) + \int_{\partial\Omega} h\frac{v^p}{u^{p-1}} \leq \|v\|_{1,p}^p = \lambda I(v) + \int_{\partial\Omega} hv. \tag{3.4}$$

Hence

$$\begin{aligned} \int_{\partial\Omega} h\frac{v^p}{u^{p-1}} &\leq \int_{\partial\Omega} hv, \\ \int_{\partial\Omega} hv\Big(1 - \frac{v^{p-1}}{u^{p-1}}\Big) &\geq 0. \end{aligned} \tag{3.5}$$

Interchanging u and v we also have

$$\int_{\partial\Omega} hu\Big(1 - \frac{u^{p-1}}{v^{p-1}}\Big) \geq 0. \tag{3.6}$$

By adding (3.5) and (3.6), we have

$$\int_{\partial\Omega} h\Big[v\Big(1 - \frac{v^{p-1}}{u^{p-1}}\Big) + u\Big(1 - \frac{u^{p-1}}{v^{p-1}}\Big)\Big] \geq 0. \tag{3.7}$$

Observe that

$$\begin{aligned} v\Big(1-\frac{v^{p-1}}{u^{p-1}}\Big)+u\Big(1-\frac{u^{p-1}}{v^{p-1}}\Big) &= \frac{vu^{p-1}-v^p}{u^{p-1}}+\frac{uv^{p-1}-u^p}{v^{p-1}} \\ &= \frac{(u^{p-1}-v^{p-1})(v^p-u^p)}{(uv)^{p-1}} \le 0. \end{aligned}$$

Thus from (3.7) we obtain

$$\int_{\partial\Omega} h\Big[v\Big(1-\frac{v^{p-1}}{u^{p-1}}\Big)+v\Big(1-\frac{v^{p-1}}{u^{p-1}}\Big)\Big]=0;$$

so $u=v$ on the set of positive measure $\{x\in\partial\Omega; h(x)\neq 0\}$. Hence, from (3.4), we obtain

$$\lambda I(v)+\int_{\partial\Omega} h\frac{v^p}{u^{p-1}}=\|v\|_{1,p}^p. \tag{3.8}$$

Then it follows from the Lemma 3.2 that $v=cu$ for some constant $c>0$. Since $u=v$ on $\{x\in\partial\Omega; h(x)\neq 0\}$ then $c=1$ and we obtain the desired result. □

Next we prove that there are no positive solutions when the parameter λ lies outside the interval $(\lambda_{-1}(m),\lambda_1(m))$.

Theorem 3.5. *Let $h\ge 0$, $h\not\equiv 0$ on $\partial\Omega$.*

(1) *Problem* (1.1) *has no solution $u\ge 0$, $u\not\equiv 0$ if $\lambda\notin[\lambda_{-1}(m),\lambda_1(m)]$.*
(2) *Problem* (1.1) *has no solution if $\lambda=\lambda_1(m)$ or $\lambda=\lambda_{-1}(m)$.*

Proof. 1. Assume by contradiction that there exists a nontrivial nonnegative solution u. We deduce from Remark 3.1 that $u>0$ in $\overline{\Omega}$ and by Lemma 3.2 one gets

$$\lambda I(\phi)+\int_{\partial\Omega} h\frac{|\phi|^p}{u^{p-1}}\le\|\phi\|_{1,p}^p \tag{3.9}$$

for all bounded $\phi\in C^1(\overline{\Omega})$ and in particular

$$\lambda I(\phi)\le\|\phi\|_{1,p}^p. \tag{3.10}$$

Then, by taking any $\phi\in C^1(\overline{\Omega})$ such that $\int_{\partial\Omega} m|\phi|^p>0$, it follows from (3.10) and the variational characterization of $\lambda_1(m)$ that $\lambda\le\lambda_1(m)$. Similarly, by taking any $\phi\in C^1(\overline{\Omega})$ satisfying $I(\phi)<0$ we have that $\lambda\ge\lambda_{-1}(m)$, a contradiction.

2. We only give the proof for the case $\lambda=\lambda_1(m)$. Assume by contradiction that there exists a solution u of (1.1) with $\lambda=\lambda_1(m)$. We claim that $u\ge 0$. Indeed, if not, we take $v=u^-\not\equiv 0$ as test function in (1.1) with $\lambda=\lambda_1(m)$ to obtain

$$\begin{aligned} \|u^-\|_{1,p}^p &= \lambda_1(m)I(u^-)-\int_{\partial\Omega} hu^- \\ &\le \lambda_1(m)I(u^-) \end{aligned} \tag{3.11}$$

and from the variational characterization of $\lambda_1(m)$ we have

$$0<\|u^-\|_{1,p}^p=\lambda_1(m)I(u^-). \tag{3.12}$$

We conclude that the infimum in (2.3) is achieved at u^- so $u^->0$ in $\overline{\Omega}$. Besides from (3.11) we obtain $\int_{\partial\Omega} hu^-=0$, a contradiction since $h\ge 0$, $h\not\equiv 0$. We have just proved that $u\ge 0$ and hence, by Remark 3.1, $u>0$ in $\overline{\Omega}$. By applying Lemma 3.2 one gets

$$\lambda_1(m)I(\phi)+\int_{\partial\Omega} h\frac{|\phi|^p}{u^{p-1}}\le\|\phi\|_{1,p}^p$$

for all $\phi \in C^1(\overline{\Omega})$.

Finally, by choosing $\phi = \varphi_1 > 0$ in $\overline{\Omega}$, we obtain $\int_{\partial\Omega} h\frac{\varphi_1^p}{u^{p-1}} \le 0$, a contradiction. □

4. Antimaximum principle

In Theorem 4.4 we will prove, for the case $p > N$, the existence of an interval of uniformity of this principle for problem (1.1).

The following result shows that the antimaximum principle holds, for a fixed $h \in C^r(\partial\Omega)$, at the right of $\lambda_1(m)$ (resp. left of $\lambda_{-1}(m)$) for λ sufficiently close to $\lambda_1(m)$ (resp. $\lambda_{-1}(m)$) and for any $p > 1$.

Theorem 4.1. *Let $h \ge 0$, $h \not\equiv 0$. Then there exists $\delta = \delta(h) > 0$ such that if $\lambda \in (\lambda_1(m), \lambda_1(m) + \delta)$ then any solution u of (1.1) satisfies $u < 0$ in $\overline{\Omega}$.*

Similarly, there exists $\delta' = \delta'(h) > 0$ such that if $\lambda \in (\lambda_{-1}(m) - \delta', \lambda_{-1}(m))$, any solution u of (1.1) satisfies $u < 0$ in $\overline{\Omega}$.

Proof. We only give the proof for the case $\lambda \in (\lambda_1(m), \lambda_1(m) + \delta)$. We assume by contradiction that there exists a sequence $(\lambda_k, u_k) \in \mathbb{R} \times W^{1,p}(\Omega)$ with $\lambda_k > \lambda_1(m)$, $\lambda_k \to \lambda_1(m)$, u_k a solution of $(\mathcal{P}_{\lambda_k,h})$ and such that

$$u_k(x_k) \ge 0 \quad \text{for some } x_k \in \overline{\Omega}.$$

Two alternatives can arise:

(a) $\|u_k\|_{\infty,\partial\Omega} \le \kappa_2$, with κ_2 some positive constant. It follows that u_k is also bounded in $L^\infty(\Omega)$ and in $C^{1,\alpha}(\overline{\Omega})$, see Remark 2.1. Then, using the compact embedding of $C^{1,\alpha}(\overline{\Omega})$ into $C^1(\overline{\Omega})$ we obtain, up to a subsequence, that $u_k \to u$ for some function u in $C^1(\overline{\Omega})$. Passing to the limit in $(\mathcal{P}_{\lambda_k,h})$ we obtain that u is a weak solution of (1.1) for $\lambda = \lambda_1(m)$, a contradiction with Theorem 3.5 (2).

(b) $\|u_k\|_{\infty,\partial\Omega} \to \infty$. Setting $w_k := \frac{u_k}{\|u_k\|_{\infty,\partial\Omega}}$, then $\|w_k\|_{\infty,\partial\Omega} = 1$ and it follows (using Remark 2.1 with $h_k := \frac{h}{\|u_k\|^{p-1}_{\infty,\partial\Omega}}$) that w_k lies in $C^{1,\alpha}(\overline{\Omega})$. Moreover there exists a constant $C > 0$ such that $\|w_k\|_{C^{1,\alpha}(\overline{\Omega})} \le C$. Thus, there exists a function w such that, for a subsequence, $w_n \to w$ in $C^1(\overline{\Omega})$. In particular $w \not\equiv 0$ since $\|w\|_{\infty,\partial\Omega} = 1$. Hence, passing to the limit, we obtain that w is an eigenfunction associated with the eigenvalue $\lambda_1(m)$ of (2.1). Consequently $w > 0$ or $w < 0$ in $\overline{\Omega}$. If $w > 0$, then for k large enough we have $u_k > 0$ and this contradicts Theorem 3.5(1). If $w < 0$, then for k large enough we have $u_k < 0$ which contradicts the existence of x_k. □

Notice that, a priori, the value δ of Theorem 4.1 depends of the function h. If this is not so, we say that the antimaximum principle is uniform on $(\lambda_1(m), \lambda_1(m)+\delta)$. In the following, we study the validity of the uniform antimaximum principle and we will give a variational characterization of the greatest value δ for which the uniform antimaximum principle holds in $(\lambda_1(m), \lambda_1(m) + \delta)$ if $p > N$.

Following [3, 4, 10] we introduce the values $\hat{\lambda}_1(m)$ and $\hat{\lambda}_{-1}(m)$:

$$\hat{\lambda}_1(m) := \inf\left\{\|u\|^p_{1,p};\ I(u) = 1 \text{ and } u \in Q\right\} \tag{4.1}$$

$$\hat{\lambda}_{-1}(m) := -\inf\left\{\|u\|^p_{1,p};\ I(u) = -1 \text{ and } u \in Q\right\}, \tag{4.2}$$

where

$$Q := \left\{u \in W^{1,p}(\Omega); \exists B(x_0, r) \text{ s.t } u_{|_{B(x_0,r)\cap\overline{\Omega}}} \equiv 0 \text{ a.e.}\right\} \quad \text{with } x_0 \in \partial\Omega$$

Clearly $\lambda_1(m) \le \hat\lambda_1(m)$ (resp. $\lambda_{-1}(m) \ge \hat\lambda_{-1}(m)$). In the following two lemmae we discuss whenever $\lambda_1(m)$ is different or equal to $\hat\lambda_1(m)$.

Lemma 4.2. *Assume $1 < p \le N$ and let $\hat\lambda_1(m), \hat\lambda_{-1}(m)$ be defined in* (4.1) *et* (4.2)*. Then $\lambda_1(m) = \hat\lambda_1(m)$ and $\lambda_{-1}(m) = \hat\lambda_{-1}(m)$.*

Proof. We distinguish two cases:

Case (i) $p < N$. We define, as in [3] or [4], the sequence of functions y_k defined for all $x \in \mathbb{R}^N$ by

$$y_k(x) := \begin{cases} 1 & \text{if } |x| \ge \frac{1}{k}, \\ 2k|x| - 1 & \text{if } \frac{1}{2k} < |x| < \frac{1}{k}, \\ 0 & \text{if } |x| \le \frac{1}{2k} \end{cases}$$

It is not difficult to prove that $y_k \to 1$ as $k \to \infty$ in $W^{1,p}_{\rm loc}(\mathbb{R}^N)$ using

$$\int_{\mathbb{R}^N} \big|\frac{\partial y_k}{\partial x_i}\big|^p = \int_{\frac{1}{2k}<|x|<\frac{1}{k}} \big|\frac{\partial y_k}{\partial x_i}\big|^p \le Ck^{p-N} \to 0 \quad \text{as } k \to \infty. \tag{4.3}$$

for some constant C.

On the other hand, let us assume without loss of generality that $0 \in \partial\Omega$ and let φ_1 be the positive eigenfunction associated with $\lambda_1(m)$ satisfying $I(\varphi_1) = 1$. Then the sequence $z_k := \varphi_1(x)y_k(x)$ vanishes in the set $B\big(0, \frac{1}{2k}\big) \cap \partial\Omega$ and clearly $z_k \in L^p(\Omega)$. Moreover, since φ_1 lies in $C^{1,\alpha}(\overline\Omega)$ and $\|\varphi_1\|_{C^{1,\alpha}(\overline\Omega)} \le C$, for some constant $C > 0$, we infer that

$$\int_\Omega \big|\frac{\partial z_k}{\partial x_i}\big|^p \le 2^p \int_\Omega \big|\frac{\partial \varphi_1}{\partial x_i} y_k\big|^p + 2^p \|\varphi_1\|_\infty^p \int_\Omega \big|\frac{\partial y_k}{\partial x_i}\big|^p,$$

and therefore $\frac{\partial z_k}{\partial x_i} \in L^p(\Omega)$ and $z_k \in W^{1,p}(\Omega)$. On the other hand, we have

$$\begin{gathered} |z_k - \varphi_1|^p = |\varphi_1 y_k - \varphi_1|^p \le \varphi_1^p \in L^1(\Omega); \\ |z_k - \varphi_1|^p = |\varphi_1 y_k - \varphi_1|^p \xrightarrow{k\to\infty} 0 \quad a.e.; \\ \Big|\frac{\partial\varphi_1}{\partial x_i} y_k - \frac{\partial\varphi_1}{\partial x_i}\Big|^p \le \Big|\frac{\partial\varphi_1}{\partial x_i}\Big|^p \in L^1(\Omega); \\ \Big|\frac{\partial\varphi_1}{\partial x_i} y_k - \frac{\partial\varphi_1}{\partial x_i}\Big|^p \to 0 \quad \text{as } k \to \infty; \end{gathered}$$

and by Lebesgue's Dominated Convergence Theorem it follows that $z_k \to \varphi_1$ in $W^{1,p}(\Omega)$ as $k \to \infty$. Hence

$$I(z_k) \to I(\varphi_1) = 1 \quad \text{as } k \to \infty$$

and in particular, for k large enough one has $I(z_k) > 0$. Then, from the definition (4.1) of $\hat\lambda_1(m)$, we have

$$\hat\lambda_1(m) \le \frac{\|z_k\|_{1,p}^p}{I(z_k)} \to \|\varphi_1\|_{1,p}^p = \lambda_1(m).$$

Case (ii) $p = N$. In this case we define instead

$$y_k(x) := \begin{cases} 1 - \frac{2}{k} & \text{if } |x| \ge \frac{1}{k}, \\ |x|^{\delta_k} - \frac{1}{k} & \text{if } \big(\frac{1}{k}\big)^{1/\delta_k} < |x| < \frac{1}{k}, \\ 0 & \text{if } |x| \le \big(\frac{1}{k}\big)^{1/\delta_k}, \end{cases}$$

where δ_k satisfies $(\frac{1}{k})^{\delta_k} = 1 - \frac{1}{k}$; ($\delta_k = 1 - \frac{\ln(k-1)}{\ln(k)} \to 0$ as $k \to \infty$). It is easy to show that $y_k \to 1$ as $k \to \infty$ a.e. in $W^{1,p}_{\rm loc}(\Omega)$ since

$$\begin{aligned}\int_{\mathbb{R}^N} \big|\frac{\partial y_k}{\partial x_i}\big|^p &= \int_{(\frac{1}{k})^{1/\delta_k}<|x|<\frac{1}{k}} \Big|\delta_k |x|^{(\delta_k-1)}\frac{x_i}{|x|}\Big|^p \\ &\le C\delta_k^p \int_{(\frac{1}{k})^{1/\delta_k}<r<\frac{1}{k}} r^{(\delta_k-1)p} r^{N-1} dr \\ &= \frac{C\delta_k^p}{(\delta_k-1)p+N}\big[(\frac{1}{k})^{\delta_k p} - (\frac{1}{k})^p\big]\end{aligned}$$

which converges to 0 as $k \to \infty$. The sequence $z_k := \varphi_1 y_k$ vanishes in the set $B\big(0, (\frac{1}{k})^{1/\delta_k}\big) \cap \partial\Omega$ and clearly z_k lies in $L^p(\Omega)$. Moreover, using the arguments of (i), we have $z_k \to \varphi_1(x)$ as $k \to \infty$ a.e. in $W^{1,p}(\Omega)$. Hence, from the definition (4.1) of $\hat\lambda_1(m)$, one deduces that

$$\hat\lambda_1(m) \le \frac{\|z_k\|_{1,p}^p}{I(z_k)} \to \|\varphi_1\|_{1,p}^p = \lambda_1(m);$$

as $k \to \infty$, and we obtain the conclusion. □

Proposition 4.3. *Let $p > N$.*

(a) It holds

$$\hat\lambda_1(m) = \inf\{\|u\|_{1,p}^p;\ I(u) = 1 \text{ and } u \in Q^0\}, \tag{4.4}$$

$$\hat\lambda_{-1}(m) = -\inf\{\|u\|_{1,p}^p;\ I(u) = -1 \text{ and } u \in Q^0\} \tag{4.5}$$

where

$$Q^0 := \{u \in W^{1,p}(\Omega); \exists x_0 \in \partial\Omega \text{ s.t. } u(x_0) = 0\}.$$

(b) The infima in (4.4) and (4.5) are achieved.

(c) $\lambda_1(m) < \hat\lambda_1(m)$ and $\hat\lambda_{-1}(m) < \lambda_{-1}(m)$.

(d) If $\hat u$ is a minimiser in (4.4) (resp. (4.5)), then $\hat u$ vanishes exactly at one point on $\partial\Omega$ and $\hat u$ does not change sign on $\partial\Omega$.

(e) $\hat\lambda_1(m) < \lambda_2(m)$ where $\lambda_2(m)$ is the eigenvalue defined in (2.6). Respectively, $\hat\lambda_{-1}(m) > \lambda_{-2}(m)$, where $\lambda_{-2}(m)$ is the eigenvalue defined in (2.7).

Proof. We only give the proofs that concern $\hat\lambda_1(m)$.

(a) Let $\phi \in C^1(\overline{\Omega})$ satisfies $I(\phi) > 0$ and assume that $\phi(x_0) = 0$ for some $x_0 \in \partial\Omega$. For any fixed $\epsilon > 0$ let us define

$$\phi_\epsilon := \max\{|\phi|, \epsilon\} - \epsilon.$$

Clearly, $\phi_\epsilon \to |\phi|$ in $W^{1,p}(\Omega)$ as $\epsilon \to 0$. By continuity, there exists $r > 0$ such that $|\phi(x)| < \epsilon$ for all $x \in B(x_0, r) \cap \overline{\Omega}$, and therefore $\phi_\epsilon(x) = 0$ for all $x \in B(x_0, r) \cap \overline{\Omega}$.

(b) The proof is standard and uses the compact embedding of $W^{1,p}(\Omega)$ in $C(\overline{\Omega})$ to assure that a weak limit of any minimizing sequence must vanish somewhere on $\partial\Omega$.

(c) Assume that $\lambda_1(m) = \hat\lambda_1(m)$. From (b), $\hat\lambda_1(m)$ is achieved at some u_0 and consequently u_0 is an eigenfunction of (2.1) associated with $\lambda_1(m)$. But this is impossible since u_0 is vanishes somewhere in $\partial\Omega$. Hence $\lambda_1(m) < \hat\lambda_1(m)$.

(d) Let us now prove that the minimiser vanishes exactly at one point on $\partial\Omega$. Set $\hat u$ the minimiser of $\hat\lambda_1(m)$ and assume that $\hat u(x_0) = 0$ for some $x_0 \in \partial\Omega$. We can assume that $\hat{u} \geq 0$ by changing $\hat{u}$ by $|\hat{u}|$ if needed. Then the definition (4.4) is equivalent to the following

$$\hat{\lambda}_1(m) := \inf_{u \in \mathscr{A}} \|u\|_{1,p}^p \tag{4.6}$$

where

$$\mathscr{A} := \{u \in W^{1,p}(\Omega); I(u) = 1 \text{ and } u(x_0) = 0\}.$$

Assume by contradiction that there exists $x_1 \neq x_0 \in \partial\Omega$ with $\hat{u}(x_1) = 0$ and set

$$\mathscr{B} := \{u \in W^{1,p}(\Omega); I(u) = 1 \text{ and } u(x_1) = 0\}$$

so we also have

$$\hat{\lambda}_1(m) = \inf_{u \in \mathscr{B}} \|u\|_{1,p}^p.$$

Let us now denote by

$$\Psi(u) := \|u\|_{1,p}^p; \quad \psi_1(u) := I(u) - 1, \quad \psi_2(u) := u(x_1).$$

By Lagrange's Multipliers Theorem there exists $(\beta_1, \beta_2) \in \mathbb{R}^2$ such that

$$\begin{aligned} \Psi'(w)(v) &= \beta_1 \psi_1'(w)(v) + \beta_2 \psi_2'(w)(v) \\ &= \beta_1 \psi_1'(w)(v) + \beta_2 v(x_1) \quad \forall v \in W^{1,p}(\Omega). \end{aligned} \tag{4.7}$$

Taking $v = w$ in (4.7) we obtain that $\beta_1 = \hat{\lambda}_1(m)$. Similarly there exists $\gamma_2 \in \mathbb{R}$ such that

$$\Psi'(w)(v) = \hat{\lambda}_1(m)\psi_1'(w)(v) + \gamma_2 v(x_0) \quad \forall v \in W^{1,p}(\Omega). \tag{4.8}$$

and therefore

$$\beta_2 v(x_1) = \gamma_2 v(x_0), \quad \forall v \in W^{1,p}(\Omega) \tag{4.9}$$

Taking $v \equiv 1$ in (4.9) one sets $\beta_2 = \gamma_2$ and since (4.9) holds for all $v \in W^{1,p}(\Omega)$, we deduce that $\beta_2 = \gamma_2 = 0$. Consequently it comes from (4.7) that $\hat{\lambda}_1(m)$ is a principal eigenvalue of (1.1) and w is a nonnegative eigenfunction associated with $\hat{\lambda}_1(m)$. By Remark 3.1, $w > 0$ in $\overline{\Omega}$, a contradiction.

We have just prove that w, and therefore $\hat{u}$, vanishes only once on $\partial\Omega$.

Now, let us show that $\hat{u}$ does not change sign on $\partial\Omega$. Assume that $\hat{u}^+ \not\equiv 0, \hat{u}^- \not\equiv 0$ and say $\hat{u}(x_1) = 0$ for some $x_1 \in \partial\Omega$. Then taking $v = \hat{u}^+$ in (4.9), one gets that

$$0 < \|\hat{u}^+\|_{1,p}^p = \hat{\lambda}_1(m) I(\hat{u}^+),$$

so the function $\frac{\hat{u}^+}{I(\hat{u}^+)^{1/p}}$ is a minimizer in (4.4). Hence $\hat{u}^+$ vanishes only at x_1 which implies $\hat{u} \geq 0$ on $\partial\Omega$.

(e) Let φ_2 be an eigenfunction associated with $\lambda_2(m)$. By (2.8) we know that φ_2 vanishes somewhere on $\partial\Omega$. Thus φ_2 is an admissible function in the definition (4.4) of $\hat{\lambda}_1(m)$ and then

$$\hat{\lambda}_1(m) \leq \frac{\|\varphi_2\|_{1,p}^p}{I(\varphi_2)} = \lambda_2(m).$$

If $\hat{\lambda}_1(m) = \lambda_2(m)$ then φ_2 would be a minimiser in (4.4) and therefore it must have a constant sign on $\partial\Omega$, according to (c), a contradiction. □

With the previous results in hand, we can give an interval where the uniform antimaximum principle holds.

Theorem 4.4. *Let $p > N$ and let $h \geq 0$, $h \not\equiv 0$. If u is a solution of (1.1) with $\lambda \in \left(\lambda_1(m), \hat{\lambda}_1(m)\right]$ then $u < 0$ in $\overline{\Omega}$. Similarly any solution u of (1.1) with $\lambda \in \left[\hat{\lambda}_{-1}(m), \lambda_{-1}(m)\right)$ is negative in $\overline{\Omega}$.*

Proof. Let u be a solution of (1.1) with $\lambda \in \left(\lambda_1(m), \hat{\lambda}_1(m)\right]$, then $u^- \not\equiv 0$ in $\overline{\Omega}$ by Theorem 3.5. Let us take $v = u^-$ as test function in (1.1) to get

$$0 < \|u^-\|_{1,p}^p = \lambda I(u^-) - \int_{\partial\Omega} hu^- \leq \lambda I(u^-). \tag{4.10}$$

In particular $I(u^-) > 0$. Let us first show that $u < 0$ on $\partial\Omega$. Indeed, if $\lambda < \hat{\lambda}_1(m)$, we have from (4.10)

$$\frac{\|u^-\|_{1,p}^p}{I(u^-)} \leq \lambda < \hat{\lambda}_1(m) = \inf_{v \in \mathscr{A}} \|v\|_{1,p}^p, \quad \forall x_0 \in \partial\Omega.$$

So $u^- \notin \mathscr{A}$ and we conclude that u^- does not vanish anywhere on $\partial\Omega$, that is, $u < 0$ on $\partial\Omega$. If $\lambda = \hat{\lambda}_1(m)$ and we assume by contradiction that u^- vanish somewhere on $\partial\Omega$, hence, from the one hand u^- is a minimizer for $\hat{\lambda}_1(m)$ according to Proposition 4.3(a) and from the other hand, using (4.10) we have

$$0 = \|u^-\|_{1,p}^p - \hat{\lambda}_1(m) I(u^-) = -\int_{\partial\Omega} hu^-.$$

We deduce from this relation that u^- vanishes on the set of positive measure $\{x \in \partial\Omega; h(x) > 0\}$ which is a contradiction with Proposition 4.3(c) (minimizers of $\hat{\lambda}_1(m)$ vanish only once).

Next we prove that $u < 0$ in $\overline{\Omega}$. Since $u < 0$ on $\partial\Omega$ one has that $u^+ \in W_0^{1,p}(\Omega)$. Take then $v := u^+$ in the weak form of (1.1) to obtain

$$\|u^+\|_{1,p}^p = \lambda I(u^+) + \int_{\partial\Omega} hu^+ = 0. \tag{4.11}$$

Consequently $u^+ \equiv 0$ in Ω and so $u \leq 0$ in $\overline{\Omega}$. Using the well know Harnack's inequality [13, Theorem 5] we deduce that $u < 0$ in Ω and then $u < 0$ in $\overline{\Omega}$. □

Finally we prove that the value $\hat{\lambda}_1(m)$ (resp. $\hat{\lambda}_{-1}(m)$.) is optimal in the sense that the antimaximum principle holds to the right of $\hat{\lambda}_1(m)$ and that the uniform antimaximum principle fails to the right of $\hat{\lambda}_1(m) + \delta$ for any $\delta > 0$.

Theorem 4.5.

(1) *For any $h \geq 0$, $h \not\equiv 0$ there exists $\delta = \delta(h) > 0$ such that if $\lambda \in \left(\hat{\lambda}_1(m), \hat{\lambda}_1(m) + \delta\right)$, every solution u of (1.1) satisfies $u < 0$ in $\overline{\Omega}$.*

(2) *Given $\delta > 0$, there exists $h \in C^r(\partial\Omega)$ satisfying $h \geq 0$, $h \not\equiv 0$ such that for all $\lambda > \hat{\lambda}_1(m) + \delta$ problem (1.1) does not admit a negative solution.*

In particular for all $\delta > 0$, the uniform antimaximum principle does not hold in $(\hat{\lambda}_1(m), \hat{\lambda}_1(m) + \delta)$.

Similar results can be stated to the left of $\hat{\lambda}_{-1}(m)$.

Proof. (1) We assume here that $p > N$ as in the case $1 < p \leq N$, $\hat{\lambda}_1(m) = \lambda_1(m)$ and the result is proved in Theorem 4.1. The proof follows the same pattern of the one in the proof of Theorem 4.1 and we just indicate the changes needed in the contradiction argument. In alternative (a), passing to the limit in $(\mathcal{P}_{\lambda_k,h})$ one gets that u is a weak solution of (1.1) for $\lambda = \hat{\lambda}_1(m)$. It follows from Theorem 4.4 that $u < 0$ in $\overline{\Omega}$ and consequently $u_k < 0$ in $\overline{\Omega}$ for k large enough (since the convergence is in $C^1(\overline{\Omega})$), a contradiction with the existence of x_k. In alternative (b), passing to the limit we obtain that w is an eigenfunction associated with $\hat{\lambda}_1(m)$. Since $\|w\|_{\infty,\partial\Omega} = 1$ then $w \not\equiv 0$ and therefore $\hat{\lambda}_1(m)$ is an eigenvalue of (1.1) and w an eigenfunction associated with $\hat{\lambda}_1(m)$, a contradiction with Proposition 4.3 (e).

(2) Let $\delta > 0$ be fixed and assume by contradiction that for any $h \geq 0$, $h \not\equiv 0$, there exists $\lambda(h) > \hat{\lambda}_1(m) + \delta$ such that $(\mathcal{P}_{\lambda(h),h})$ admits a solution $u_h \in C^1(\overline{\Omega})$ such that $u_h < 0$ in $\overline{\Omega}$. Let $\phi \in W^{1,p}(\Omega)$ satisfies $I(\phi) > 0$ and assume that there exists $x_0 \in \partial\Omega$ and there exists $r > 0$ such that $\phi(x) = 0$ a.e. in $B(x_0, r) \cap \overline{\Omega}$. Choose $h \geq 0, h \not\equiv 0$ satisfying

$$\operatorname{supp}_{\partial\Omega} h \subset B(x_0, r) \cap \partial\Omega. \tag{4.12}$$

By applying Lemma 3.2 to $v = -u_h > 0$ (which is a solution of problem (1.1) with $\lambda = \lambda(h)$ and $-h$ instead of h, we obtain

$$(\hat{\lambda}_1(m) + \delta)I(\phi) < \lambda(h)I(\phi) \leq \|\phi\|_{1,p}^p$$

which implies

$$\hat{\lambda}_1(m) + \delta \leq \frac{\|\phi\|_{1,p}^p}{I(\phi)},$$

and taking the infimum over all $\phi \in W^{1,p}(\Omega)$ satisfying $I(\phi) > 0$ and vanishing on $B(x_0, r) \cap \overline{\Omega}$, for some $x_0 \in \partial\Omega$, we obtain

$$\hat{\lambda}_1(m) + \delta \leq \hat{\lambda}_1(m),$$

which is a contradiction. □

5. Spectra in dimension 1

5.1. **Case $p = 2$.** A simple computation shows that in the case $N = 1$, $p = 2$, there are only two eigenvalues for the Steklov problem. Take for instance $\Omega = (0, 1)$ and

$$m(x) = \begin{cases} -1 & \text{if } x = 0; \\ 1 & \text{if } x = 1. \end{cases}$$

Hence the only eigenvalues of the eigenvalue problem

$$\begin{gathered} -u'' + u = 0 \quad \text{in } (0,1); \\ -u'(0) = \lambda m(0)u(0), \\ u'(1) = \lambda m(1)u(1), \end{gathered}$$

are $\lambda_{-1}(m) = -1$ and $\lambda_1(m) = 1$. Let

$$\alpha = \inf\{\|u\|_{1,2}^2; u \in \mathcal{H}_1 \text{ and } u(1) = 1\}, \quad \beta = \inf\{\|u\|_{1,2}^2; u \in \mathcal{H}_2 \text{ and } u(0) = 1\},$$

where

$$\begin{aligned} \mathcal{H}_1 &= \{u \in H^1((0,1)); I(u) = 1 \text{ and } u(0) = 0\}, \\ \mathcal{H}_2 &= \{u \in H^1((0,1)); I(u) = 1 \text{ and } u(1) = 0\}. \end{aligned}$$

Then

$$\hat{\lambda}_1(m) = \inf\{\|u\|_{1,2}^2; u \in H^1((0,1)), I(u) = 1, u(0) = 0 \text{ or } u(1) = 0\} = \min\{\alpha, \beta\}$$

By a simple computation we obtain

$$\alpha = \frac{e+e^{-1}}{e-e^{-1}} = \beta = \hat{\lambda}_1(m).$$

Furthermore, if $h > 0$ is a function defined on the boundary of $\Omega = (0,1)$ by

$$h(x) = \begin{cases} a & \text{if } x = 0; \\ b & \text{if } x = 1, \end{cases}$$

then it results that, if $\lambda > 1$, the (unique) solution u of

$$\begin{gathered} -u'' + u = 0 \quad \text{in } (0,1); \\ -u'(0) = \lambda m(0)u(0) + a, \\ u'(1) = \lambda m(1)u(1) + b. \end{gathered}$$

is non-positive if and only if

$$1 < \lambda < \frac{2b}{a(e-e^{-1})} + \frac{e+e^{-1}}{e-e^{-1}}.$$

Then there is an uniform antimaximum principle for $\lambda \in \big(1, \frac{e+e^{-1}}{e-e^{-1}}\big]$.

5.2. **General case.** Let us consider (1.1) in dimension 1 for the weight $m \equiv 1$, i.e.

$$\begin{gathered} (|u'|^{p-2}u')'|u|^{p-2}u \quad \text{in } \Omega = (0,1) \\ |u'(0)|^{p-2}u'(0) = -\lambda|u(0)|^{p-2}u(0) \\ |u'(1)|^{p-2}u'(1) = \lambda|u(1)|^{p-2}u(1) \end{gathered} \tag{5.1}$$

First we look for positive solution u of (5.1) such that

$$u(0) = u(1), \quad u'(\frac{1}{2}) = 0.$$

From (5.1) we obtain

$$-\frac{|u'(t)|^p}{p'} + \frac{|u(t)|^p}{p} = C, \quad \forall t \in (0,1) \tag{5.2}$$

where $p' = \frac{p}{p-1}$ and the constant C is such that

$$\begin{aligned} C &= -\frac{|u'(0)|^p}{p'} + \frac{|u(0)|^p}{p} = (u(0))^p\big[\frac{1}{p} - \frac{\lambda^{\frac{p}{p-1}}}{p'}\big] \\ &= -\frac{|u'(1/2)|^p}{p'} + \frac{|u(1/2)|^p}{p} \end{aligned} \tag{5.3}$$

Let us assume that $u(1/2) = 1$. Then $C = \frac{1}{p}$ and

$$u(0) = \Big(1 - (p-1)\lambda^{\frac{p}{p-1}}\Big)^{-1/p}. \tag{5.4}$$

Moreover, using the fact that $u'(t) < 0$ for all $t \in (0, \frac{1}{2})$, from (5.2) we obtain

$$-\frac{du}{(|u|^p - 1)^{1/p}} = (p-1)^{-1/p}dt \tag{5.5}$$

Hence

$$\int_1^{u(0)} \frac{dz}{(|z|^p - 1)^{1/p}} = \frac{1}{2}(p-1)^{-1/p}$$

or equivalently

$$u(0)=\Lambda_p[\frac{1}{2}(p-1)^{-1/p}] \tag{5.6}$$

where we denote by $\Lambda_p:\mathbb{R}\to\mathbb{R}$ the function defined implicitly by

$$\Lambda_p(t)=y \iff t=\int_1^y \frac{dz}{(|z|^p-1)^{1/p}} \tag{5.7}$$

From (5.4) and (5.6) we obtain

$$\lambda_1=\lambda=\Big[\frac{1}{p-1}\Big(1-\Big(\Lambda_p\big[\frac{1}{2}(p-1)^{-1/p}\big]\Big)^{-p}\Big)\Big]^{\frac{p-1}{p}} \tag{5.8}$$

On another hand, since $u'(t)>0$ for all $t\in(\frac{1}{2},1)$, we deduce from (5.2) that

$$u(s)=\varphi_1(s)=\Lambda_p\big[(p-1)^{-1/p}|s-\frac{1}{2}|\big],\quad \forall s\in(0,1)$$

Now we look for a solution $u=\varphi_2$ of (5.1) which changes sign on $(0,1)$ such that $u(1/2)=0$ and $u'(1/2)=1$. From (5.2) we deduce that

$$\begin{aligned}-\frac{|u'(t)|^p}{p'}+\frac{|u(t)|^p}{p}&=-\frac{1}{p'}=-\frac{|u'(0)|^p}{p'}+\frac{|u(0)|^p}{p}\\&=|u(0)|^p[\frac{1}{p}-\frac{\lambda^{\frac{p}{p-1}}}{p'}]\end{aligned} \tag{5.9}$$

and consequently, since $u'(t)>0$ for all $t\in(0,1)$, we have

$$u'(t)=\big(1+\frac{|u(t)|^p}{p-1}\big)^{1/p}.$$

Hence

$$\begin{aligned}\frac{1}{2}&=\int_{u(0)}^0\frac{dz}{(1+\frac{|z|^p}{p-1})^{1/p}}\\&=(p-1)^{1/p}\int_0^{-u(0)(p-1)^{-1/p}}\frac{dz}{(1+|z|^p)^{1/p}}\end{aligned} \tag{5.10}$$

Similarly we define $\Phi_p(t)=y$ implicitly by

$$\Phi_p(t)=y \iff t=\int_0^y\frac{dv}{(1+|v|^p)^{1/p}}=(p-1)^{-1/p}\int_0^{(p-1)^{1/p}y}\frac{dv}{(1+\frac{|v|^p}{p-1})^{1/p}}.$$

Hence from (5.9) and (5.10) we deduce that

$$\big[(p-1)\lambda^{\frac{p}{p-1}}-1\big]^{-1/p}=-u(0)(p-1)^{-1/p}=\Phi_p\big[\frac{1}{2}(p-1)^{-1/p}\big],$$

so

$$\lambda_2=\lambda=\Big\{\frac{1}{p-1}\Big[1+\Big(\Phi_p\big[\frac{1}{2}(p-1)^{-1/p}\big]\Big)^{-p}\Big]\Big\}^{\frac{p-1}{p}},$$

$$\varphi_2(s)=u(s)=(p-1)^{1/P}\Phi_p[(p-1)^{1/P}(s-\frac{1}{2})],\quad \forall s\in(0,1).$$

It remains to explain the value of $\hat\lambda_1(m)$. Since

$$\hat\lambda_1(m)=\inf\{\|u\|_{1,p}^p; u(0)=0 \text{ and } u(1)=1\}=\|\hat u\|_{1,p}^p$$

it follows that $\hat{u}$ is solution of the problem

$$\begin{gathered}(|\hat{u}'|^{p-2}\hat{u}')' = |\hat{u}|^{p-2}\hat{u} \\ \hat{u}(0)=0, \hat{u}(1)=1\end{gathered} \tag{5.11}$$

Hence

$$-\frac{|\hat{u}'|^p}{p'} + \frac{|\hat{u}|^p}{p} = C = -\frac{|\hat{u}'(0)|^p}{p'} = \frac{1}{p} - \frac{|\hat{u}'(1)|^p}{p'} \tag{5.12}$$

From (5.12) we obtain

$$1 = \int_0^1 \frac{du}{(\frac{|u|^p}{p-1} - Cp')^{1/p}} = (p-1)^{1/p} \int_0^{(-Cp)^{-1/p}} \frac{dt}{(|t|^p+1)^{1/p}},$$

which is equivalent to

$$\Phi_p[(p-1)^{-1/p}] = (-Cp)^{-1/p} \tag{5.13}$$

Multiplying (5.11) by $\hat{u}$, integrating by parts and using (5.12) we have

$$\begin{aligned}\hat{\lambda}_1(m) &= \|\hat{u}\|_{1,p}^p \\ &= (\hat{u}'(1))^{p-1} \\ &= (p-1)^{-(p-1)/p}\,(1-Cp)^{(p-1)/p}\end{aligned}$$

Finally (5.13) leads to

$$\Phi_p[(p-1)^{-1/p}] = \big[-1 + (p-1)\big(\hat{\lambda}_1(m)\big)^{\frac{p}{p-1}}\big]^{-1/p}$$

and hence

$$\hat{\lambda}_1(m) = \Big\{\frac{1}{p-1}\Big[1 + \Big(\Phi_p[(p-1)^{-1/p}]\Big)^{-p}\Big]\Big\}^{\frac{p-1}{p}}$$

Some properties of Φ_p and Λ_p can be found in [8, 9, 14]

Acknowledgements. P. Nshimirimana was supported by the Centre d'Excellence Africain en Sciences Mathématiques et Applications (CEA-SMA) through IMSP. This work was partially carried out while the she was visiting the LMPA of the Université du Littoral Côte d'Opale (ULCO). She would like to express her gratitude to these institutions.

References

[1] W. Allegretto, Y. X. Huang; *A Picone's identity for the p-Laplacian an applications*, Nonlinear Anal. 32 (7) (1998) 819-830. doi: 10.1016/s0362-54X(97)00530-0.

[2] A. Anane, O. Chakrone, N. Moradi; *Regularity of the solutions to a nonlinear boundary problem with indefinite weight*, Bol. Soc. Parana. Mat. (3) 29 (1) (2011) 17-23. doi: 10.5269/bspm.v29i1.11402.

[3] A. Anane, O. Chakrone, N. Moradi; *Maximum and anti-maximum principle for the p-Laplacian with a nonlinear boundary condition*, Proceedings of the 2005 Oujda International Conference on Nonlinear Analysis, Vol. 14, Electron. J. Differ. Equ., Conf. 14 (2006), pp. 95-107.

[4] M. Arias, J. Campos, J.P. Gossez; *On the antimaximum principle and the Fučik spectrum for the Neumann p-Laplacian*, Differential Integral Equations 13 (1-3) (2000) 217-226.

[5] J. Fernández Bonder, J. D. Rossi; *A nonlinear eigenvalue problem with indefinite weights related to the Sobolev trace embedding*, Publ. Mat. 46 (1) (2002) 221-235.

[6] M. Cuesta, L. A. Leadi, P. Nshimirimana; *Bifurcation from the first eigenvalue of the p-Laplacian with nonlinear boundary condition*, Electron. J. Differential Equations, 2019 (2019) No. 32, 1–29.

[7] M. Cuesta, L. Leadi; *Weighted eigenvalue problems for quasilinear elliptic operators with mixed Robin-Dirichlet boundary conditions*, J. Math. Anal. Appl., 422 (1) (2015) 1-26. doi:10.1016/j.jmaa.2014.08.015.

[8] M. A. del Pino, R. F. Manásevich, A.E. Murúa; *Existence and multiplicity of solutions with prescribed period for a second order quasilinear ODE*, Nonlinear Anal., 18 (1) (1992), 79-92. doi:10.1016/0362-546X(92)90048-J.

[9] P. Drábek; *Solvability and bifurcations of nonlinear equations*, Vol. 264 of Pitman Research Notes in Mathematics Series, Logman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1992.

[10] T. Godoy, J. P. Gossez, S. Paczka; *On the antimaximum principle for the p-Laplacian with indefinite weight*, Nonlinear Anal. 51 (3) (2002) 449-467. doi:10.1016/s0362-546X(01)00839-2.

[11] T. Godoy, J. P. Gossez, S. Paczka; *Antimaximum principle for elliptic problems with weight*, Electron. J. Differential Equations, 1999 (1999) No. 22, 1–15.

[12] L. Leadi, A. Marcos; *A weighted eigencurve for Steklov problems with a potential*, NoDEA Nonlinear Differential Equations Appl. 20 (3) (2013) 687-713. doi:10.1007/s00030-012-0175-0.

[13] J. Serrin; *Local behavior of solutions of quasi-linear equations*, Acta Math. 111 (1964) 247-302. doi:10.1007/BF02391014.

[14] Y.-Q. Song, Y.-M. Chu, B.-Y. Liu, M.-K. Wang; *A note on generalized trigonometric and hyperbolic functions*, J. Math. Inequal., 8 (3) (2014) 630-642. doi:10.7153/jmi-08-46.

[15] J. L. Vázquez; *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim. 12 (3) (1984) 191-202. doi:10.1007/BF01449041.

Mabel Cuesta
Université du Littoral ULCO, LMPA, 50 rue F. Buisson 62220 Calais, France
Email address: mabel.cuesta@univ-littoral.fr

Liamidi Leadi
Université d'Abomey Calavi, FAST, IMSP, Porto-Novo, Bénin
Email address: leadiare@imsp-uac.org

Pascaline Nshimirimana
Université d'Abomey Calavi, FAST, IMSP, Porto-Novo, Bénin
Email address: pascaline.nshimirimana@imsp-uac.org