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# NONLOCAL PROBLEMS FOR HYPERBOLIC EQUATIONS FROM THE VIEWPOINT OF STRONGLY REGULAR BOUNDARY CONDITIONS

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Abstract. In this article, we consider a nonlocal problem for hyperbolic equation with integral conditions and show their close connection with the notion of strongly regular boundary conditions. This has an important bearing on the method of the study of solvability. We propose also a new approach which enables us to prove a unique solvability of the nonlocal problem with integral condition.

## 1. Introduction

In this article, we consider the nonlocal problem for hyperbolic equations

$$\mathcal{L}u \equiv u_{tt} - (a(x,t)u_x)_x + c(x,t)u = f(x,t). \tag{1.1}$$

The question is to find a solution of (1.1) in $Q_T = (0,l) \times (0,T)$, with $l, T < \infty$, satisfying the initial conditions

$$u(x,0) = 0, \quad u_t(x,0) = 0 \tag{1.2}$$

and nonlocal conditions

$$\int_0^l K_i(x)u(x,t)dx = 0, \quad i = 1,2. \tag{1.3}$$

Various phenomena of modern natural science often lead to nonlocal problems on mathematical modeling, and nonlocal models turn out to be often more precise that local conditions; see [5]. Nonlocal problems form a relatively new division of differential equations theory and generate a need in developing some new methods of research [30]. Nowadays various nonlocal problems for partial differential equations are actively studied and one can find a lot of papers dealing with them; see [2, 9, 14, 13, 18] and references therein. We focus our attention on nonlocal problems with integral conditions for hyperbolic equations which have been studied in [1, 3, 4, 6, 12, 9, 25, 17, 19, 23, 27, 28]. Systematic studies of nonlocal problems with integral conditions originated with the papers by Cannon [10] and Kamynin [16]. These and further investigations of nonlocal problems show that classical methods most widely used to prove solvability of initial-boundary problems break down when applied to

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nonlocal problems. Nowadays several methods have been devised for overcoming the difficulties arising because of nonlocal conditions.

It appears that conditions for the existence and uniqueness of a solution to the nonlocal problem are closely related to the notion of regular boundary conditions [7, 8, 32]. It is known that the system of root functions of an ordinary differential operator with strongly regular boundary conditions form a Riesz basis in $L_2(0,1)$. This property is particularly useful for obtaining results on solvability of boundary problems. For convenience we state here a criterium for strong regularity of boundary-value conditions for $n=2$ in an easy-to-use form [24, pp. 72-73].

*Sturm-Liuville problem:* Consider

$$y'' + \lambda y = 0$$

with the conditions

$$\begin{aligned} a_1y'(0) + b_1y'(l) + a_0y(0) + b_0y(l) = 0,\\ c_1y'(0) + d_1y'(l) + c_0y(0) + d_0y(l) = 0. \end{aligned} \tag{1.4}$$

If the coefficients in (1.4) satisfy one of the following sets of conditions

(I) $a_1d_1 - b_1c_1 \neq 0$;
(II) $a_1d_1 - b_1c_1 = 0$, $|a_1| + |b_1| > 0$, $b_1c_0 + a_1d_0 \neq 0$;
(III) $a_1 = b_1 = c_1 = d_1 = 0$, $a_0d_0 - b_0c_0 \neq 0$,

then (1.4) are strongly regular.

Before we return to the main problem (1.1)–(1.3), we mention one of initial works dealing with nonlocal problems. In 1897 Steclov [31] considered the problem for the heat equation with nonlocal boundary conditions

$$\begin{aligned} a_1u_x(0,t) + b_1u_x(l,t) + a_0u(0,t) + b_0u(l,t) = 0,\\ c_1u_x(0,t) + d_1u_x(l,t) + c_0u(0,t) + d_0u(l,t) = 0. \end{aligned} \tag{1.5}$$

It is obvious that separating of variables in (1.5) leads to (1.4). Much later nonlocal problems with conditions of the form (1.5) were studied in [15, 14, 22] and other papers. A feature of the problems with nonlocal conditions is that operator generated by such conditions, in particular (1.5), is not self-adjoint. But if nonlocal conditions of the form (1.4) (as a result of the separation of variables in (1.5)) are strongly regular then there exists a unique solution to the nonlocal problem (see [15]).

Now let us return to problem (1.1)–(1.3). Note that (1.3) are first-kind integral conditions as both of them include only integral terms. (The kind of a nonlocal integral condition depends on presence or lack of terms containing a trace of the required solution or its derivative outside the integral). Such conditions cause a considerable difficulties when we try to show that (1.1)–(1.3) is solvable. One method has been advanced for overcoming this difficulty [27]. Its essential idea is as follows. We reduce the first-kind integral conditions to the second-kind ones. To do this, we suppose that $u(x,t)$ is a solution to (1.1)–(1.3), multiply (1.1) by $K_i(x)$

and integrate over $(0, l)$. As a result we obtain

$$\begin{aligned} &K_i(0)a(0,t)u_x(0,t) - K_i(l)a(l,t)u_x(l,t) - K_i'(0,)a(0,t)u(0,t) \\ &+ K_i'(l)a(l,t)u(l,t) - \int_0^l [(K_i'(x)a(x,t))_x - K_i(x)c(x,t)]u(x,t)dx \\ &= \int_0^l K_i(x)f(x,t)dx. \end{aligned} \tag{1.6}$$

Let us denote

$$\begin{gathered} a_1(t) = K_1(0)a(0,t), \quad b_1(t) = -K_1(l)a(l,t), \\ a_0(t) = -K_1'(0)a(0,t), \quad b_0(t) = K_1'(l)a(l,t), \\ c_1(t) = K_2(0)a(0,t), \quad d_1(t) = -K_2(l)a(l,t), \\ c_0(t) = -K_2'(0)a(0,t), \quad d_0(t) = K_2'(l)a(l,t), \\ H_i(x,t) = (K_i'(x)a(x,t))_x - K_i(x)c(x,t), \quad g_i(t) = \int_0^l K_i(x)f(x,t)dx \end{gathered}$$

and write now (1.6) (omitting the arguments of coefficients) as

$$\begin{aligned} a_1u_x(0,t) + b_1u_x(l,t) + a_0u(0,t) + b_0u(l,t) - \int_0^l H_1(x,t)u(x,t)dx = g_1(t), \\ c_1u_x(0,t) + d_1u_x(l,t) + c_0u(0,t) + d_0u(l,t) - \int_0^l H_2(x,t)u(x,t)dx = g_2(t). \end{aligned} \tag{1.7}$$

This system may be interpreted as perturbed Steclov conditions (1.5) (see [31, 29]). Thus, we establish certain formal connections between (1.7) and (1.4). We will consider it essentially in the next section and show that the nonlocal problem has a unique solution if coefficients of non-perturbed part satisfy one of conditions (I)–(III). The choice of a method depends on a particular criterion.

## 2. Solvability of nonlocal problems

2.1. **Formulation of the problem.** It was mentioned in the introduction that integral conditions (1.3) can be reduced to second-kind integral conditions (1.7). As problems (1.1)–(1.3) and (1.1)–(1.2), and (1.7) are equivalent [27], we will consider the problem with integral conditions (1.7). For convenience we formulate this problem here with a new indexing: find in $Q_T$ a solution of the hyperbolic equation

$$u_{tt} - (a(x,t)u_x)_x + c(x,t)u = f(x,t) \tag{2.1}$$

satisfying the initial conditions

$$u(x,0) = 0, \quad u_t(x,0) = 0, \quad x \in [0,l] \tag{2.2}$$

and nonlocal conditions ($t \in [0,T]$),

$$\begin{aligned} a_1u_x(0,t) + b_1u_x(l,t) + a_0u(0,t) + b_0u(l,t) - \int_0^l H_1(x,t)u(x,t)dx = g_1(t), \\ c_1u_x(0,t) + d_1u_x(l,t) + c_0u(0,t) + d_0u(l,t) - \int_0^l H_2(x,t)u(x,t)dx = g_2(t). \end{aligned} \tag{2.3}$$

Note that there is no loss of generality in supposing that initial conditions are homogeneous.

2.2. **Criterium I.** For all $t \in [0,T]$,

$$\Delta_1 \equiv a_1 d_1 - b_1 c_1 \neq 0. \tag{2.4}$$

Solving (2.3) as a system with respect to $u_x(0,t), u_x(l,t)$, we obtain

$$\begin{gathered}
a(0,t)u_x(0,t) + \alpha_{11}(t)u(0,t) + \alpha_{12}(t)u(l,t) + \int_0^l H_{11}(x,t)u(x,t)dx = g_{11}(t),\\
a(l,t)u_x(l,t) + \alpha_{21}(t)u(0,t) + \alpha_{22}(t)u(l,t) + \int_0^l H_{12}(x,t)u(x,t)dx = g_{12}(t),
\end{gathered} \tag{2.5}$$

$$\begin{gathered}
\alpha_{11}(t) = \frac{a_0 d_1 - c_0 b_1}{\Delta_1} a(0,t), \quad \alpha_{12}(t) = \frac{b_0 d_1 - d_0 b_1}{\Delta_1} a(0,t),\\
\alpha_{21}(t) = \frac{a_0 c_1 - c_0 a_1}{\Delta_1} a(l,t), \quad \alpha_{22}(t) = \frac{b_0 c_1 - d_0 a_1}{\Delta_1} a(l,t),\\
H_{11}(x,t) = \frac{d_1 H_1(x,t) - b_1 H_2(x,t)}{\Delta_1} a(0,t),\\
H_{12}(x,t) = \frac{c_1 H_1(x,t) - a_1 H_2(x,t)}{\Delta_1} a(l,t),\\
g_{11}(t) = \frac{(d_1 g_1(t) - b_1 g_2(t))a(0,t)}{\Delta_1}, \quad g_{12}(t) = \frac{(c_1 g_1(t) - a_1 g_2(t))a(l,t)}{\Delta_1}.
\end{gathered}$$

This form of integral conditions enables to apply, with only little modifications, a well-known method for boundary-value problem [21], based on a priori estimates. In our view, this approach is effective for studying nonlocal problems with conditions of the form (2.5). It was used for some particular cases [6, 27] so we will not demonstrate it here in detail.

**Problem 1.** Find a solution $u(x,t)$ to (2.1) satisfying (2.2) and (2.5).

We consider the Sobolev space $W_2^1(Q_T)$ and denote

$$\widehat{W}_2^1(Q_T) = \{v(x,t) : v \in W_2^1(Q_T),\ v(x,T) = 0\}.$$

Let $u(x,t)$ be a solution to the Problem I and $v \in \widehat{W}_2^1(Q_T)$. Using a standard method [21, p. 92] and taking into account (2.5) and $u_t(x,0) = 0$ we derive the equality

$$\begin{aligned}
&\int_0^T \int_0^l (-u_t v_t + a u_x v_x + cuv)\,dx\,dt\\
&- \int_0^T v(0,t)[\alpha_{11}u(0,t) + \alpha_{12}u(l,t)]\,dt + \int_0^T v(l,t)[\alpha_{21}u(0,t) + \alpha_{22}u(l,t)]\,dt\\
&- \int_0^T v(0,t)\int_0^l H_{11}(x,t)u(x,t)\,dx\,dt + \int_0^T v(l,t)\int_0^l H_{12}(x,t)u(x,t)\,dx\,dt\\
&= \int_0^T \int_0^l fv\,dx\,dt + \int_0^T v(0,t)g_{11}(t)\,dt - \int_0^T v(l,t)g_{12}(t)\,dt.
\end{aligned} \tag{2.6}$$

A function $u \in W_2^1(Q_T)$ is said to be a weak solution to the Problem I if $u(x,0) = 0$ and for every $v \in \widehat{W}(Q_T)$ the identity (2.6) holds.

**Theorem 2.1.** *Assume that*

(i) $a \in C(\bar{Q}_T) \cap C^1(Q_T)$, $c \in C(\bar{Q}_T)$, $a(x,t) > 0$ *for all* $(x,t) \in \bar{Q}_T$*;*

(ii) $H_{1i}, H_{1it} \in C(\bar{Q}_T)$, $f \in L_2(Q_T)$, $g_{1i} \in W_2^1(0,T)$, $i = 1,2$;
(iii) $\alpha_{12} + \alpha_{21} = 0$, $\alpha_{11}\xi^2 + 2\alpha_{12}\xi\eta - \alpha_{22}\eta^2 \le 0$.

*Then there exists a unique weak solution to Problem I.*

The proof of this theorem for $a(x,t) = 1$ one can find in [27]. It is not too difficult to show this theorem for $a(x,t)$ not constant.

2.3. **Criterium II.** Now let $\Delta \equiv a_1d_1 - b_1c_1 = 0$ and $|a_1| + |b_1| > 0$. We will not loss too much generality if suppose that the coefficients in (2.1) do not depend on $t$. This assumption simplifies the computational work. Then (2.3) can be written as

$$\begin{gathered} a_1u_x(0,t) + b_1u_x(l,t) + a_0u(0,t) + b_0(t)u(l,t) - \int_0^l H_1(x)u(x,t)dx = g_1(t), \\ c_0u(0,t) + d_0u(l,t) - \int_0^l H_2(x)u(x,t)dx = g_2(t). \end{gathered} \tag{2.7}$$

**Problem 2.** Find a solution $u(x,t)$ to (2.1) satisfying initial conditions

$$u(x,0) = 0, \quad u_t(x,0) = 0$$

and nonlocal conditions (2.7).

We can not give at once a definition of a weak solution to this problem as for Problem 1. In response to this, the following can be done.

Let $u(x,t)$ be a solution to the Problem 2. Differentiating the second relation of (2.7) with respect to $t$ twice we obtain:

$$c_0u_{tt}(0,t) + d_0u_{tt}(l,t) + \int_0^l H_2(x)u_{tt}dx = g_2''(t).$$

As $u(x,t)$ satisfies (2.1), we have

$$\int_0^l H_2(x)u_{tt}(x,t)dx = \int_0^l H_2(x)[(au_x)_x - cu + f]dx.$$

After some manipulation,

$$\begin{aligned} \int_0^l H_2(x)(au_x)_x dx &= H_2(l)a(l)u_x(l,t) - H_2(0)a(0)u_x(0,t) - H_2'(l)a(l)u(l,t) \\ &\quad + H_2'(0)a(0)u(0,t) + \int_0^l (H_2'(x)a(x))_x u(x,t)dx. \end{aligned}$$

Then the second relation in (2.7) becomes

$$\begin{aligned} &c_0u_{tt}(0,t) + d_0u_{tt}(l,t) - H_2(l)a(l)u_x(l,t) + H_2(0)a(0)u_x(0,t) + H_2'(l)a(l)u(l,t) \\ &- H_2'(0)a(0)u(0,t) - \int_0^l [(H_2'(x)a(x))_x - H_2(x)c(x)]u(x,t)dx = g_{22}(t) \end{aligned}$$

where $g_{22}(t) = g_2''(t) + \int_0^l H_2(x)f(x,t)dx$. Equation (2.7) can be written as

$$
\begin{aligned}
&a_1 u_x(0,t) + b_1 u_x(l,t) + a_0 u(0,t) + b_0(t)u(l,t) - \int_0^l H_1 u\,dx = g_1(t),\\
&- H_2(0)a(0)u_x(0,t) + H_2(l)a(l)u_x(l,t) + H_2'(0)a(0)u(0,t) - H_2'(l)a(l)u(l,t)\\
&+ c_0 u_{tt}(0,t) + d_0 u_{tt}(l,t) - \int_0^l [(H_2'(x)a(x))_x - H_2(x)c(x)]u(x,t)dx\\
&= g_{22}(t).
\end{aligned} \tag{2.8}
$$

If $\Delta_2 = a_1 H_2(l)a(l) + b_1 H_2(0)a(0) \neq 0$, then we can solve system (2.8) with respect to $u_x(0,t)$ and $u_x(l,t)$:

$$
\begin{aligned}
a(0)u_x(0,t) &= \alpha_{11}u(0,t) + \alpha_{12}u(l,t) + \beta_{11}u_{tt}(0,t)\\
&\quad + \beta_{12}u_{tt} + \int_0^l P_1 u\,dx + G_1(t),\\
a(l)u_x(l,t) &= \alpha_{21}u(0,t) + \alpha_{22}u(l,t) + \beta_{21}u_{tt}(0,t)\\
&\quad + \beta_{22}u_{tt} + \int_0^l P_2 u\,dx + G_2(t),
\end{aligned} \tag{2.9}
$$

where $\alpha_{ij}, \beta_{ij}, P_i(x), G_i(t)$ $i,j = 1,2$ can be find easily, for example,

$$
\begin{gathered}
\alpha_{11} = \frac{H_2'(0)b_1 a(0) - H_2(l)a_0 a(l)}{\Delta_2}a(0), \quad \beta_{11} = \frac{c_0 b_1}{\Delta_2}a(0),\\
P_1(x) = \frac{H_1(x)H_2(l)a(l) + (H_2'(x)a(x))_x b_1 - H_2(x)c(x)b_1}{\Delta_2}a(0).
\end{gathered}
$$

(We do not cite all formulas because of their length). Conditions (2.9)) are known as dynamical conditions [11, 20, 33].

Thus if (2.7) holds then (2.9) also holds. The converse is also true if $g_2(0) = g_2'(0) = 0$. Indeed, let $u(x,t)$ be a solution of (2.1) and let (2.9) hold. Then (2.8) holds. Integrating $\int_0^l (H_2'(x)a(x))_x u(x,t)dx$ by parts and taking into account that $u(x,t)$ is a solution to (2.1) we easily arrive to

$$
\frac{d^2}{dt^2}[c_0 u(0,t) + d_0 u(l,t) + \int_0^l H_2(x)u(x,t)dx - g_2(t)] = 0.
$$

Integrating this equality with respect to $t$ twice, taking into account homogeneous initial data $c_0 u(0,0) + d_0 u(l,0) + \int_0^l H_2(x)u(x,0)dx - g_2(0) = 0$, $c_0 u_t(0,0) + d_0 u_t(l,0) + \int_0^l H_2(x)u_t(x,0)dx - g_2'(0) = 0$ we obtain (2.7).

Thus the nonlocal conditions (2.7) and (2.9) are equivalent, so we will consider the Problem 2 as follows: find a solution $u(x,t)$ to (2.1) satisfying (2.2) and (2.9).

This form of nonlocal conditions enables us to introduce a notation of a weak solution. Following [21, p. 92], we obtain

$$\begin{aligned}
&\int_0^T\int_0^l(-u_tv_t+au_xv_x+cuv)\,dx\,dt+\int_0^T v(0,t)[\alpha_{11}u(0,t)+\alpha_{12}u(l,t)]\,dt\\
&-\int_0^T v_t(0,t)[\beta_{11}u_t(0,t)+\beta_{12}u_t(l,t)]\,dt+\int_0^T v(0,t)\int_0^l P_1(x)u(x,t)\,dx\,dt\\
&-\int_0^T v(l,t)[\alpha_{21}u(0,t)+\alpha_{22}u(l,t)]\,dt\\
&+\int_0^T v_t(l,t)[\beta_{21}u_t(0,t)+\beta_{22}u_t(l,t)]\,dt-\int_0^T v(l,t)\int_0^l P_2(x)u(x,t)\,dx\,dt\\
&=\int_0^T\int_0^l f(x,t)v(x,t)\,dx\,dt-\int_0^T v(0,t)G_1(t)\,dt+\int_0^T v(l,t)G_2(t)\,dt.
\end{aligned}\tag{2.10}$$

Let us denote

$$\begin{gathered}
\Gamma_0=\{(x,t):x=0,t\in[0,T]\},\quad \Gamma_l=\{(x,t):x=l,t\in[0,T]\},\quad \Gamma=\Gamma_0\cup\Gamma_l,\\
W(Q_T)=\{u:u\in W_2^1(Q_T),\quad u_t\in L_2(\Gamma)\},\\
\widehat{W}(Q_T)=\{v(x,t):v(x,t)\in W(Q_T),\quad v(x,T)=0\}.
\end{gathered}$$

A function $u\in W(Q_T)$ is said to be a weak solution to the Problem 2 if $u(x,0)=0$ and for every $v\in\widehat{W}(Q_T)$ the (2.10) holds.

**Theorem 2.2.** *Assume the following conditions*

- (i) $a\in C(\bar Q_T)$, $a(x,t)>0$ *for all* $(x,t)\in\bar Q_T$, $c\in C(\bar Q_T)$;
- (ii) $P_i\in C(\bar Q_T)$, $f\in L_2(Q_T)$, $f_t\in L_2(Q_T)$, $G_i\in C[0,T]\cap C^1(0,T)$;
- (iii) $\beta_{11}\xi^2+2\beta_{21}\xi\eta-\beta_{22}\eta^2\ge 0$;
- (iv) $\alpha_{12}+\alpha_{21}=0$, $\beta_{12}+\beta_{21}=0$;
- p(v) $\beta_{11}>0$, $\beta_{22}<0$, $\beta_{11}-|\beta_{21}|>0$, $-\beta_{22}-|\beta_{21}|>0$.

*Then there exists a unique weak solution to Problem 2.*

*Proof.* **Uniqueness.** Suppose that $u_1$ and $u_2$ are two solutions to Problem 2. Then $u=u_1-u_2$ satisfies initial condition $u(x,0)=0$, and the equation

$$\begin{aligned}
&\int_0^T\int_0^l(-u_tv_t+au_xv_x+cuv)\,dx\,dt+\int_0^T v(0,t)[\alpha_{11}u(0,t)+\alpha_{12}u(l,t)]\,dt\\
&-\int_0^T v_t(0,t)[\beta_{11}u_t(0,t)+\beta_{12}u_t(l,t)]\,dt+\int_0^T v(0,t)\int_0^l P_1(x)u(x,t)\,dx\,dt\\
&-\int_0^T v(l,t)[\alpha_{21}u(0,t)+\alpha_{22}u(l,t)]\,dt\\
&+\int_0^T v_t(l,t)[\beta_{21}u_t(0,t)+\beta_{22}u_t(l,t)]\,dt-\int_0^T v(l,t)\int_0^l P_2(x)u(x,t)\,dx\,dt=0.
\end{aligned}$$

Setting

$$v(x,t)=\begin{cases}\int_\tau^t u(x,\eta)d\eta, & 0\le t\le\tau,\\ 0, & \tau\le t\le T,\end{cases}$$

where $\tau \in [0, T]$ is arbitrary, and after some manipulation we obtain

$$\begin{aligned} &\int_0^l [u^2(x,\tau) + a(x)v_x^2(x,0)]dx \\ &= 2\int_0^\tau \int_0^l cuv\,dx\,dt - \beta_{11}u^2(0,\tau) + 2\beta_{21}u(0,\tau)u(l,\tau) + \beta_{22}u^2(l,\tau) \\ &\quad + \alpha_{22}v^2(l,0) + 2\alpha_{21}v(0,0)v(l,0) - \alpha_{11}v^2(0,0) \\ &\quad + 2\int_0^\tau (\alpha_{12} + \alpha_{21})v(0,t)v_t(l,t)\,dt - 2\int_0^\tau (\beta_{12} + \beta_{21})u(0,t)u_t(l,t)\,dt \\ &\quad + \int_0^\tau v(0,t)\int_0^l P_1(x)u(x,t)\,dx\,dt - \int_0^\tau v(l,t)\int_0^l P_2(x)u(x,t)\,dx\,dt. \end{aligned} \tag{2.11}$$

Taking into account condition (iii), namelly $\beta_{11}\xi^2 + 2\beta_{21}\xi\eta - \beta_{22}\eta^2 \geq 0$, and (iiii) of Theorem 2.2 we obtain

$$\begin{aligned} &\int_0^l [u^2(x,\tau) + a(x)v_x^2(x,0)]dx \\ &\leq \Big|2\int_0^\tau \int_0^l c(x)u(x,t)v(x,t)\,dx\,dt + a_{22}v^2(l,0) + 2\alpha_{21}v(0,0)v(l,0) \\ &\quad - \alpha_{11}v^2(0,0) + \int_0^\tau v(0,t)\int_0^l P_1(x)u(x,t)\,dx\,dt \\ &\quad - \int_0^\tau v(l,t)\int_0^l P_2(x)u(x,t)\,dx\,dt\Big|. \end{aligned} \tag{2.12}$$

Note that under conditions of Theorem 2.2 there exist positive numbers $a_0, c_0, p$ such that

$$\max_{[0,l]} |c(x)| \leq c_0, \quad a(x) \geq a_0, \quad \max_i \int_0^l P_i^2(x)dx \leq p.$$

Let us denote $A = \max_{ij} |\alpha_{ij}|$. Now we estimate right-hand side of (2.12). Firstly, we use Cauchy and Cauchy-Bunyakovskii-Schwartz inequalities to obtain

$$\begin{aligned} &\int_0^l [u^2(x,\tau) + a_0 v_x^2(x,0)]dx \\ &\leq \int_0^l [u^2(x,\tau) + a(x)v_x^2(x,0)]dx \\ &\leq c_0\int_0^\tau \int_0^l [u^2(x,t) + v^2(x,t)]\,dx\,dt + A[v^2(0,0) + v^2(l,0)] \\ &\quad + \int_0^\tau [v^2(0,\tau) + v^2(l,\tau)]\,dt + 2p\int_0^\tau \int_0^l u^2(x,t)\,dx\,dt. \end{aligned}$$

Using trace inequalities,

$$v^2(z_i,t) \leq 2l\int_0^l v_x^2(x,t)dx + \frac{2}{l}\int_0^l v^2(x,t)dx, \quad z_1 = 0,\ z_2 = l,$$

(both are derived from $v(z_i,t) = \int_x^{z_i} v_\xi(\xi,t)d\xi + v(x,t)$), we obtain

$$\int_0^\tau [v^2(0,\tau) + v^2(l,\tau)]\,dt \leq 4l\int_0^\tau \int_0^l v_x^2(x,t)\,dx\,dt + \frac{4}{l}\int_0^\tau \int_0^l v^2(x,t)\,dx\,dt.$$

To estimate $A[v^2(0,0)+v^2(l,0)]$ we use the following inequalities (a partial case for $n=1$ in [21, p.77]):

$$v^2(z_i,t)\le \varepsilon\int_0^l v_x^2(x,t)dx + c(\varepsilon)\int_0^l v^2(x,t)dx,$$

where $z_1=0$, $z_2=l$ and $t\in[0,\tau]$. Then we obtain

$$v^2(0,0)\le \varepsilon\int_0^l v_x^2(x,0)dx + c(\varepsilon)\int_0^l v^2(x,0)dx,$$
$$v^2(l,0)\le \varepsilon\int_0^l v_x^2(x,0)dx + c(\varepsilon)\int_0^l v^2(x,0)dx,$$

where $c(\varepsilon)=(l+\varepsilon)/l\varepsilon$. We note also that from representation of $v(x,t)$ it follows that

$$v^2(x,t)\le \tau\int_0^\tau u^2(x,t)\,dt.$$

Hence,

$$\int_0^\tau\int_0^l v^2(x,t)\,dx\,dt \le \tau^2\int_0^\tau\int_0^l u^2(x,t)\,dx\,dt,$$
$$\int_0^l v^2(x,0)dx\le \tau\int_0^\tau\int_0^l u^2(x,t)\,dx\,dt.$$

Choosing $\varepsilon$ with due care ($\varepsilon=a_0/4$, then $a_0-2\varepsilon>0$) we obtain

$$\int_0^l [u^2(x,\tau)+\frac{a_0}{2}v_x^2(x,0)]dx \le M\int_0^\tau\int_0^l (u^2(x,t)+v_x^2(x,t))\,dx\,dt, \tag{2.13}$$

where $M=\max\{c_0+2p,(c_0+\frac{4}{l})\tau^2, A, 4l\}$.

Let $w(x,t)=\int_0^t u_x(x,\eta)d\eta$. Then

$$v_x(x,t)=w(x,t)-w(x,\tau),\quad v_x(x,0)=-w(x,\tau).$$

With the aid of these equalities we obtain

$$\int_0^l [u^2(x,\tau)+\frac{a_0}{2}w^2(x,\tau)]dx \le 2M\int_0^\tau\int_0^l [u^2+w^2]\,dx\,dt + 2M\tau\int_0^l w^2(x,\tau)\,dx\,dt.$$

As $\tau$ is arbitrary we choose it so that $a_0-4M\tau>0$. To be specific, let $a_0-4M\tau\ge\frac{a_0}{2}$. Then for all $\tau\in[0,\frac{a_0}{8M}]$

$$m_1\int_0^l [u^2(x,\tau)+w^2(x,\tau)]dx \le 2M\int_0^\tau\int_0^l (u^2+w^2)\,dx\,dt,$$

where $m_1=\min\{1,a_0/4\}$.

From Gronwall's lemma it follows that $\int_0^l [u^2(x,\tau)+w^2(x,\tau)]dx=0$. Hence $u(x,\tau)=0$ for all $\tau\in[0,\frac{a_0}{8M}]$. Using the same reasoning as in [21, p.212], we obtain $u(x,t)=0$ in $Q_T$. It means that there cannot be more than one weak solution to the Problem II.

**Existence.** First, we construct approximations of the weak solution by the Faedo-Galerkin method. Let $w_k(x)\in C^2[0,l]$ be a basis in $W_2^1(0,l)$. We define the approximations as follows

$$u^m(x,t)=\sum_{k=1}^m c_k(t)w_k(x) \tag{2.14}$$

and shall seek $c_k(t)$ from the equations

$$
\begin{aligned}
&\int_0^l (u^m_{tt}w_j + au^m_x w'_j + cu^m w_j)dx + w_k(0)\Big[\alpha_{11}u^m(0,t) + \alpha_{12}u^m(l,t)\\
&+ \beta_{11}u^m_{tt}(0,t) + \beta_{12}u^m(l,t) + \int_0^l P_1(x)u(x,t)dx\Big]\\
&- w_k(l)\Big[\alpha_{21}u^m(0,t) + \alpha_{22}u^m(l,t) + \beta_{21}u^m_{tt}(0,t) + \beta_{22}u^m(l,t)\\
&+ \int_0^l P_2(x)u(x,t)dx\Big]\\
&= \int_0^l f(x,t)w_j(x)dx - w_j(0)G_1(t) + w_j(l)G_2(t).
\end{aligned}
\tag{2.15}
$$

For every $m$, (2.15) represents a system of second-order ODE's with respect to $c_k(t)$,

$$
\sum_{k=1}^m A_{kj}c''_k(t) + \sum_{k=1}^m B_{kj}c_k(t) = f_j(t), \tag{2.16}
$$

where

$$
\begin{gathered}
A_{kj} = \int_0^l w_k(x)w_j(x)dx + \beta_{11}w_k(0)w_j(0) + \beta_{12}w_k(l)w_j(0)\\
- \beta_{21}w_k(0)w_j(l) - \beta_{22}w_k(l)w_j(l),
\end{gathered}
$$

$$
\begin{aligned}
B_{kj} = &\int_0^l (a(x)w'_k(x)w'_j(x) + c(x)w_k(x)w_j(x))dx + w_k(0)\int_0^l P_1(x)w_j(x)dx\\
&- w_k(l)\int_0^l P_2(x)w_j(x)dx + \alpha_{11}w_k(0)w_j(0) + \alpha_{12}w_k(l)w_j(0)\\
&- \alpha_{21}w_k(0)w_j(l) - \alpha_{22}w_k(0)w_j(0),
\end{aligned}
$$

$$
f_j(t) = \int_0^l f(x,t)w_j(x)dx - w_j(0)G_1(t) + w_j(l)G_2(t).
$$

Firstly we prove that this system is solvable with respect to $c''_k(t)$. To this end consider the matrix $\{A_{ij}\}$ and show that it is positive definite.

Consider the quadratic form $q = \sum_{i,j=1}^m A_{kj}\xi_k\xi_j$ and denote $z(x) = \sum_{k=1}^m \xi_k w_k(x)$. After substituting $A_{kj}$ in $q$ we obtain

$$
q = \int_0^l |z(x)|^2dx + \beta_{11}|z(0)|^2 + 2\beta_{12}|z(0)||z(l)| - \beta_{22}|z(l)|^2.
$$

From (iii) of Theorem 2.2, $q \geq 0$. As $q = 0$ if and only if $z = 0$ and $\{w_k\}$ is linearly independent then $\xi_k = 0$ for $k = 1,\dots,m$, therefore $q$ is positive definite. Hence (2.16) is solvable with respect to $c''_k(t)$. Thus we can state that under conditions of Theorem 2.2 Cauchy problem for (2.16) with initial conditions $c_k(0) = 0$, $c'_k(0) = 0$ has a solution for every $m$ and $\{u^m\}$ is constructed.

To derive the estimate we multiply (2.15) by $c'_j(t)$, sum over $j = 1,\dots,m$ and integrate over $(0,\tau)$, where $\tau \in [0,T]$ is arbitrary:

$$
\int_0^\tau \int_0^l (u^m_{tt}u^m_t + au^m_x u^m_{xt} + cu^m u^m_t)\,dx\,dt
$$

$$
\begin{aligned}
&+\int_0^\tau u_t^m(0,t)[\alpha_{11}u^m(0,t)+\alpha_{12}u^m(l,t)+\beta_{11}u_{tt}^m(0,t)+\beta_{12}u_{tt}^m(l,t)]\,dt\\
&+\int_0^\tau u_t^m(0,t)\int_0^l P_1(x)u^m(x,t)\,dx\,dt\\
&-\int_0^\tau u_t^m(l,t)[\alpha_{21}u^m(0,t)+\alpha_{22}u^m(l,t)+\beta_{21}u_{tt}^m(0,t)+\beta_{22}u_{tt}^m(l,t)]\,dt\\
&-\int_0^\tau u_t^m(l,t)\int_0^l P_2(x)u^m(x,t)\,dx\,dt\\
&=\int_0^\tau\int_0^l f(x,t)u_t^m(x,t)\,dx\,dt-\int_0^\tau u_t^m(0,t)G_1(t)\,dt+\int_0^\tau u_t^m(l,t)G_2(t)\,dt.
\end{aligned}
$$

Integration by parts and condition (iiii) lead to

$$
\begin{aligned}
&\int_0^l[(u_t^m(x,\tau))^2+a(x)(u_x^m(x,\tau))^2]dx+\beta_{11}(u_t^m(0,\tau))^2-\beta_{22}(u_t^m(l,\tau))^2\\
&=2\beta_{21}u_t^m(0,\tau)u_t^m(l,\tau)-[\alpha_{11}(u^m(0,\tau))^2+2\alpha_{21}u^m(0,\tau)u^m(l,\tau)\\
&\quad-\alpha_{22}(u^m(l,\tau))^2]-2\int_0^\tau\int_0^l cu^mu_t^m\,dx\,dt\\
&\quad+2\int_0^\tau u^m(0,t)\int_0^l P_1(x)u_t^m\,dx\,dt-2u^m(0,\tau)\int_0^l P_1(x)u^m(x,\tau)dx\\
&\quad-2\int_0^\tau u^m(l,t)\int_0^l P_2(x)u_t^m\,dx\,dt+2u^m(l,\tau)\int_0^l P_2(x)u^m(x,\tau)dx\\
&\quad+2\int_0^\tau\int_0^l fu_t^m\,dx\,dt+2\int_0^\tau u^m(0,t)G_{1t}(t)\,dt-2\int_0^\tau u^m(l,t)G_{2t}(t)\,dt\\
&\quad+2u^m(0,\tau)G_1(\tau)-2u^m(l,\tau)G_2(\tau).
\end{aligned} \tag{2.17}
$$

As $\beta_{11}>0$ and $\beta_{22}<0$, By (iiii) of Theorem 2.2, the left-hand side of (2.17) is positive. To estimate right-hand side of (2.17) we use the same technique as in the subsection for uniqueness. Therefore we demonstrate this procedure briefly. Using Cauchy and Cauchy-Bunyakovskii-Schwartz inequalities we obtain

$$
\begin{aligned}
&\int_0^l[(u_t^m(x,\tau))^2+a_0(u_x^m(x,\tau))^2]dx+\beta_{11}(u_t^m(0,\tau))^2-\beta_{22}(u_t^m(l,\tau))^2\\
&\le C_1\int_0^\tau\int_0^l[(u^m(x,t))^2+(u_t^m(x,t))^2]\,dx\,dt+2p\int_0^l(u^m(x,\tau))^2dx\\
&\quad+2\int_0^\tau[(u^m(0,t))^2+(u^m(l,t))^2]\,dt\\
&\quad+(2+\sqrt{|a_{21}|})[(u^m(0,\tau))^2+(u^m(l,\tau))^2]\\
&\quad+\sqrt{|b_{21}|}[(u_t^m(0,\tau))^2+(u_t^m(l,\tau))^2]+\int_0^\tau\int_0^l f^2(x,t)\,dx\,dt\\
&\quad+\int_0^\tau[(G_{1t}(t))^2+(G_{2t}(t))^2]\,dt+G_1^2(\tau)+G_2^2(\tau).
\end{aligned} \tag{2.18}
$$

where $C_1$ depends on $a_0, c_0, p, l, T$ and do not depend on $m$. Using the inequality

$$v^2(z_i,t) \le \varepsilon \int_0^l v_x^2(x,t)dx + c(\varepsilon)\int_0^l v^2(x,t)dx, \quad z_1 = 0,\ z_2 = l$$

we obtain

$$\begin{aligned}
&(u^m(0,\tau))^2 + (u^m(l,\tau))^2 \le 2\varepsilon \int_0^l (u_x^m(x,\tau))^2 dx + 2c(\varepsilon)\int_0^l (u^m(x,\tau))^2 dx,\\
&\int_0^\tau [(u^m(0,\tau))^2 + (u^m(l,\tau))^2]\,dt\\
&\le 2\varepsilon \int_0^\tau\int_0^l (u_x^m(x,\tau))^2\,dx\,dt + 2c(\varepsilon)\int_0^\tau\int_0^l (u^m(x,\tau))^2\,dx\,dt.
\end{aligned}$$

Taking into account (iv) in Theorem 2.2, $(u^m(x,\tau))^2 \le \tau\int_0^\tau (u_t^m(x,t))^2 dt$, and adding the inequality $\int_0^l (u^m(x,\tau))^2 dx \le \tau\int_0^\tau\int_0^l (u_t^m(x,t))^2\,dx\,dt$ to the both sides of (2.18), we obtain

$$\begin{aligned}
&\int_0^l [(u^m(x,\tau))^2 + (u_t^m(x,\tau))^2 + a_0(u_x^m(x,\tau))^2]dx\\
&+ (\beta_{11} - \sqrt{|b_{21}|})(u_t^m(0,\tau))^2 + (-\beta_{22} - \sqrt{|b_{21}|})(u_t^m(l,\tau))^2\\
&\le C_2\int_0^\tau\int_0^l [(u^m(x,t))^2 + (u_t^m(x,t))^2 + (u_x^m(x,t))^2]\,dx\,dt\\
&\quad + 2\sqrt{|a_{21}|}\varepsilon\int_0^l (u_x^m(x,\tau))^2 dx + \int_0^\tau\int_0^l f^2(x,t)\,dx\,dt\\
&\quad + \int_0^\tau [(G_{1t}(t))^2 + (G_{2t}(t))^2]\,dt + G_1^2(\tau) + G_2^2(\tau).
\end{aligned} \tag{2.19}$$

Choosing $\varepsilon$ such that $\nu = a_0 - 2\sqrt{|a_{21}|}\varepsilon > 0$, we can carry $2\sqrt{|a_{21}|}\varepsilon\int_0^l (u_x^m(x,\tau))^2 dx$ to the left-hand side of (2.19). Consequently,

$$\begin{aligned}
&\int_0^l [(u^m(x,\tau))^2 + (u_t^m(x,\tau))^2 + (u_x^m(x,\tau))^2]dx + [(u_t^m(0,\tau))^2 + (u_t^m(l,\tau))^2]\\
&\le C_3\int_0^\tau\int_0^l [(u^m(x,t))^2 + (u_t^m(x,t))^2 + (u_x^m(x,t))^2]\,dx\,dt\\
&\quad + C_4\Big(\int_0^\tau\int_0^l f^2(x,t)\,dx\,dt + \int_0^\tau [(G_{1t}(t))^2 + (G_{2t}(t))^2]\,dt\\
&\quad + G_1^2(\tau) + G_2^2(\tau)\Big).
\end{aligned} \tag{2.20}$$

In particular,

$$\begin{aligned}
&\int_0^l [(u^m(x,\tau))^2 + (u_t^m(x,\tau))^2 + (u_x^m(x,\tau))^2]dx\\
&\le C_3\int_0^\tau\int_0^l [(u^m(x,t))^2 + (u_t^m(x,t))^2 + (u_x^m(x,t))^2]\,dx\,dt\\
&\quad + C_4\Big(\int_0^\tau\int_0^l f^2(x,t)\,dx\,dt + \int_0^\tau [(G_{1t}(t))^2 + (G_{2t}(t))^2]\,dt
\end{aligned}$$

$$+ G_1^2(\tau) + G_2^2(\tau)\Big).$$

Applying Gronwall's lemma to the above inequality, after integrating over $(0,T)$, we obtain

$$\|u^m\|_{W_2^1(Q_T)} \le r_1,$$
$$r_1 = C_4 T e^{C_3 T}(\|f\|^2_{L_2(Q_T)} + \|G_1\|^2_{W_2^1(0,T)} + \|G_2\|^2_{W_2^1(0,T)}).$$

Moreover, it follows also from (2.20) that

$$\begin{aligned}(u_t^m(0,\tau))^2 + (u_t^m(l,\tau))^2 &\le C_3\|u\|^2_{W_2^1(Q_T)} + C_4\int_0^\tau\int_0^l f^2(x,t)\,dx\,dt \\ &\quad + C_4\Big(\int_0^\tau [(G_{1t}(t))^2 + (G_{2t}(t))^2]\,dt + G_1^2(\tau) + G_2^2(\tau)\Big).\end{aligned}$$

Then

$$\|u^m\|_{L_2(\Gamma)} \le r_2,$$
$$r_2 = TC_3 r_1 + TC_4(\|f\|^2_{L_2(Q_T)} + \|G_1\|^2_{W_2^1(0,T)} + \|G_2\|^2_{W_2^1(0,T)}).$$

Thus we have *a priori estimate*,

$$\|u^m\|_{W(Q_T)} \le R, \quad R = \max_i\{r_i\},\ i = 1,2. \tag{2.21}$$

Because of (2.21) we can extract a subsequence $\{u^\mu\}$ from $\{u^m\}$ such that as $\mu \to \infty$ $\{u^\mu\}$ converges weakly to $u \in W(Q_T)$. This enables us to use standard technique [21, pp. 214-215] and show that the limit of $\{u^\mu\}$ is the required weak solution to Problem 2. □

2.4. **Criterium III.** Let $a_1 = b_1 = c_1 = d_1 = 0$, $\Delta_3 = a_0 d_0 - b_0 c_0 \ne 0$. Then (2.3) can be write as

$$\begin{gathered} u(0,t) + \int_0^l S_1(x,t)u(x,t)dx = g_{31}(t), \\ u(l,t) + \int_0^l S_2(x,t)u(x,t)dx = g_{32}(t), \end{gathered} \tag{2.22}$$

where

$$S_1(x,t) = \frac{b_0 H_2(x,t) - d_0 H_1(x,t)}{\Delta_3}, \quad S_2(x,t) = \frac{c_0 H_1(x,t) - a_0 H_2(x,t)}{\Delta_3},$$
$$g_{31}(t) = \frac{d_0 g_1(t) - b_0 g_2(t)}{\Delta_3}, \quad g_{32}(t) = \frac{a_0 g_2(t) - c_0 g_1(t)}{\Delta_3}.$$

**Problem 3.** Find a solution $u(x,t)$ to equation (2.1) satisfying (2.2) and (2.22).

We can use at least two methods to show the solvability of the Problem 3. One of them may be considered as a particular case of the method used in earlier section. Namely, we differentiate both equations in (2.22) with respect to $t$ twice and arrive at dynamic nonlocal conditions. This method works for for a partial case: $S_i$ does not depend on $t$ [28]. However, this method is not effective when $S_i$ depend on $t$ also.

The second method we can apply is a particular case of the technique initiated in [17]. The main idea of this method is the following: we introduce a new unknown function $v(x,t)$ and arrive at the boundary-value problem for a loaded equation with respect to $v(x,t)$ and can use the technique represented in [17].

Here we propose a third way. Using the idea in [17] to form a new unknown function. We suggest a different method.

A function $u(x,t)$ is said to be the solution to the Problem 3 if it satisfies equation (2.1) for almost all $(x,t) \in Q_T$, the initial condition (2.2), and conditions (2.22) in the $L_2(0,T)$ sense.

**Theorem 2.3.** *Assume that: $a, a_t, a_x, a_{tt}, c, c_t \in C(\bar{Q}_T)$, $a_0, b_0, c_0, d_0 \in C^2[0,T]$,*

$$S_i, S_{it} \in C^2(\bar{Q}_T), \quad S_i(0,t) = S_i(l,t) = 0, \quad \frac{2l}{3}\int_0^l (S_1 + S_2)^2 d\xi < 1$$

*for all $t \in [0,T]$, $g_{3i} \in C^3[0,T]$, $g_{3i}(x,0) = g'_{3i}(x,0) = 0$, andf, $f_t \in L_2(Q_T)$. Then there exists a unique solution $u(x,t)$ to the problem 3.*

The proof is rather long, so we break it up into 3 steps.

**Step 1.** Reduction to a problem for a loaded equation. Let $u(x,t)$ be a solution to the Problem 3. We introduce a new function

$$v(x,t) = u(x,t) + \int_0^l \tilde{S}(x,\xi,t)u(\xi,t)d\xi - \tilde{g}(x,t)$$

where

$$\tilde{S}(x,\xi,t) = \frac{l-x}{l}S_1(\xi,t) + \frac{x}{l}S_2(\xi,t), \quad \tilde{g}(x,t) = \frac{l-x}{l}g_{31}(t) + \frac{x}{l}g_{32}(t).$$

Then $v(x,t)$ satisfies the equation

$$\begin{aligned} &v_{tt} - (av_x)_x + cv - \int_0^l (\tilde{S}u)_{tt}d\xi + (a(x,t)\int_0^l \tilde{S}_x(x,\xi,t)u(\xi,t))_x d\xi \\ &- c(x,t)\int_0^l \tilde{S}(x,\xi,t)u(\xi,t)d\xi \\ &= f + \tilde{g}_{tt} + c\tilde{g} + a_x\tilde{g}_x. \end{aligned}$$

As $u(x,t)$ satisfies (2.1), then

$$\int_0^l (\tilde{S}u)_{tt}d\xi = \int_0^l \tilde{S}_{tt}d\xi + 2\int_0^l \tilde{S}_t u_t d\xi + \int_0^l \tilde{S}[(au_\xi)_\xi - c(\xi,t)u + f]d\xi.$$

After little manipulations and taking into account the assumptions of Theorem 2.3 we obtain

$$\begin{aligned} &v_{tt} - (av_x)_x + cv \\ &= \int_0^l M(x,\xi,t)u(\xi,t)d\xi + 2\int_0^l \tilde{S}_t u_t d\xi \\ &\quad + \tilde{S}_\xi(x,0,t)a(0,t)u(0,t - \tilde{S}_\xi(x,l,t)a(l,t)u(l,t) + G(x,t), \end{aligned} \tag{2.23}$$

where

$$\begin{aligned} M(x,\xi,t) &= \tilde{S}_{tt}(x,\xi,t) + (a(\xi,t)\tilde{S}_\xi(x,\xi,t))_\xi - (a(x,t)\tilde{S}_x(x,\xi,t))_x \\ &\quad + [c(x,t) - c(\xi,t)]\tilde{S}(x,\xi,t), \\ G(x,t) &= f(x,t) + \tilde{g}_{tt}(x,t) + c(x,t)\tilde{g}(x,t) + a_x(x,t)\tilde{g}_x(x,t) \\ &\quad + \int_0^l \tilde{S}(x,\xi,t)f(\xi,t)d\xi. \end{aligned}$$

It is easy to see that

$$v(x,0) = v_t(x,0) = 0, \quad v(0,t) = v(l,t) = 0 \tag{2.24}$$

and we arrive to the next problem.

**Problem 4.** Find a solution to equation (2.23) satisfying (2.24) Note that we are required to find not only $v(x,t)$, but also $u(x,t)$. Let us denote

$$\begin{gathered} W^1_{2,0}(Q_T) = \{v(x,t) : v \in W^1_2(Q_T),\ v(0,t) = v(l,t) = 0\}, \\ \widehat{W}^1_{2,0}(Q_T) = \{\eta(x,t) : \eta \in W^1_{2,0}(Q_T),\ \eta(x,T) = 0\}. \end{gathered}$$

A pair $(u,v)$ is said to be a weak solution to Problem 4 if $u \in W^1_2(Q_T)$, $v \in W^1_{2,0}(Q_T)$, $v(x,0) = 0$, for every $\eta \in \widehat{W}^1_{2,0}(Q_T)$:

$$\begin{aligned} &\int_0^T \int_0^l (-v_t\eta_t + av_x\eta_x + cv\eta)\,dx\,dt \\ &= \int_0^T \int_0^l \eta(x,t) \int_0^l M(x,\xi,t)u(\xi,t)d\xi\,dx\,dt \\ &\quad + \int_0^T \int_0^l \eta(x,t)[\tilde{S}_\xi(x,0,t)a(0,t)u(0,t) - \tilde{S}_\xi(x,l,t)a(l,t)u(l,t)]\,dx\,dt \\ &\quad + 2\int_0^T \int_0^l \eta(x,t) \int_0^l \tilde{S}_t u_t d\xi\,dx\,dt + \int_0^T \int_0^l G(x,t)\eta(x,t)\,dx\,dt \end{aligned} \tag{2.25}$$

and $u, v$ are related by

$$v(x,t) = u(x,t) + \int_0^l \tilde{S}(x,\xi,t)u(\xi,t)d\xi - \tilde{g}(x,t). \tag{2.26}$$

**Step 2.** Solvability of Problem 4.

**Theorem 2.4.** *Under the assumptions of Theorem 2.3 there exists a unique weak solution $(u,v)$ to Problem 4.*

*Proof.* We approximate our weak solution as follows. Let $u^0 = 0$ and define $(u^n, v^n)$ by

$$\begin{aligned} &\int_0^T \int_0^l (-v^n_t\eta_t + av^n_x\eta_x + cv^n\eta)\,dx\,dt \\ &= \int_0^T \int_0^l \eta(x,t) \int_0^l M(x,\xi,t)u^{n-1}(\xi,t)d\xi\,dx\,dt \\ &\quad + \int_0^T \int_0^l \eta(x,t)[\tilde{S}_\xi(x,0,t)a(0,t)u^{n-1}(0,t) - \tilde{S}_\xi(x,l,t)a(l,t)u^{n-1}(l,t)]\,dx\,dt \\ &\quad + 2\int_0^T \int_0^l \eta(x,t) \int_0^l \tilde{S}_t u^{n-1}_t d\xi\,dx\,dt + \int_0^T \int_0^l G(x,t)\eta(x,t)\,dx\,dt, \end{aligned} \tag{2.27}$$

$$v^n(x,t) = u^n(x,t) + \int_0^l \tilde{S}(x,\xi,t)u^n(\xi,t)d\xi - \tilde{g}(x,t). \tag{2.28}$$

As $u_0 = 0$, then for $v^1$ we have

$$\int_0^T \int_0^l (-v^1_t\eta_t + av^1_x\eta_x + cv^1\eta)\,dx\,dt = \int_0^T \int_0^l G\eta\,dx\,dt.$$

This means that $v^1(x,t)$ is a weak solution of the first initial boundary problem for

$$v_{tt} - (av_x)_x + cv = G(x,t). \tag{2.29}$$

It is known [21, pp. 213-215] that this solution is unique and $\|v^1\|_{W^1_{2,0}}(Q_T) \le C\|G\|_{L_2(Q_T)}$. Moreover, as $G_t \in L_2(Q_T)$ and $a, a_t, a_{tt}, c_t$ are bounded then $v^1 \in W^2_2(Q_T)$ [21, pp. 216-219].

Now we can find $u^1(x,t)$ from (2.28) as under assumptions of Theorem 2.4 (2.28) is a second kind Fredholm integral equation with $\|\tilde S\| < 1$. Then we find $v^2(x,t)$ as a solution of the first initial boundary problem for the equation of the form (2.29) with

$$\begin{aligned} G_2(x,t) = \int_0^l M(x,\xi,t)u^1(\xi,t)d\xi + \big[\tilde S_\xi(x,0,t)a(0,t)u^1(0,t) \\ - \tilde S_\xi(x,l,t)a(l,t)u^1(l,t)\big] + 2\int_0^l \tilde S_t(x,\xi,t)u^1_t(\xi,t)d\xi + G(x,t). \end{aligned}$$

Proceeding as above we obtain $u^n(x,t)$ and $v^n(x,t)$. The conditions of Theorem 2.4 provide that for every $n$, $G_n, G_{nt} \in L_2(Q_T)$. So $u^n, v^n \in W^2_2(Q_T)$ and the sequence of pairs $(u^n, v^n)$ is well defined.

Now we show that this sequence converges as $n \to \infty$ in $W^1_{2,0}$ and the limit pair $(u,v)$ is the weak solution of the Problem 4.

Let $z^n = v^{n+1} - v^n$, $r^n = u^{n+1} - u^n$. From (2.27) and (2.28) we have

$$\begin{aligned} &\int_0^T \int_0^l (-z^n_t\eta_t + az^n_x\eta_x + cz^n\eta)\,dx\,dt \\ &= \int_0^T \int_0^l \eta(x,t) \int_0^l M(x,\xi,t)r^{n-1}(\xi,t)d\xi\,dx\,dt \\ &\quad + \int_0^T \int_0^l \eta(x,t)[\tilde S_\xi(x,0,t)a(0,t)r^{n-1}(0,t) - \tilde S_\xi(x,l,t)a(l,t)r^{n-1}(l,t)]\,dx\,dt \\ &\quad + 2\int_0^T \int_0^l \eta(x,t) \int_0^l \tilde S_t r^{n-1}_t d\xi\,dx\,dt, \end{aligned} \tag{2.30}$$

$$z^n(x,t) = r^n(x,t) + \int_0^l \tilde S(x,\xi,t)r^n(\xi,t)d\xi. \tag{2.31}$$

The assumptions of Theorem 2.4 provide the existence of positive number $s_0$ such that

$$\max_{\bar Q_T}\{\int_0^l \tilde S^2 d\xi, \int_0^l \tilde S^2_t d\xi, \int_0^l \tilde S^2_\xi d\xi\} \le s_0.$$

Then for $1 - 2s_0 l > 0$ from (2.31), we obtain

$$\|r^n\|^2_{L_2(Q_T)} \le \frac{2}{1-2s_0 l}\|z^n\|^2_{L_2(Q_T)}. \tag{2.32}$$

From $z_t^n = r_t^n + \int_0^l \tilde{S}_t r^n d\xi + \int_0^l \tilde{S} r_t^n d\xi$ and $z_x^n = r_x^n + \int_0^l \tilde{S}_x r^n d\xi$, for $1 - 3s_0 l > 0$, we obtain

$$\|r_t^n\|^2_{L_2(Q_T)} \le \frac{3}{1-s_0 l}\|z_t^n\|^2_{L_2(Q_T)} + \frac{6s_0 l}{(1-2s_0 l)(1-3s_0 l)}\|z^n\|^2_{L_2(Q_T)}, \tag{2.33}$$

$$\|r_x^n\|^2_{L_2(Q_T)} \le \|z_t^n\|^2_{L_2(Q_T)} + \frac{4s_0 l}{(1-2s_0 l)^2}\|z^n\|^2_{L_2(Q_T)}. \tag{2.34}$$

Then from (2.32)–(2.34),

$$\|r^n\|^2_{W_2^1(Q_T)} \le A\|z^n\|^2_{W_2^1(Q_T)} \tag{2.35}$$

where $A$ depends on $a_0, a_1, c_0, s_0, l, T$. To proceed further we prove the following statement.

If $v \in W_2^2(Q_T)$ is a solution to the first initial-boundary problem in (2.29), then for almost all $\tau \in [0,T]$,

$$\begin{aligned} &\int_0^l [(v^2(x,\tau))^2 + a(x,\tau)v_x^2(x,\tau)]dx \\ &= \int_0^\tau \int_0^l a_t v_x^2\, dx\, dt - 2\int_0^\tau \int_0^l cvv_t\, dx\, dt + 2\int_0^\tau \int_0^l Gv_t\, dx\, dt. \end{aligned} \tag{2.36}$$

We obtain (2.36) after integrating the equality $(v_{tt} - (av_x)_x + cv)v_t = G(x,t)v_t$ over $Q_\tau = (0,l) \times (0,\tau)$.

Note that we seek $z^n$ as a solution of (2.29) with

$$\begin{aligned} G(x,t) = \int_0^l M(x,\xi,t) r^{n-1}(\xi,t)d\xi + \Big[\tilde{S}_\xi(x,0,t)a(0,t)r^{n-1}(0,t) \\ - \tilde{S}_\xi(x,l,t)a(l,t)r^{n-1}(l,t)\Big] + 2\int_0^l \tilde{S}_t(x,\xi,t) r_t^{n-1}(\xi,t)d\xi. \end{aligned}$$

Under the assumptions of Theorem 2.4 there exist positive numbers $a_0, a_1, c_0, s_1$ such that $a(x,t) \ge a_0$, $|c(x,t)| \le c_0$, $|a(x,t), a_x(x,t), a_t(x,t)| \le a_1$,

$$s_1 = \max\big\{\max_{\bar{Q}_T} \int_0^l (\tilde{S}_{tt})^2 d\xi,\ \max_{\bar{Q}_T} \int_0^l (\tilde{S}_{\xi\xi})^2 d\xi\big\}.$$

From Theorem 2.1 and some manipulations (integrating and using Cauchy and Cauchy "with $\varepsilon$" inequalities) we obtain

$$\begin{aligned} &\int_0^l [(z^n(x,\tau))^2 + a(x,\tau)(z_x^n(x,\tau))^2]dx \\ &\le A_1 \int_0^\tau \int_0^l [(z^n)^2 + (z_t^n)^2 + (x_x^n)^2]\, dx\, dt \\ &\quad + c(\varepsilon)\int_0^\tau \int_0^l (z_t^n)^2\, dx\, dt + \varepsilon \int_0^\tau \int_0^l G^2(x,t)\, dx\, dt. \end{aligned}$$

Estimating the last term with the help of Cauchy-Bunyakovskii-Schwartz inequality, and the trace inequalities

$$(r^{n-1}(z_i,t))^2 \le 2l\int_0^l (r_x^{n-1}(x,t))^2 dx + \frac{2}{l}\int_0^l (r^{n-1}(x,t))^2 dx, \quad z_1 = 0,\ z_2 = l,$$

we obtain

$$\begin{aligned}
&\int_0^l [(z^n(x,\tau))^2 + a(x,\tau)(z^n_x(x,\tau))^2]dx \\
&\le A_2 \int_0^\tau \int_0^l [(z^n)^2 + (z^n_t)^2 + (x^n_x)^2]\,dx\,dt \\
&\quad + \varepsilon A_3 \int_0^\tau \int_0^l [(r^{n-1})^2 + (r^{n-1}_t)^2 + (r^{n-1}_x)^2]\,dx\,dt.
\end{aligned}$$

From Gronwall's lemma and integrating over $(0,T)$ it follows that

$$\|z^n\|^2_{W^1_2(Q_T)} \le \varepsilon B \|r^{n-1}\|^2_{W^1_2(Q_T)}, \tag{2.37}$$

where $B = TA_3 e^{A_2 T}$, $A_i$, $i = 1,2,3$, depend only on $a_0, a_1, c_0, s_1, l, T$. From (2.35) and (2.37) it follows that

$$\|z^n\|^2_{w^1_2(Q_T)} \le AB\varepsilon \|z^{n-1}\|^2_{w^1_2(Q_T)}, \quad \|r^n\|^2_{w^1_2(Q_T)} \le AB\varepsilon \|r^{n-1}\|^2_{w^1_2(Q_T)}. \tag{2.38}$$

We select a small $\varepsilon$ such that $0 < \varepsilon AB < 1$. Hence, $\{u^n, v^n\}$ is a Cauchy sequence in $W^1_2(Q_T)$. Thus, there exists a unique pair $(u,v) \in W^1_2(Q_T)$ such that $u^n \to u$, $v^n \to v$. Let $n \to \infty$ in (2.27), (2.28). From the converges of $\{u^n, v^n\}$ we see that $(u,v)$ is the required solution of the Problem 4. □

**Step 3.** Solvability of the Problem 3. As we noted above, under assumptions of Theorem 2.4, the weak solution $(u,v)$ belongs to $W^2_2(Q_T)$. That is why we can rewrite (2.25) as

$$\begin{aligned}
&\int_0^T \int_0^l (v_{tt} - (av_x)_x + cv)\eta\,dx\,dt \\
&= \int_0^T \int_0^l \eta(x,t) \int_0^l M(x,\xi,t)u(\xi,t)d\xi\,dx\,dt \\
&\quad + \int_0^T \int_0^l \eta(x,t)[\tilde S_\xi(x,0,t)a(0,t)u(0,t) - \tilde S_\xi(x,l,t)a(l,t)u(l,t)]\,dx\,dt \\
&\quad + 2\int_0^T \int_0^l \eta(x,t) \int_0^l \tilde S_t u_t d\xi\,dx\,dt + \int_0^T \int_0^l G(x,t)\eta(x,t)\,dx\,dt.
\end{aligned} \tag{2.39}$$

Substituting $v(x,t)$ represented by (2.26) into (2.39), after some manipulations, we obtain

$$\int_0^T \int_0^l (u_{tt} - (au_x)_x + cu)\eta(x,t)\,dx\,dt = \int_0^T \int_0^l f(x,t)\eta(x,t)\,dx\,dt$$

for all $\eta \in \mathring{W}^2_2(Q_T)$. So, $u(x,t)$ is the solution to the (2.1). Obviously, the conditions (2.22) are fulfilled. This completes the proof of Theorem 2.3.

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## References


[1] Ashyralyev, A.; Aggez, N.; *On the Solution of Multipoint NBVP for Hyperbolic Equation with Integral Condition*, Malaysian Journal of Mathematical Sciences 2012, 6(S), p.111-121.

[2] Ashyralyev, A.; Sharifov, Y. A.; *Optimal control problems for impulsive systems with integral boundary conditions*, Electron. J. Differential Equ. Vol. 2013 (2013), no. 80, p. 1-11.

[3] Avalishvili, G.; Avalishvili, M.; Gordeziani, D.; *On Integral Nonlocal Boundary Value Problems for some Partial Differential Equations*, Bulletin of the Georgian National Academy of Sciences, 2011, vol. 5, no. 1, p. 31-37.

[4] Assanova, A. T.; *Nonlocal problem with integral conditions for the system of hyperbolic equations in the characteristic rectangle*, Russian Math., 2017, no. 5, pp. 7–20.

[5] Bažant, Zdeněk, P.; Jirásek, Milan; *Nonlocal Integral Formulation of Plasticity And Damage: Survey of Progress*, American Society of Civil Engineers. Journal of Engineering Mechanics, 2002, pp. 1119-1149.

[6] Beilin, S. A.; *On a mixed nonlocal problem for a wave equation*, Electron. J. Differential Equ., Vol. 2006 (2006), No. 103, pp. 1-10.

[7] Birkhoff, G. D.; *On the asymptotic character of the solutions of certain linear differential equations containing a parameter*, Trans. Am. Math. Soc., 1908, no. 9, pp. 21–231.

[8] Birkhoff, G. D.; *Boundary value and expansion problems of ordinary linear differential equations*, Trans. Am. Math. Soc., 1908, no. 9, pp. 373–395.

[9] Bouziani, A.; *On the solvability of parabolic and hyperbolic problems with a boundary integral condition*, IJMMS journal, Volume 31 (2002), Issue 4, Pages 201-213.

[10] Cannon, J. R.; *The solution of the heat equation subject to the specification of energy*, Quart. Appl. Math. 1963. no. 21. P. 155–160.

[11] Doronin, G. G.; Larkin, N. A.; Souza, A. J.; *A hyperbolic problem with nonlinear second-order boundary damping*, Electron. J. Differential Equ., Vol.1998 (1998), no. 28, pp. 1-10.

[12] Gordeziani, D. G.; Avalishvili, G. A.; *Solutions of Nonlocal Problems for One-dimensional Oscillations of the Medium*, Mat. Modelir., 2000, vol. 12, no.1, p.94-103.

[13] F. F. Ivanauskas, F. F; Novitski, Yu, A.; Sapagovas, M. P.; *On the stability of an explicit difference scheme for hyperbolic equations with nonlocal boundary conditions*, Differential equations, 2013 vol. 49, no. 7, pp. 849-856.

[14] Ionkin, N. I.; *A Solution of Certain Boundary-Value Problem of Heat Conduction with Nonclassical Boundary Condition*, Diff. Equations, 1977, Vol. 13, no.2, pp. 294–301 (in Russian)

[15] Ionkin, N. I.; Moiseev, E. I.; *On a Problem for Heat Equation with Two-point Boundary Conditions*, Differential equations, 1979 Vol. 15, no. 7, pp. 294–304. (in Russian)

[16] Kamynin, L. I.; *On certain boundary problem of heat conduction with nonclassical boundary conditions*, J. of Comp. Math. and Math. Physics, 1964, Vol. 4, no. 6, p. 1006–1024.

[17] Kozhanov, A. I.; Pulkina, L. S.; *On the Solvability of Boundary Value Problems with a Nonlocal Boundary Condition of Integral Form for Multidimentional Hyperbolic Equations*, Differential Equations, 2006, vol. 42, no. 9, pp. 1233–1246.

[18] Kozhanov, A. I.; *On the solvability of certain spatially nonlocal boundary-value problems for linear hyperbolic equations of second order*, Mathematical Notes, 2011, 90:2, p. 238–249.

[19] Korzyuk, V. I.; Kozlovskaya, I. S.; Naumavets, S. N.; *Classical solution of a problem with integral congitions of the second kind for the one-dimensional wave equation*, Differential Equations, 2019, V.55, Issue 3, pp 353–362.

[20] Korpusov, O. M.; *Blow-up in nonclassical wave equations*, Moscow, URSS. 2010.

[21] Ladyzhenskaya, O. A.; *Boundary-value problems of mathematical physics*, Moscow: Nauka, 1973.

[22] Lažetič, N. L.; *On the Classical Solvability of the Mixed Problem for a Second-Order One-Dimensional Hyperbolic Equation*, Differential equations, Vol. 42, No. 8, 2006 p.1134–1139.

[23] Moiseev, E. I.; Korzyuk, V. I.; Kozlovskaya, I. S.; *Classical Solution of a Problem with an Integral Congition for the One-Dimensional Wave Equation*, Differential equations, 2014, Vol. 50, Issue 10, p. 1364–1377.

[24] Naimark, M. A.; *Linear differential operators.* Moscow, 1969.

[25] Oussaeif, T.-E.; Bouziani, A.; *Existence and uniqueness of solutions to parabolic fractional differential equations with integral conditions*, Electron. J. Differential Equ., Vol. 2014 (2014), No. 179, pp. 1-10.

[26] Pulkina, L. S.; *Initial-Boundary Value Problem with a Nonlocal Boundary Condition for a Multidimensional Hyperbolic equation*, Differential equations, 2008, vol. 44, no. 8, p. 1119-1125.

[27] Pulkina, L. S.; *Boundary value problems for a hyperbolic equation with nonlocal conditions of the I and II kind*, Russian Mathematics (Iz.VUZ) 2012, vol. 56, no. 4, p. 62–69.

[28] Pulkina, L. S.; Savenkova, A. E.; *A problem with second kind integral conditions for hyperbolic equation*, Vestnik of Samara University. Natural science series. no. 1-2. 2016. P. 33-45.

[29] Orazov, I.; Sadybekov, M. A.; *One Nonlocal Problem of Determination of the Temperature and Density of Heat Sources*, Russian Mathematics, V. 56, no. 2, p. 60–65.

[30] Samarskii, A. A.; *On Certain Problems of the Modern Theory of Differential equations*, Differ. Uravn., 1980, vol.16, no. 11, p. 1221-1228 (in Russian).

[31] Steclov, V. A.; *A Promlem on Cooling of a Solid*, Comm. of Kharkov Math. Society, 1896, Vol. 5, no. 3-4, p. 136–181.

[32] Tamarkin, J. D.; *Some general problems of the theory of linear differential equations and expansions of an arbitrary function in series*, Petrograd, 1917.

[33] Zhifei, Zhang; *Stabilization of the wave equation with variable coefficients and a dynamical boundary control*, Electron. J. Differential Equ., vol. 2016 (2016), no. 27, p. 1–10.



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