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ASYMPTOTICALLY ALMOST PERIODICITY OF DELAYED NICHOLSON-TYPE SYSTEM INVOLVING PATCH STRUCTURE

CHUANGXIA HUANG, JIAFU WANG, LIHONG HUANG

ABSTRACT. In this article we study a delayed Nicholson-type system involving patch structure. We apply differential inequality techniques to establish a sufficient condition for the existence of positive asymptotically almost periodic solutions. By constructing suitable Lyapunov functions, we obtain a new criterion for the uniqueness and global attractivity of the asymptotically almost periodic solutions.

1. INTRODUCTION

Several classes of differential equations models arising from biological mathematics have been intensively investigated, the hot topics include: stability, limit cycles, bifurcation and periodic solutions [2, 5, 7, 9, 8, 10, 13, 14, 18]. Although periodicity is important in real surroundings and world, when adding the factors of the environmental vary, almost periodicity is always more accurate, more realistic and more general than periodicity. As given in [1, 3, 4, 23] in comparison with periodic effects, almost periodic effects are more frequent in lots of real world applications. In particular, the existence and global stability of almost periodic solutions for the famous scalar Nicholson's blowflies equation

$$x'(t) = -a(t)x(t) + \sum_{j=1}^{m} \beta_j(t)x(t-\tau_j(t))e^{-\gamma_j(t)x(t-\tau_j(t))}, \tag{1.1}$$

and the Nicholson's blowflies systems with patch structure

$$\begin{aligned} x_i'(t) = -a_{ii}(t)x_i(t) &+ \sum_{j=1,j\neq i}^{n} a_{ij}(t)x_j(t) \\ &+ \sum_{j=1}^{m} \beta_{ij}(t)x_i(t-\tau_{ij}(t))e^{-\gamma_{ij}(t)x_i(t-\tau_{ij}(t))}, \end{aligned} \tag{1.2}$$

for $i \in Q := \{1, 2, \dots, n\}$, has been extensively investigated [12, 19, 20]. Here, the scalar Nicholson's blowflies equation (1.1) is a special case of Nicholson's blowflies system (1.2). $x_i(t)$ denotes the density of the ith population at time t, $a_{ij}(t)(i \neq j)$ is the rate of the population moving from class j to class i at time t, $a_{ii}(t)$ is the coefficient of instantaneous loss for class i at time t (which integrates both the death

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rate and the dispersal rates of the population in class i moving to the other classes), $\beta_{ij}(t)x_i(t-\tau_{ij}(t))e^{-\gamma_{ij}(t)x_i(t-\tau_{ij}(t))}$ is the birth function for class i at time t, $\beta_{ij}(t)$ is the per capita daily adult death rate for the species in the patch i at time t and $1/\gamma_{ij}(t)$ is the size at which the ith-population reproduces at its maximum rate in time t and $\tau_{ij}(t)$ is the generation time of ith-population at time t and $i \in Q$. In particular, the results of [19] complement previously known results [12, 20].

It should be mentioned that Wang et al [19] established the existence and global convergence of almost periodic solutions for Nicholson's blowflies systems (1.2) under the additional conditions that there exists a positive constant $M > \kappa$ such that

$$\gamma_{ij}(t)M \le \widetilde{\kappa}, \quad \text{for all } t \in \mathbb{R}, i \in Q,\ j \in I = \{1, 2, \dots, m\}, \tag{1.3}$$

$$\sup_{t\in\mathbb{R}}\{-a_{ii}(t) + \sum_{j=1,j\neq i}^{n} a_{ij}(t) + \frac{1}{eM}\sum_{j=1}^{m}\frac{\beta_{ij}(t)}{\gamma_{ij}(t)}\} < 0, \quad i \in Q, \tag{1.4}$$

$$\inf_{t\in\mathbb{R}}\{-a_{ii}(t) + \sum_{j=1,j\neq i}^{n} a_{ij}(t) + \sum_{j=1}^{m}\frac{\beta_{ij}(t)}{\gamma_{ij}(t)}e^{-\kappa}\} > 0, \quad i \in Q, \tag{1.5}$$

$$\inf_{t\ge t_0}\sum_{j=1}^{m}\frac{\beta_{ij}(t)}{\gamma_{ij}(t)} > 0, \quad i \in Q,\ j \in I, \tag{1.6}$$

where

$$\begin{gathered}\kappa \in (0,1), \quad \frac{1-\kappa}{e^{\kappa}} = \frac{1}{e^2}, \quad \widetilde{\kappa} \in (1, +\infty), \\ \kappa e^{-\kappa} = \widetilde{\kappa} e^{-\tilde{\kappa}}, \quad \kappa \approx 0.7215, \quad \widetilde{\kappa} \approx 1.3423\,.\end{gathered} \tag{1.7}$$

Obviously, (1.6) can be obtained from (1.4) and (1.5). Unfortunately, the technical conditions (1.3)–(1.5) on the coefficient functions are limited in the whole real axis, which are clearly not consistent with the biological background in the considered systems. Obviously, according to the biological interpretation of Nicholson's blowflies models in [22, 17], it is necessary to relax the above technical conditions as follows:

$$M \limsup_{t\to+\infty} \gamma_{ij}(t) \le \widetilde{\kappa}, \quad \text{for all } i \in Q,\ j \in I, \tag{1.8}$$

$$\sup_{t\in[t_0,+\infty)}\{-a_{ii}(t) + \sum_{j=1,j\neq i}^{n} a_{ij}(t) + \frac{1}{eM}\sum_{j=1}^{m}\frac{\beta_{ij}(t)}{\gamma_{ij}(t)}\} < 0, i \in Q, \tag{1.9}$$

$$\liminf_{t\to+\infty}\{-a_{ii}(t) + \sum_{j=1,j\neq i}^{n} a_{ij}(t) + \sum_{j=1}^{m}\frac{\beta_{ij}(t)}{\gamma_{ij}(t)}e^{-\kappa}\} > 0, \quad i \in Q. \tag{1.10}$$

Motivated by the above discussions, in this paper, we apply a novel proof to establish the existence and global attractivity of positive asymptotically almost periodic solutions for system (1.2) with weaker conditions (1.8)–(1.10).

This article is organized as follows: In Section 2, some necessary definitions, lemmas, assumptions are presented. In Section 3, the existence and global attractivity of positive asymptotically almost periodic solutions are demonstrated by virtue of some differential inequalities and analytic techniques. To verify our theoretical results, a numerical experiment is carried out in Section 4. Conclusions are drawn in Section 5.

2. Preliminary results

Throughout this paper, we assume that there exists $\widetilde{t_0} > t_0$ such that

$$\sigma_i = \max_{j\in I} \sup_{t\in\mathbb{R}} \tau_{ij}(t) > 0, \quad \inf_{t\geq \widetilde{t_0}} \gamma_{ij}(t) \geq 1, \quad i \in Q,\ j \in I, \tag{2.1}$$

which is a weaker condition than that $\inf_{t\in\mathbb{R}} \gamma_{ij}(t) \geq 1$ adopted in [18, 8, 7]. For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, define $|x| = (|x_1|, \dots, |x_n|)$ and $\|x\| = \max_{i\in Q} |x_i|$. Let $\mathbb{R}^+ = [0, +\infty)$, and $C_+ = \prod_{i=1}^n C([-\sigma_i, 0], \mathbb{R}^+)$. For $\mathbb{J}, \mathbb{J}_1, \mathbb{J}_2 \subseteq \mathbb{R}$, denote

$$W_0(\mathbb{R}^+, \mathbb{J}) = \{\nu : \nu \in C(\mathbb{R}^+, \mathbb{J}), \lim_{t\to+\infty} \nu(t) = 0\},$$

and let $BC(\mathbb{J}_1, \mathbb{J}_2)$ be the set of bounded and continuous functions from $\mathbb{J}_1$ to $\mathbb{J}_2$.

Definition 2.1 ([13, 10])**.** A subset P of $\mathbb{R}$ is said to be relatively dense in $\mathbb{R}$ if there exists a number $l > 0$ such that $[t, t+l] \cap P \neq \emptyset$ $(t \in \mathbb{R})$. $u \in BC(\mathbb{R}, \mathbb{J})$ is said to be almost periodic on $\mathbb{R}$ if, for any $\epsilon > 0$, the set $T(u, \epsilon) = \{\delta : |u(t+\delta) - u(t)| < \epsilon,\ \forall t \in \mathbb{R}\}$ is relatively dense.

Definition 2.2 ([23, 4])**.** $u \in C(\mathbb{R}^+, \mathbb{J})$ is said to be asymptotically almost periodic if there exist an almost periodic function h and a continuous function $g \in W_0(\mathbb{R}^+, \mathbb{J})$ such that $u = h + g$.

For $\mathbb{J} \subseteq \mathbb{R}$, we denote the set of the almost periodic functions from $\mathbb{R}$ to $\mathbb{J}$ by $AP(\mathbb{R}, \mathbb{J})$. The collection of the asymptotically almost periodic functions will be denoted by $AAP(\mathbb{R}, \mathbb{J})$. In addition, $AP(\mathbb{R}, \mathbb{J})$ is a proper subspace of $AAP(\mathbb{R}, \mathbb{J})$ [23, 4].

The decomposition $u = h + g$ given in Definition 2.2 is unique; see [23, Remark 5.16]. Hereafter, we assume that $a_{ii}, \gamma_{ij} \in AAP(\mathbb{R}, (0, +\infty))$, $a_{ij} (i \neq j), \beta_{ij}, \tau_{ij} \in AAP(\mathbb{R}, \mathbb{R}^+)$ and

$$a_{ij} = a_{ij}^h + a_{ij}^g, \quad \beta_{ij} = \beta_{ij}^h + \beta_j^g, \quad \gamma_{ij} = \gamma_j^h + \gamma_j^g, \quad \tau_{ij} = \tau_{ij}^h + \tau_{ij}^g,$$

where $a_{ii}^h, \gamma_{ij}^h \in AP(\mathbb{R}, (0, +\infty))$, $a_{ij}^h (i \neq j), \beta_{ij}^h, \tau_{ij}^h \in AP(\mathbb{R}, \mathbb{R}^+)$, $a_{ij}^g, \beta_{ij}^g, \gamma_{ij}^g, \tau_{ij}^g \in W_0(\mathbb{R}^+, \mathbb{R}^+)$, and $i \in Q$, $j \in I$.

To proceed further, we need to introduce a nonlinear almost periodic differential system:

$$\begin{aligned} x_i'(t) &= -a_{ii}^h(t) x_i(t) + \sum_{j=1, j\neq i}^n a_{ij}^h(t) x_j(t) \\ &\quad + \sum_{j=1}^m \beta_{ij}^h(t) x_i(t - \tau_{ij}^h(t)) e^{-\gamma_{ij}^h(t) x_i(t - \tau_{ij}^h(t))}, \quad i \in Q, \end{aligned} \tag{2.2}$$

We will consider the admissible initial conditions

$$x_i(t_0 + \theta) = \varphi_i(\theta), \quad \theta \in [-\sigma_i, 0], \quad \varphi = (\varphi_1, \dots, \varphi_n) \in C_+, \quad \varphi_i(0) > 0, \tag{2.3}$$

for $i \in Q$.

Lemma 2.3. *Let $x(t; t_0, \varphi)$ be a solution of the initial value problem* (2.2) *and* (2.3)*. Suppose that there exists a positive constant $M > \kappa$ such that* (1.8)*,* (1.10) *and*

$$\sup_{t\in[t_0, +\infty)} \Big\{ -a_{ii}^h(t) + \sum_{j=1, j\neq i}^n a_{ij}^h(t) + \frac{1}{eM} \sum_{j=1}^m \frac{\beta_{ij}^h(t)}{\gamma_{ij}^h(t)} \Big\} < 0, \quad i \in Q. \tag{2.4}$$

hold. Then, $x(t) = x(t; t_0, \varphi)$ exists on $[t_0, +\infty)$, and there is $t_\varphi \in [t_0, +\infty)$ such that

$$\kappa < x_i(t) < M \quad \text{for all } t \in [t_\varphi, +\infty),\ i \in Q. \tag{2.5}$$

Proof. First, we claim that

$$x_i(t) > 0 \quad \text{for all } t \in [t_0, \eta(\varphi)),\ i \in Q, \tag{2.6}$$

where $[t_0, \eta(\varphi))$ is the maximal right existence interval of $x(t)$. Otherwise, we can find $i_0 \in Q$ and $\bar{t}_{i_0} \in (t_0, \eta(\varphi))$ that satisfy

$$x_{i_0}(\bar{t}_{i_0}) = 0, \quad x_j(t) > 0 \quad \text{for all } t \in [t_0, \bar{t}_{i_0}),\ j \in Q.$$

From the facts that $x_{i_0}(t_0) = \varphi_{i_0}(0) > 0$ and

$$x'_{i_0}(t) \ge -a^h_{i_0 i_0}(t) x_{i_0}(t) + \sum_{j=1}^{m} \beta^h_{i_0 j}(t) x_{i_0}(t - \tau^h_{i_0 j}(t)) e^{-\gamma^h_{i_0 j}(t) x_{i_0}(t - \tau^h_{i_0 j}(t))},$$

for $t \in [t_0, \bar{t}_{i_0})$, we obtain

$$\begin{aligned}
0 &= x_{i_0}(\bar{t}_{i_0}) \\
&\ge e^{-\int_{t_0}^{\bar{t}_{i_0}} a^h_{i_0 i_0}(u) du} x_{i_0}(t_0) + e^{-\int_{t_0}^{\bar{t}_{i_0}} a^h_{i_0 i_0}(u) du} \int_{t_0}^{\bar{t}_{i_0}} e^{\int_{t_0}^{s} a^h_{i_0 i_0}(v) dv} \\
&\quad \times \sum_{j=1}^{m} \beta^h_{i_0 j}(s) x_{i_0}(s - \tau^h_{i_0 j}(s)) e^{-\gamma^h_{i_0 j}(s) x_{i_0}(s - \tau^h_{i_0 j}(s))} ds \\
&> 0,
\end{aligned}$$

which is a contradiction.

Now, we demonstrate that $x(t)$ is bounded on $[t_0, \eta(\varphi))$. For $t \in [t_0 - \sigma_i, \eta(\varphi))$ and $i \in Q$, we define

$$M_i(t) = \max\{\xi : \xi \le t, x_i(\xi) = \max_{t_0 - \sigma_i \le s \le t} x_i(s)\}.$$

Suppose that $x(t)$ is unbounded on $[t_0, \eta(\varphi))$. Then, we can choose $i^* \in Q$ and a strictly monotone increasing sequence $\{\zeta_n\}_{n=1}^{+\infty}$ such that

$$x_{i^*}(M_{i^*}(\zeta_n)) = \max_{j \in Q}\{x_j(M_j(\zeta_n))\}, \quad \lim_{n \to +\infty} x_{i^*}(M_{i^*}(\zeta_n)) = +\infty,$$
$$\lim_{n \to +\infty} \zeta_n = \eta(\varphi),$$

and

$$\lim_{n \to +\infty} M_{i^*}(\zeta_n) = \eta(\varphi). \tag{2.7}$$

It follows that there exists $n^* > 0$ satisfying

$$M_{i^*}(\zeta_n) > t_0, \quad x_{i^*}(M_{i^*}(\zeta_n)) > M \quad \text{for all } n > n^*.$$

From $\sup_{u \ge 0} u e^{-u} = \frac{1}{e}$, (2.2), (2.4) and (2.7) it follows that, for all $n > n^*$,

$$\begin{aligned}
0 &\le x'_{i^*}(M_{i^*}(\zeta_n)) \\
&= -a^h_{i^* i^*}(M_{i^*}(\zeta_n)) x_{i^*}(M_{i^*}(\zeta_n)) + \sum_{j=1, j \ne i}^{n} a^h_{i^* j}(M_{i^*}(\zeta_n)) x_j(M_{i^*}(\zeta_n)) \\
&\quad + \sum_{j=1}^{m} \frac{\beta^h_{i^* j}(M_{i^*}(\zeta_n))}{\gamma^h_{i^* j}(M_{i^*}(\zeta_n))} \gamma^h_{i^* j}(M_{i^*}(\zeta_n)) x_{i^*}(M_{i^*}(\zeta_n) - \tau^h_{i^* j}(M_{i^*}(\zeta_n)))
\end{aligned}$$

$$
\begin{aligned}
&\quad \times e^{-\gamma^h_{i^*j}(M_{i^*}(\zeta_n))x_{i^*}(M_{i^*}(\zeta_n)-\tau^h_{i^*j}(M_{i^*}(\zeta_n)))}\\
&\le [-a^h_{i^*i^*}(M_{i^*}(\zeta_n)) + \sum_{j=1,j\neq i}^{n} a^h_{i^*j}(M_{i^*}(\zeta_n))]x_{i^*}(M_{i^*}(\zeta_n))\\
&\quad + \sum_{j=1}^{m} \frac{\beta^h_{i^*j}(M_{i^*}(\zeta_n))}{\gamma^h_{i^*j}(M_{i^*}(\zeta_n))}\frac{1}{e}\\
&\le -\frac{1}{eM}\sum_{j=1}^{m} \frac{\beta^h_{i^*j}(M_{i^*}(\zeta_n))}{\gamma^h_{i^*j}(M_{i^*}(\zeta_n))}x_{i^*}(M_{i^*}(\zeta_n)) + \sum_{j=1}^{m} \frac{\beta^h_{i^*j}(M_{i^*}(\zeta_n))}{\gamma^h_{i^*j}(M_{i^*}(\zeta_n))}\frac{1}{e} < 0,
\end{aligned}
$$

which is absurd and suggests that $x(t)$ is bounded on $[t_0, \eta(\varphi))$. By [6, Theorem 2.3.1], we easily show $\eta(\varphi) = +\infty$. □

Hereafter, we assume that (2.5) is true. Designate $i^l, i^L \in Q$ such that

$$
l = \liminf_{t\to+\infty} x_{i^l}(t) = \min_{i\in Q}\liminf_{t\to+\infty} x_i(t),\ \ L = \limsup_{t\to+\infty} x_{i^L}(t) = \max_{i\in Q}\limsup_{t\to+\infty} x_i(t).
$$

By the fluctuation lemma (see [15, Lemma A.1]), we can select a sequence $\{t^*_k\}_{k=1}^{+\infty}$ satisfying

$$
\lim_{k\to+\infty} t^*_k = +\infty, \quad \lim_{k\to+\infty} x_{i^L}(t^*_k) = L = \limsup_{t\to+\infty} x_{i^L}(t), \quad \lim_{k\to+\infty} x'_{i^L}(t^*_k) = 0. \tag{2.8}
$$

From the almost periodicity of (2.2), we can select a subsequence of $\{k\}_{k\ge1}$, still denoted by $\{k\}_{k\ge1}$, such that $\lim_{k\to+\infty} a^h_{i^Lj}(t^*_k)$, $\lim_{k\to+\infty} b^h_{i^Lj}(t^*_k)$, $\lim_{k\to+\infty} \beta^h_{i^Lq}(t^*_k)$, $\lim_{k\to+\infty} \gamma^h_{i^Lq}(t^*_k)$, $\lim_{k\to+\infty} x_j(t^*_k)$ and $\lim_{k\to+\infty} x_{i^L}(t^*_k - \tau^h_{i^Lq}(t^*_k))$ exist for all $j \in Q$ and all $q \in I$. Furthermore, by taking limits, we have from (2.2) and (2.8) that

$$
\begin{aligned}
0 &= \lim_{k\to+\infty} x'_{i^L}(t^*_k)\\
&= -\lim_{k\to+\infty} a^h_{i^Li^L}(t^*_k)L + \sum_{j=1,j\neq i^L}^{n} \lim_{k\to+\infty} a^h_{i^Lj}(t^*_k)\lim_{k\to+\infty} x_j(t^*_k)\\
&\quad + \sum_{j=1}^{m} \lim_{k\to+\infty}\frac{\beta^h_{i^Lj}(t^*_k)}{\gamma^h_{i^Lj}(t^*_k)}\lim_{k\to+\infty}\gamma^h_{i^Lj}(t^*_k)x_{i^L}(t^*_k - \tau^h_{i^Lj}(t^*_k))\\
&\quad \times e^{-\lim_{k\to+\infty}\gamma^h_{i^Lj}(t^*_k)\lim_{k\to+\infty} x_{i^L}(t^*_k-\tau^h_{i^Lj}(t^*_k))}\\
&\le -\lim_{k\to+\infty} a^h_{i^Li^L}(t^*_k)L + \sum_{j=1,j\neq i^L}^{n} \lim_{k\to+\infty} a^h_{i^Lj}(t^*_k)L + \sum_{j=1}^{m}\lim_{k\to+\infty}\frac{\beta^h_{i^Lj}(t^*_k)}{\gamma^h_{i^Lj}(t^*_k)}\frac{1}{e}\\
&= \lim_{k\to+\infty} L[-a^h_{i^Li^L}(t^*_k) + \sum_{j=1,j\neq i^L}^{n} a^h_{i^Lj}(t^*_k) + \sum_{j=1}^{m}\frac{\beta^h_{i^Lj}(t^*_k)}{\gamma^h_{i^Lj}(t^*_k)}\frac{1}{eL}]\\
&\le \sup_{t\in[t_0,+\infty)}\Big\{-a^h_{i^Li^L}(t)L + \sum_{j=1,j\neq i^L}^{n} a^h_{i^Lj}(t)L + \sum_{j=1}^{m}\frac{\beta^h_{i^Lj}(t)}{\gamma^h_{i^Lj}(t)}\frac{1}{e}\Big\},
\end{aligned}
$$

which, together with (2.4), entails that

$$
\sup_{t\in[t_0,\ +\infty)}\Big\{-a^h_{i^Li^L}(t)M + \sum_{j=1,j\neq i^L}^{n} a^h_{i^Lj}(t)M + \sum_{j=1}^{m}\frac{\beta^h_{i^Lj}(t)}{\gamma^h_{i^Lj}(t)}\frac{1}{e}\Big\}
$$

$$
\begin{aligned}
&= M \sup_{t\in[t_0,+\infty)} \{-a^h_{i^L i^L}(t) + \sum_{j=1,j\neq i^L}^{n} a^h_{i^L j}(t) + \sum_{j=1}^{m} \frac{\beta^h_{i^L j}(t)}{\gamma^h_{i^L j}(t)} \frac{1}{Me}\} \\
&< 0 \\
&\le \sup_{t\in[t_0,+\infty)} \Big\{ - a^h_{i^L i^L}(t)L + \sum_{j=1,j\neq i^L}^{n} a^h_{i^L j}(t)L + \sum_{j=1}^{m} \frac{\beta^h_{i^L j}(t)}{\gamma^h_{i^L j}(t)} \frac{1}{e}\Big\},
\end{aligned}
$$

and then $L < M$. Consequently, there exists $t_0^* \ge t_0$ such that

$$x_i(t) < M, \quad \text{for all } t \ge t_0^*,\ i \in Q.$$

Next, we show that $l > 0$. By way of contradiction, we assume that

$$\liminf_{t\to+\infty} x_{i^l}(t) = \min_{i\in Q} \liminf_{t\to+\infty} x_i(t) = 0. \tag{2.9}$$

Let $\omega_i(t) = \max\{\xi : \xi \le t, x_i(\xi) = \min_{t_0\le s\le t} x_i(s)\}$ for each $t \ge t_0$. From (2.9), we can choose $i^{**} \in Q$ and a strictly monotone increasing sequence $\{\xi_n\}_{n=1}^{+\infty}$ such that

$$x_{i^{**}}(\omega_{i^{**}}(\xi_n)) = \min_{j\in Q}\{x_j(\omega_j(\xi_n))\}, \quad \lim_{n\to+\infty} x_{i^{**}}(\omega_{i^{**}}(\xi_n)) = 0, \quad \lim_{n\to+\infty} \xi_n = +\infty,$$

and then

$$\lim_{n\to+\infty} \omega_{i^{**}}(\xi_n) = +\infty.$$

According to (1.8), (2.1), (2) and $L < M$, one can find there exists $n^{**} > 0$ such that, for $n > n^{**}$ and $j \in I$,

$$\omega_{i^{**}}(\xi_n) > t_0^* + \sigma_{i^{**}}, \quad x_{i^{**}}(\omega_{i^{**}}(\xi_n)) < \kappa, quad \gamma^h_{i^{**}j}(\omega_{i^{**}}(\xi_n)) \ge 1, \tag{2.10}$$

$$x_{i^{**}}(\omega_{i^{**}}(\xi_n)) \le \gamma^h_{i^{**}j}(\omega_{i^{**}}(\xi_n)) x_{i^{**}}(\omega_{i^{**}}(\xi_n) - \tau^h_{i^{**}j}(\omega_{i^{**}}(\xi_n))) \le \widetilde{\kappa}. \tag{2.11}$$

It follows from (1.2), (2.10) and (2.11) that

$$
\begin{aligned}
&0 \ge x'_{i^{**}}(\omega_{i^{**}}(\xi_n)) \\
&= -a^h_{i^{**}i^{**}}(\omega_{i^{**}}(\xi_n)) x_{i^{**}}(\omega_{i^{**}}(\xi_n)) + \sum_{j=1,j\neq i^{**}}^{n} a^h_{i^{**}j}(\omega_{i^{**}}(\xi_n)) x_j(\omega_{i^{**}}(\xi_n)) \\
&\quad + \sum_{j=1}^{m} \frac{\beta^h_{i^{**}j}(\omega_{i^{**}}(\xi_n))}{\gamma^h_{i^{**}j}(\omega_{i^{**}}(\xi_n))} \gamma^h_{i^{**}j}(\omega_{i^{**}}(\xi_n)) x_{i^{**}}(\omega_{i^{**}}(\xi_n) - \tau^h_{i^{**}j}(\omega_{i^{**}}(\xi_n))) \\
&\quad \times e^{-\gamma^h_{i^{**}j}(\omega_{i^{**}}(\xi_n)) x_{i^{**}}(\omega_{i^{**}}(\xi_n) - \tau^h_{i^{**}j}(\omega_{i^{**}}(\xi_n)))} \\
&\ge -a^h_{i^{**}i^{**}}(\omega_{i^{**}}(\xi_n)) x_{i^{**}}(\omega_{i^{**}}(\xi_n)) + \sum_{j=1,j\neq i^{**}}^{n} a^h_{i^{**}j}(\omega_{i^{**}}(\xi_n)) x_j(\omega_{i^{**}}(\xi_n)) \\
&\quad + \sum_{j=1}^{m} \frac{\beta^h_{i^{**}j}(\omega_{i^{**}}(\xi_n))}{\gamma^h_{i^{**}j}(\omega_{i^{**}}(\xi_n))} x_{i^{**}}(\omega_{i^{**}}(\xi_n)) e^{-x_{i^{**}}(\omega_{i^{**}}(\xi_n))}, \quad n > n^{**},
\end{aligned}
$$

and

$$
\begin{aligned}
&a^h_{i^{**}i^{**}}(\omega_{i^{**}}(\xi_n)) \\
&\ge \sum_{j=1,j\neq i^{**}}^{n} a^h_{i^{**}j}(\omega_{i^{**}}(\xi_n)) \frac{x_j(\omega_{i^{**}}(\xi_n))}{x_{i^{**}}(\omega_{i^{**}}(\xi_n))} + \sum_{j=1}^{m} \frac{\beta^h_{i^{**}j}(\omega_{i^{**}}(\xi_n))}{\gamma^h_{i^{**}j}(\omega_{i^{**}}(\xi_n))} e^{-x_{i^{**}}(\omega_{i^{**}}(\xi_n))} \\
&\ge \sum_{j=1,j\neq i^{**}}^{n} a^h_{i^{**}j}(\omega_{i^{**}}(\xi_n)) + \sum_{j=1}^{m} \frac{\beta^h_{i^{**}j}(\omega_{i^{**}}(\xi_n))}{\gamma^h_{i^{**}j}(\omega_{i^{**}}(\xi_n))} e^{-x_{i^{**}}(\omega_{i^{**}}(\xi_n))},\ n > n^{**},
\end{aligned}
$$

This inequality and (1.10), yield

$$
\begin{aligned}
0 &\ge \liminf_{n\to+\infty}\{-a^h_{i^{**}i^{**}}(\omega_{i^{**}}(\xi_n)) + \sum_{j=1,j\neq i^{**}}^{n} a^h_{i^{**}j}(\omega_{i^{**}}(\xi_n)) \\
&\quad + \sum_{j=1}^{m}\frac{\beta^h_{i^{**}j}(\omega_{i^{**}}(\xi_n))}{\gamma^h_{i^{**}j}(\omega_{i^{**}}(\xi_n))}e^{-x_{i^{**}}(\omega_{i^{**}}(\xi_n))} \\
&\ge \liminf_{t\to+\infty}\{-a^h_{i^{**}i^{**}}(t) + \sum_{j=1,j\neq i^{**}}^{n} a^h_{i^{**}j}(t) + \sum_{j=1}^{m}\frac{\beta^h_{i^{**}j}(t)}{\gamma^h_{i^{**}j}(t)}\} \\
&\ge \liminf_{t\to+\infty}\{-a^h_{i^{**}i^{**}}(t) + \sum_{j=1,j\neq i^{**}}^{n} a^h_{i^{**}j}(t) + \sum_{j=1}^{m}\frac{\beta^h_{i^{**}j}(t)}{\gamma^h_{i^{**}j}(t)}e^{-\kappa}\} \\
&= \liminf_{t\to+\infty}\{-a_{i^{**}i^{**}}(t) + \sum_{j=1,j\neq i^{**}}^{n} a_{i^{**}j}(t) + \sum_{j=1}^{m}\frac{\beta_{i^{**}j}(t)}{\gamma_{i^{**}j}(t)}e^{-\kappa}\} > 0.
\end{aligned}
$$

This is a clear contradiction and proves that $l > 0$.

Finally, we show that $l > \kappa$. Again from the fluctuation lemma [15, Lemma A.1] and the almost periodicity of (2.2), we can pick a sequence $\{t^{**}_k\}_{k=1}^{+\infty}$ such that

$$
\lim_{k\to+\infty} t^{**}_k = +\infty, \quad \lim_{k\to+\infty} x_{i^l}(t^{**}_k) = l = \liminf_{t\to+\infty} x_{i^l}(t), \quad \lim_{k\to+\infty} x'_{i^l}(t^{**}_k) = 0 \tag{2.12}
$$

and $\lim_{k\to+\infty} a^h_{i^l j}(t^{**}_k)$, $\lim_{k\to+\infty} b^h_{i^l j}(t^{**}_k)$, $\lim_{k\to+\infty} \beta^h_{i^l q}(t^{**}_k)$, $\lim_{k\to+\infty} \gamma^h_{i^l q}(t^{**}_k)$, $\lim_{k\to+\infty} x_j(t^{**}_k)$, $\lim_{k\to+\infty} x_{i^l}(t^{**}_k - \tau^h_{i^l q}(t^{**}_k))$ exist for all $j \in Q$ and all $q \in I$. Furthermore,

$$
\begin{gathered}
l \le \lim_{k\to+\infty} x_j(t^{**}_k) \le L < M, \\
l \le \lim_{k\to+\infty} \gamma^h_{i^l q}(t^{**}_k) \lim_{k\to+\infty} x_{i^l}(t^{**}_k - \tau^h_{i^l q}(t^{**}_k)) \le \widetilde{\kappa}, \quad \forall j \in Q,\ q \in I.
\end{gathered} \tag{2.13}
$$

By way of contradiction, we assume that $0 < l \le \kappa$. With the help of (1.10), (2.12) and (2.13), we have

$$
\begin{aligned}
0 &= \lim_{k\to+\infty} x'_{i^l}(t^{**}_k) \\
&\ge -\lim_{k\to+\infty} a^h_{i^l i^l}(t^{**}_k) l + \sum_{j=1,j\neq i^l}^{n} \lim_{k\to+\infty} a^h_{i^l j}(t^{**}_k) l \\
&\quad + \sum_{j=1}^{m}\frac{\lim_{k\to+\infty}\beta^h_{i^l j}(t^{**}_k)}{\lim_{k\to+\infty}\gamma^h_{i^l j}(t^{**}_k)} \lim_{k\to+\infty}\gamma^h_{i^l j}(t^{**}_k) x_{i^l}(t^{**}_k - \tau^h_{i^l j}(t^{**}_k)) \\
&\quad \times e^{-\lim_{k\to+\infty}\gamma^h_{i^l j}(t^{**}_k) x_{i^l}(t^{**}_k - \tau^h_{i^l j}(t^{**}_k))} \\
&\ge -\lim_{k\to+\infty} a^h_{i^l i^l}(t^{**}_k) l + \sum_{j=1,j\neq i^l}^{n} \lim_{k\to+\infty} a^h_{i^l j}(t^{**}_k) l + \sum_{j=1}^{m}\frac{\lim_{k\to+\infty}\beta^h_{i^l j}(t^{**}_k)}{\lim_{k\to+\infty}\gamma^h_{i^l j}(t^{**}_k)} l e^{-l} \\
&\ge l \liminf_{t\to+\infty}\{-a^h_{i^l i^l}(t) + \sum_{j=1,j\neq i^l}^{n} a^h_{i^l j}(t) + \sum_{j=1}^{m}\frac{\beta^h_{i^l j}(t)}{\gamma^h_{i^l j}(t)}e^{-l}\} \\
&\ge l \liminf_{t\to+\infty}\{-a^h_{i^l i^l}(t) + \sum_{j=1,j\neq i^l}^{n} a^h_{i^l j}(t) + \sum_{j=1}^{m}\frac{\beta^h_{i^l j}(t)}{\gamma^h_{i^l j}(t)}e^{-\kappa}\}
\end{aligned}
$$

$$= l \liminf_{t\to+\infty}\{-a_{i^l i^l}(t) + \sum_{j=1,j\neq i^l}^{n} a_{i^l j}(t) + \sum_{j=1}^{m} \frac{\beta_{i^l j}(t)}{\gamma_{i^l j}(t)} e^{-\kappa}\} > 0,$$

which results in a contradiction. This proves that $l > \kappa$. Hence, there exits $t_\varphi > t_0$ such that

$$\kappa < x_i(t; t_0, \varphi) < M \quad \text{for all } t \geq t_\varphi,\ i \in Q.$$

The proof is now complete.

By using a similar argument as in Lemma 2.3, we can show the following result.

Lemma 2.4. *Let $x(t) = x(t; t_0, \varphi)$ be a solution of the initial value problem* (1.2) *and* (2.3)*. Suppose that there exists a positive constant $M > \kappa$ such that* (1.8)*,* (1.9) *and* (1.10) *hold. Then, $x(t)$ exists on $[t_0, +\infty)$,*

$$\kappa < \min_{i\in Q} \liminf_{t\to+\infty} x_i(t) \leq \max_{i\in Q} \limsup_{t\to+\infty} x_i(t) < M,$$

and there is $t^*_\varphi \in [t_0, +\infty)$ such that

$$\kappa < x_i(t) < M \quad \textit{for all } t \in [t^*_\varphi, +\infty),\ i \in Q. \tag{2.14}$$

Lemma 2.5. *Suppose that there exists a positive constant $M > \kappa$ such that* (1.8)*,* (1.10) *and* (2.4) *hold. Moreover, assume that $x(t) = x(t; t_0, \varphi)$ is a solution of equation* (2.2) *and* (2.3)*. Then, for any $\epsilon > 0$, we can choose a relatively dense subset P_ϵ of $\mathbb{R}$ with the property that, for each $\delta \in P_\epsilon$, there exists $T = T(\delta) > 0$ satisfying*

$$\|x(t+\delta) - x(t)\| < \frac{\epsilon}{2}, \quad \textit{for all } t > T.$$

Proof. With the help of Lemma 2.3, (1.8), (2.1) and (2.4), we can choose positive constants $T_1 > \max\{0,\ t_\varphi\}$ and ζ such that for all $t \geq T_1$ and $i \in Q$,

$$\gamma^h_{ij}(t) x_i(t - \tau^h_{ij}(t)) > \kappa, \quad 1 \leq \frac{\widetilde{\kappa}}{M\gamma^h_{ij}(t)}, \quad 1 \leq \frac{\widetilde{\kappa}}{M\gamma_{ij}(t)}, \quad \frac{\widetilde{\kappa}}{e} < 1,$$

and

$$\begin{aligned}
& -a^h_{ii}(t) + \sum_{j=1,j\neq i}^{n} a^h_{ij}(t) + \frac{1}{e^2} \sum_{j=1}^{m} \beta^h_{ij}(t) \\
&\leq -a^h_{ii}(t) + \sum_{j=1,j\neq i}^{n} a^h_{ij}(t) + \frac{1}{eM} \sum_{j=1}^{m} \frac{\beta^h_{ij}(t)}{\gamma^h_{ij}(t)} \frac{\widetilde{\kappa}}{e} \\
&\leq -a^h_{ii}(t) + \sum_{j=1,j\neq i}^{n} a^h_{ij}(t) + \frac{1}{eM} \sum_{j=1}^{m} \frac{\beta^h_{ij}(t)}{\gamma^h_{ij}(t)} \\
&< -\zeta.
\end{aligned}$$

Then there exist two constants $\eta > 0$ and $\lambda \in (0, 1]$ such that, for $i \in Q$,

$$\sup_{t\in[T_1,\ +\infty)} \Big\{ -[a^h_{ii}(t) - \lambda] + \sum_{j=1,j\neq i}^{n} a^h_{ij}(t) + \sum_{j=1}^{m} \beta^h_{ij}(t) \frac{1}{e^2} e^{\lambda\sigma_i} \Big\} < -\eta. \tag{2.15}$$

Define

$$x_i(t) \equiv x_i(t_0 - \sigma_i), \quad \text{for all } t \in (-\infty, t_0 - \sigma_i],\ i \in Q, \tag{2.16}$$

and

$$
\begin{aligned}
A_i(\delta,t) &= -[a_{ii}^h(t+\delta)-a_{ii}^h(t)]x_i(t+\delta)+\sum_{j=1,j\neq i}^{n}[a_{ij}^h(t+\delta)-a_{ij}^h(t)]x_j(t+\delta)\\
&\quad+\sum_{j=1}^{m}[\beta_{ij}^h(t+\delta)-\beta_{ij}^h(t)]x_i(t+\delta-\tau_{ij}^h(t+\delta))e^{-\gamma_{ij}^h(t+\delta)x_i(t+\delta-\tau_{ij}^h(t+\delta))}\\
&\quad+\sum_{j=1}^{m}\beta_{ij}^h(t)[x_i(t+\delta-\tau_{ij}^h(t+\delta))e^{-\gamma_{ij}^h(t+\delta)x_i(t+\delta-\tau_{ij}^h(t+\delta))}\\
&\quad-x_i(t-\tau_{ij}^h(t)+\delta)e^{-\gamma_{ij}^h(t+\delta)x_i(t-\tau_{ij}^h(t)+\delta)}]\\
&\quad+\sum_{j=1}^{m}\beta_{ij}^h(t)[x_i(t-\tau_{ij}^h(t)+\delta)e^{-\gamma_{ij}^h(t+\delta)x_i(t-\tau_{ij}^h(t)+\delta)}\\
&\quad-x_i(t-\tau_{ij}^h(t)+\delta)e^{-\gamma_{ij}^h(t)x_i(t-\tau_{ij}^h(t)+\delta)}],\quad\text{for all } t\in\mathbb{R},\ i\in Q.
\end{aligned}
$$

From Lemma 2.3, one can see that $x(t)$ is bounded and the right-hand side of (2.2) is also bounded. It follows from (2.16) that $x(t)$ is uniformly continuous on $\mathbb{R}$. Therefore, for any $\epsilon>0$, we can choose a sufficiently small constant $\epsilon^*>0$ such that

$$
\begin{gathered}
|a_{ij}^h(t)-a_{ij}^h(t+\delta)|<\epsilon^*,|\beta_{ij}^h(t)-\beta_{ij}^h(t+\delta)|<\epsilon^*,\\
|\gamma_{ij}^h(t)-\gamma_{ij}^h(t+\delta)|<\epsilon^*,|\tau_{ij}^h(t)-\tau_{ij}^h(t+\delta)|<\epsilon^*.
\end{gathered}
$$

It follows that

$$
|A_i(\delta,t)|<\frac{1}{2}\eta\epsilon, \tag{2.17}
$$

where $t\in\mathbb{R}$, $i\in Q$ and $j\in I$.

Furthermore, for $\epsilon^*>0$, from the uniformly almost periodic family theory in [4, p. 19, Corollary 2.3], one can choose a relatively dense subset P_{ϵ^*} of $\mathbb{R}$ such that

$$
\begin{gathered}
|a_{ij}^h(t)-a_{ij}^h(t+\delta)|<\epsilon^*,\quad|\beta_{ij}^h(t)-\beta_{ij}^h(t+\delta)|<\epsilon^*,\\
|\gamma_{ij}^h(t)-\gamma_{ij}^h(t+\delta)|<\epsilon^*,\quad|\tau_{ij}^h(t)-\tau_{ij}^h(t+\delta)|<\epsilon^*,
\end{gathered} \tag{2.18}
$$

$\delta\in P_{\epsilon^*}$, $t\in\mathbb{R}$, $i\in Q$, and $j\in I$.

Let $P_\epsilon=P_{\epsilon^*}$. Then for any $\delta\in P_\epsilon$, from (2.17) and (2.18), we have

$$
|A_i(\delta,t)|<\frac{1}{2}\eta\epsilon,\quad\text{for all } t\in\mathbb{R},\ i\in Q. \tag{2.19}
$$

Let

$$
\Lambda_0\geq\max\big\{|t_0|+T_1+\max_{i\in Q}\sigma_i,\ |t_0|+T_1+\max_{i\in Q}\sigma_i-\delta\big\}.
$$

For $t\in\mathbb{R}$, denote

$$
\begin{gathered}
u(t)=(u_1(t),u_2(t),\dots,u_n(t)),\quad u_i(t)=x_i(t+\delta)-x_i(t),\\
U(t)=(U_1(t),U_2(t),\dots,U_n(t)),\quad U_i(t)=e^{\lambda t}u_i(t),
\end{gathered}
$$

where $i\in Q$. Let i_t be an index such that

$$
|U_{i_t}(t)|=\|U(t)\|. \tag{2.20}
$$

Then, for all $t \geq \Lambda_0$, we have

$$
\begin{aligned}
u_i'(t) &= -a_{ii}^h(t)[x_i(t+\delta) - x_i(t)] + \sum_{j=1,j\neq i}^{n} a_{ij}^h(t)[x_j(t+\delta) - x_j(t)] \\
&\quad + \sum_{j=1}^{m} \beta_{ij}^h(t)[x_i(t-\tau_{ij}^h(t)+\delta)e^{-\gamma_{ij}^h(t)x_i(t-\tau_{ij}^h(t)+\delta)} \\
&\quad - x_i(t-\tau_{ij}^h(t))e^{-\gamma_{ij}^h(t)x_i(t-\tau_{ij}^h(t))}] + A_i(\delta,t).
\end{aligned}
$$

From the above equality, (2.4) and

$$
|\alpha e^{-\alpha} - \beta e^{-\beta}| \leq \frac{1}{e^2}|\alpha - \beta| \quad \text{where } \alpha, \beta \in [\kappa, +\infty), \tag{2.21}
$$

we obtain

$$
\begin{aligned}
& D^-(|U_{i_s}(s)|)|_{s=t} \\
& \leq \lambda e^{\lambda t}|u_{i_t}(t)| + e^{\lambda t}\Big\{ - a_{i_t i_t}^h(t)[x_{i_t}(t+\delta) - x_{i_t}(t)]\operatorname{sgn}(x_{i_t}(t+\delta) - x_{i_t}(t)) \\
& \quad + \sum_{j=1,j\neq i_t}^{n} a_{i_t j}^h(t)|x_j(t+\delta) - x_j(t)| + \sum_{j=1}^{m}\beta_{i_t j}^h(t)\big|x_{i_t}(t-\tau_{i_t j}^h(t)+\delta) \\
& \quad \times e^{-\gamma_{i_t j}^h(t)x_{i_t}(t-\tau_{i_t j}^h(t)+\delta)} - x_{i_t}(t-\tau_{i_t j}^h(t))e^{-\gamma_{i_t j}^h(t)x_{i_t}(t-\tau_{i_t j}^h(t))}\big| \\
& \quad + |A_{i_t}(\delta,t)|\Big\} \\
& = \lambda e^{\lambda t}|u_{i_t}(t)| + e^{\lambda t}\Big\{ - a_{i_t i_t}^h(t)[x_{i_t}(t+\delta) - x_{i_t}(t)]\operatorname{sgn}(x_{i_t}(t+\delta) - x_{i_t}(t)) \\
& \quad + \sum_{j=1,j\neq i_t}^{n} a_{i_t j}^h(t)|x_j(t+\delta) - x_j(t)| + \sum_{j=1}^{m}\frac{\beta_{i_t j}^h(t)}{\gamma_{i_t j}^h(t)} \\
& \quad \times |\gamma_{i_t j}^h(t)x_{i_t}(t-\tau_{i_t j}^h(t)+\delta)e^{-\gamma_{i_t j}^h(t)x_{i_t}(t-\tau_{i_t j}^h(t)+\delta)} \\
& \quad - \gamma_{i_t j}^h(t)x_{i_t}(t-\tau_{i_t j}^h(t))e^{-\gamma_{i_t j}^h(t)x_{i_t}(t-\tau_{i_t j}^h(t))}| + |A_{i_t}(\delta,t)|\Big\} \\
& \leq \lambda e^{\lambda t}|u_{i_t}(t)| + e^{\lambda t}\Big\{ - a_{i_t i_t}^h(t)|u_{i_t}(t)| + \sum_{j=1,j\neq i_t}^{n} a_{i_t j}^h(t)|u_j(t)| \\
& \quad + \sum_{j=1}^{m}\beta_{i_t j}^h(t)\frac{1}{e^2}|u_{i_t}(t-\tau_{i_t j}^h(t))| + |A_{i_t}(\delta,t)|\Big\} \\
& = -[a_{i_t i_t}^h(t) - \lambda]|U_{i_t}(t)| + \sum_{j=1,j\neq i_t}^{n} a_{i_t j}^h(t)|U_j(t)| \\
& \quad + \sum_{j=1}^{m}\beta_{i_t j}^h(t)\frac{1}{e^2}e^{\lambda\tau_{i_t j}^h(t)}|U_{i_t}(t-\tau_{i_t j}^h(t))| + e^{\lambda t}|A_{i_t}(\delta,t)| \quad \text{for all } t \geq \Lambda_0.
\end{aligned} \tag{2.22}
$$

Let

$$
E(t) = \sup_{-\infty < s \leq t}\{e^{\lambda s}\|u(s)\|\}.
$$

It is obvious that $e^{\lambda t}\|u(t)\| \leq E(t)$, and $E(t)$ is non-decreasing. The remaining of the proof will be divided into two steps.

Step 1. If $E(t) > e^{\lambda t}\|u(t)\|$ for all $t \ge \Lambda_0$, we assert that

$$E(t) \equiv \|U(\Lambda_0)\| \quad \text{for all } t \ge \Lambda_0. \tag{2.23}$$

In the opposite case, one can pick $\Lambda_1 > \Lambda_0$ such that $E(\Lambda_1) > E(\Lambda_0)$. Because

$$e^{\lambda t}\|u(t)\| \le E(\Lambda_0) \quad \text{for all } t \le \Lambda_0,$$

there must exist $\beta^* \in (\Lambda_0, \Lambda_1)$ such that

$$e^{\lambda \beta^*}\|u(\beta^*)\| = E(\Lambda_1) \ge E(\beta^*),$$

which contradicts that $E(\beta^*) > e^{\lambda\beta^*}\|u(\beta^*)\|$ and proves the above assertion. Then, we can select $\Lambda_2 > \Lambda_0$ satisfying

$$\|u(t)\| \le e^{-\lambda t}E(t) = e^{-\lambda t}E(\Lambda_0) < \frac{\varepsilon}{2} \quad \text{for all } t \ge \Lambda_2. \tag{2.24}$$

Step 2. If there exists $\varsigma \ge \Lambda_0$ such that $E(\varsigma) = e^{\lambda\varsigma}\|u(\varsigma)\|$, we can have from (2.22) and the definition of $E(t)$ that

$$\begin{aligned}
0 &\le D^-(|U_{i_s}(s)|)|_{s=\varsigma} \\
&\le -[a^h_{i_\varsigma i_\varsigma}(t) - \lambda]|U_{i_\varsigma}(\varsigma)| + \sum_{j=1, j\ne i_\varsigma}^{n} a^h_{i_\varsigma j}(t)|U_j(\varsigma)| \\
&\quad + \sum_{j=1}^{m} \beta^h_{i_\varsigma j}(\varsigma)\frac{1}{e^2}e^{\lambda\tau^h_{i_\varsigma j}(\varsigma)}|U_{i_\varsigma}(\varsigma - \tau^h_{i_\varsigma j}(\varsigma))| + e^{\lambda\varsigma}|A_{i_\varsigma}(\delta,\varsigma)| \\
&\le \Big\{ -[a^h_{i_\varsigma i_\varsigma}(t) - \lambda] + \sum_{j=1, j\ne i_\varsigma}^{n} a^h_{i_\varsigma j}(t) + \sum_{j=1}^{m} \beta^h_{i_\varsigma j}(\varsigma)\frac{1}{e^2}e^{\lambda\tau^h_{i_\varsigma j}(\varsigma)}\Big\}E(\varsigma) + \frac{1}{2}\eta\varepsilon e^{\lambda\varsigma} \\
&< -\eta E(\varsigma) + \frac{1}{2}\eta\varepsilon e^{\lambda\varsigma},
\end{aligned}$$

which leads to

$$e^{\lambda\varsigma}\|u(\varsigma)\| = E(\varsigma) < \frac{\varepsilon}{2}e^{\lambda\varsigma}, \quad \text{and } \|u(\varsigma)\| < \frac{\varepsilon}{2}. \tag{2.25}$$

For any $t > \varsigma$ satisfying $E(t) = e^{\lambda t}\|u(t)\|$, by the same method as that in the derivation of (2.25), we can show

$$e^{\lambda t}\|u(t)\| < \frac{\varepsilon}{2}e^{\lambda t}, \quad \text{and} \quad \|u(t)\| < \frac{\varepsilon}{2}. \tag{2.26}$$

Furthermore, if $E(t) > e^{\lambda t}\|u(t)\|$ and $t > \varsigma$, one can pick $\Lambda_3 \in [\varsigma,\ t)$ such that

$$E(\Lambda_3) = e^{\lambda\Lambda_3}\ |u(\Lambda_3)\|, \quad E(s) > e^{\lambda s}\|u(s)\| \quad \text{for all } s \in (\Lambda_3, t],$$

which, together with (2.25) and (2.26), suggest that

$$\|u(\Lambda_3)\| < \frac{\varepsilon}{2}. \tag{2.27}$$

With a similar reasoning as that in the proof of Step one, we can entail that

$$E(s) \equiv E(\Lambda_3) \quad \text{is a constant for all } s \in (\Lambda_3, t],$$

which, together with (2.27), follows that

$$\|u(t)\| < e^{-\lambda t}E(t) = e^{-\lambda t}E(\Lambda_3) = \|u(\Lambda_3)\|e^{-\lambda(t-\Lambda_3)} < \frac{\varepsilon}{2}.$$

Finally, the above discussion infers that there exists $\hat{\Lambda} > \max\{\varsigma, \Lambda_0, Lambda_2\}$ satisfying

$$\|u(t)\| \leq \frac{\varepsilon}{2} < \varepsilon \quad \text{for all } t > \hat{\Lambda},$$

which completes the proof of Lemma 2.5. □

3. Main results

The main result in this article reads as follows.

Theorem 3.1. *Assume that there exists a positive constant $M > \kappa$ such that* (1.7), (1.8), (1.9), (1.10) *and* (2.4) *hold. Then system* (2.2) *has exactly one positive almost periodic solution $x^*(t)$, and every solution of* (1.2) *with initial condition* (2.3) *is asymptotically almost periodic on $\mathbb{R}^+$, and converges to $x^*(t)$ as $t \to +\infty$.*

Proof. Let $v(t)$ be a solution of system (2.2) with initial function φ satisfying (2.3), and

$$v_i(t) \equiv v_i(t_0 - \sigma_i), \quad \text{for all } t \in (-\infty, t_0 - \sigma_i],\ i \in Q.$$

Also we define

$$\begin{aligned} B_i(q,t) = &-[a_{ii}^h(t+t_q) - a_{ii}^h(t)]v_i(t+t_q) + \sum_{j=1, j\neq i}^{n} [a_{ij}^h(t+t_q) - a_{ij}^h(t)]v_j(t+t_q) \\ &+ \sum_{j=1}^{m} [\beta_{ij}^h(t+t_q) - \beta_{ij}^h(t)] v_i(t+t_q - \tau_{ij}^h(t+t_q)) e^{-\gamma_{ij}^h(t+t_q) v_i(t+t_q-\tau_{ij}^h(t+t_q))} \\ &+ \sum_{j=1}^{m} \beta_{ij}^h(t)[v_i(t+t_q - \tau_{ij}^h(t+t_q)) e^{-\gamma_{ij}^h(t+t_q) v_i(t+t_q-\tau_{ij}^h(t+t_q))} \\ &- v_i(t - \tau_{ij}^h(t) + t_q) e^{-\gamma_{ij}^h(t+t_q) v_i(t-\tau_{ij}^h(t)+t_q)}] \\ &+ \sum_{j=1}^{m} \beta_{ij}^h(t)[v_i(t - \tau_{ij}^h(t) + t_q) e^{-\gamma_{ij}^h(t+t_q) v_i(t-\tau_{ij}^h(t)+t_q)} \\ &- v_i(t - \tau_{ij}^h(t) + t_q) e^{-\gamma_{ij}^h(t) v_i(t-\tau_{ij}^h(t)+t_q)}], \quad \text{for all } t \in \mathbb{R},\ i \in Q. \end{aligned}$$

where $\{t_q\}_{q\geq 1} \subseteq \mathbb{R}$ is a sequence. Then

$$\begin{aligned} v_i'(t+t_q) = &-a_{ii}^h(t) v_i(t+t_q) + \sum_{j=1, j\neq i}^{n} a_{ij}^h(t) v_j(t+t_q) \\ &+ \sum_{j=1}^{m} \beta_{ij}^h(t) v_i(t - \tau_{ij}^h(t) + t_q) e^{-\gamma_{ij}^h(t) v_i(t-\tau_{ij}^h(t)+t_q)} + B_i(q,t), \end{aligned} \tag{3.1}$$

for all $t + t_q \geq t_0,\ i \in Q$. By using a similar proof as in Lemma 2.5, we can choose $\{t_q\}_{q\geq 1}$ such that

$$|B_i(q,t)| < \frac{1}{q} \quad \text{for all } i, q, t. \tag{3.2}$$

By Arzala-Ascoli Lemma and the fact that the function sequence $\{v(t+t_q)\}_{q\geq 1}$ is uniformly bounded and equiuniformly continuous, we can choose a subsequence $\{t_{q_j}\}_{j\geq 1}$ of $\{t_q\}_{q\geq 1}$, such that $\{v(t+t_{q_j})\}_{j\geq 1}$ converges uniformly to a continuous function $x^*(t) = (x_1^*(t), x_2^*(t), \dots, x_n^*(t))$ on any compact set of $\mathbb{R}$ (for convenience, we denote this subsequence by $\{v(t+t_q)\}_{q\geq 1}$).

Then, from Lemma 2.3, we have

$$\kappa < \min_{i\in Q}\liminf_{t\to+\infty} v_i(t) \le x_i^*(t) \le \max_{i\in Q}\limsup_{t\to+\infty} v_i(t) < M \quad \text{for all } t\in\mathbb{R},\ i\in Q, \tag{3.3}$$

and

$$\begin{aligned}
-a_{ii}^h(t)v_i(t+t_q) &\Rightarrow -a_{ii}^h(t)x_i^*(t), \quad i\in Q,\\
\sum_{j=1,j\neq i}^{n} a_{ij}^h(t)v_j(t+t_q) &\Rightarrow \sum_{j=1,j\neq i}^{n} a_{ij}^h(t)x_j^*(t), \quad i\in Q,\\
\sum_{j=1}^{m}\beta_{ij}^h(t)v_i(t-\tau_{ij}^h(t)+t_q)&e^{-\gamma_{ij}^h(t)v_i(t-\tau_{ij}^h(t)+t_q)}\\
&\Rightarrow \sum_{j=1}^{m}\beta_{ij}^h(t)x_i^*(t-\tau_{ij}^h(t))e^{-\gamma_{ij}^h(t)x^*(t-\tau_{ij}^h(t))}, \quad i\in Q,
\end{aligned}\tag{3.4}$$

as $q\to+\infty$, on any compact set of $\mathbb{R}$, where $\Rightarrow$ denotes uniformly converge. Thus, for $i\in Q$, (3.1), (3.2) and (3.4) produce that $\{v_i'(t+t_q)\}_{q\ge1}$ converges uniformly to

$$-a_{ii}^h(t)x_i^*(t)+\sum_{j=1,j\neq i}^{n} a_{ij}^h(t)x_j^*(t)+\sum_{j=1}^{m}\beta_{ij}^h(t)x_i^*(t-\tau_{ij}^h(t))e^{-\gamma_{ij}^h(t)x^*(t-\tau_{ij}^h(t))}$$

on any compact subset of $\mathbb{R}$. According to the properties of the uniform convergence function sequence, we obtain that $x^*(t)$ is a solution of (2.2) and

$$\begin{aligned}
(x_i^*(t))' &= -a_{ii}^h(t)x_i^*(t)+\sum_{j=1,j\neq i}^{n} a_{ij}^h(t)x_j^*(t)\\
&\quad+\sum_{j=1}^{m}\beta_{ij}^h(t)x_i^*(t-\tau_{ij}^h(t))e^{-\gamma_{ij}^h(t)x^*(t-\tau_{ij}^h(t))}, \quad \text{for all } t\in\mathbb{R},\ i\in Q.
\end{aligned}$$

Now, from Lemma 2.5, for any $\epsilon>0$, we can choose a relatively dense subset P_ϵ of $\mathbb{R}$ with the property that, for each $\delta\in P_\epsilon$, there exists $T=T(\delta)>0$ satisfying

$$\|v(s+t_q+\delta)-v(s+t_q)\| < \frac{\epsilon}{2}, \quad \text{for all } s+t_q>T,$$

$$\lim_{q\to+\infty}\|v(s+t_q+\tau)-v(s+t_q)\| = \|x^*(s+\delta)-x^*(s)\| \le \frac{\epsilon}{2} < \epsilon \quad \text{for all } s\in\mathbb{R},$$

which implies that $x^*(t)$ is a positive almost periodic solution of (2.2).

Next, we show that all solutions of (1.2) converge to $x^*(t)$ as $t\to+\infty$. Let $x(t)$ be an arbitrary solution of system (1.2) with initial value φ satisfies (2.3). Define $y(t)=x(t)-x^*(t)$, add the definition of $x_i(t)$ with $x_i(t)\equiv x_i(t_0-\sigma_i)$ for all $t\in(-\infty,t_0-\sigma_i]$, and let

$$\begin{aligned}
F_i(t) &= -[(a_{ii}^h(t)+a_{ii}^g(t))x_i(t)-a_{ii}^h(t)x_i(t)]\\
&\quad+\sum_{j=1,j\neq i}^{n}[(a_{ij}^h(t)+a_{ij}^g(t))x_j(t)-a_{ij}^h(t)x_j(t)]\\
&\quad+\sum_{j=1}^{m}[(\beta_{ij}^h(t)+\beta_{ij}^g(t))x_i(t-(\tau_{ij}^h(t)+\tau_{ij}^g(t)))\\
&\quad\times e^{-(\gamma_{ij}^h(t)+\gamma_{ij}^g(t))x_i(t-(\tau_{ij}^h(t)+\tau_{ij}^g(t)))}-\beta_{ij}^h(t)x_i(t-\tau_{ij}^h(t))e^{-\gamma_{ij}^h(t)x_i(t-\tau_{ij}^h(t))}].
\end{aligned}$$

Then

$$
\begin{aligned}
y_i'(t) &= -a_{ii}^h(t)y_i(t) + \sum_{j=1, j\neq i}^{n} a_{ij}^h(t)y_j(t) \\
&\quad + \sum_{j=1}^{m} \beta_{ij}^h(t)[x_i(t-\tau_{ij}^h(t))e^{-\gamma_{ij}^h(t)x_i(t-\tau_{ij}^h(t))} \\
&\quad - x_i^*(t-\tau_{ij}^h(t))e^{-\gamma_{ij}^h(t)x_i^*(t-\tau_{ij}^h(t))}] + F_i(t), \quad \text{for all } t \geq t_0,\ i \in Q.
\end{aligned} \tag{3.5}
$$

For any $\epsilon > 0$, in view of the global existence and uniform continuity of x and the fact that $a_{ij}^g, \beta_{ij}^g, \gamma_{ij}^g, \tau_{ij}^g \in W_0(\mathbb{R}^+, \mathbb{R}^+)$, we can choose a constant $T_\varphi^{**} > \max\{T_1,\ t_\varphi^*\}$ such that

$$
|F_i(t)| < \eta\frac{\epsilon}{2}, \quad \text{for all } t > T_\varphi^{**}. \tag{3.6}
$$

Set

$$
G(t) = \sup_{-\infty < s \leq t}\{e^{\lambda s}\|y(s)\|\}, \quad \text{for all } t \in \mathbb{R},
$$

and index i_t such that

$$
e^{\lambda t}|y_{i_t}(t)| = \|e^{\lambda t}y(t)\|.
$$

According to (1.8), (2.1), (3.3), Lemma 2.4, one can find $T_{\varphi,x^*} > T_\varphi^{**}$ such that

$$
\kappa < x_i(t),\ x_i^*(t),\ \gamma_{ij}^h(t)x_i(t-\tau_{ij}^h(t)) \leq \widetilde{\kappa} \quad \text{for all } t > T_{\varphi,x^*},\ i \in Q. \tag{3.7}
$$

In view of (2.21), (3.5) and (3.7), we have

$$
\begin{aligned}
&D^-(e^{\lambda s}|y_{i_s}(s)|)|_{s=t} \\
&\leq -[a_{i_ti_t}^h(t) - \lambda]e^{\lambda t}|y_{i_t}(t)| + \sum_{j=1, j\neq i_t}^{n} a_{i_tj}^h(t)e^{\lambda t}|y_j(t)| + \sum_{j=1}^{m}\beta_{i_tj}^h(t) \\
&\quad \times \frac{1}{e^2}e^{\lambda\tau_{i_tj}^h(t)}e^{\lambda(t-\tau_{i_tj}^h(t))}|y_{i_t}(t-\tau_{i_tj}^h(t))| + e^{\lambda t}|F_{i_t}(t)|
\end{aligned} \tag{3.8}
$$

for all $t \geq T_{\varphi,x^*}$ and $i \in Q$.

Then, from (2.15), (3.6) and (3.8), by employing the argument of Lemma 2.5, we know that there is a constant $\widetilde{T} \geq T_{\varphi,x^*}$ such that

$$
\|y(t)\| < \frac{\epsilon}{2} \quad \text{for all } t \geq \widetilde{T},
$$

which yields

$$
\lim_{t\to+\infty} x(t) = x^*(t), \quad \text{and} \quad x(t) \in AAP(\mathbb{R}, \mathbb{R}^n).
$$

It follows from the uniqueness of the limit function that (2.2) has exactly one positive almost periodic solution $x^*(t)$. The proof is complete. □

Remark 3.2. Under the conditions in Lemma 2.5, according to Lemma 2.3 and Lemma 2.5, by applying a similar way as in [19, Theorem 3.2], one can show that the solution $x(t; t_0, \varphi)$ of (2.2) converges exponentially to $x^*(t)$ as $t \to +\infty$. Since all conditions in (1.7)–(1.10) are weaker than those in (1.3)–(1.5), one can easily see that all results on almost periodicity of (2.2) in [12, 19, 20] are special cases of Theorem 3.1 in this article.

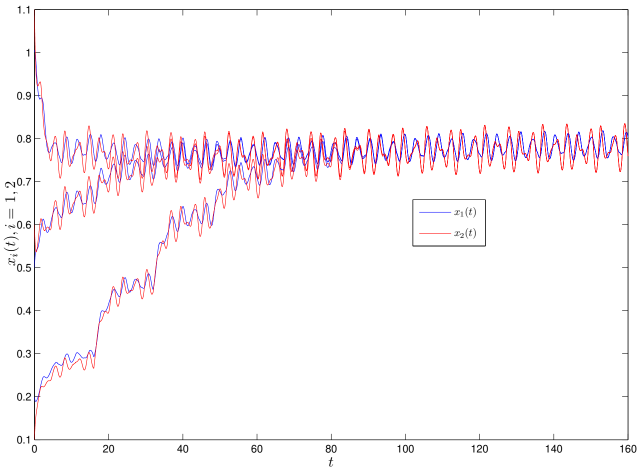

FIGURE 1. Numerical solutions of (4.1) for different initial values: (0.5,0,6), (1,1.1), (0,2,0.1).

4. Numerical simulation

We consider the delayed Nicholson-type system with patch structure

$$\begin{aligned} x_1'(t) &= -(1.85 + 0.1|\sin\sqrt{2}t| + \frac{100t}{1+t^2})x_1(t) + (1.5 + 0.1|\sin t| + \frac{1}{800+t^2})x_2(t) \\ &\quad + (e0.4 + \frac{1}{100+t^2})x_1(t - 2\cos^2\sqrt{3}t - 2)e^{-(1.01 - \frac{2\cos t}{100+t^2})x_1(t-2\cos^2\sqrt{3}t-2)} \\ &\quad + (0.4 + \frac{1}{100+t^2})x_1(t - 2\sin^2\sqrt{3}t - 2)e^{-(1.01 - \frac{2\cos t}{100+t^4})x_1(t-2\sin^2\sqrt{3}t-2)}, \\ x_2'(t) &= -(2.85 + 0.3|\cos t| + \frac{100t}{2+t^2})x_2(t) + (2.5 + 0.3\sin^2\sqrt{2}t + \frac{1}{100+t^4})x_1(t) \\ &\quad + (0.4 + \frac{1}{100+t^4})x_2(t - \cos t - 5)e^{-(1.01 - \frac{2\cos t}{100+t^4})x_2(t-\cos t-5)} \\ &\quad + (0.4 + \frac{1}{100+t^4})x_2(t - \cos t - 15)e^{-(1.01 - \frac{2\cos t}{100+t^4})x_2(t-\cos t-15)}, \end{aligned} \tag{4.1}$$

where $t_0 = 0$.

Take $M = 1.301$, we can find that (1.8), (1.9), (1.10), (2.1) and (2.4) are satisfied. By Theorem 3.1, all solutions of (4.1) are asymptotically almost periodic functions on $\mathbb{R}^+$, and converge to a same almost periodic function as $t \to +\infty$. This fact can be presented in the Figure 1.

In system (4.1),

$$\Big\{ - a_{ii}(t) + \sum_{j=1, j\neq i}^{2} a_{ij}(t) + \frac{1}{eM}\sum_{j=1}^{2}\frac{\beta_{ij}(t)}{\gamma_{ij}(t)}\Big\}\Big|_{t=-1, M>\kappa} > 10, \quad i \in Q = \{1, 2\},$$

and

$$\Big\{ -a_{ii}(t) + \sum_{j=1,j\neq i}^{n} a_{ij}(t) + \sum_{j=1}^{m} \frac{\beta_{ij}(t)}{\gamma_{ij}(t)} e^{-\kappa} \Big\}\Big|_{t=1} < -12, \quad i \in Q = \{1,2\},$$

imply that (4.1) does not satisfy conditions (1.4) and (1.5) in [12, 19, 20]. In addition, the asymptotically almost periodic dynamics of delayed Nicholson-type system with patch structure was not studied in [16, 17, 21, 22]. Hence, it is not hard to see that all results in the references [12, 17, 19, 20, 22] and [16, 11, 21] cannot be applied to conclude that all solutions of (4.1) converge globally are almost periodic solutions.

Conclusions. In this paper, we combine the Lyapunov function method with the differential inequality method to establish some new criteria ensuring the existence and attractivity of positive asymptotically almost periodic solutions for a class of a class of delayed Nicholson's blowflies systems with patch structure. The assumptions adopted in this present paper are weaker than some previously known literature. Numerical simulations have been given to illustrate the obtained results. The approach presented in this article can be used as a possible way to study the asymptotically almost periodic patch structure population models, for example, neoclassical growth model, Mackey-Glass model, epidemic system or age-structured population model and so on. We leave this as our future work.

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Chuangxia Huang
School of Mathematics and Statistics, Changsha University of Science and Technology, Hunan Provincial Key Laboratory of Mathematical Modeling and Analysis in Engineering, Changsha, Hunan 410114, China
Email address: `cxiahuang@amss.ac.cn`

Jiafu Wang
School of Mathematics and Statistics, Changsha University of Science and Technology, Hunan Provincial Key Laboratory of Mathematical Modeling and Analysis in Engineering, Changsha, Hunan 410114, China
Email address: `jfwang@csust.edu.cn`

Lihong Huang (corresponding author)
School of Mathematics and Statistics, Changsha University of Science and Technology, Hunan Provincial Key Laboratory of Mathematical Modeling and Analysis in Engineering, Changsha, Hunan 410114, China
Email address: `lhhuang@csust.edu.cn`