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HÖLDER CONTINUITY FOR VECTOR-VALUED MINIMIZERS OF QUADRATIC FUNCTIONALS

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Abstract. In this article we give a sufficient condition for interior everywhere Hölder continuity of weak minimizers of a class of quadratic functionals with coefficients $A_{ij}^{\alpha\beta}(\cdot,u)$ belonging to the VMO-class, uniformly with respect to $u\in\mathbb{R}^N$, and continuous with respect to u. The condition is global. It is typical for the functionals belonging to the class that the continuity moduli of their coefficients become slowly growing sufficiently far from zero. Some features of the main result are illustrated by examples.

1. Introduction

The aim of this article is to study the interior everywhere regularity of functions minimizing variational integrals

$$\mathcal{A}(u;\Omega)=\int_\Omega A_{ij}^{\alpha\beta}(x,u)D_\alpha u^i D_\beta u^j\,dx \tag{1.1}$$

where $u:\Omega\to\mathbb{R}^N$, $N>1$, $\Omega\subset\mathbb{R}^n$, $n\geq 3$ is a bounded open set, $x=(x_1,\dots,x_n)\in\Omega$, $u(x)=(u^1(x),\dots,u^N(x))$, $Du=\{D_\alpha u^i\}$, $D_\alpha=\partial/\partial x_\alpha$, $\alpha=1,\dots,n$, $i=1,\dots,N$.

Throughout the whole text we use the summation convention over repeated indices. We call a function $u\in W^{1,2}(\Omega,\mathbb{R}^N)$ is a minimizer of the functional $\mathcal{A}(u;\Omega)$ if and only if $\mathcal{A}(u;\Omega)\leq\mathcal{A}(v;\Omega)$ for every $v\in W^{1,2}(\Omega,\mathbb{R}^N)$ such that $u-v\in W_0^{1,2}(\Omega,\mathbb{R}^N)$. For more information see [5, 10].

On the functional $\mathcal{A}$ we assume:

(i) $A_{ij}^{\alpha\beta}=A_{ji}^{\beta\alpha}$, $A_{ij}^{\alpha\beta}$ are continuous functions in $u\in\mathbb{R}^N$ for every $x\in\Omega$ and there exists $M>0$ such that $\sum_{i,j,\alpha,\beta}|A_{ij}^{\alpha\beta}(x,u)|\leq M$, for all $x\in\Omega$, and all $u\in\mathbb{R}^N$.

(ii) (ellipticity) There exists $\nu>0$ such that

$$A_{ij}^{\alpha\beta}(x,u)\xi_\alpha^i\xi_\beta^j\geq\nu|\xi|^2,\quad\forall x\in\Omega,\,\forall u\in\mathbb{R}^N,\,\forall\xi\in\mathbb{R}^{nN}. \tag{1.2}$$

(iii) (oscillation of coefficients) There exists a real function ω continuous on $[0,\infty)$, which is bounded, nondecreasing, concave, $\omega(0)=0$ and such that

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for all $x \in \Omega$ and u, $v \in \mathbb{R}^N$

$$\sum_{i,j,\alpha,\beta} |A_{ij}^{\alpha\beta}(x,u) - A_{ij}^{\alpha\beta}(x,v)| \le \omega\left(|u-v|\right). \tag{1.3}$$

We set $\omega_\infty = \lim_{t\to\infty} \omega(t) \le 2M$.

(iv) For all $u \in \mathbb{R}^N$, $A_{ij}^{\alpha\beta}(\cdot,u) \in VMO(\Omega)$ (uniformly with respect to $u \in \mathbb{R}^N$).

Assumptions (i) and (ii) allow us to conclude that if $u \in W^{1,2}(\Omega,\mathbb{R}^N)$ is a minimizer of (1.1) then for any admissible function $v \in W^{1,2}(\Omega,\mathbb{R}^N)$

$$\int_\Omega |Du|^2\,dx \le \frac{M}{\nu}\int_\Omega |Dv|^2\,dx\,. \tag{1.4}$$

Concerning the assumption (iii) it is worth to point out (see [5, p.169]) that for uniformly continuous coefficients $A_{ij}^{\alpha\beta}$ there exists a real function ω satisfying the assumption (iii) and, viceversa, (iii) implies the uniform continuity of coefficients and absolute continuity of ω on $[0,\infty)$.

In this paper we will consider the continuous function

$$\omega(t) = \begin{cases} \omega_0(t) & \text{for } 0 \le t < t_0,\ t_0 \ge 0 \\ \omega_1(t) \le \omega_\infty, & \text{for } t_0 \le t < \infty \end{cases} \tag{1.5}$$

where ω_0 is an arbitrary continuous, concave, nondecreasing function, increasing on a neighbourhood of zero such that $\omega_0(0) = 0$ and the point t_0 and the function ω_1 are chosen in such a way that ω preserves its continuity and concavity on $[0,\infty)$.

With respect to (iv) it is worth to recall that since the space of continuous functions is a proper subset of VMO, the continuity of coefficients $A_{ij}^{\alpha\beta} = A_{ij}^{\alpha\beta}(x,u)$ with respect to x is not supposed. In the linear case, when the coefficients $A_{ij}^{\alpha\beta} = A_{ij}^{\alpha\beta}(x)$ belong to $C^{0,\gamma}(\overline{\Omega})$ the regularity of minimizers of functionals as (1.1) is well understood (see [5, Thorems 3.1, 3.2 on p.87, 88]). These results were later generalized to the case where the above coefficients are in VMO, hence possibly discontinuous (see [4, 19] and references therein).

It is well known that even in the continuous case the dependence of coefficients $A_{ij}^{\alpha\beta}$ on u leads to weaker regularity results for minimizers. In dimension $n \ge 3$ there are examples of vectorial quadratic functionals ($N > 1$) with analytic coefficients $A_{ij}^{\alpha\beta} = A_{ij}^{\alpha\beta}(u)$ whose minimizers are discontinuous (see [10, p. 317], [11]). For the analytic coefficients $A_{ij}^{\alpha\beta} = A_{ij}^{\alpha\beta}(x,u)$ see counterexample in [18]. These examples indicate that, in general, only partial regularity results can be achieved for minimizers of vectorial functionals. For detailed information on this topic we refer to sources [5]-[10] for classic results and to [13, 15, 19] for recent results.

Besides the partial regularity results, a few everywhere regularity results were obtained for some special types of vectorial functionals (see [10, 15]). Our paper deals just with the last mentioned type of regularity results. In the recent papers [1, 3] conditions guaranteeing the local Hölder continuity of minimizers of functional (1.1) in Ω are given. Because the paper [3] extends the results of [1], we mention only [3] in more detail. Main results of the paper [3] are stated in two theorems. The first of them refers that if a quantity expressed by means of parameters ω_∞/ν and M/ν is small enough, the minimizers of (1.1) are regular. This result is not very surprising but, moreover, an upper bound (although probably not optimal) of the above mentioned quantity is designed. In a case when the mentioned condition is not fulfilled a sufficient condition for regularity of minimizers of functional (1.1) is stated as well. A basic advantage of the second condition in the paper [3] is, that it admits (for sufficiently big ellipticity constant ν) an arbitrary growth of the continuity modulus $\omega = \omega(t)$ when t is near by zero. Here it is needful to note that the second condition works likewise when ν is small but, in this case, the modulus of continuity ω has to grow slowly enough. A disadvantage of the condition is its "local character", analogous to the regularity conditions in partial regularity theory. The present paper essentially extends results of [1] and [3]. Here we study the regularity for variational integrals, coefficients of which satisfy (iii) with modulus of continuity given by (1.5). Together with more delicate estimates and careful designing of some parameters in proof, it allows us to state the regularity condition preserving all the advantages of the previous mentioned conditions from [1, 3] and, moreover, the condition is formulated much simpler and more exactly than the previous ones in [1, 3]. Consequently, it improves the possibility of immediate application (it is well visible mainly in the case of the Dirichlet problem - see Remark 1.4 below). It is worth to mention that the regularity condition (expressed by (1.6), (1.7), (1.8)) has, compared to that one from [3, Thm. 2], global features. The methods of proving the main results are based on those that were developed in the classic partial regularity theory (see for example [5, 10]), but they are essentially modified. In Remark 4.2 it is shown that, in a case of split coefficients, joining the results of this paper with those from [12], we are able to guarantee the regularity of minimizers of (1.1) in $\overline{\Omega}$. Now we can formulate the main result.

Theorem 1.1. *Let $\Omega_0 \subset\subset \Omega$, $n-2 \le \vartheta < n$ be given and the coefficients $A_{ij}^{\alpha\beta}$ of the functional* (1.1) *satisfy (i), (ii), (iii) and (iv). There exists a positive constant $\mathcal{M}$ such that if the minimizer u of the functional* (1.1) *satisfies the condition*

$$\frac{1}{|\Omega|^{1-2/n}} \int_\Omega |Du|^2\, dy \le \frac{1}{\mathcal{M}^2} \tag{1.6}$$

then u belongs to $C^{0,(\vartheta-n+2)/2}(\Omega_0, \mathbb{R}^{nN})$ when $\vartheta > n-2$ and to $BMO(\Omega_0, \mathbb{R}^{nN})$ when $\vartheta = n-2$. Here

$$\mathcal{M} = \sup_{t_0<t<\infty} \frac{\widetilde{\Psi}\left(\frac{\omega(t)}{\varepsilon}\right) - \widetilde{\Psi}\left(\frac{\omega(t_0)}{\varepsilon}\right)}{t-t_0} \quad \text{and} \quad \widetilde{\Psi}\left(\frac{\omega(t_0)}{\varepsilon}\right) \le 2^{n+2}\sqrt{C_2}\,. \tag{1.7}$$

Remark 1.2. In the foregoing formula the function $\widetilde{\Psi}(u) = u e^{(u/2\sqrt{\mu})^{2/(2\mu-1)}}$ (for further properties of $\widetilde{\Psi}$ see (2.1) below), $t_0 \ge 0$ (t_0 is the parameter from the definition of ω, see (1.5)), $\varepsilon = \omega_\infty / C_\mu^\rho$, $C_\mu = (\mu/((p-1)e))^\mu$, the constants $\mu \ge 6$ and $\rho > 1/p$ are such that

$$C_\mu^{\rho p-1} \ge K\, C_1^{2p} C_2^{(p+1)/2}\, L^{p\vartheta/(n-\vartheta)} \left(\frac{\omega_\infty}{\nu}\right)^p \left(\frac{|\Omega|^{1-2/n}}{(2d)^{n-2}}\right)^{(p-1)/2} \tag{1.8}$$

in the case when the coefficients $A_{ij}^{\alpha\beta}$ depend only on u. Here $p > 1$ is from Lemma 2.9, $K = 2^{(n+11+(n+3)\vartheta/(n-\vartheta))p-(2n+5)}\, \kappa_n^{1-p}$, L is the constant from Lemma 2.7 below, C_1, C_2 are the constants from Lemma 2.9 and 2.10 respectively, $d = \operatorname{dist}(\Omega_0, \partial\Omega)/2 > 0$ and the symbol $|\cdot|$ stands for the n-dimensional Lebesgue measure (κ_n is the Lebesgue measure of the unit ball in $\mathbb{R}^n$).

If $A_{ij}^{\alpha\beta} = A_{ij}^{\alpha\beta}(x,u)$ then, formally, the constant K on the right-hand side of (1.8) is substituted by $2K$ (here, as it is visible at the end of the proof of Theorem 1.1, the multiplier 2 could be substituted by another one, bigger than 1). It is important to release that the dependence of the coefficients $A_{ij}^{\alpha\beta}$ on variable x tends to the choice $d = \min\{R_0, \operatorname{dist}(\Omega_0, \partial\Omega)/2\}$ (for definition of R_0 see (3.25) below) and so d and, consequently, the value of the constant $C_\mu^{\rho p-1}$ from (1.8) depend on "VMO-quality" of x-dependence of coefficients $A_{ij}^{\alpha\beta}$ as well. Broadly speaking, the bigger R_0 is, the better regularity result one can obtain.

Remark 1.3. It is easily seen that instead of the assumption (iv) in the foregoing Theorem 1.1 one can suppose the coefficients $A_{ij}^{\alpha\beta}$ of the functional (1.1) to be of BMO-class with suitable small BMO semi-norms (see (3.25) below).

Remark 1.4. It is a consequence of the estimate (1.4) that if $u \in W^{1,2}(\Omega, \mathbb{R}^N)$, mentioned in the foregoing theorem, is such that $u - g \in W_0^{1,2}(\Omega, \mathbb{R}^N)$ for some $g \in W^{1,2}(\Omega, \mathbb{R}^N)$ (the Dirichlet problem for functional (1.1)), then the left-hand side of (1.6) can be replaced by the term

$$\frac{M}{\nu|\Omega|^{1-2/n}} \int_\Omega |Dg|^2 \, dy \, .$$

The regularity theorem, we formulated above, can be illustrated with two samples of the function ω, defined by (1.5), for which we give estimates of the parameter $\mathcal{M}$. Broadly speaking, if the coefficients of the functional satisfy (iii) with some ω given below and (1.8) is fulfilled, we have the regularity.

Example 1.5. Let

$$\omega(t) = \begin{cases} \omega_0(t) & \text{for } 0 \le t < t_0, \\ \omega_\infty \ln\Big(1 + \frac{e^{\varepsilon/\omega_\infty} - 1}{t_0^\gamma} t^\gamma\Big) & \text{for } t_0 \le t \le t_1,\ 0 < \gamma \le 1, \\ \omega_\infty & \text{for } t > t_1 \end{cases} \tag{1.9}$$

where ω_0 is an arbitrary continuous, concave, nondecreasing function such that $\omega_0(0) = 0$ and the points t_0, t_1 are chosen so that ω is continuous and concave on $[0, \infty)$. If we put $\varepsilon = \omega_\infty / C_\mu^\rho$ in (1.9) then the right-hand side of (1.6) can be chosen in the form (see Appendix for more information)

$$\frac{1}{\mathcal{M}^2} = \Big(\frac{t_0}{10 C_\mu^{\frac{2}{2\mu-1}\rho}} \min\Big\{1, \frac{3 C_\mu^{\frac{2}{2\mu-1}\rho}}{e^{C_\mu^{\frac{2}{2\mu-1}\rho}}}\Big\}\Big)^2 . \tag{1.10}$$

Here $\mu \ge 6$, $\rho > 1/p$ and $t_0 > 0$.

Example 1.6. Let

$$\omega(t) = \frac{2\omega_\infty}{\pi} \arctan\Big(\frac{t}{C_\mu^\tau}\Big) \quad \text{for } 0 \le t < \infty \tag{1.11}$$

then the constant from (1.6) can have the form (in this case $t_0 = 0$, see Appendix as well)

$$\frac{1}{\mathcal{M}^2} = \Bigg(\frac{C_\mu^{\tau-\rho}}{e^{\big(\frac{C_\mu^\rho}{2\sqrt{\mu}}\big)^{\frac{2}{2\mu-1}}}} \Bigg)^2 . \tag{1.12}$$

Here $\tau > \rho > 1/p$, $\mu \ge 6$ satisfy (1.8) and $\widetilde{\Psi}(\omega(t_0)/\varepsilon) = 0$.

2. Preliminaries

If $x \in \mathbb{R}^n$ and r is a positive real number, we set $B_r(x) = \{y \in \mathbb{R}^n : |y-x| < r\}$, $\Omega_r(x) = \Omega \cap B_r(x)$. Denote by

$$u_{x,r} = \frac{1}{|\Omega_r(x)|} \int_{\Omega_r(x)} u(y)\,dy = \fint_{\Omega_r(x)} u(y)\,dy$$

the mean value of the function $u \in L^1(\Omega, \mathbb{R}^N)$ over the set $\Omega_r(x)$ where the symbol $|\cdot|$ denotes the n-dimensional Lebesgue measure. Moreover, we set $\phi(r) = \phi(x,r) = \int_{B_r(x)} |Du(y)|^2\,dy$, $U_r = U_r(x) = r^{2-n}\phi(x,r)$ for $B_r(x) \subset \Omega$. Beside the standard space $C_0^\infty(\Omega, \mathbb{R}^N)$, Hölder space $C^{0,\alpha}(\Omega, \mathbb{R}^N)$ and Sobolev spaces $W^{k,p}(\Omega, \mathbb{R}^N)$, $W_0^{k,p}(\Omega, \mathbb{R}^N)$ we use Morrey spaces $L^{q,\lambda}(\Omega, \mathbb{R}^N)$ (see, e.g. [5, 14]). We will denote by $X_{\rm loc}(\Omega, \mathbb{R}^N)$ the space of all functions which belong to $X(\widetilde{\Omega}, \mathbb{R}^N)$ for any bounded subdomain $\widetilde{\Omega}$ with smooth boundary which is compactly embedded in Ω.

We recall a definition of VMO - spaces and a few properties of Morrey spaces. We set for $f \in L^1(\Omega)$, $0 < a < \infty$

$$\mathcal{N}_a(f, \Omega) := \sup_{x\in\Omega, r<a} \fint_{\Omega_r(x)} |f(y) - f_{x,r}|\,dy.$$

Definition 2.1 (see [20]). A function $f \in L^1(\Omega)$ is said to belong to $BMO(\Omega)$ if

$$\mathcal{N}_{\operatorname{diam}\Omega}(f, \Omega) < \infty.$$

A function $f \in L^1(\Omega)$ is said to belong to $VMO(\Omega)$ if

$$\lim_{a\to 0} \mathcal{N}_a(f, \Omega) = 0.$$

Proposition 2.2. *For a bounded domain $\Omega \subset \mathbb{R}^n$ with the Lipschitz boundary, for $q \in (1, \infty)$ and $0 < \lambda < \mu < \infty$ we have the following:*

(a) $L^{q,\mu}(\Omega, \mathbb{R}^N) \subset L^{q,\lambda}(\Omega, \mathbb{R}^N)$.
(b) *If $u \in W^{1,2}_{\rm loc}(\Omega, \mathbb{R}^N)$ and $Du \in L^{2,\lambda}_{\rm loc}(\Omega, \mathbb{R}^{nN})$, $n-2 < \lambda < n$ then $u \in C^{0,(\lambda-n+2)/2}(\Omega, \mathbb{R}^N)$.*
(c) *If $u \in W^{1,2}_{\rm loc}(\Omega, \mathbb{R}^N)$ and $Du \in L^{2,n-2}_{\rm loc}(\Omega, \mathbb{R}^{nN})$ then $u \in BMO_{\rm loc}(\Omega, \mathbb{R}^N)$.*
(d) *$L^{q,n}(\Omega, \mathbb{R}^N)$ is isomorphic to the $L^\infty(\Omega, \mathbb{R}^N)$.*
(e) *$L^\infty(\Omega, \mathbb{R}^N) \subsetneqq BMO(\Omega, \mathbb{R}^N)$.*

Let now Φ, Ψ be a pair of complementary Young functions

$$\begin{gathered} \Phi(u) = u \ln_+^\mu(au) \quad \text{for} \quad u \ge 0, \\ \Psi(u) \le \overline{\Psi}(u) = \frac{1}{a} u e^{(\frac{u}{2\sqrt{\mu}})^{2/(2\mu-1)}} = \frac{1}{a}\widetilde{\Psi}(u) \quad \text{for} \quad u \ge 0 \end{gathered} \tag{2.1}$$

where $a > 0$, $\mu \ge 2$ are constants, and

$$\ln_+(au) = \begin{cases} 0 & \text{for } 0 \le u < 1/a, \\ \ln(au) & \text{for } u \ge 1/a. \end{cases} \tag{2.2}$$

Then the Young inequality for Φ and Ψ reads

$$uv \le \Phi(u) + \Psi(v), \quad u, v \ge 0. \tag{2.3}$$

Lemma 2.3 ([21, p.37]). *Let $\phi : [0,\infty) \to [0,\infty)$ be a non decreasing function which is absolutely continuous on every closed interval of finite length, $\phi(0) = 0$. If $w \geq 0$ is measurable and $l(t) = \{y \in \mathbb{R}^n : w(y) > t\}$ then*

$$\int_{\mathbb{R}^n} \phi \circ w \, dy = \int_0^\infty |l(t)| \, \phi'(t) \, dt.$$

Lemma 2.4. *Let $v \geq 0$, $b > 0$, $\mu > 0$ and $q > 1$ be arbitrary. Then*

$$v \ln_+^\mu(bv) \leq C_\mu \, b^{q-1} \, v^q \tag{2.4}$$

where $C_\mu = \big(\frac{\mu}{(q-1)\mathrm{e}}\big)^\mu$.

For a proof of the above lemma, calculate $\sup\big\{\frac{\ln_+^\mu(bv)}{v^{q-1}}; v \in (0,\infty)\big\}$. The next Lemma is taken from [1, Lemma 6].

Lemma 2.5. *Let A, $R_0 \leq R_1$ be positive numbers, $n-2 \leq \vartheta < n$, η a nonnegative and nondecreasing function on $(0,\infty)$. Then there exist ϵ_0, c positive so that for any nonnegative, nondecreasing function ϕ defined on $[0, 2R_1]$ and satisfying with $(B_1 + B_2\eta(U_{2R_0})) \in [0, \epsilon_0]$ the inequality*

$$\phi(\sigma) \leq \left\{ A\left(\frac{\sigma}{R}\right)^n + \frac{1}{2}\left(1 + A\left(\frac{\sigma}{R}\right)^n\right)[B_1 + B_2\eta(U_{2R})] \right\} \phi(2R) \tag{2.5}$$

for all σ, R such that $0 < \sigma < R \leq R_0$, it holds

$$\phi(\sigma) \leq c\sigma^\vartheta \phi(2R_0), \quad \forall \sigma : 0 < \sigma \leq R_0. \tag{2.6}$$

Remark 2.6. Note that we can take

$$\epsilon_0 = \frac{1}{2(2^{n+1}A)^{\frac{\vartheta}{n-\vartheta}}}, \quad c = \Big(\frac{(2^{n+1}A)^{\frac{1}{n-\vartheta}}}{2R_0}\Big)^\vartheta.$$

Lemma 2.7 ([5, p.78]). *Given the system*

$$-D_\alpha\Big(A_{ij}^{\alpha\beta} D_\beta u^j\Big) = 0, \quad i = 1, \dots, N$$

where $A_{ij}^{\alpha\beta}$ are constants satisfying (i) and (ii). There exists a constant $L = L(n, N, M/\nu) \geq 1$ such that for every weak solution $u \in W^{1,2}(\Omega, \mathbb{R}^N)$, for every $x \in \Omega$ and $0 < \sigma \leq R \leq \operatorname{dist}(x, \partial\Omega)$ the following estimate holds,

$$\int_{B_\sigma(x)} |Du(y)|^2 \, dy \leq L\left(\frac{\sigma}{R}\right)^n \int_{B_R(x)} |Du(y)|^2 \, dy \,.$$

Remark 2.8. Note that

$$L = c(n, N)\Big(\frac{M}{\nu}\Big)^{2k}, \quad k = 1 + \big[\frac{n}{2}\big]$$

and for $n = 3$ and $N = 2$ it holds

$$L < 10^4\Big(\frac{M}{\nu}\Big)^4. \tag{2.7}$$

One of the tools for the proof of our main result is the following reverse Hölder inequality that is standard in our setting .

Lemma 2.9 (see [5, 10])**.** *Let $u \in W^{1,2}(\Omega,\mathbb{R}^N)$ be a minimum of the functional* (1.1) *under the assumptions (i) and (ii). Then $Du \in L^{2p}_{\rm loc}(\Omega,\mathbb{R}^{nN})$ for some $p>1$ and there exists a constant $C_1 = C_1(n,N,M/\nu)$ such that for all balls $B_{2R}(x) \subset \Omega$,*

$$\Big(\fint_{B_R(x)} |Du|^{2p}\,dy\Big)^{1/2p} \le C_1\Big(\fint_{B_{2R}(x)} |Du|^2\,dy\Big)^{1/2}.$$

Let x_0 be any fixed point of Ω, $0<R\le \operatorname{dist}(x_0,\partial\Omega)$. We set

$$A^{\alpha\beta}_{ij}(u_{x_0,R})_{x_0,R} = \fint_{B_R(x_0)} A^{\alpha\beta}_{ij}(y,u_{x_0,R})\,dy\,.$$

Asolution to the system

$$\begin{gathered} D_\alpha\Big(A^{\alpha\beta}_{ij}(u_{x_0,R})_{x_0,R} D_\beta v^j\Big) = 0 \quad \text{in } B_R(x_0),\\ v-u \in W^{1,2}_0(B_R(x_0),\mathbb{R}^N) \end{gathered} \tag{2.8}$$

posses the following property.

Lemma 2.10 (see [5, 6, 10])**.** *Let $v \in W^{1,2}(B_R(x_0),\mathbb{R}^N)$ be a solution to* (2.8) *with $u \in W^{1,2p}(B_R(x_0),\mathbb{R}^N)$, $p\ge 1$. Then*

$$\int_{B_R(x_0)} |Dv|^{2p}\,dy \le C_2 \int_{B_R(x_0)} |Du|^{2p}\,dy.$$

Here $C_2 := C_2(M/\nu)$.

Remark 2.11. Revising proofs of Lemmas 2.9 and 2.10 one can see that the constants from the foregoing estimates depend increasingly on M/ν. Moreover, in a case $p=1$, the constant C_2 from Lemma 2.10 can be computed as $C_2 = 2\left[1+(M/\nu)^2\right]$.

In the proof of Theorem 1.1 we use an inequality which is a consequence of the Natanson's Lemma (see e.g. [17, pg. 262]). It reads as follows.

Lemma 2.12 (see [2, Lemma 3.7])**.** *Let $f:[a,\infty)\to\mathbb{R}$ be a nonnegative function which is integrable on $[a,b]$ for all $a<b<\infty$ and*

$$\mathcal{N} = \sup_{0<h<\infty} \frac{1}{h}\int_a^{a+h} f(t)\,dt < \infty\,.$$

Let $g:[a,\infty)\to\mathbb{R}$ be an arbitrary nonnegative, non-increasing and integrable function. Then $\int_a^\infty f(t)g(t)\,dt$ exists and

$$\int_a^\infty f(t)g(t)\,dt \le \mathcal{N}\int_a^\infty g(t)\,dt.$$

The next two propositions will be used in the proof of Theorem 1.1.

Proposition 2.13. *Let $u \in W^{1,2}(\Omega,\mathbb{R}^N)$ be a minimizer of the functional* (1.1) *under the assumptions (i) and (ii). Then for every ball $B_{2R}(x) \subset \Omega$, arbitrary constants $b>0$, $\mu\ge 2$ and the constant $p>1$ from Lemma 2.9 we have*

$$\int_{B_R(x)} |Du|^2 \ln_+^\mu(b|Du|^2)\,dy \le 2^{-n} C_1^{2p} C_\mu \Big(b\fint_{B_{2R}(x)} |Du|^2\,dy\Big)^{p-1}\int_{B_{2R}(x)} |Du|^2\,dy$$

where C_1 is the constant from Lemma 2.9.

The above proposition is a straightforward consequence of Lemmas 2.4 and 2.9.

Proposition 2.14. *Let $v \in W^{1,2}(B_R(x_0),\mathbb{R}^N)$ be a weak solution to* (2.8) *where $u \in W^{1,2}(\Omega,\mathbb{R}^N)$ be a minimizer of the functional* (1.1) *under the assumptions (i) and (ii). Then for ball $B_{2R}(x_0) \subset \Omega$, arbitrary constants $b > 0$, $\mu \geq 2$ and the constant $p > 1$ from Lemma 2.9 we have*

$$\begin{aligned} &\int_{B_R(x_0)} |Dv|^2 \ln_+^{\mu}\left(b|Dv|^2\right) dx \\ &\leq 2^{-n} C_1^{2p} C_2 C_\mu \Big(b \fint_{B_{2R}(x_0)} |Du|^2\, dx\Big)^{p-1} \int_{B_{2R}(x_0)} |Du|^2\, dx \end{aligned} \tag{2.9}$$

where C_2 is the constant from Lemma 2.10.

The proof of the above proposition is a consequence of Lemmas 2.4, 2.10 and 2.9.

3. Proof of Theorem 1.1

We divide the proof into two parts. In the first part of the proof we assume that the coefficients $A_{ij}^{\alpha\beta}$ of the functional (1.1) depend only on u, and the second part we consider the proof of the theorem in its full generality.

Case $A_{ij}^{\alpha\beta} = A_{ij}^{\alpha\beta}(u)$. We set $\phi(r) = \phi(x,r) = \int_{B_r(x)} |Du|^2\, dy$ and $U_r = U_r(x) = r^{2-n}\phi(x,r)$ for $B_r(x) \subset \Omega$. Now let x be any fixed point of $\overline{\Omega}_0 \subset \Omega$, $\operatorname{dist}(\Omega_0,\partial\Omega) = 2d > 0$, $B_{2R}(x) \subset \Omega$, $0 < R \leq d$ and v be a minimizer of the frozen functional

$$\mathcal{A}^0(v; B_R(x)) = \int_{B_R(x)} A_{ij}^{\alpha\beta}(u_R) D_\alpha v^i D_\beta v^j\, dy$$

among all the functions in $W^{1,2}(B_R(x),\mathbb{R}^N)$ taking the values u on $\partial B_R(x)$.

From the Euler equation for v and from Lemma 2.7 we have

$$\int_{B_\sigma(x)} |Dv|^2\, dy \leq L\left(\frac{\sigma}{R}\right)^n \int_{B_R(x)} |Dv|^2\, dy, \quad \text{for } 0 < \sigma \leq R. \tag{3.1}$$

Put $w = u - v$. It is clear that $w \in W_0^{1,2}(B_R(x),\mathbb{R}^N)$. Using (3.1) by standard arguments we obtain

$$\begin{aligned} &\int_{B_\sigma(x)} |Du|^2\, dy \\ &\leq 2\left(1 + 2L\left(\frac{\sigma}{R}\right)^n\right) \int_{B_R(x)} |Dw|^2\, dy + 4L\left(\frac{\sigma}{R}\right)^n \int_{B_R(x)} |Du|^2\, dy. \end{aligned} \tag{3.2}$$

Now we estimate the first integral on the right-hand side of (3.2). From [7, Lemma 2.1] we have

$$
\begin{aligned}
\int_{B_R(x)} |Dw|^2\,dy &\le \frac{2}{\nu}\left(\mathcal{A}^0\left(u;B_R(x)\right)-\mathcal{A}^0\left(v;B_R(x)\right)\right)\\
&\le \frac{2}{\nu}\Big\{\int_{B_R(x_0)}\Big(A_{ij}^{\alpha\beta}(u_R)-A_{ij}^{\alpha\beta}(u)\Big)D_\alpha u^i D_\beta u^j\,dx\\
&\quad+\int_{B_R(x_0)}\Big(A_{ij}^{\alpha\beta}(v)-A_{ij}^{\alpha\beta}(u_R)\Big)D_\alpha v^i D_\beta v^j\,dx\\
&\quad+\mathcal{A}\left(u;B_R(x_0)\right)-\mathcal{A}\left(v;B_R(x_0)\right)\Big\}\\
&=\frac{2}{\nu}\left\{I+II+\mathcal{A}\left(u;B_R(x)\right)-\mathcal{A}\left(v;B_R(x)\right)\right\}\\
&\le \frac{2}{\nu}\left(I+II\right).
\end{aligned}
\tag{3.3}
$$

Note that $\mathcal{A}\left(u;B_R(x)\right)-\mathcal{A}\left(v;B_R(x)\right)\le 0$, since u is a minimizer. Now we estimate terms I and II from (3.3).

Assumption (iii) and the Young inequality (2.3) give

$$
\begin{aligned}
|I| &\le \int_{B_R(x)}\omega\left(|u-u_R|\right)|Du|^2\,dy\\
&\le \int_{B_R(x)}\Phi\left(\varepsilon|Du|^2\right)\,dy+\int_{B_R(x)}\overline{\Psi}\left(\frac{1}{\varepsilon}\omega\left(|u-u_R|\right)\right)\,dy\\
&= I_1+I_2.
\end{aligned}
\tag{3.4}
$$

By Proposition 2.13 we have

$$
\begin{aligned}
I_1 &= \varepsilon\int_{B_R(x)}|Du|^2\ln_+^{\mu}\left(a\varepsilon|Du|^2\right)\,dy\\
&\le \varepsilon\,2^{-n}C_1^{2p}C_\mu\Big(a\varepsilon\fint_{B_{2R}(x)}|Du|^2\,dy\Big)^{p-1}\phi(2R).
\end{aligned}
\tag{3.5}
$$

According to Lemma 2.3 (see (2.1) as well) we have

$$
I_2=\int_{B_R(x)}\overline{\Psi}\Big(\frac{1}{\varepsilon}\omega\left(|u-u_R|\right)\Big)\,dy=\frac{1}{a}\int_0^\infty\frac{d}{dt}\widetilde{\Psi}\Big(\frac{\omega(t)}{\varepsilon}\Big)m_R(t)\,dt=\frac{1}{a}\widetilde{I}_2 \tag{3.6}
$$

where $m_R(t)=|\left\{y\in B_R(x):|u(y)-u_R|>t\right\}|$.

Estimating the term $\widetilde{I}_2$ we use the fact that $m_R(t) \le \kappa_n R^n$ and the constant from the Poincaré inequality on the ball equals to 2^{2n}. By Lemma 2.12 we obtain

$$\begin{aligned}
\widetilde{I}_2 &\le \int_0^{t_0} \frac{d}{dt}\widetilde{\Psi}\Big(\frac{\omega(t)}{\varepsilon}\Big) m_R(t)\,dt + \int_{t_0}^{\infty} \frac{d}{dt}\widetilde{\Psi}\Big(\frac{\omega(t)}{\varepsilon}\Big) m_R(t)\,dt \\
&\le \kappa_n R^n \int_0^{t_0} \frac{d}{dt}\widetilde{\Psi}\Big(\frac{\omega(t)}{\varepsilon}\Big)\,dt \\
&\quad + \sup_{t_0<t<\infty}\Big(\frac{1}{t-t_0}\int_{t_0}^{t} \frac{d}{ds}\widetilde{\Psi}\Big(\frac{\omega(s)}{\varepsilon}\Big)\,ds\Big) \int_{t_0}^{\infty} m_R(s)\,ds \\
&\le \kappa_n \widetilde{\Psi}\Big(\frac{\omega(t_0)}{\varepsilon}\Big) R^n + \sup_{t_0<t<\infty}\Big[\frac{\widetilde{\Psi}\big(\frac{\omega(t)}{\varepsilon}\big) - \widetilde{\Psi}\big(\frac{\omega(t_0)}{\varepsilon}\big)}{t-t_0}\Big] \int_{B_R(x)} |u-(u)_R|\,dy \qquad (3.7) \\
&\le \frac{4\kappa_n \widetilde{\Psi}\Big(\frac{\omega(t_0)}{\varepsilon}\Big) R^2}{2^n U_{2R}} \phi(2R) + \frac{\sqrt{\kappa_n}2^n}{2^{1+n/2}}(2R)^{1+n/2}\mathcal{M}\phi^{1/2}(2R) \\
&\le 4R^2\Big(\frac{\kappa_n\widetilde{\Psi}\big(\frac{\omega(t_0)}{\varepsilon}\big)}{2^n U_{2R}} + \frac{\sqrt{\kappa_n}2^n\mathcal{M}}{2^{1+n/2}\sqrt{U_{2R}}}\Big)\phi(2R) \\
&< 4R^2\Big(\frac{\widetilde{\Psi}\big(\frac{\omega(t_0)}{\varepsilon}\big)}{U_{2R}} + \frac{2^{n-1}\mathcal{M}}{\sqrt{U_{2R}}}\Big)\phi(2R)
\end{aligned}$$

where

$$\mathcal{M} = \sup_{t_0<t<\infty} \frac{\widetilde{\Psi}\big(\frac{\omega(t)}{\varepsilon}\big) - \widetilde{\Psi}\big(\frac{\omega(t_0)}{\varepsilon}\big)}{t-t_0}. \qquad (3.8)$$

The above estimate leads to

$$\begin{aligned}
|I| &\le \varepsilon\, 2^{-n} C_1^{2p} C_\mu \Big(a\varepsilon \fint_{B_{2R}(x)} |Du|^2\,dy\Big)^{p-1}\phi(2R) \\
&\quad + \frac{4R^2}{a}\Big(\frac{\widetilde{\Psi}\big(\frac{\omega(t_0)}{\varepsilon}\big)}{U_{2R}} + \frac{2^{n-1}\mathcal{M}}{\sqrt{U_{2R}}}\Big)\phi(2R).
\end{aligned} \qquad (3.9)$$

A technique similar to the previous one yields the estimate:

$$\begin{aligned}
|II| &\le \int_{B_R(x)} \omega\left(|v-u_R|\right)|Dv|^2\,dy \\
&\le \int_{B_R(x)} \Phi\left(\varepsilon|Dv|^2\right)\,dy + \int_{B_R(x)} \overline{\Psi}\Big(\frac{1}{\varepsilon}\omega\left(|v-u_R|\right)\Big)\,dy = J_1 + J_2.
\end{aligned}$$

By Proposition 2.14 we obtain

$$\begin{aligned}
J_1 &= \varepsilon \int_{B_R(x)} |Dv|^2 \ln_+^{\mu}\left(a\varepsilon|Dv|^2\right)\,dy \\
&\le \varepsilon\, 2^{-n} C_2 C_1^{2p} C_\mu \Big(a\varepsilon \fint_{B_{2R}(x)} |Du|^2\,dy\Big)^{p-1}\phi(2R).
\end{aligned} \qquad (3.10)$$

Applying Lemma 2.3 to the second integral J_2 we have

$$J_2 = \int_{B_R(x)} \overline{\Psi}\Big(\frac{1}{\varepsilon}\omega\left(|v-u_R|\right)\Big)\,dy = \frac{1}{a}\int_0^{\infty} \frac{d}{dt}\widetilde{\Psi}\Big(\frac{\omega(t)}{\varepsilon}\Big) m_R(t)\,dt = \frac{1}{a}\widetilde{J}_2 \qquad (3.11)$$

where the function $\widetilde{\Psi}$ is the same as in (3.6).

Using Poincaré inequality and a formula on [16, pg. 98] we obtain

$$
\begin{aligned}
&\int_{B_R(x)} |v-u_R|\,dy\\
&\le \int_{B_R(x)} |u-u_R|\,dy + \int_{B_R(x)} |v-u|\,dy\\
&= \int_{B_R(x)} |u-u_R|\,dy + \int_{B_R(x)} |w|\,dy\\
&\le \Big[\Big(\int_{B_R(x)} |u-u_R|^2\,dy\Big)^{1/2} + \Big(\int_{B_R(x)} |w|^2\,dy\Big)^{1/2}\Big]\kappa_n^{1/2}R^{n/2}\\
&\le \Big[\sqrt{\kappa_n}\Big(2^n + \frac{2}{\pi\sqrt{n}}\Big)(1+\sqrt{C_2})\Big]R^{1+n/2}\phi^{1/2}(R)\\
&\le 2^{n+2}\sqrt{C_2}R^{1+n/2}\phi^{1/2}(R),
\end{aligned}
$$

and we can estimate $\widetilde{J}_2$ in the same way as in the case of $\widetilde{I}_2$ (see (3.7), (3.8)):

$$
\widetilde{J}_2 < 4R^2\Big(\frac{\widetilde{\Psi}\big(\frac{\omega(t_0)}{\varepsilon}\big)}{U_{2R}} + \frac{2^{n+2}\sqrt{C_2}\mathcal{M}}{\sqrt{U_{2R}}}\Big)\phi(2R).
$$

The last consideration leads to the analogous estimate as for I, we obtain

$$
\begin{aligned}
|II| \le\ & 2^{-n}C_1^{2p}C_2C_\mu\varepsilon\Big(a\varepsilon ⨍_{B_{2R}(x)} |Du|^2\,dy\Big)^{p-1}\phi(2R)\\
&+ \frac{4R^2}{a}\Big(\frac{\widetilde{\Psi}\big(\frac{\omega(t_0)}{\varepsilon}\big)}{U_{2R}} + \frac{2^{n+2}\sqrt{C_2}\mathcal{M}}{\sqrt{U_{2R}}}\Big)\phi(2R).
\end{aligned}
\tag{3.12}
$$

Substituting (3.9) and (3.12) into (3.3) gives

$$
\begin{aligned}
\int_{B_R(x)} |Dw|^2\,dy \le & \frac{1}{\nu}\Big[2^{2-n}C_1^{2p}C_2C_\mu\,\varepsilon\Big(a\varepsilon ⨍_{B_{2R}(x)} |Du|^2\,dy\Big)^{p-1}\\
&+ \frac{16R^2}{a}\Big(\frac{\widetilde{\Psi}\big(\frac{\omega(t_0)}{\varepsilon}\big)}{U_{2R}} + \frac{2^{n+2}\sqrt{C_2}\mathcal{M}}{\sqrt{U_{2R}}}\Big)\Big]\phi(2R)\,.
\end{aligned}
\tag{3.13}
$$

From (3.2), (3.13) and from the assumptions of Theorem 1.1 we obtain

$$
\begin{aligned}
\phi(\sigma) \le\ & 4L\big(\frac{\sigma}{R}\big)^n\phi(2R) + 2\left(1 + 2L\big(\frac{\sigma}{R}\big)^n\right)\\
&\times\Big[\frac{2^{2-n}C_1^{2p}C_2C_\mu}{\nu}\,\varepsilon\Big(a\varepsilon ⨍_{B_{2R}(x)} |Du|^2\,dy\Big)^{p-1}\\
&+ \frac{16R^2}{a\nu}\Big(\frac{\widetilde{\Psi}\big(\frac{\omega(t_0)}{\varepsilon}\big)}{U_{2R}} + \frac{2^{n+2}\sqrt{C_2}}{\sqrt{U_{2R}}}\mathcal{M}\Big)\Big]\phi(2R)
\end{aligned}
\tag{3.14}
$$

for all $\sigma < R \le d$.

Now, in (3.14), we can choose the constants ε and a in the following way:

$$
\varepsilon = \frac{\omega_\infty}{C_\mu^\rho}, \quad a = \frac{2^{n+10}\sqrt{C_2}R^2}{\nu^\delta\, U_{2R}}\Big(\frac{|\Omega|^{1-2/n}}{(2d)^{n-2}}\Big)^{1/2} \quad \text{for } U_{2R} > 0 \tag{3.15}
$$

where $\delta = 1 + \ln \epsilon_0 / \ln \nu$, $\epsilon_0 = \frac{1}{2(2^{n+3}L)^{\vartheta/(n-\vartheta)}}$, $C_\mu = \big(\frac{\mu}{(p-1)e}\big)^\mu$ and ρ, $\mu \in \mathbb{R}$ are suitable constants. We obtain

$$\begin{aligned} \phi(\sigma) \le & 4L\left(\frac{\sigma}{R}\right)^n \phi(2R) + \frac{1}{2}\left(1 + 2L\left(\frac{\sigma}{R}\right)^n\right) \\ & \times \Big[\frac{2^{(n+10)p-2(n+3)}(PC_1^2)^p C_2^{(p+1)/2}}{\kappa_n^{p-1}\epsilon_0^{p-1}C_\mu^{p\rho-1}}\Big(\frac{|\Omega|^{1-2/n}}{(2d)^{n-2}}\Big)^{(p-1)/2} \\ & + \epsilon_0\Big(\frac{(2d)^{n-2}}{|\Omega|^{1-2/n}}\Big)^{1/2}\Big(\frac{1}{2^{n+4}\sqrt{C_2}}\widetilde{\Psi}\Big(\frac{\omega(t_0)}{\varepsilon}\Big) + \frac{1}{4}\mathcal{M}\sqrt{U_{2R}}\Big)\Big]\phi(2R), \end{aligned} \tag{3.16}$$

where $P = \omega_\infty/\nu$.

The constants $\rho > 1/p$ and $\mu \ge 6$ can be always chosen in such a way that

$$\frac{2^{(n+10)p-2(n+3)}(PC_1^2)^p C_2^{(p+1)/2}}{\kappa_n^{p-1}\epsilon_0^{p-1}C_\mu^{p\rho-1}}\Big(\frac{|\Omega|^{1-2/n}}{(2d)^{n-2}}\Big)^{(p-1)/2} \le \frac{1}{2}\epsilon_0,$$

which is equivalent to the estimate

$$C_\mu^{\rho p-1} \ge \frac{2^{(n+10)p-(2n+5)}(PC_1^2)^p C_2^{(p+1)/2}}{\kappa_n^{p-1}\epsilon_0^p}\Big(\frac{|\Omega|^{1-2/n}}{(2d)^{n-2}}\Big)^{(p-1)/2}.$$

Using the second term in (1.7) and taking into account that $((2d)^{n-2}/|\Omega|^{1-2/n})^{1/2} \le 1$, and we obtain

$$\begin{aligned} \phi(\sigma) \le & \, 4L\left(\frac{\sigma}{R}\right)^n \phi(2R) + \frac{1}{2}\left(1 + 2L\left(\frac{\sigma}{R}\right)^n\right) \\ & \times \Big[\frac{3}{4} + \frac{1}{4}\Big(\frac{(2d)^{n-2}}{|\Omega|^{1-2/n}}\Big)^{1/2}\mathcal{M}\sqrt{U_{2R}}\Big]\epsilon_0\,\phi(2R), \quad \text{for } 0 < \sigma \le R \le d. \end{aligned} \tag{3.17}$$

For $R = d$ by (1.6) we obtain

$$\Big(\frac{(2d)^{n-2}}{|\Omega|^{1-2/n}}\Big)^{1/2}\mathcal{M}\sqrt{U_{2d(x)}} \le \mathcal{M}\Big(\frac{1}{|\Omega|^{1-2/n}}\int_\Omega |Du|^2\,dy\Big)^{1/2} \le 1\,.$$

Putting $A = 4L$, $B_1 = 3\epsilon_0/4$, and $B_2 = \epsilon_0/4$ in (3.17) and using Lemma 2.5, we can conclude that

$$\phi(\sigma) \le c\sigma^\vartheta\phi(2R), \quad \text{for } 0 < \sigma \le R\,.$$

Now, the result follows from Proposition 2.2.

Case $A_{ij}^{\alpha\beta} = A_{ij}^{\alpha\beta}(x,u)$**.** Let x be any fixed point of $\overline{\Omega}_0 \subset \Omega$, $\operatorname{dist}(\Omega_0, \partial\Omega) = 2d_0 > 0$, $B_{2R}(x) \subset \Omega$, $0 < R \le d_0$ and v be a minimizer of the functional

$$\mathcal{A}^0(v; B_R(x)) = \int_{B_R(x)} A_{ij}^{\alpha\beta}(u_R)_R D_\alpha v^i D_\beta v^j\,dy$$

among all the functions in $W^{1,2}(B_R(x), \mathbb{R}^N)$ taking the values u on $\partial B_R(x)$ where

$$A_{ij}^{\alpha\beta}(z)_R = \fint_{B_R(x)} A_{ij}^{\alpha\beta}(y,z)\,dy.$$

Arguments, analogous to those at the beginning of the proof of Theorem 1.1, give us

$$\begin{aligned} & \int_{B_\sigma(x)} |Du|^2\,dy \\ & \le 2\left(1 + 2L\left(\frac{\sigma}{R}\right)^n\right)\int_{B_R(x)} |Dw|^2\,dy + 4L\left(\frac{\sigma}{R}\right)^n\int_{B_R(x)} |Du|^2\,dy \end{aligned} \tag{3.18}$$

where $w=(u-v)\in W_0^{1,2}(B_R(x),\mathbb{R}^N)$. Now we estimate the first integral on the right hand side of (3.2). From [7, Lemma 2.1] we have

$$
\begin{aligned}
\int_{B_R(x)}|Dw|^2\,dy &\le \frac{2}{\nu}\left(\mathcal{A}^0\left(u;B_R(x)\right)-\mathcal{A}^0\left(v;B_R(x)\right)\right)\\
&\le \frac{2}{\nu}\Big\{\int_{B_R(x)}\Big(A_{ij}^{\alpha\beta}(u_R)_R-A_{ij}^{\alpha\beta}(y,u_R)\Big)D_\alpha u^iD_\beta u^j\,dy\\
&\quad+\int_{B_R(x)}\Big(A_{ij}^{\alpha\beta}(y,u_R)-A_{ij}^{\alpha\beta}(y,u)\Big)D_\alpha u^iD_\beta u^j\,dy\\
&\quad+\int_{B_R(x)}\Big(A_{ij}^{\alpha\beta}(y,u_R)-A_{ij}^{\alpha\beta}\left(u_R\right)_R\Big)D_\alpha v^iD_\beta v^j\,dy\\
&\quad+\int_{B_R(x)}\Big(A_{ij}^{\alpha\beta}(y,v)-A_{ij}^{\alpha\beta}(y,u_R)\Big)D_\alpha v^iD_\beta v^j\,dy\\
&\quad+\mathcal{A}\left(u;B_R(x)\right)-\mathcal{A}\left(v;B_R(x)\right)\Big\}\\
&=\frac{2}{\nu}\left\{I+II+III+IV+\mathcal{A}\left(u;B_R(x)\right)-\mathcal{A}\left(v;B_R(x)\right)\right\}\\
&\le\frac{2}{\nu}\left(I+II+III+IV\right).
\end{aligned}
\tag{3.19}
$$

Notice that $\mathcal{A}\left(u;B_R(x)\right)-\mathcal{A}\left(v;B_R(x)\right)\le 0$, since u is a minimizer. Now we will estimate the terms I, II, III and IV from (3.19). In the following we will denote $A:=(A_{ij}^{\alpha\beta})$. Using Hölder inequality, higher integrability of gradient of minima (Lemma 2.9, $p>1$, $p'=p/(p-1)$) we obtain

$$
\begin{aligned}
|I| &\le |B_R(x)|^{1/p}\Big(\int_{B_R(x)}|A(u_R)_R-A(y,u_R)|^{p'}\,dy\Big)^{1/p'}\Big(\fint_{B_R(x)}|Du|^{2p}\,dy\Big)^{1/p}\\
&\le C_1^2|B_R(x)|^{1/p}\Big(\int_{B_R(x)}|A(u_R)_R-A(y,u_R)|^{p'}\,dy\Big)^{1/p'}\fint_{B_{2R}(x)}|Du|^2\,dy.
\end{aligned}
$$

Taking into account assumptions (i), (iv) and Definition 2.1 we obtain

$$
\Big(\fint_{B_R(x)}|A(u_R)_R-A(y,u_R)|^{p'}\,dy\Big)^{1/p'}\le(2M)^{1/p}\left(\mathcal{N}_R\left(A(\cdot,u_R)\right)\right)^{1-1/p}
$$

and then, using the above two estimates, we have

$$
|I|\le 2^{-n}C_1^2(2M)^{1/p}\left(\mathcal{N}_R\left(A(\cdot,u_R)\right)\right)^{1-1/p}\phi(2R). \tag{3.20}
$$

A similarity of the terms I and III enables us to write (by Lemma 2.10) the inequality

$$
|III|\le 2^{-n}C_1^2C_2^{1/p}(2M)^{1/p}\left(\mathcal{N}_R\left(A(\cdot,u_R)\right)\right)^{1-1/p}\phi(2R). \tag{3.21}
$$

Now it remains to estimate the terms II and IV from (3.19). Estimating these two terms is step by step the same as estimating the terms I and II from (3.3) in the previous part of the proof. So we have

$$
\begin{aligned}
|II| &\le \varepsilon 2^{-n}C_1^{2p}C_\mu\Big(a\varepsilon\fint_{B_{2R}(x)}|Du|^2\,dy\Big)^{p-1}\phi(2R)\\
&\quad+\frac{4R^2}{a}\Big(\frac{\widetilde{\Psi}\left(\frac{\omega(t_0)}{\varepsilon}\right)}{U_{2R}}+\frac{2^{n-1}\mathcal{M}}{\sqrt{U_{2R}}}\Big)\phi(2R)
\end{aligned}
\tag{3.22}
$$

and

$$\begin{aligned} |IV| \le 2^{-n} C_1^{2p} C_2 C_\mu\, \varepsilon \Big(a\varepsilon \fint_{B_{2R}(x)} |Du|^2\, dy \Big)^{p-1} \phi(2R) \\ + \frac{4R^2}{a} \Big(\frac{\widetilde{\Psi}\big(\frac{\omega(t_0)}{\varepsilon}\big)}{U_{2R}} + \frac{2^{n+2}\sqrt{C_2}\mathcal{M}}{\sqrt{U_{2R}}} \Big) \phi(2R). \end{aligned} \tag{3.23}$$

Substituting (3.20)–(3.23) into (3.19) and, consequently, (3.19) into (3.18) we obtain

$$\begin{aligned} \phi(\sigma) \le 4L \left(\frac{\sigma}{R}\right)^n \phi(2R) + 2\Big(1 + 2L\left(\frac{\sigma}{R}\right)^n\Big) \Big[\frac{K_1(R)}{\nu} \epsilon_0 \\ + \frac{2^{2-n} C_1^{2p} C_2 C_\mu}{\nu}\, \varepsilon \Big(a\varepsilon \fint_{B_{2R}(x)} |Du|^2\, dy \Big)^{p-1} \\ + \frac{16R^2}{a\nu} \Big(\frac{\widetilde{\Psi}\big(\frac{\omega(t_0)}{\varepsilon}\big)}{U_{2R}} + \frac{2^{n+2}\sqrt{C_2}}{\sqrt{U_{2R}}} \mathcal{M} \Big) \Big] \phi(2R) \end{aligned} \tag{3.24}$$

for $\sigma < R \le d_0$, where

$$\epsilon_0 = \frac{1}{2(2^{n+3}L)^{\frac{\vartheta}{n-\vartheta}}}, \quad K_1(R) = \frac{C_1^2 (MC_2)^{1/p} \left(\mathcal{N}_R\left(A(\cdot, u_R)\right)\right)^{1-1/p}}{2^{n-3}\epsilon_0},$$

see Remark 2.6 and Lemma 2.7 as well.

Assumption (iv) implies that there exists $R_0 > 0$, such that

$$\frac{K_1(R)}{\nu} \le \frac{1}{16} \iff \mathcal{N}_R\left(A(\cdot, u_R)\right) \le M\Big(\frac{2^{n-7}\epsilon_0 \nu}{C_1^2 C_2^{1/p} M} \Big)^{p/(p-1)} \tag{3.25}$$

for $0 < R \le R_0$ (here we recall that the choice of the constant R_0 does not depend on $x \in \overline{\Omega}_0$). Let us put $d = \min\{d_0, R_0\}$. Then, in the estimate (3.24), we can choose the constants ε and a in the following way:

$$\varepsilon = \frac{\omega_\infty}{C_\mu^\rho}, \quad a = \frac{2^{n+10}\sqrt{C_2}R^2}{\nu^\delta\, U_{2R}} \Big(\frac{|\Omega|^{1-2/n}}{(2d)^{n-2}} \Big)^{1/2} \quad \text{for } U_{2R} > 0, \tag{3.26}$$

where $\delta = 1 + \ln \epsilon_0 / \ln \nu$, $C_\mu = \big(\frac{\mu}{(p-1)\mathrm{e}}\big)^\mu$ and ρ, $\mu \in \mathbb{R}$ are suitable constants. We obtain

$$\begin{aligned} \phi(\sigma) \le 4L \left(\frac{\sigma}{R}\right)^n \phi(2R) + \frac{1}{2}\left(1 + 2L\left(\frac{\sigma}{R}\right)^n\right) \\ \times \Big[\frac{1}{4}\epsilon_0 + \frac{2^{(n+10)p-2(n+3)} (PC_1^2)^p C_2^{(p+1)/2}}{\kappa_n^{p-1} \epsilon_0^{p-1} C_\mu^{p\rho-1}} \Big(\frac{|\Omega|^{1-2/n}}{(2d)^{n-2}} \Big)^{(p-1)/2} \\ + \epsilon_0 \Big(\frac{(2d)^{n-2}}{|\Omega|^{1-2/n}} \Big)^{1/2} \Big(\frac{1}{2^{n+4}\sqrt{C_2}} \widetilde{\Psi}\Big(\frac{\omega(t_0)}{\varepsilon}\Big) + \frac{1}{4}\mathcal{M}\sqrt{U_{2R}} \Big) \Big] \phi(2R), \end{aligned} \tag{3.27}$$

where $P = \omega_\infty / \nu$.

The constants $\rho > 1/p$ and $\mu \ge 6$ can be always chosen in such a way that

$$\frac{2^{(n+10)p-2(n+3)} (PC_1^2)^p C_2^{(p+1)/2}}{\kappa_n^{p-1} \epsilon_0^{p-1} C_\mu^{p\rho-1}} \Big(\frac{|\Omega|^{1-2/n}}{(2d)^{n-2}} \Big)^{(p-1)/2} \le \frac{1}{4}\epsilon_0$$

which is equivalent to the estimate

$$C_\mu^{\rho p-1} \ge \frac{2^{(n+10)p-2(n+2)} (PC_1^2)^p C_2^{(p+1)/2}}{\kappa_n^{p-1} \epsilon_0^{p}} \Big(\frac{|\Omega|^{1-2/n}}{(2d)^{n-2}} \Big)^{(p-1)/2}.$$

Using the second term in (1.7) and taking into account $((2d)^{n-2}/|\Omega|^{1-2/n})^{1/2} \le 1$, we obtain

$$\begin{aligned}\phi(\sigma) &\le 4L\Big(\frac{\sigma}{R}\Big)^n \phi(2R)\\ &\quad + \frac{1}{2}\Big(1+2L\Big(\frac{\sigma}{R}\Big)^n\Big)\Big[\frac{3}{4}+\frac{1}{4}\Big(\frac{(2d)^{n-2}}{|\Omega|^{1-2/n}}\Big)^{1/2}\mathcal{M}\sqrt{U_{2R}}\Big]\epsilon_0\,\phi(2R),\end{aligned}$$

for $0 < \sigma \le R \le d$. The above estimate is formally the same as (3.17) in the first part of the proof. So, one can see that the result follows in the same way as it is demonstrated at the end of the previous case.

4. Illustrating examples and comments

Here, for simplicity, we consider $A_{ij}^{\alpha\beta} = A_{ij}^{\alpha\beta}(u)$.

Example 4.1. Let $\Omega = B_R(0) \subset \mathbb{R}^n$ in Theorem 1.1, the function g, mentioned in Remark 1.4, belong to $W^{1,2}_{\rm loc}(\mathbb{R}^n,\mathbb{R}^N)$, and, for $n-2 \le \lambda \le n$ it satisfy the condition $\sup_{0<\sigma\le R}\sigma^{-\lambda}\int_{B_\sigma(0)}|Dg(y)|^2\,dy \le c_\lambda$, $c_\lambda > 0$. Then choosing $\Omega_0 = B_{R/2}(0)$, $d = R/4$ and using (1.4), condition (1.6) will have the form

$$\frac{c_\lambda M}{\kappa_n^{1-2/n}\,\nu}R^{\lambda-n+2} \le \frac{1}{\mathcal{M}^2}\,. \tag{4.1}$$

So, for sufficiently small R we obtain regularity of minimizer u in $B_{R/2}(0)$ by Theorem 1.1 in the case when $\lambda > n-2$.

The case $\lambda = n-2$ leads to the regularity condition that depends only on the parameters of functional (1.1) and the function g.

For sufficiently big R we obtain regularity of minimizer u in $B_{R/2}(0)$ by Theorem 1.1 in the case when $0 < \lambda < n-2$.

Remark 4.2. Theorem 1.1 with a result from [12] can guarantee the everywhere regularity up to the boundary in a specific case. More precisely, if we consider $\Omega = B_R(0)$, split coefficients $A_{ij}^{\alpha\beta}(u) = \gamma^{\alpha\beta}a_{ij}(u)$ in (1.1), and suppose that $A_{ij}^{\alpha\beta}$ are uniformly continuous on $\mathbb{R}^N$ with the modulus of continuity (1.5), the function g, introduced in Remark 1.4, belongs to $W^{1,s}(B_R(0),\mathbb{R}^N)$, $s > n$ and the minimizer u is bounded, then, according to [12], there exists a constant $0 < R_1 < R$ such that $u \in C^{0,1-n/s}\big(\overline{B_R(0)}\setminus\overline{B_{R_1}(0)},\mathbb{R}^N\big)$. Now, choosing in Theorem 1.1 $\vartheta = n-2n/s$, $d = (R-R_2)/2$, $0 < R_1 < R_2 < R$, if condition (1.6) is fulfilled, then $u \in C^{0,1-n/s}(\overline{B_R(0)},\mathbb{R}^N)$.

Example 4.3. In $\Omega = B_R(0) \subset \mathbb{R}^3$ we consider the quasilinear variational integral

$$\mathcal{A}(u;\Omega) = \int_\Omega A_{ij}^{\alpha\beta}(u)D_\alpha u^i D_\beta u^j\,dx$$

where

$$A_{ij}^{\alpha\beta}(u) = a\delta_{ij}\delta_{\alpha\beta} + b\Big(\delta_{i\alpha}\arctan\frac{|u^i|}{C_\mu^\tau} + \delta_{j\beta}\arctan\frac{|u^j|}{C_\mu^\tau}\Big)$$

for $\alpha,\beta = 1,2,3$, $i,j = 1,2$, $u-g \in W_0^{1,2}(B_R(0),\mathbb{R}^2)$, $g \in W^{1,2}_{\rm loc}(\mathbb{R}^3,\mathbb{R}^2)$ $a > 6\pi b > 0$ (C_μ is from Remark 1.2 and $\tau > \rho > 0$). In this case we have

$$M = 6a + 10\pi b,\quad \nu = a - 6\pi b,\quad \omega_\infty = \pi\, b$$

and the modulus of continuity ω is given in Example 1.6. This is a sample of functional, regularity properties of which could be well understood through Theorem 1.1.

Example 4.4. To complete reader's notion of practical consequences of the results formulated in Theorem 1.1, we give two charts of possible values of the basic parameters appearing in the theorem. The first chart corresponds to the function ω defined by (1.11) and the second one corresponds to (1.9). For the simplicity, we put $\Omega = B_R(0) \subset \mathbb{R}^3$ and $\Omega_0 = B_{R/2}(0)$ (we use the same denotation as in Example 4.3). Choosing in the previous example $a = 16\pi\, b$ we have $M/\nu = 10.6$, $P = \omega_\infty/\nu = 0.1$, $C_1 = 10^4$, $C_2 = 10^2$, from (2.7) we obtain $L = 1.2 \cdot 10^8$, by means of Remark 2.8 we have $\epsilon_0 = 2.3 \cdot 10^{-6}$. In this case the function ω is defined by (1.11) and choosing $p = 1.5$, $\vartheta = 1.05$ we can present the following chart.

ν	$=$	10^{30}	10^{40}	10^{50}	10^{60}	10^{70}
ω_∞	$=$	10^{29}	10^{39}	10^{49}	10^{59}	10^{69}
$\omega(\omega_\infty)$	$\approx$	10^{8}	10^{27}	10^{44}	10^{59}	10^{69}
t_1	$\approx$	10^{51}	10^{51}	10^{55}	10^{59}	10^{65}
real value $\frac{1}{\mathcal{M}^2}$	$\approx$	10^{4}	10^{6}	10^{9}	10^{14}	10^{22}
estimate $\frac{1}{\mathcal{M}^2}$ by means of (1.12)	$\approx$	10^{2}	10^{4}	10^{7}	10^{13}	10^{21}
ρ	$=$	1.32	1.27	1.25	1.14	1.1
τ	$=$	2	1.9	1.9	1.7	1.7
μ	$=$	21	22	23	26.5	28.5

where t_1 is the point for which $\omega(t_1) = 0.95 \cdot \omega_\infty$.

In the case when the function ω is defined by (1.9), for the foregoing parameters we obtain the following chart.

ω_∞	$=$	10^{30}	10^{40}	10^{50}	10^{60}	10^{70}
t_0	$=$	10^{7}	10^{10}	10^{13}	10^{16}	10^{19}
$\omega(t_0)$	$\approx$	1	10^{10}	10^{19}	10^{30}	10^{40}
$\omega(\omega_\infty)$	$\approx$	10^{15}	10^{28}	10^{42}	10^{57}	10^{70}
t_1	$\approx$	10^{56}	10^{60}	10^{62}	10^{66}	10^{68}
real value $\frac{1}{\mathcal{M}^2}$	$\approx$	10^{11}	10^{17}	10^{22}	10^{28}	10^{35}
estimate $\frac{1}{\mathcal{M}^2}$ by means of (1.10)	$\approx$	10	10^{7}	10^{11}	10^{18}	10^{24}
ρ	$=$	1.51	1.51	1.51	1.5	1.49
γ	$=$	0.61	0.61	0.62	0.62	0.62
μ	$=$	17.7	17.9	18	18	18.1

where t_1 is the point for which $\omega(t_1) = \omega_\infty$. We note that for above mentioned parameters the second condition from (1.7) is satisfied.

5. Appendix

We give estimates of the constant $\mathcal{M}$ from (1.7) where ω is defined by Examples 1.5 and 1.6.

$$
\begin{aligned}
\frac{\widetilde{\Psi}\big(\frac{\omega(t)}{\varepsilon}\big) - \widetilde{\Psi}\big(\frac{\omega(t_0)}{\varepsilon}\big)}{t - t_0} &= \Big(\frac{d}{dt}\widetilde{\Psi}\Big(\frac{\omega(t)}{\varepsilon}\Big)\Big)|_{t=\xi} \\
&= \frac{\omega'(\xi)}{\varepsilon}\Big[1 + \frac{2}{2\mu - 1}\Big(\frac{1}{2\sqrt{\mu}}\frac{\omega(\xi)}{\varepsilon}\Big)^{\frac{2}{2\mu-1}}\Big]\mathrm{e}^{\big(\frac{1}{2\sqrt{\mu}}\frac{\omega(\xi)}{\varepsilon}\big)^{\frac{2}{2\mu-1}}},
\end{aligned}
$$

for $t_0 < \xi < t \le t_1$.

(a) Estimate of $\mathcal{M}$ related to the function ω from Example 1.5. Here we consider $\mu \geq 6$, $\rho > 1/p$, $0 < \gamma < 1$, $t_0 > 0$, $C_\mu > 1$.

$$\begin{aligned}
\mathcal{M} &= \sup_{t_0<t<t_1} \frac{\widetilde{\Psi}\big(\frac{\omega(t)}{\varepsilon}\big) - \widetilde{\Psi}\big(\frac{\omega(t_0)}{\varepsilon}\big)}{t-t_0} \\
&= \sup_{t_0<t<t_1} \Big(\frac{\omega'(t)}{\varepsilon}\Big) \mathrm{e}^{(\frac{1}{2\sqrt{\mu}}\frac{\omega(t)}{\varepsilon})^{\frac{2}{2\mu-1}}} \Big[1 + \frac{2}{2\mu-1}\Big(\frac{1}{2\sqrt{\mu}}\frac{\omega(t)}{\varepsilon}\Big)^{\frac{2}{2\mu-1}}\Big] \\
&= \sup_{t_0<t<t_1} \Big(\gamma C_\mu^\rho \frac{\mathrm{e}^{1/C_\mu^\rho}-1}{\big[t_0^\gamma + \big(\mathrm{e}^{1/C_\mu^\rho}-1\big)t^\gamma\big]} t^{\gamma-1}\Big) \mathrm{e}^{\big(\frac{C_\mu^\rho}{2\sqrt{\mu}}\ln\big(1+\frac{\mathrm{e}^{1/C_\mu^\rho}-1}{t_0^\gamma}t^\gamma\big)\big)^{\frac{2}{2\mu-1}}} \\
&\quad \times \Big[1 + \frac{2}{2\mu-1}\Big(\frac{C_\mu^\rho}{2\sqrt{\mu}}\ln\Big(1+\frac{\mathrm{e}^{1/C_\mu^\rho}-1}{t_0^\gamma}t^\gamma\Big)\Big)^{\frac{2}{2\mu-1}}\Big] \\
&\leq \sup_{t_0<t<t_1} \Big(\gamma C_\mu^\rho \frac{\mathrm{e}^{1/C_\mu^\rho}-1}{\big[t_0^\gamma + \big(\mathrm{e}^{1/C_\mu^\rho}-1\big)t^\gamma\big]}\Big) \sup_{t_0<t<t_1} \Big(t^{\gamma-1} \mathrm{e}^{\big(\frac{C_\mu^\rho}{2\sqrt{\mu}}\ln\big(1+\frac{\mathrm{e}^{1/C_\mu^\rho}-1}{t_0^\gamma}t^\gamma\big)\big)^{\frac{2}{2\mu-1}}}\Big) \\
&\quad \times \Big[1 + \frac{2}{2\mu-1}\sup_{t_0<t<t_1}\Big(\frac{C_\mu^\rho}{2\sqrt{\mu}}\ln\Big(1+\frac{\mathrm{e}^{1/C_\mu^\rho}-1}{t_0^\gamma}t^\gamma\Big)\Big)^{\frac{2}{2\mu-1}}\Big] \\
&\quad = S_1 S_2\Big(1 + \frac{2}{2\mu-1}S_3\Big).
\end{aligned} \tag{5.1}$$

The estimates of S_1, S_2 and S_3 are as follows.

$$S_1 \leq \gamma C_\mu^\rho \frac{\mathrm{e}^{1/C_\mu^\rho}-1}{t_0^\gamma} \leq \frac{\gamma(\mathrm{e}-1)}{t_0^\gamma}, \quad \forall t_0 \leq t \leq t_1;$$

$$S_2 \leq \sup_{t_0<t<t_1} \frac{\mathrm{e}^{\big(\frac{1}{\sqrt{\mu}}(\frac{t}{t_0})^\gamma\big)^{\frac{2}{2\mu-1}}}}{t^{1-\gamma}}.$$

If we define

$$f(t) = \frac{\mathrm{e}^{\big(\frac{1}{\sqrt{\mu}}(\frac{t}{t_0})^\gamma\big)^{\frac{2}{2\mu-1}}}}{t^{1-\gamma}}, \quad t \in (0,\infty),$$

then the standard method of differential calculus gives us the estimate

$$\begin{aligned}
S_2 &\leq \max\{f(t_0), f(t_1)\} \leq \max\Big\{\frac{\mathrm{e}^{\big(\frac{1}{\sqrt{\mu}}\big)^{\frac{2}{2\mu-1}}}}{t_0^{1-\gamma}}, \frac{\mathrm{e}^{\big(\frac{2C_\mu^\rho}{\sqrt{\mu}}\big)^{\frac{2}{2\mu-1}}}}{C_\mu^{\frac{1-\gamma}{\gamma}\rho} t_0^{1-\gamma}}\Big\} \\
&\leq \frac{1}{t_0^{1-\gamma}} \max\Big\{3, \frac{\mathrm{e}^{\big(\frac{2C_\mu^\rho}{\sqrt{\mu}}\big)^{\frac{2}{2\mu-1}}}}{C_\mu^{\frac{1-\gamma}{\gamma}\rho}}\Big\}.
\end{aligned}$$

Finally,

$$S_3 \leq \Big(\frac{C_\mu^\rho}{2\sqrt{\mu}}\Big)^{\frac{2}{2\mu-1}}, \quad \forall t_0 \leq t \leq t_1.$$

Inserting the above estimates into (5.1), we obtain

$$\begin{aligned} \mathcal{M} &\le \frac{\gamma(\mathrm{e}-1)}{t_0}\Big(1+\frac{2}{2\mu-1}\Big(\frac{C_\mu^\rho}{2\sqrt{\mu}}\Big)^{\frac{2}{2\mu-1}}\Big)\max\Big\{3, \frac{\mathrm{e}^{\big(\frac{2C_\mu^\rho}{\sqrt{\mu}}\big)^{\frac{2}{2\mu-1}}}}{C_\mu^{\frac{1-\gamma}{\gamma}\rho}}\Big\} \\ &\le \frac{10C_\mu^{\frac{2}{2\mu-1}\rho}}{t_0}\max\Big\{1, \frac{\mathrm{e}^{C_\mu^{\frac{2}{2\mu-1}\rho}}}{3C_\mu^{\frac{1-\gamma}{\gamma}\rho}}\Big\}. \end{aligned} \tag{5.2}$$

The term $\widetilde{\Psi}(\frac{\omega(t_0)}{\varepsilon})$ from the definition of $\mathcal{M}$ we can estimate as

$$\widetilde{\Psi}\Big(\frac{\omega(t_0)}{\varepsilon}\Big) = \frac{\omega(t_0)}{\varepsilon}\mathrm{e}^{\big(\frac{\omega(t_0)}{2\sqrt{\mu}\,\varepsilon}\big)^{2/(2\mu-1)}} \le \mathrm{e}^{\big(\frac{1}{2\sqrt{\mu}}\big)^{\frac{2}{2\mu-1}}} \le 3, \quad \forall t_0 > 0.$$

(b) Estimate of $\mathcal{M}$ for ω from Example 1.6:

$$\mathcal{M} \le \frac{\mathrm{e}^{\big(\frac{C_\mu^\rho}{2\sqrt{\mu}}\big)^{\frac{2}{2\mu-1}}}}{C_\mu^{\tau-\rho}}, \quad \tau > \rho > \frac{1}{p} \tag{5.3}$$

and $\widetilde{\Psi}(\omega(t_0)/\varepsilon) = 0$.

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