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EXISTENCE AND STABILITY OF TRAVELING WAVES FOR A COMPETITIVE-COOPERATIVE RECURSION SYSTEM

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Abstract. This article concerns the existence and global stability of bistable traveling waves for a competitive-cooperative recursion system. We first show that the spatially homogeneous system associated with the competitive-cooperative recursion system admits a bistable structure. Then using the theory of bistable waves for monotone semiflows and a dynamical system approach, we prove that there exists an unique and global stable traveling wave solution connecting two stable equilibria for such recursion system under appropriate conditions.

1. Introduction

In this article, we consider the existence and global stability of bistable traveling waves of the three-species competitive-cooperative recursion model

$$\begin{aligned} u_{n+1}(x) &= \int_{\mathbb{R}} \frac{(1+r_1)u_n(x-y)}{1+r_1(a_1u_n(x-y)+b_1v_n(x-y)+c_1w_n(x-y))}k_1(y)\,dy \\ v_{n+1}(x) &= \int_{\mathbb{R}} \frac{(1+r_2)v_n(x-y)}{1+r_2(a_2v_n(x-y)-b_2w_n(x-y)+c_2u_n(x-y))}k_2(y)\,dy, \\ w_{n+1}(x) &= \int_{\mathbb{R}} \frac{(1+r_3)w_n(x-y)}{1+r_3(a_3w_n(x-y)-b_3v_n(x-y)+c_3u_n(x-y))}k_3(y)\,dy, \end{aligned} \tag{1.1}$$

for $n \geq 0$ and $x \in \mathbb{R}$. Here $u_n(x)$, $v_n(x)$ and $w_n(x)$ are the population densities of three species u, v and w, respectively, at time n and position $x \in \mathbb{R}$; r_i, a_i, b_i, c_i $(i = 1, 2, 3)$ are positive constants; $k_i(y)$ $(i = 1, 2, 3)$ represents the dispersal kernel of three species. In (1.1), the variables v and w denote the densities of two species that work together in a mutualistic way, at the same time, the species v and w compete with u.

Traveling wave solutions of recursion systems $\mathbf{u}_{n+1} = \mathbf{Q}[\mathbf{u}_n]$ have been widely studied, see for example [2, 6, 7, 12, 13, 14, 15, 16, 17, 19, 20, 21, 24, 25, 26, 27, 30, 31] and references therein. Weinberger [24] studied the existence of asymptotic speeds for a scalar discrete-time recursion with that $\mathbf{Q}$ is a translation invariant order-preserving operator. Lui[19] extended the results in [24] to a multi-species verison of recursion system. Weinberger [25] also developed the theory in [19, 24] to the order-preserving operator with a periodic habitat. Weinberger et al. [26] further extended the results in [19, 24] so that they can be applied to invasion processes of

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cooperative or competitive models among multiple species. In fact, when specie v or w vanishes in (1.1), model (1.1) reduces to the classical two species competitive system. Lewis et al. [11] studied the linear determinacy of spreading speed for monotone discrete-time recursion system and applied their results to discrete-time two species competitive recursion model in the monostable case. Lin et al. [14] studied the spreading speed and traveling wave solutions of general discrete time recursion systems in the monostable case. Zhang and Zhao [29] also established the existence and global stability of bistable waves for discrete-time two species competition recursion systems with bistable structure. Recently, Wu and Zhao[28] studied the existence of spatially periodic traveling wave, single spreading speed and the linear determinacy for a class of intergrodifference competition models in a periodic habitat. For the competitive-cooperative reaction-diffusion system with nonlocal delays,

$$\begin{aligned}
\frac{\partial u}{\partial t} &= D_1\Delta u + r_1 u(1 - a_1 u - b_1 g_1 * v - c_1 g_2 * w),\\
\frac{\partial v}{\partial t} &= D_2\Delta v + r_2 v(1 - a_2 v + b_2 g_3 * w - c_2 g_4 * u),\\
\frac{\partial w}{\partial t} &= D_3\Delta w + r_3 w(1 - a_3 w + b_3 g_5 * v - c_3 g_6 * u),
\end{aligned} \tag{1.2}$$

where the general kernel function $g_i(t,x)$ $(i = 1,\dots,6)$ satisfy

$$g_i * z(t,x) = \int_0^{+\infty}\int_{-\infty}^{+\infty} g_i(s,y) z(t-s, x-y)\,ds\,dy, \quad \forall t > 0,\ x \in \mathbb{R}.$$

Tian and Zhao [23] have established the existence and global stability of bistable traveling wave for (1.2) with the infinite delay case by the finite-delay approximation approach and global convergence results for monotone semiflows. We refer the readers to [1, 4, 5, 8, 9] and references therein for the traveling waves and spreading speed of three-species competition system.

To consider the internal interaction and propagation phenomenon of three-competitive and cooperative species in discrete-time case, we are interested in the study of traveling waves for competitive-cooperative system (1.1). Assume that the kernel function $k_i(y)$ $(i = 1, 2, 3)$ is a continuous and nonnegative function satisfying

(H1) $k_i(-y) = k_i(y)$, $\int_{\mathbb{R}} k_i(y)\,dy = 1$, $\int_{\mathbb{R}} e^{\alpha y} k_i(y)\,dy < \infty$ for all $\alpha \in \mathbb{R}$ and $i = 1, 2, 3$.

The symmetric property of the kernel functions $k_i(y)$ in (H1) implies that the dispersal of three species is isotropic and that the growth and dispersal properties are the same at each point.

The spatially homogeneous system associated with (1.1) is

$$\begin{aligned}
u_{n+1} &= \frac{(1+r_1)u_n}{1 + r_1(a_1 u_n + b_1 v_n + c_1 w_n)},\\
v_{n+1} &= \frac{(1+r_2)v_n}{1 + r_2(a_2 v_n - b_2 w_n + c_2 u_n)},\\
w_{n+1} &= \frac{(1+r_3)w_n}{1 + r_3(a_3 w_n - b_3 v_n + c_3 u_n)},
\end{aligned} \tag{1.3}$$

for $n \ge 0$. It is easy to see that $(0,0,0)$, $(1/a_1,0,0)$, $(0,1/a_2,0)$ and $(0,0,1/a_3)$ are four boundary equilibria of (1.3). If $a_2a_3 - b_2b_3 > 0$, there is a nonnegative equilibrium

$$(0, v^+, w^+) = \Big(0, \frac{a_3 + b_2}{a_2 a_3 - b_2 b_3}, \frac{a_2 + b_3}{a_2 a_3 - b_2 b_3}\Big).$$

If $\frac{a_3 - c_1}{a_1 a_3 - c_1 c_3} > 0$, $\frac{a_1 - c_3}{a_1 a_3 - c_1 c_3} > 0$ and $a_1 a_3 \neq c_1 c_3$, then there is a nonnegative equilibrium

$$(\widehat{u}^+, 0, \widehat{w}^+) = \Big(\frac{a_3 - c_1}{a_1 a_3 - c_1 c_3}, 0, \frac{a_1 - c_3}{a_1 a_3 - c_1 c_3}\Big).$$

If $\frac{a_2 - b_1}{a_1 a_2 - b_1 c_2} > 0$, $\frac{a_1 - c_2}{a_1 a_2 - b_1 c_2} > 0$ and $a_1 a_2 \neq b_1 c_2$, then there is a nonnegative equilibrium

$$(\breve{u}^+, \breve{v}^+, 0) = \Big(\frac{a_2 - b_1}{a_1 a_2 - b_1 c_2}, \frac{a_1 - c_2}{a_1 a_2 - b_1 c_2}, 0\Big).$$

If $\operatorname{sign}(1 - b_1 \frac{a_3 + b_2}{a_2 a_3 - b_2 b_3} - c_1 \frac{a_2 + b_3}{a_2 a_3 - b_2 b_3}) = \operatorname{sign}(|A|)$ and $|A| \neq 0$, then there is a positive equilibrium (u^*, v^*, w^*), where

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ c_2 & a_2 & -b_2 \\ c_3 & -b_3 & a_3 \end{pmatrix}.$$

In this article, we study of the existence and stability of bistable traveling waves for system (1.1) in the case where the corresponding spatially homogeneous system admits a bistable structure. Though bistable waves in two species competition recursion system have been studied before (see Zhang and Zhao [29]), here we would like to emphasize that there is no result about the bistable waves of three species competitive-cooperative recursion model. We use the theory of monotone semiflows and squeezing technique to prove the existence and stability of bistable traveling waves. However, comparison to two species competition system, there exists eight equilibria for three species competitive-cooperative system, it is difficult to show the stability of these eight equilibria and the counter-propagation phenomenon between two different equilibria (see (A6) in Section 2). In this paper, we will show that the equilibria $(\frac{1}{a_1}, 0, 0)$ and $(0, v^+, w^+)$ are stable and the other equilibria are unstable. We first transfer system (1.1) into a cooperative system. By the changes of variables

$$\widetilde{u}_n = \frac{1}{a_1} - u_n, \quad \widetilde{v}_n = v_n \quad \text{and} \quad \widetilde{w}_n = w_n. \tag{1.4}$$

Dropping the tilde, we have

$$\begin{aligned}
&u_{n+1}(x)\\
&= \int_{\mathbb{R}} \frac{\frac{1}{a_1} r_1 (b_1 v_n(x-y) + c_1 w_n(x-y)) + u_n(x-y)}{1 + r_1(1 - a_1 u_n(x-y) + b_1 v_n(x-y) + c_1 w_n(x-y))} k_1(y)\,dy\\
&v_{n+1}(x)\\
&= \int_{\mathbb{R}} \frac{(1 + r_2) v_n(x-y)}{1 + r_2(a_2 v_n(x-y) - b_2 w_n(x-y) + \frac{c_2}{a_1} - c_2 u_n(x-y))} k_2(y)\,dy,\\
&w_{n+1}(x)\\
&= \int_{\mathbb{R}} \frac{(1 + r_3) w_n(x-y)}{1 + r_3(a_3 w_n(x-y) - b_3 v_n(x-y) + \frac{c_3}{a_1} - c_3 u_n(x-y))} k_3(y)\,dy.
\end{aligned} \tag{1.5}$$

By (1.4), the equilibria $(0,0,0)$, $(1/a_1, 0, 0)$, $(0, 1/a_2, 0)$, $(0, 0, 1/a_3)$, $(0, v^+, w^+)$, $(\widehat{u}^+, 0, \widehat{w}^+)$, $(\breve{u}^+, \breve{v}^+, 0)$ and (u^*, v^*, w^*) become $(1/a_1, 0, 0)$, $(0,0,0)$, $(1/a_1, 1/a_2, 0)$,

$(1/a_1, 0, 1/a_3)$, $(1/a_1, v^+, w^+)$, $(\frac{1}{a_1} - \widehat{u}^+, 0, \widehat{w}^+)$, $(\frac{1}{a_1} - \breve{u}^+, \breve{v}^+, 0)$ and $(\frac{1}{a_1} - u^*, v^*, w^*)$, respectively.

Let

$$\mathbf{0} = (0,0,0), \quad \mathbf{v}_1 := (\widehat{v}_1^+, 0, \widehat{v}_3^+) = (\frac{1}{a_1} - \widehat{u}^+, 0, \widehat{w}^+),$$
$$\mathbf{v}_2 := (\breve{v}_1^+, \breve{v}_2^+, 0) = (\frac{1}{a_1} - \breve{u}^+, \breve{v}^+, 0),$$
$$\mathbf{v}^+ := (v_1^+, v_2^+, v_3^+) = (\frac{1}{a_1}, v^+, w^+),$$
$$\mathbf{v}^* := (v_1^*, v_2^*, v_3^*) = (\frac{1}{a_1} - u^*, v^*, w^*)$$

and define the set

$$E = \Big\{ \Big(\frac{1}{a_1}, 0, 0\Big), \mathbf{0}, \Big(\frac{1}{a_1}, \frac{1}{a_2}, 0\Big), \Big(\frac{1}{a_1}, 0, \frac{1}{a_3}\Big), \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}^+, \mathbf{v}^* \Big\}.$$

Note that (1.5) is a cooperative system. To study the traveling wave solution of system (1.1) connecting $(\frac{1}{a_1}, 0, 0)$ and $(0, v^+, w^+)$, it is equivalent to study the traveling wave solution of (1.5) connecting $\mathbf{0}$ and $\mathbf{v}^+ = (\frac{1}{a_1}, v^+, w^+)$.

We assume (H1) and that the parameters in (1.1) satisfy $c_2/a_1 > 1$, $c_3/a_1 > 1$, $a_1a_3 < c_1c_3$, $a_1a_2 < b_1c_2$ and

$$1 - b_1 \frac{a_3 + b_2}{a_2a_3 - b_2b_3} - c_1 \frac{a_2 + b_3}{a_2a_3 - b_2b_3} < 0.$$

Then we have the following results on traveling wave solution for system (1.5):

- (Existence) There is $c \in \mathbb{R}$ such that system (1.5) admits a nondecreasing traveling wave solution $\mathbf{\Phi}(x - cn) = (\Phi_1(x - cn), \Phi_2(x - cn), \Phi_3(x - cn))$ with speed c satisfies $\mathbf{\Phi}(-\infty) = (0,0,0)$ and $\mathbf{\Phi}(+\infty) = (v_1^+, v_2^+, v_3^+)$ (see Theorem 3.5).
- (Stability) If the initial value $\boldsymbol{\psi}(\cdot) \in \mathcal{X}_{[\mathbf{0}, \mathbf{v}^+]}$ satisfies one of the following two cases: Case (i) $\boldsymbol{\psi}(\cdot)$ is nondecreasing and satisfies
$$\liminf_{\xi \to +\infty} \psi_i(\xi) > v_i^* > \limsup_{\xi \to -\infty} \psi_i(\xi)$$
for $i = 1, 2, 3$; Case (ii) the kernel k_i $(i = 1, 2, 3)$ has a compact support and $\boldsymbol{\psi}(\xi)$ satisfies $\liminf_{\xi \to +\infty} \psi_i(\xi) > v_i^* > \limsup_{\xi \to -\infty} \psi_i(\xi)$ for $i = 1, 2, 3$, then there exists $s_{\boldsymbol{\psi}}$ such that $\lim_{n \to +\infty} \|\mathbf{U}_n(x, \boldsymbol{\psi}) - \mathbf{\Phi}(x - cn + s_{\boldsymbol{\psi}})\| = 0$ uniformly for $x \in \mathbb{R}$ (see Theorem 3.7).
- (Uniqueness) Any monotone traveling wave solutions of (1.5) connecting $\mathbf{0}$ and $\mathbf{v}^+$ is a translation of $\mathbf{\Phi}(\cdot)$ (see Corollary 3.8).

We end the introduction with the following remarks. By (1.4), for (1.2), there is an unique traveling wave solution $\hat{\mathbf{\Phi}}(x - cn) = (\hat{\Phi}_1(x - cn), \hat{\Phi}_2(x - cn), \hat{\Phi}_3(x - cn))$ connecting two stable points $(\frac{1}{a_1}, 0, 0)$ and $(0, v^+, w^+)$. Thus $\hat{\mathbf{\Phi}}(x - cn) \to (\frac{1}{a_1}, 0, 0)$ as $n \to \infty$ for $c > 0$, which implies that specie u will persistent and species v, w will extinct. If $c < 0$, $\hat{\mathbf{\Phi}}(x - cn) \to (0, v^+, w^+)$ as $n \to \infty$, which implies that species v and w are persistent and specie u will go to extinct. Hence, the traveling wave solution $\hat{\mathbf{\Phi}}(x - cn)$ can be used to determine the winner of such a competition-cooperative system in the presence of spatial diffusion and discrete time and the the sign of the wave speed c plays an important role, which will be considered in future.

The rest of this paper is organized as follows. In Section 2, we will present some preliminaries for system (1.5). In Section 3, we will establish the existence and global stability of bistable traveling waves for system (1.5) by appealing the theory of bistable waves for monotone semiflows in [3] and a dynamical system approach.

2. Preliminary

In this section, we introduce notation and show that system (1.3) admits a bistable structure. Let $\mathcal{C} := C(\mathbb{R}, \mathbb{R}^3)$ be the set of all bounded and continuous functions from $\mathbb{R}$ to $\mathbb{R}^3$ equipped with the compact open topology. Let $\mathcal{C}_+ = \{(\phi_1, \phi_2, \phi_3) \in \mathcal{C} : \phi_i(x) \geq 0, \forall x \in \mathbb{R}, i = 1, 2, 3\}$. Define $\mathcal{C}_r := \{\phi \in \mathcal{C} : 0 \leq \phi \leq r\}$ and $\mathcal{C}_{[a,b]} := \{\phi \in \mathcal{C} : a \leq \phi \leq b\}$ for any $a, b, r \in \mathbb{R}^3$ with $a \leq b$ and $r \gg 0$.

Define an operator $\mathbf{Q} = (Q_1, Q_2, Q_3)$ on $\mathcal{C}$ by

$$
\begin{aligned}
Q_1[u, v, w](x) &= \int_{\mathbb{R}} \frac{\frac{1}{a_1} r_1(b_1 v + c_1 w) + u}{1 + r_1(1 - a_1 u + b_1 v + c_1 w)} k_1(y)\, dy, \\
Q_2[u, v, w](x) &= \int_{\mathbb{R}} \frac{(1 + r_2)v}{1 + r_2(a_2 v(x) - b_2 w + \frac{c_2}{a_1} - c_2 u)} k_2(y)\, dy, \\
Q_3[u, v, w](x) &= \int_{\mathbb{R}} \frac{(1 + r_3)w}{1 + r_3(a_3 w(x) - b_3 v + \frac{c_3}{a_1} - c_3 u)} k_3(y)\, dy,
\end{aligned}
$$

Then system (1.5) can be expressed as

$$
\mathbf{U}_{n+1}(x) = \mathbf{Q}[\mathbf{U}_n](x), \quad \mathbf{U}_n := (u_n, v_n, w_n), \quad n \geq 0.
$$

In this article, we mainly consider the bistable structure of system (1.5). It is then needed to show that the fixed points $\mathbf{0}$ and $\mathbf{v}^+$ are stable and others are unstable. Let $\widehat{\mathbf{Q}}$ be the spatially homogeneous operator of $\mathbf{Q}$ to $[\mathbf{0}, \mathbf{v}^+]$, where $\widehat{\mathbf{Q}} = (\widehat{Q}_1, \widehat{Q}_2, \widehat{Q}_3)$ and

$$
\begin{aligned}
\widehat{Q}_1[w_1, w_2, w_3] &= \frac{\frac{1}{a_1} r_1(b_1 w_2 + c_1 w_3) + w_1}{1 + r_1(1 - a_1 w_1 + b_1 w_2 + c_1 w_3)}, \\
\widehat{Q}_2[w_1, w_2, w_3] &= \frac{(1 + r_2)w_2}{1 + r_2(a_2 w_2 - b_2 w_3 + \frac{c_2}{a_1} - c_2 w_1)}, \\
\widehat{Q}_3[w_1, w_2, w_3] &= \frac{(1 + r_3)w_3}{1 + r_3(a_3 w_3 - b_3 w_2 + \frac{c_3}{a_1} - c_3 w_1)},
\end{aligned}
\tag{2.1}
$$

To obtain the Jacobian matrices of $\widehat{\mathbf{Q}}[w_1, w_2, w_3]$ at point (w_1, w_2, w_3), we list the first row of Jacobian matrix as follows

$$
\begin{aligned}
\frac{\partial \widehat{Q}_1}{\partial w_1} &= \frac{1 + r_1(1 - a_1 w_1 + b_1 w_2 + c_1 w_3) + r_1 a_1 [\frac{r_1}{a_1}(b_1 w_2 + c_1 w_3) + w_1]}{[1 + r_1(1 - a_1 w_1 + b_1 w_2 + c_1 w_3)]^2}, \\
\frac{\partial \widehat{Q}_1}{\partial w_2} &= \frac{\frac{r_1}{a_1} b_1(1 + r_1(1 - a_1 w_1 + b_1 w_2 + c_1 w_3)) - [\frac{r_1}{a_1}(b_1 w_2 + c_1 w_3) + w_1] r_1 b_1}{[1 + r_1(1 - a_1 w_1 + b_1 w_2 + c_1 w_3)]^2}, \\
\frac{\partial \widehat{Q}_1}{\partial w_3} &= \frac{\frac{r_1}{a_1} c_1(1 + r_1(1 - a_1 w_1 + b_1 w_2 + c_1 w_3)) - [\frac{r_1}{a_1}(b_1 w_2 + c_1 w_3) + w_1] r_1 c_1}{[1 + r_1(1 - a_1 w_1 + b_1 w_2 + c_1 w_3)]^2};
\end{aligned}
$$

the second row of Jacobian matrix is

$$
\frac{\partial \widehat{Q}_2}{\partial w_1} = \frac{(1 + r_2) w_2 r_2 c_2}{[1 + r_2(a_2 w_2 - b_2 w_3 + \frac{c_2}{a_1} - c_2 w_1)]^2},
$$

$$\frac{\partial \widehat{Q}_2}{\partial w_2} = \frac{(1+r_2)(1+r_2(a_2w_2-b_2w_3+\frac{c_2}{a_1}-c_2w_1))-(1+r_2)w_2r_2a_2}{[1+r_2(a_2w_2-b_2w_3+\frac{c_2}{a_1}-c_2w_1)]^2},$$

$$\frac{\partial \widehat{Q}_2}{\partial w_3} = \frac{(1+r_2)w_2r_2b_2}{[1+r_2(a_2w_2-b_2w_3+\frac{c_2}{a_1}-c_2w_1)]^2};$$

and the third row is

$$\frac{\partial \widehat{Q}_3}{\partial w_1} = \frac{(1+r_3)w_3r_3c_3}{[1+r_3(a_3w_3-b_3w_2+\frac{c_3}{a_1}-c_3w_1)]^2},$$

$$\frac{\partial \widehat{Q}_3}{\partial w_2} = \frac{(1+r_3)w_3r_3b_3}{[1+r_3(a_3w_3-b_3w_2+\frac{c_3}{a_1}-c_3w_1)]^2},$$

$$\frac{\partial \widehat{Q}_3}{\partial w_3} = \frac{(1+r_3)(1+r_3(a_3w_3-b_3w_2+\frac{c_3}{a_1}-c_3w_1))-(1+r_3)w_3r_3a_3}{[1+r_3(a_3w_3-b_3w_2+\frac{c_3}{a_1}-c_3w_1)]^2}.$$

Thus the Jacobian matrix of $\widehat{\mathbf{Q}}$ at $\mathbf{0}$ is

$$J_{\mathbf{0}} = \begin{pmatrix} \frac{1}{1+r_1} & \frac{r_1}{a_1}\frac{b_1}{1+r_1} & \frac{r_1}{a_1}\frac{c_1}{1+r_1} \\ 0 & \frac{1+r_2}{1+\frac{r_2c_2}{a_1}} & 0 \\ 0 & 0 & \frac{1+r_3}{1+\frac{r_3c_3}{a_1}} \end{pmatrix}$$

and the characteristic equation of $J_{\mathbf{0}}$ is

$$\Big(\lambda-\frac{1}{1+r_1}\Big)\Big(\lambda-\frac{1+r_2}{1+\frac{r_2c_2}{a_1}}\Big)\Big(\lambda-\frac{1+r_3}{1+\frac{r_3c_3}{a_1}}\Big)=0.$$

It is obvious that $J_{\mathbf{0}}$ has three positive eigenvalues

$$\lambda_1=\frac{1}{1+r_1} \quad \lambda_2=\frac{1+r_2}{1+\frac{r_2c_2}{a_1}}, \quad \lambda_3=\frac{1+r_3}{1+\frac{r_3c_3}{a_1}}\,.$$

If $c_2/a_1>1$ and $c_3/a_1>1$, we obtain $\lambda_1<1$, $\lambda_2<1$ and $\lambda_3<1$. Then the fixed point $\mathbf{0}$ is stable (see [10, Chapter 1, Section 9]).

Consider $\mathbf{v}^+=(1/a_1,v_2^+,v_3^+)$. Note that $\mathbf{v}^+$ is positive fixed point of (2.1) and

$$1=\frac{1+r_2}{1+r_2(a_2v_2^+-b_2v_3^+)} \quad \text{and} \quad 1=\frac{1+r_3}{1+r_3(a_3v_3^+-b_3v_2^+)}.$$

In this case, it is easy to check that $b_1v_2^++c_1v_3^+\neq 1$. Thus the Jacobian matrix of $\widehat{\mathbf{Q}}$ at $\mathbf{v}^+$ is

$$J_{\mathbf{v}^+} = \begin{pmatrix} \frac{1+r_1}{1+r_1(b_1v_2^++c_1v_3^+)} & 0 & 0 \\ \frac{r_2c_2v_2^+}{1+r_2} & 1-\frac{r_2a_2v_2^+}{1+r_2} & \frac{r_2b_2v_2^+}{1+r_2} \\ \frac{r_3c_3v_3^+}{1+r_3} & \frac{r_3b_3v_3^+}{1+r_3} & 1-\frac{r_3a_3v_3^+}{1+r_3} \end{pmatrix}.$$

Then the characteristic equation at $\mathbf{v}^+$ is

$$\begin{aligned} &\Big(\lambda-\frac{1+r_1}{1+r_1(b_1v_2^++c_1v_3^+)}\Big)\Big(\Big(\lambda-1+\frac{r_2a_2v_2^+}{1+r_2}\Big)\Big(\lambda-1+\frac{r_3a_3v_3^+}{1+r_3}\Big) \\ &-\frac{r_2b_2v_2^+}{1+r_2}\frac{r_3b_3v_3^+}{1+r_3}\Big)=0 \end{aligned} \tag{2.2}$$

Recall that $v^+ = \frac{a_3+b_2}{a_2a_3-b_2b_3}$ and $w^+ = \frac{a_2+b_3}{a_2a_3-b_2b_3}$. Thus,

$$1 - b_1\frac{a_3+b_2}{a_2a_3-b_2b_3} - c_1\frac{a_2+b_3}{a_2a_3-b_2b_3} < 0$$

implies $b_1v_2^+ + c_1v_3^+ > 1$. It then follows that the Jacobian matrix $J_{\mathbf{v}^+}$ has three positive eigenvalues λ_i $(i = 1, 2, 3)$ and $\lambda_i < 1$ for $i = 1, 2, 3$. It then follows that the fixed point $\mathbf{v}^+$ is stable.

For $\mathbf{v}_1 = (v_1, v_2, v_3) = \big(\frac{1}{a_1} - \widehat{u}^+, 0, \widehat{w}^+\big)$, we have

$$a_1v_1 = c_1v_3 \quad \text{and} \quad a_3v_3 + \frac{c_3}{a_1} - c_3v_1 = 1.$$

Then we have that the Jacobian matrixes of $\widehat{\mathbf{Q}}$ at $\mathbf{v}_1$ is

$$J_{\mathbf{v}_1} = \begin{pmatrix} \frac{1+r_1c_1v_3}{1+r_1} & \frac{\frac{r_1b_1}{a_1}(1-c_1v_3)}{1+r_1} & \frac{\frac{r_1a_1}{a_1}(1-c_1v_3)}{1+r_1} \\ 0 & 1 & 0 \\ \frac{r_3c_3v_3}{1+r_3} & \frac{r_3b_3v_3}{1+r_3} & 1-\frac{r_3a_3v_3}{1+r_3} \end{pmatrix}.$$

Assume $a_1a_3 < c_1c_3$. Since $\widehat{u}^+ = \frac{a_3-c_1}{a_1a_3-c_1c_3} > 0$, we obtain that $a_3 - c_1 < 0$ and $c_1v_3 = c_1\frac{a_1-c_3}{a_1a_3-c_1c_3} > 1$. Thus it is easy to see that there is an eigenvalue $\lambda_1 > 1$ for $J_{\mathbf{v}_1}$. Hence $\mathbf{v}_1$ is unstable.

Similarly, for $\mathbf{v}_2 = (\breve{v}_1, \breve{v}_2, \breve{v}_3) = \big(\frac{1}{a_1} - \breve{u}^+, \breve{v}^+, 0\big)$, we have $a_1\breve{v}_1 = b_1\breve{v}_2$, $a_2\breve{v}_2 + \frac{c_2}{a_1} - c_2\breve{v}_1 = 1$, and

$$J_{\mathbf{v}_2} = \begin{pmatrix} \frac{1+r_1b_1\breve{v}_2}{1+r_1} & \frac{\frac{r_1b_1}{a_1}(1-b_1\breve{v}_2)}{1+r_1} & \frac{\frac{r_1c_1}{a_1}(1-b_1\breve{v}_2)}{1+r_1} \\ \frac{r_2c_2\breve{v}_2}{1+r_2} & 1-\frac{r_2b_2\breve{v}_2}{1+r_2} & \frac{r_2a_2\breve{v}_2}{1+r_2} \\ 0 & 0 & 1 \end{pmatrix}.$$

If $a_1a_3 < b_1c_2$, we have $a_2 < b_1$ and $b_1\breve{v}_2 > 1$. Thus the point $\mathbf{v}_2$ is also unstable.

For $(1/a_1, 0, 0)$, $(1/a_1, 1/a_2, 0)$ and $(1/a_1, 0, 1/a_3)$, the Jacobian matrices of $\mathbf{Q}$ at these three points are

$$J_{(\frac{1}{a_1},0,0)} = \begin{pmatrix} 1+r_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad J_{(\frac{1}{a_1},\frac{1}{a_2},0)} = \begin{pmatrix} \frac{1+r_1}{1+r_1\frac{b_1}{a_2}} & 0 & 0 \\ \frac{\frac{1}{a_2}r_2c_2}{1+r_2} & \frac{1}{1+r_2} & \frac{\frac{1}{a_1}r_2b_2}{1+r_2} \\ 0 & 0 & 1 \end{pmatrix}$$

$$J_{(\frac{1}{a_1},0,\frac{1}{a_3})} = \begin{pmatrix} \frac{1+r_1}{1+r_1\frac{c_1}{a_3}} & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\frac{1}{a_3}r_3c_3}{1+r_3} & \frac{\frac{1}{a_3}r_3b_3}{1+r_3} & \frac{1}{1+r_3} \end{pmatrix},$$

respectively. Note that $a_1a_3 < c_1c_3$ and $a_1a_3 < b_1c_2$ imply $a_3 < c_1$ and $a_2 < b_1$. Hence $(\frac{1}{a_1}, 0, 0)$, $(\frac{1}{a_1}, \frac{1}{a_2}, 0)$ and $(\frac{1}{a_1}, 0, \frac{1}{a_3})$ are unstable.

Next, we consider $\mathbf{v}^* = (v_1^*, v_2^*, v_3^*) = (\frac{1}{a_1} - u^*, v^*, w^*)$. In this case, we have

$$\begin{gathered} b_1v_2^* + c_1v_3^* = a_1v_1^*, \\ a_2v_2^* - b_2v_3^* + \frac{c_2}{a_1} - c_2v_1^* = 1, \\ a_3v_3^* - b_3v_3^* + \frac{c_3}{a_1} - c_3v_1^* = 1. \end{gathered} \tag{2.3}$$

The Jacobian matrix of $\mathbf{Q}$ at $\mathbf{v}^*$ is

$$J_{\mathbf{v}^*} = \begin{pmatrix} \frac{1+r_1a_1v_1^*}{1+r_1} & \frac{b_1r_1}{1+r_1}(\frac{1}{a_1}-v_1^*) & \frac{c_1r_1}{1+r_1}(\frac{1}{a_1}-v_1^*) \\ \frac{r_2c_2v_2^*}{1+r_2} & 1-\frac{r_2a_2v_2^*}{1+r_2} & \frac{r_2b_2v_2^*}{1+r_2} \\ \frac{r_3c_3v_3^*}{1+r_3} & \frac{r_3b_3v_3^*}{1+r_3} & 1-\frac{r_3a_3v_3^*}{1+r_3} \end{pmatrix}.$$

Then

$$F(\lambda) = |\lambda I - J_{\mathbf{v}^*}| = \begin{vmatrix} \lambda-\frac{1+r_1a_1v_1^*}{1+r_1} & -\frac{b_1r_1}{1+r_1}(\frac{1}{a_1}-v_1^*) & -\frac{c_1r_1}{1+r_1}(\frac{1}{a_1}-v_1^*) \\ -\frac{r_2c_2v_2^*}{1+r_2} & \lambda-1+\frac{r_2a_2v_2^*}{1+r_2} & -\frac{r_2b_2v_2^*}{1+r_2} \\ -\frac{r_3c_3v_3^*}{1+r_3} & -\frac{r_3b_3v_3^*}{1+r_3} & \lambda-1+\frac{r_3a_3v_3^*}{1+r_3} \end{vmatrix}.$$

Note that $F(+\infty) = +\infty$, and

$$\begin{aligned} F(1) &= \begin{vmatrix} 1-\frac{1+r_1a_1v_1^*}{1+r_1} & -\frac{b_1r_1}{1+r_1}(\frac{1}{a_1}-v_1^*) & -\frac{c_1r_1}{1+r_1}(\frac{1}{a_1}-v_1^*) \\ -\frac{r_2c_2v_2^*}{1+r_2} & \frac{r_2a_2v_2^*}{1+r_2} & -\frac{r_2b_2v_2^*}{1+r_2} \\ -\frac{r_3c_3v_3^*}{1+r_3} & -\frac{r_3b_3v_3^*}{1+r_3} & \frac{r_3a_3v_3^*}{1+r_3} \end{vmatrix} \\ &= \frac{r_1u^*}{1+r_1}\frac{r_2v^*}{1+r_2}\frac{r_3w^*}{1+r_3}\begin{vmatrix} a_1 & b_1 & c_1 \\ c_2 & a_2 & -b_2 \\ c_3 & -b_3 & a_3 \end{vmatrix}. \end{aligned}$$

Since

$$\operatorname{sign}(1-b_1\frac{a_3+b_2}{a_2a_3-b_2b_3}-c_1\frac{a_2+b_3}{a_2a_3-b_2b_3}) = \operatorname{sign}(|A|)$$

and $1-b_1\frac{a_3+b_2}{a_2a_3-b_2b_3}-c_1\frac{a_2+b_3}{a_2a_3-b_2b_3} < 0$, we have $F(1) < 0$. Then there is $\lambda' > 1$ such that $F(\lambda') = 0$, which implies that $\mathbf{v}^*$ is unstable.

From above all calculations, we have the following lemma.

Lemma 2.1. *The following statements are valid.*

(1) *If $c_2/a_1 > 1$, $c_3/a_1 > 1$, then $\mathbf{0}$ is stable.*
(2) *If*
$$1-b_1\frac{a_3+b_2}{a_2a_3-b_2b_3}-c_1\frac{a_2+b_3}{a_2a_3-b_2b_3} < 0, \tag{2.4}$$
then $\mathbf{v}^+$ is stable.
(3) *If (2.4) hold, then $\mathbf{v}^*$ is unstable.*
(4) *If $a_1a_3 < c_1c_3$, then $\mathbf{v}_1$ is unstable.*
(5) *If $a_1a_3 < b_1c_2$, then $\mathbf{v}_2$ is unstable.*

By Lemma 2.1, if the parameters satisfy $c_2/a_1 > 1$, $c_3/a_1 > 1$, $a_1a_3 < c_1c_3$, $a_1a_2 < b_1c_2$ and $1-b_1\frac{a_3+b_2}{a_2a_3-b_2b_3}-c_1\frac{a_2+b_3}{a_2a_3-b_2b_3} < 0$, then system (1.5) is of the bistable structure. According to [3, Theorem 3.1], if the operator $\mathbf{Q}$ satisfies the following conditions:

(A1) (Translation invariance) $T_y \circ \mathbf{Q}[\Phi] = Q \circ T_y[\Phi]$, for all $\Phi \in \mathcal{C}_{\mathbf{v}^+}$, $y \in \mathbb{R}$, where $T_y[\Phi](x) = \Phi(x-y)$.
(A2) (Continuity) $Q : \mathcal{C}_{\mathbf{v}^+} \to \mathcal{C}_{\mathbf{v}^+}$ is continuous with respect to the compact open topology.
(A3) (Monotonicity) $\mathbf{Q}$ is order preserving in the sense that $\mathbf{Q}[\Phi] \geq \mathbf{Q}[\Psi]$ whenever $\Phi \geq \Psi$ in $\mathcal{C}_{\mathbf{v}^+}$.
(A4) (Compactness) $Q : \mathcal{C}_{\mathbf{v}^+} \to \mathcal{C}_{\mathbf{v}^+}$ is compact with respect to the compact open topology.

(A5) (Bistability) Two fixed points $\mathbf{0}$ and $\mathbf{v}^+$ are strongly stable from above and below, respectively, for the map $Q : [\mathbf{0}, \mathbf{v}^+] \to [\mathbf{0}, \mathbf{v}^+]$, that is, there exist a number $\delta > 0$ and unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2 \in Int(\mathbb{R}^3)$ such that

$$\mathbf{Q}[\eta \mathbf{e}_1] \ll \eta \mathbf{e}_1, \quad \mathbf{Q}[\mathbf{v}^+_\tau - \eta \mathbf{e}_2] \gg \mathbf{v}^+ - \eta \mathbf{e}_2, \quad \forall \eta \in (0, \delta)$$

and the set $E \setminus \{\mathbf{0}, \mathbf{v}^+\}$ is totally unordered.

(A6) (Counter-propagation) For each $\alpha \in E \setminus \{\mathbf{0}, \mathbf{v}^+\}$, $c^*_-(\alpha, \mathbf{v}^+) + c^*_+(\mathbf{0}, \alpha) > 0$, where $c^*_-(\alpha, \mathbf{v}^+)$ and $c^*_+(\mathbf{0}, \alpha)$ represent the leftward and rightward spreading speeds of monotone subsystem $\{Q^n\}_{n \ge 0}$ restricted on $[\alpha, \mathbf{v}^+]$ and $[\mathbf{0}, \alpha]$, respectively.

then there exists a nondecreasing traveling wave solution $\mathbf{\Phi}(x - cn) = (\Phi_1(x - cn), \Phi_2(x - cn), \Phi_3(x - cn))$ with speed $c \in \mathbb{R}$ and connecting two bistable points $\mathbf{0}$ and $\mathbf{v}^+$. Hence, in section 3, we will verify that operator $\mathbf{Q}$ given by system (1.5) satisfies assumptions (A1)–(A6).

3. Existence and globally stability of traveling wave

In this section, we establish the existence and stability of bistable traveling waves for system (1.5). Since (1.5) is cooperative, it is easy to verify that the map $\mathbf{Q}$ satisfies (A1)–(A4). In the following, we show that (A5) and (A6) also hold.

Lemma 3.1. *Assume that $c_2/a_1 > 1$, $c_3/a_1 > 1$, $a_1 a_3 < c_1 c_3$, $a_1 a_2 < b_1 c_2$ and* (2.4) *holds. Then $\mathbf{Q}$ satisfies* (A5).

Proof. From Lemma 2.1, we know that $\mathbf{0}$ and $\mathbf{v}^+$ are stable. We now prove that $\mathbf{0}$ is strongly stable from above and $\mathbf{v}^+$ is strongly stable from below. Since $c_2/a_1 > 1$ and $c_3/a_1 > 1$, then the Jacobian matrix $J_{\mathbf{0}}$ has three eigenvalues λ_i, $i = 1, 2, 3$. If $1 > \max\{\lambda_2, \lambda_3\} > \lambda_1$, then $J_{\mathbf{0}}$ has a unit eigenvector $\mathbf{e}_0 > 0$ associated with $\max\{\lambda_2, \lambda_3\}$ such that

$$J_{\mathbf{0}}(\mathbf{e}_0) = \max\{\lambda_2, \lambda_3\} \mathbf{e}_0 \ll \mathbf{e}_0.$$

If $1 > \lambda_1 > \max\{\lambda_2, \lambda_3\}$, take $k \in (\lambda_1, 1)$, $\varepsilon_0 \in \big(0, \frac{a_1(1+r_1)}{r_1 b_1}\big)$, $\eta_0 \in \big(0, \frac{a_1(1+r_1)}{r_1 c_1}\big)$ and unit vector

$$\mathbf{e}_0 = \Big(\frac{\varepsilon_0}{\sqrt{1 + \varepsilon_0^2 + \eta_0^2}}, \frac{\eta_0}{\sqrt{1 + \varepsilon_0^2 + \eta_0^2}}, \frac{1}{\sqrt{1 + \varepsilon_0^2 + \eta_0^2}} \Big)$$

such that

$$J_{\mathbf{0}}(\mathbf{e}_0) \ll k \mathbf{e}_0 \ll \mathbf{e}_0.$$

By the continuous differentiality of $\widehat{\mathbf{Q}}$, there exists $\delta > 0$ such that

$$\widehat{\mathbf{Q}}(\eta \mathbf{e}_0) = \widehat{\mathbf{Q}}(\mathbf{0}) + \int_0^1 D\widehat{\mathbf{Q}}(t\eta \mathbf{e}_0) \eta \mathbf{e}_0 \, dt = \eta \int_0^1 D\widehat{\mathbf{Q}}(t\eta \mathbf{e}_0) \mathbf{e}_0 \, dt \le \eta k \mathbf{e}_0 \ll \eta \mathbf{e}_0$$

for all $\eta \in (0, \delta]$ and hence $\mathbf{0}$ is strongly stable from above for the map $\widehat{\mathbf{Q}}$. By similar argument, we also have that $\mathbf{v}^+$ is strongly stable form below.

From above arguments, we have that $\mathbf{0}$ is strongly stable from above and $\mathbf{v}^+$ is strongly stable from below. Next, we mainly show that $E \setminus \{\mathbf{0}, \mathbf{v}^+\}$ are totally unordered.

We first show $\widehat{v}_3^+ > v_3^*$ if $a_1 a_3 < c_1 c_3$. From (2.1), we have

$$\widehat{v}_1^+ = \frac{\frac{1}{a_1} r_1 (0 + c_1 \widehat{v}_3^+) + \widehat{v}_1^+}{1 + r_1 (1 - a_1 \widehat{v}_1^+ + c_1 \widehat{v}_3^+)},$$

$$\widehat{v}_3^+ = \frac{(1+r_3)\widehat{v}_3^+}{1+r_3(a_3\widehat{v}_3^+ + \frac{c_3}{a_1} - c_3\widehat{v}_1^+)},$$

that is, $a_1\widehat{v}_1^+ - c_1\widehat{v}_3^+ = 0$ and $a_3\widehat{v}_3^+ + \frac{c_3}{a_1} - c_3\widehat{v}_1^+ = 1$. Then

$$\widehat{v}_3^+ = \frac{1-\frac{c_3}{a_1}}{a_1 - \frac{c_1c_3}{a_1}} = \frac{a_1 - c_3}{a_1a_3 - c_1c_3}.$$

On the other hand,

$$\begin{gathered} a_1v_1^* - c_1v_3^* = b_2v_3^* > 0, \\ 1 - a_3v_3^* - \frac{c_3}{a_1} + c_3v_1^* = -b_3v_2^* < 0, \end{gathered}$$

which implies that

$$v_3^* < \frac{a_1 - c_3}{a_1a_3 - c_1c_3}$$

for $a_1a_3 < c_1c_3$. Thus $v_3^* < \widehat{v}_3^+$ if $a_1a_3 < c_1c_3$. By the similar way, we have $\check{v}_2^+ > v_2^*$ if $a_1a_2 < b_1c_2$. It follows that the set $E \setminus \{\mathbf{0}, \mathbf{v}^+\}$ are totally unordered and $\mathbf{Q}$ satisfies (A5). □

Lemma 3.2. $c^*(\mathbf{0}, \mathbf{v}^*) + c^*(\mathbf{v}^*, \mathbf{v}^+) > 0$.

Proof. Recall that $\mathbf{v}^* = (v_1^*, v_2^*, v_3^*)$ satisfies (2.3). To consider $c^*(\mathbf{v}^*, \mathbf{v}^+)$, let $\widetilde{u}_n = u_n - v_1^*$, $\widetilde{v}_n = v_n - v_2^*$ and $\widetilde{w}_n = w_n - v_3^*$. Then system (1.5) becomes

$$\begin{aligned} &\widetilde{u}_{n+1}(x) \\ &= -v_1^* + \int_{\mathbb{R}} \frac{\frac{1}{a_1}r_1(b_1\widetilde{v}_n(x-y) + c_1\widetilde{w}_n(x-y)) + \widetilde{u}_n(x-y) + (1+r_1)v_1^*}{1 + r_1(1 - a_1\widetilde{u}_n(x-y) + b_1\widetilde{v}_n(x-y) + c_1\widetilde{w}_n(x-y))} k_1(y)\,dy, \\ &\widetilde{v}_{n+1}(x) \\ &= -v_2^* + \int_{\mathbb{R}} \frac{(1+r_2)(\widetilde{v}_n(x-y) + v_2^*)}{1 + r_2(1 + a_2\widetilde{v}_n(x-y) - b_2\widetilde{w}_n(x-y) - c_2\widetilde{u}_n(x-y))} k_2(y)\,dy, \\ &\widetilde{w}_{n+1}(x) \\ &= -v_3^* + \int_{\mathbb{R}} \frac{(1+r_3)(\widetilde{w}_n(x-y) + v_3^*)}{1 + r_3(1 + a_3\widetilde{w}_n(x-y) - b_3\widetilde{v}_n(x-y) - c_3\widetilde{u}_n(x-y))} k_3(y)\,dy. \end{aligned} \tag{3.1}$$

It is easy to verify that system (3.1) is cooperative and positively invariant in $\mathcal{C}_{[\mathbf{0},\beta]} = \{\psi \in \mathcal{C} : \mathbf{0} \le \psi \le \beta\}$, where $\beta = \mathbf{v}^+ - \mathbf{v}^* \gg \mathbf{0}$. The spatially homogeneous system

$$\begin{aligned} \widetilde{u}_{n+1} &= -v_1^* + \frac{\frac{1}{a_1}r_1(b_1\widetilde{v}_n + c_1\widetilde{w}_n) + \widetilde{u}_n + (1+r_1)v_1^*}{1 + r_1(1 - a_1\widetilde{u}_n + b_1\widetilde{v}_n + c_1\widetilde{w}_n)} \\ \widetilde{v}_{n+1} &= -v_2^* + \frac{(1+r_2)(\widetilde{v}_n + v_2^*)}{1 + r_2(1 + a_2\widetilde{v}_n - b_2\widetilde{w}_n - c_2\widetilde{u}_n)}, \\ \widetilde{w}_{n+1} &= -v_3^* + \frac{(1+r_3)(\widetilde{w}_n + v_3^*)}{1 + r_3(1 + a_3\widetilde{w}_n - b_3\widetilde{v}_n - c_3\widetilde{u}_n)} \end{aligned} \tag{3.2}$$

has stable equilibrium β and unstable one $\mathbf{0}$, and there are no other equilibria between these two equilibria in $[\mathbf{0}, \beta] \in \mathbb{R}^3$.

Now we consider the linearization of (3.1) at $\mathbf{0}$,

$$
\begin{aligned}
&\widetilde{u}_{n+1}(x)\\
&= \int_{\mathbb{R}} \Big(\frac{(1 + r_1 a_1 v_1^*)}{1 + r_1} \widetilde{u}_n + \frac{(\frac{1}{a_1} - v_1^*) r_1 b_1}{1 + r_1} \widetilde{v}_n + \frac{(\frac{1}{a_1} - v_1^*) r_1 c_1}{1 + r_1} \widetilde{w}_n \Big) k_1(y)\,dy\\
&\widetilde{v}_{n+1}(x) = \int_{\mathbb{R}} \Big(\frac{r_2 c_2 v_2^*}{1 + r_2} \widetilde{u}_n + \frac{(1 + r_1) - r_2 a_2 v_2^*}{1 + r_2} \widetilde{v}_n + \frac{r_2 b_2 v_3^*}{1 + r_2} \widetilde{w}_n \Big) k_2(y)\,dy,\\
&\widetilde{w}_{n+1}(x) = \int_{\mathbb{R}} \Big(\frac{r_3 c_3 v_3^*}{1 + r_3} \widetilde{u}_n + \frac{r_3 b_3 v_2^*}{1 + r_3} \widetilde{v}_n + \frac{(1 + r_3) - r_3 a_3 v_3^*}{1 + r_3} \widetilde{w}_n \Big) k_3(y)\,dy.
\end{aligned}
\tag{3.3}
$$

For any $\mu \in \mathbb{R}_+$, let $\widehat{u}_n(x) = e^{-\mu x}\alpha_n$, $\widehat{v}_n(x) = e^{-\mu x}\beta_n$ and $\widehat{w}_n(x) = e^{-\mu x}\gamma_n$ for $n \geq 0$. Then α_n, β_n and γ_n satisfies

$$
\begin{aligned}
&\alpha_{n+1}\\
&= \frac{(1 + r_1 a_1 v_1^*)}{1 + r_1} K_1(\mu)\alpha_n + \frac{(\frac{1}{a_1} - v_1^*) r_1 b_1}{1 + r_1} K_1 \beta_n + \frac{(\frac{1}{a_1} - v_1^*) r_1 c_1}{1 + r_1} K_1(\mu)\gamma_n,\\
&\beta_{n+1} = \frac{r_2 c_2 v_2^*}{1 + r_2} K_2(\mu)\alpha_n + \frac{(1 + r_1) - r_2 a_2 v_2^*}{1 + r_2} K_2(\mu)\beta_n + \frac{r_2 b_2 v_3^*}{1 + r_2} K_2(\mu)\gamma_n,\\
&\gamma_{n+1} = \frac{r_3 c_3 v_3^*}{1 + r_3} K_3(\mu)\alpha_n + \frac{r_3 b_3 v_2^*}{1 + r_3} K_2(\mu)\beta_n + \frac{(1 + r_3) - r_3 a_3 v_3^*}{1 + r_3} K_2(\mu)\gamma_n,
\end{aligned}
\tag{3.4}
$$

where

$$
K_i(\mu) := \int_{-\infty}^{\infty} e^{\mu y} k_i(y)\,dy > \int_{-\infty}^{\infty} k_i(y)\,dy = 1, \quad i = 1, 2, 3. \tag{3.5}
$$

We define the matrix

$$
B_\mu := \begin{pmatrix}
\frac{(1+r_1 a_1 v_1^*)}{1+r_1} K_1(\mu) & \frac{(\frac{1}{a_1} - v_1^*) r_1 b_1}{1+r_1} K_1(\mu) & \frac{(\frac{1}{a_1} - v_1^*) r_1 c_1}{1+r_1} K_1(\mu)\\
\frac{r_2 c_2 v_2^*}{1+r_2} K_2(\mu) & \frac{(1+r_1) - r_2 a_2 v_2^*}{1+r_2} K_2(\mu) & \frac{r_2 b_2 v_3^*}{1+r_2} K_2(\mu)\\
\frac{r_3 c_3 v_3^*}{1+r_3} K_3(\mu) & \frac{r_3 b_3 v_2^*}{1+r_3} K_2(\mu) & \frac{(1+r_3) - r_3 a_3 v_3^*}{1+r_3} K_2(\mu)
\end{pmatrix}.
$$

Note that each entry of B_μ is positive for any $\mu \geq 0$, then B_μ is positive. Let $\lambda(\mu)$ be the principal eigenvalue of B_μ. By [22, Theorem A4], $\lambda(\mu)$ is positive with a strongly positive eigenvector. Let $\lambda(0)$ be the principle eigenvalue of B_μ with $\mu = 0$, we have

$$
B_0 := \begin{pmatrix}
\frac{(1+r_1 a_1 v_1^*)}{1+r_1} & \frac{(\frac{1}{a_1} - v_1^*) r_1 b_1}{1+r_1} & \frac{(\frac{1}{a_1} - v_1^*) r_1 c_1}{1+r_1}\\
\frac{r_2 c_2 v_2^*}{1+r_2} & \frac{(1+r_1) - r_2 a_2 v_2^*}{1+r_2} & \frac{r_2 b_2 v_3^*}{1+r_2}\\
\frac{r_3 c_3 v_3^*}{1+r_3} & \frac{r_3 b_3 v_2^*}{1+r_3} & \frac{(1+r_3) - r_3 a_3 v_3^*}{1+r_3}
\end{pmatrix}.
$$

Since $\mathbf{0}$ is unstable, then $\lambda(0) > 1$. By $K_i(\mu) > 1$, $\forall \mu > 0$, $i = 1, 2, 3$, we have $B_\mu > B_0$ for any $\mu > 0$. The monotonicity of the principle eigenvalue with respect to the positive matrix implies that

$$
\lambda(\mu) > \lambda(0) > 1 \quad \text{for any } \mu > 0.
$$

Let $\Psi(\mu) := \frac{\ln \lambda(\mu)}{\mu}$, then $\Psi(\mu) > 0$ for any $\mu > 0$ and $\lim_{\mu \to 0^+} \Psi(\mu) = +\infty$, we also have

$$
\begin{aligned}
\liminf_{\mu \to \infty} \Psi(\mu) &= \liminf_{\mu \to \infty} \frac{\ln \lambda(\mu)}{\mu} \geq \liminf_{\mu \to} \frac{\ln (tr B_\mu)}{\mu}\\
&\geq \liminf_{\mu \to} \frac{\ln \frac{(1+r_1 a_1 v_1^*)}{1+r_1} e^{\mu y_0} \int_{y_0}^{+\infty} k_1(y)\,dy}{\mu} = y_0 > 0.
\end{aligned}
$$

It then follows that $\bar{c} := \inf_{\mu>0} \Psi(\mu) > 0$. By [16, Theorem 3.10], we have $c^*(\mathbf{0}, \beta) \geq \bar{c}$. Then $c^*(\mathbf{v}^*, \mathbf{v}^+) = c^*(\mathbf{0}, \beta) \geq \bar{c} > 0$.

To compute $c^*(\mathbf{0}, \mathbf{v}^*)$, let $\widetilde{u}_n = -u_n + v_1^*$, $\widetilde{v}_n = -v_n + v_2^*$ and $\widetilde{w}_n = -w_n + v_3^*$. Then system (1.5) becomes

$$
\begin{aligned}
&\widetilde{u}_{n+1}(x)\\
&= v_1^* - \int_{\mathbb{R}} \frac{-\frac{1}{a_1} r_1(b_1\widetilde{v}_n(x-y) + c_1\widetilde{w}_n(x-y)) - \widetilde{u}_n(x-y) + (1+r_1)v_1^*}{1 + r_1(1 + a_1\widetilde{u}_n(x-y) + b_1\widetilde{v}_n(x-y) + c_1\widetilde{w}_n(x-y))} k_1(y)\, dy\\
&\widetilde{v}_{n+1}(x)\\
&= v_2^* - \int_{\mathbb{R}} \frac{(1+r_2)(-\widetilde{v}_n(x-y) + v_2^*)}{1 + r_2(1 - a_2\widetilde{v}_n(x-y) + b_2\widetilde{w}_n(x-y) + c_2\widetilde{u}_n(x-y))} k_2(y)\, dy,\\
&\widetilde{w}_{n+1}(x)\\
&= v_3^* - \int_{\mathbb{R}} \frac{(1+r_3)(-\widetilde{w}_n(x-y) + v_3^*)}{1 + r_3(1 - a_3\widetilde{w}_n(x-y) + b_3\widetilde{v}_n(x-y) + c_3\widetilde{u}_n(x-y))} k_3(y)\, dy.
\end{aligned}
\tag{3.6}
$$

Note that this system is cooperative and the spatially homogeneous system of (3.6) has unstable equilibrium $\mathbf{0}$ and stable equilibrium $\mathbf{v}^* \gg \mathbf{0}$ in $[\mathbf{0}, \mathbf{v}^*] \subset \mathbb{R}^3$. By the similar arguments as for system (3.1), we have $c^*(\mathbf{0}, \mathbf{v}^*) > 0$. Therefore, $c^*(\mathbf{0}, \mathbf{v}^*) + c^*(\mathbf{v}^*, \mathbf{v}^+) > 0$. ☐

Lemma 3.3. *Let* $\mathbf{v}_0 = (1/a_1, 0, 0)$. *Then* $c^*(\mathbf{0}, \mathbf{v}_0) + c^*(\mathbf{v}_0, \mathbf{v}^+) > 0$.

Proof. To calculate the speed $c^*(\mathbf{0}, \mathbf{v}_0)$, we only need to consider the following one-dimensional monotone subsystem of (1.1)

$$
u_{n+1}(x) = \int_{\mathbb{R}} \frac{(1+r_1)u_n(x-y)}{1 + r_1 a_1 u_n(x-y)} k_1(y)\, dy, \quad n \geq 0. \tag{3.7}
$$

By [7, Theorem 2.1], (3.7) has a monotone traveling wave connecting 0 and 1 with the minimal wave c^*, where

$$
c^* = \inf_{\mu>0} \Big\{ \frac{1}{\mu} \ln \Big(\frac{1+r_1}{1+r_1 a_1} \int_{\mathbb{R}} e^{\mu y} k_1(y)\, dy \Big) \Big\}
$$

is the spreading speed and $c^*(\mathbf{0}, \mathbf{v}_0) = c^*$ (also see [26, Lemma 2.3]. As the proof of [29, Lemma 2.1], we have $c^*(\mathbf{0}, \mathbf{v}_0) > 0$.

Next, we consider $c^*(\mathbf{v}_0, \mathbf{v}^+)$. We consider the two-dimensional monotone system

$$
\begin{aligned}
v_{n+1}(x) &= \int_{\mathbb{R}} \frac{(1+r_2)v_n}{1 + r_2(a_2 v_n(x-y) - b_2 w_n(x-y))} k_2(y)\, dy,\\
w_{n+1}(x) &= \int_{\mathbb{R}} \frac{(1+r_3)w_n}{1 + r_3(a_3 w_n(x-y) - b_3 v_n(x-y))} k_3(y)\, dy.
\end{aligned}
\tag{3.8}
$$

Note that

$$
\begin{aligned}
v_{n+1}(x) &\geq \int_{\mathbb{R}} \frac{(1+r_2)v_n}{1 + r_2 a_2 v_n(x-y)} k_2(y)\, dy,\\
w_{n+1}(x) &\geq \int_{\mathbb{R}} \frac{(1+r_3)w_n}{1 + r_3 a_3 w_n(x-y)} k_3(y)\, dy.
\end{aligned}
$$

Thus, $c^*(\mathbf{v}_0, \mathbf{v}^+) \geq \min\{c_v^*, c_w^*\}$, where

$$
c_v^* = \inf_{\mu>0} \Big\{ \frac{1}{\mu} \ln \Big(\frac{1+r_2}{1+r_2 a_2} \int_{\mathbb{R}} e^{\mu y} k_1(y)\, dy \Big) \Big\} > 0,
$$

$$c_w^* = \inf_{\mu>0}\Big\{\frac{1}{\mu}\ln\Big(\frac{1+r_3}{1+r_3a_3}\int_{\mathbb{R}} e^{\mu y}k_1(y)\,dy\Big)\Big\} > 0.$$

It follows that $c^*(\mathbf{v}_0, \mathbf{v}^+) > 0$. Therefore, $c^*(\mathbf{0}, \mathbf{v}_0)+c^*(\mathbf{v}_0, \mathbf{v}^+) > 0$. This completes the proof. □

As in Lemmas 3.2 and 3.3, for the other equilibria in $E \setminus \{\mathbf{0}, \mathbf{v}^+\}$, we have the following results.

Lemma 3.4. *Assume that $c_2/a_1 > 1$, $c_3/a_1 > 1$, $a_1a_3 < c_1c_3$, $a_1a_2 < b_1c_2$ and* (2.4) *hold. Then $\{Q_n\}_{n=0}^{\infty}$ satisfies* (A6).

As a consequence of Lemmas 3.1–3.4 and [3, Theorem 3.1], we have the following results.

Theorem 3.5. *Assume that $c_2/a_1 > 1$, $c_3/a_1 > 1$, $a_1a_3 < c_1c_3$, $a_1a_2 < b_1c_2$ and* (2.4) *hold. Then system* (1.5) *admits a nondecreasing traveling wave solution $\mathbf{\Phi}(x - cn) = (\Phi_1(x - cn), \Phi_2(x - cn), \Phi_3(x - cn))$ with speed $c \in \mathbb{R}$ and connecting two stable equilibria $\mathbf{0}$ and $\mathbf{v}^+ = (v_1^+, v_2^+, v_3^+)$.*

Next, we study the global stability and uniqueness of the bistable traveling waves $\mathbf{\Phi}(x-cn) = (\Phi_1(x-cn), \Phi_2(x-cn), \Phi_3(x-cn))$ for system (1.5). Let $z = x-c(n+1)$. Thus (1.5) can be transformed into the system

$$\overline{\mathbf{U}}_{n+1}(z) = \mathbf{Q}[\overline{\mathbf{U}}_n](z + c), \quad n \geq 0. \tag{3.9}$$

In the following, $\overline{\mathbf{U}}_n(z, \boldsymbol{\psi})$ denotes the solution of (3.9) with the initial value $\overline{\mathbf{U}}_0(z, \boldsymbol{\psi}) = \boldsymbol{\psi} := (\psi_1, \psi_2, \psi_3)$. Then $\mathbf{\Phi}(z)$ is an equilibrium solution of system (3.9), that is,

$$\mathbf{\Phi}(z) = \mathbf{Q}[\mathbf{\Phi}](z + c), \quad \forall z \in \mathbb{R}.$$

Clearly, the solution $\mathbf{U}_n(x, \boldsymbol{\psi})$ of (1.5) with initial value is given by

$$\mathbf{U}_n(x, \boldsymbol{\psi}) = \overline{\mathbf{U}}_n(x - cn, \boldsymbol{\psi}).$$

We choose $0 < \widehat{\delta}_i < \widetilde{\delta} < 1$ $(i = 1, 2)$ with $\widehat{\delta}_2/\widehat{\delta}_1$ and $\widehat{\delta}_3/\widehat{\delta}_1$ sufficiently small such that

$$\frac{b_1}{a_1}\frac{\widehat{\delta}_2}{\widehat{\delta}_1} + \frac{c_1}{a_1}\frac{\widehat{\delta}_3}{\widehat{\delta}_1} < 1. \tag{3.10}$$

We choose $\breve{\delta}_2, \breve{\delta}_3 < 1$ such that $b_3/a_3 <$
$breve\delta_3/\breve{\delta}_2 < a_2/b_2$ and choose $\breve{\delta}_1 < \widetilde{\delta}$ small enough to satisfy

$$c_2\frac{\breve{\delta}_1}{\breve{\delta}_2} - a_2 + b_2\frac{\breve{\delta}_3}{\breve{\delta}_2} < 0 \quad \text{and} \quad c_3\frac{\breve{\delta}_1}{\breve{\delta}_3} - a_3 + b_3\frac{\breve{\delta}_2}{\breve{\delta}_3} < 0. \tag{3.11}$$

Define a continuous function $\boldsymbol{\delta}(\xi) = (\delta_1(\xi), \delta_2(\xi), \delta_3(\xi))$ by

$$\delta_i(\xi) = \begin{cases} \widehat{\delta}_i, & \xi < \widehat{\xi} < 0, \\ \text{nondecreasing}, & \widehat{\xi} \leq \xi \leq \breve{\xi}, \\ \breve{\delta}_i, & \xi > \breve{\xi} > 0, \end{cases}$$

for $i = 1, 2, 3$. Moreover, we define $\mathbf{W}_n^{\pm}(z) = (U_n^{\pm}, V_n^{\pm}, W_n^{\pm})$ as follows

$$\begin{aligned} U_n^{\pm}(z) &= \Phi_1\big(z \pm \widehat{z} \pm \varepsilon(1 - e^{-\sigma n})\big) \pm \varepsilon\delta_1(z \pm \widehat{z})e^{-\sigma n}, \\ V_n^{\pm}(z) &= \Phi_2\big(z \pm \widehat{z} \pm \varepsilon(1 - e^{-\sigma n})\big) \pm \varepsilon\delta_2(z \pm \widehat{z})e^{-\sigma n}, \\ W_n^{\pm}(z) &= \Phi_3\big(z \pm \widehat{z} \pm \varepsilon(1 - e^{-\sigma n})\big) \pm \varepsilon\delta_3(z \pm \widehat{z})e^{-\sigma n}. \end{aligned}$$

The following Lemma shows that $\mathbf{W}_n^{\pm}(z) = (U_n^{\pm}, V_n^{\pm}, W_n^{\pm})$ is the upper and lower solution of (3.9).

Lemma 3.6. *Assume that $c_2/a_1 > 1$, $c_3/a_1 > 1$, $a_1a_3 < c_1c_3$, $a_1a_2 < b_1c_2$ and (2.4) hold. There exist positive number σ and $\varepsilon_0 \in (0,1)$ such that for any $\widehat{z}$ and $\varepsilon \in (0, \varepsilon_0)$, $(U_n^{\pm}, V_n^{\pm}, W_n^{\pm})$ are upper solution and lower solution of system (3.9) for $n \geq 0$ and $z \in \mathbb{R}$, respectively.*

The proof of the above lemma is similar to that of [29, Lemma 3.2]; and we omit it here. With the help of Lemma 3.6, the bistable traveling waves $\mathbf{\Phi}(z)$ is a Lyapunov stable equilibrium of system (3.9). Let $\mathcal{X} = BUC(\mathbb{R}, \mathbb{R}^3)$ be the Banach space of all bounded and uniformly continuous functions from $\mathbb{R}$ to $\mathbb{R}^3$ with the usual supreme norm. Let $\mathcal{X}_+ = \{(\psi_1, \psi_2, \psi_3) \in \mathcal{X} : \psi_i(x) \geq 0, \forall x \in \mathbb{R},\ i = 1, 2, 3\}$. Now we are ready to prove the global stability of the bistable wave $\mathbf{\Phi}(z)$.

Theorem 3.7. *Let $\mathbf{\Phi}(x - cn)$ be a monotone traveling wave solutions of (1.5) and $\mathbf{U}_n(x, \boldsymbol{\psi})$ be the solution of (1.5) with $\mathbf{U}_0(\cdot, \boldsymbol{\psi}) = \boldsymbol{\psi}(\cdot) := (\psi_1(\cdot), \psi_2(\cdot), \psi_3(\cdot)) \in \mathcal{X}_{[\mathbf{0},\mathbf{v}^+]}$. Let the initial value $\boldsymbol{\psi}$ satisfy one of the following statements:*

(i) *$\boldsymbol{\psi} \in \mathcal{X}_{[\mathbf{0},\mathbf{v}^+]}$ is nondecreasing and satisfies*
$$\liminf_{\xi \to +\infty} \psi_i(\xi) > v_i^* > \limsup_{\xi \to -\infty} \psi_i(\xi) \quad \text{for } i = 1, 2, 3;$$

(ii) *the kernel k_i $(i = 1, 2, 3)$ has a compact support and $\boldsymbol{\psi}(\xi)$ satisfies*
$$\liminf_{\xi \to +\infty} \psi_i(\xi) > v_i^* > \limsup_{\xi \to -\infty} \psi_i(\xi) \quad \text{for } i = 1, 2, 3,$$

then there exists $s_{\boldsymbol{\psi}} \in \mathbb{R}$ such that
$$\lim_{n \to +\infty} \|\mathbf{U}_n(x, \boldsymbol{\psi}) - \mathbf{\Phi}(x - cn + s_{\boldsymbol{\psi}})\| = 0 \tag{3.12}$$
uniformly for $x \in \mathbb{R}$.

Proof. Let $\varepsilon \in (0, \varepsilon_0)$ small. For any given nondecreasing initial value $\boldsymbol{\psi} \in \mathcal{X}_{[\mathbf{0},\mathbf{v}^+]}$ satisfying $\liminf_{\xi \to +\infty} \psi_i(\xi) > v_i^*$ and $v_i^* > \limsup_{\xi \to -\infty} \psi_i(\xi)$ for $i = 1, 2, 3$, we can show that there exist $\hat{z} = \hat{z}(\varepsilon, \boldsymbol{\psi}) > 0$ such that
$$\mathbf{\Phi}(z - \hat{z}) - \varepsilon \mathbf{e} \leq \overline{\mathbf{U}}_0(z, \boldsymbol{\psi}) \leq \mathbf{\Phi}(z + \hat{z}) + \varepsilon \mathbf{e}, \quad \forall z \in \mathbb{R}. \tag{3.13}$$
By [32, Theorem 2.2.4] and the same strategy as in [29, Theorem 3.1], we have that (3.12) holds true. In fact, by the comparison principle and the definition of upper and lower solutions $\mathbf{W}_n^{\pm}(z)$ in Lemma 3.6, we have
$$\mathbf{W}_n^-(z) \leq \overline{\mathbf{U}}_n(z, \overline{\mathbf{U}}_0(\cdot)) \leq \mathbf{W}_n^+(z), \quad \forall z \in \mathbb{R},\ n \in \mathbb{N},$$
which implies
$$\mathbf{\Phi}(z - \widehat{z} - \varepsilon_0) - \varepsilon \boldsymbol{\delta}(z - \widehat{z}) e^{-\sigma n} \leq \overline{\mathbf{U}}_n(z, \boldsymbol{\psi}) \leq \mathbf{\Phi}(z + \widehat{z} + \varepsilon_0) + \varepsilon \boldsymbol{\delta}(z + \widehat{z}) e^{-\sigma n}. \tag{3.14}$$
We define the semiflow solution of (3.9) as
$$\mathbf{\Psi}_n(\psi) := \overline{\mathbf{U}}_n(z, \boldsymbol{\psi}), \quad \forall \boldsymbol{\psi} \in \mathcal{X},\ n \in \mathbb{N}^+.$$
Since $\lim_{z \to -\infty} \mathbf{\Phi}(z) = \mathbf{0}$ and $\lim_{z \to +\infty} \mathbf{\Phi}(z) = \mathbf{v}^+$, by (3.14), we have
$$\gamma^+(\boldsymbol{\psi}) := \{\mathbf{\Psi}_n(\boldsymbol{\psi}) : n \geq 0\}$$
is bounded in $\mathcal{X}$. The Ascoli-Arzelà theorem implies that $\gamma^+(\boldsymbol{\psi})$ is precompact in $\mathcal{X}$ and the omega limit set $\omega(\boldsymbol{\psi})$ is nonempty, compact and invariant. Let $z_0 = \widehat{z} + \varepsilon_0$ and $n \to \infty$ in (3.14), we have the omega limit set $\omega(\boldsymbol{\psi}) \subset I :=$

$[\mathbf{\Phi}(\cdot - z_0), \mathbf{\Phi}(\cdot + z_0)]_{\mathcal{X}}$. Let $h(s) = \mathbf{\Phi}(\cdot + s)$, for all $s \in [-z_0, z_0]$. Then h is a monotone homeomorphism from $[-z_0, z_0]$ onto a subset $\widehat{I} \subset I$. Then $\mathbf{\Psi}_n : \mathcal{X}_{[\mathbf{0},\mathbf{v}^+]} \to \mathcal{X}_{[\mathbf{0},\mathbf{v}^+]}$ is a monotone autonomous semiflow and each $h(s)$ is stable equilibrium for $\mathbf{\Psi}_n$. Clearly, each $\boldsymbol{\psi} \in \widehat{I}$ satisfies $\liminf_{\xi\to+\infty} \psi_i(\xi) > v_i^* > \limsup_{\xi\to-\infty} \psi_i(\xi)$ and then $\gamma^+(\boldsymbol{\psi})$ is precompact. By the similar process in [29, Theorem 3.1] and [32, Theorem 2.2.4], there is $s_{\boldsymbol{\psi}} \in [-z_0, z_0]$ such that $\omega(\boldsymbol{\psi}) = h(s_{\boldsymbol{\psi}}) = \mathbf{\Phi}(\cdot + s_{\boldsymbol{\psi}})$. Then $\lim_{n\to\infty} \mathbf{\Psi}_n(\boldsymbol{\psi}) = \mathbf{\Phi}(\cdot + s_{\boldsymbol{\psi}})$. Since $\mathbf{U}_n(x, \boldsymbol{\psi}) = \overline{\mathbf{U}}_n(x - cn, \boldsymbol{\psi}) = \mathbf{\Psi}_n(\boldsymbol{\psi})(x - cn)$, we obtain

$$\lim_{n\to+\infty} \|\mathbf{U}_n(x, \boldsymbol{\psi}) - \mathbf{\Phi}(x - cn + s_{\boldsymbol{\psi}})\| = 0$$

uniformly for $x \in \mathbb{R}$.

If the kernel k_i $(i = 1, 2, 3)$ has a compact support, then for any $\varepsilon > 0$ and any $\boldsymbol{\psi}(\xi)$ satisfies $\liminf_{\xi\to+\infty} \psi_i(\xi) > v_i^* > \limsup_{\xi\to-\infty} \psi_i(\xi)$, then there exist $\hat{z} = \hat{z}(\varepsilon, \boldsymbol{\psi}) > 0$ and a large time $n_0 \in \mathbb{N}^+$ such that $\mathbf{\Phi}(z - \hat{z}) - \varepsilon\mathbf{e} \le \overline{\mathbf{U}}_{n_0}(z, \boldsymbol{\psi}) \le \mathbf{\Phi}(z + \hat{z}) + \varepsilon\mathbf{e}$. Then there are n_0 and $\hat{z}$ such that for any $z \in \mathbb{R}$,

$$\begin{gathered}
\overline{\mathbf{U}}_{n_0}(z, \boldsymbol{\psi}) \le \mathbf{\Phi}(z + \widehat{z}) + \varepsilon\widehat{\boldsymbol{\delta}} \le \mathbf{\Phi}(z + \widehat{z}) + \varepsilon\widehat{\boldsymbol{\delta}}(z + \widehat{z}) = \mathbf{W}_0^+(z + \widehat{z}), \\
\overline{\mathbf{U}}_{n_0}(z, \boldsymbol{\psi}) \ge \mathbf{\Phi}(z - \widehat{z}) - \varepsilon\widehat{\boldsymbol{\delta}} \ge \mathbf{\Phi}(z - \widehat{z}) - \varepsilon\widehat{\boldsymbol{\delta}}(z - \widehat{z}) = \mathbf{W}_0^-(z + \widehat{z})
\end{gathered}$$

It follows that $\mathbf{W}_n^-(z) \le \overline{\mathbf{U}}_n(z, \overline{\mathbf{U}}_{n_0}(\cdot)) \le \mathbf{W}_n^+(z)$ for $z \in \mathbb{R}$ and $n \in \mathbb{N}$. Note that $\overline{\mathbf{U}}_n(z, \overline{\mathbf{U}}_{n_0}(\cdot)) = \overline{\mathbf{U}}_{n+n_0}(z, \boldsymbol{\psi})$ for any $z \in \mathbb{R}$ and $n \in \mathbb{N}$. It then follows that

$$\mathbf{\Phi}(z - \widehat{z} - \varepsilon_0) - \varepsilon\boldsymbol{\delta}(z - \widehat{z})e^{-\sigma n} \le \overline{\mathbf{U}}_{n+n_0}(z, \boldsymbol{\psi}) \le \mathbf{\Phi}(z + \widehat{z} + \varepsilon_0) + \varepsilon\boldsymbol{\delta}(z + \widehat{z})e^{-\sigma n}.$$

Similar to case (i), (3.12) also holds for case (ii). This completes the proof. □

Corollary 3.8. *Let $\widetilde{\mathbf{\Phi}}(x - \widetilde{c}n)$ be a monotone traveling wave solution of system* (1.5) *satisfying $\widetilde{\mathbf{\Phi}}(-\infty) = \mathbf{0}$ and $\widetilde{\mathbf{\Phi}}(+\infty) = \mathbf{v}^+$. Then there exists $\widetilde{s} \in \mathbb{R}$ such that*

$$\widetilde{\mathbf{\Phi}}(\cdot) \equiv \mathbf{\Phi}(\cdot + \widetilde{s}) \quad \text{and} \quad \widetilde{c} = c,$$

where $\mathbf{\Phi}(x - cn)$ is defined by Theorem 3.5.

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