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# GLOBAL SOLUTIONS TO A QUASILINEAR HYPERBOLIC EQUATION

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Abstract. This article concerns the existence and decay of solutions of a mixed problem for a quasilinear hyperbolic equation which has its motivation in a mathematical model that describes the nonlinear vibrations of the cross-section of a bar.

## 1. Introduction

Milla Miranda et al [16] presented a mathematical model for the small longitudinal vibrations of the cross sections of a bar of length $L$ which is clamped on one end and the other end is glued in a mass $M$. This model has the form

$$\begin{gathered} u''(x,t) - \frac{\partial}{\partial x}\sigma(u_x(x,t)) = 0, \quad 0 < x < L,\ t > 0; \\ u(0,t) = 0, \quad Mu''(L,t) + \sigma(u_x(L,t)) = 0, \quad t > 0; \\ u(x,0) = u^0(x), \quad u'(x,0) = u^1(x), \quad 0 < x < L, \end{gathered} \tag{1.1}$$

where $u(x,t)$ denotes the displacement of the cross section $x$ of the bar at time $t$, and $u' = \frac{\partial u}{\partial t}$.

To obtain (1.1) we use Hooke's law $\tau(x,t) = \sigma(u_x(x,t))$ in which $\tau(x,t)$ and $u_x(x,t)$ are the tension and the deformation of the bar at $(x,t)$, respectively, and $\sigma(s)$ is a real function. The linear version of Problem (1.1) can be found in Timoshenko et al [18, p 387].

For a zero Dirichlet boundary conditions in (1.1), there are a lot of papers investigating the existence and decay of solutions of this problem, among of them we can mention [2, 4, 5, 11, 12]. MacCamy and Mizel [11] proved that for some functions $\sigma(s)$ this problem has solutions that blow up in finite time. Dafermos [3] consider (1.1) with $\sigma(u_x, u_x')$ and the boundary conditions

$$\begin{gathered} \sigma(u_x(0,t), u_x'(0,t)) = \sigma_0(t), \quad t \in [0,T]; \\ \sigma(u_x(L,t), u_x'(L,t)) = \sigma_1(t), \quad t \in [0,T]. \end{gathered}$$

Then Dafermos [3] obtained the existence and decay of solutions.

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We focus our attention on Problem (1.1) with

$$\sigma(s) = |s|^p s, \quad \text{with } p > 0, \tag{1.2}$$

$M = 1$, and internal damping. More precisely, we consider the problem

$$\begin{gathered} u''(x,t) - \frac{\partial}{\partial x}\Big(|\frac{\partial u}{\partial x}(x,t)|^p \frac{\partial u}{\partial x}(x,t) + \frac{\partial u}{\partial x}(x,t)\Big) = 0, \quad 0 < x < L,\ t > 0; \\ u(0,t) = 0, \quad u''(L,t) + |\frac{\partial u}{\partial x}(L,t)|^p \frac{\partial u}{\partial x}(L,t) + \frac{\partial u}{\partial x}(L,t) = 0 \quad t > 0; \\ u(x,0) = u^0(x), \quad \frac{\partial u}{\partial t}(x,0) = u^1(x), \quad 0 < x < L. \end{gathered} \tag{1.3}$$

We observe that the function $\sigma(s)$ given in (1.2) is different from the $\sigma(s)$ considered in the above papers. Note also that the existence of global solutions of (1.3) with zero Dirichlet boundary conditions and without internal damping is an open problem (cf. J. L. Lions [9]). This justifies the introduction of the internal damping for obtain the existence of global solutions of (1.3).

Tsutsumi [19] and Giorgio and Matarazzo [4] considered Problem (1.3) with zero Dirichlet boundary conditions. They obtain global solutions for in an $n$-dimensional case. Later Maia and Milla Miranda [13] analyzed Problem (1.3) with zero Dirichlet boundary conditions in an abstract framework. The authors obtained global solutions and decay of solutions for this problem and generalized the papers [4, 19].

Maia and Milla Miranda [13] found an estimate for $(u''_m)$, where $u_m$ is an approximate solution of (1.3), to apply the theory of monotone operators. For that, the eigenvectors of a positive self-adjoint operator of a Hilbert space and the projection method are used. This approach does not work in Problem (1.3) because of the boundary conditions $(1.3)_2$

To overcome the above difficulty, the authors in [16] introduced in equation $(1.3)_1$ the internal damping $u'_{xxxx}$ to obtain the existence and decay of solutions of (1.3).

Our objective in this article is not introduce new internal damping in $(1.3)_1$, but decrease the class of functions $\sigma(s)$ given in (1.2) to obtain global solutions of (1.3). More precisely, considering the truncated of functions $|s|^p s$ (see Examples in Section 6), we succeed in to obtain the existence, uniqueness and exponential decay of solutions of Problem (1.3) in an $n$-dimensional case.

In our approach to prove the existence of solutions, we use the Faedo-Galerkin method with a special basis, the theory of monotone operators (cf. J. L. Lions [9] and Medeiros and Pereira [15]) and results on the trace of non-smooth functions. The estimate for $(u''_m)$ is obtained thanks to the truncation of the functions $|s|^p s$ and the special basis. In the decay of solutions is used a Liapunov functional (cf. Komornik and Zuazua [8] and Komornik [7])

We note that it is not usual for hyperbolic problems to have an equation at the boundary which contains a nonlinear term of the normal derivative and the second derivative with respect to $t$, respectively, of the solution.

As far as we know, the only results on the existence of global solutions of (1.3) are given in the present paper and in Milla Miranda et al [16]. In this case the existence of solution for the linear case can also be obtained using semigroup theory as in Goldstein [6].

## 2. Notation and main results

Let $\Omega$ be open bounded set of $\mathbb{R}^n$ whose boundary $\Gamma$ is constituted of two parts $\Gamma_0$ and $\Gamma_1$ such that $\Gamma = \Gamma_0 \cup \Gamma_1$ and $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$. With $\nu(x)$ is denoted the unit exterior normal at $x \in \Gamma_1$.

The scalar product and norm of $L^2(\Omega)$ are denoted, respectively, by $(u, v)$ and $|u|$. Let

$$H^1_{\Gamma_0} = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_0\}$$

equipped with the scalar product

$$((u, v)) = \sum_{i=1}^{n} \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx$$

and norm $\|u\| = ((u, u))^{1/2}$. Its dual is denoted by $H^{-1}_{\Gamma_0}(\Omega)$. The notations and results on Functional Analysis and Sobolev Spaces can be seen in Brezis [1], J. L. Lions [10] and Medeiros and Milla Miranda [14].

We consider the functions $\sigma_i : \mathbb{R} \to \mathbb{R}$ $(i = 1, 2, \dots, n)$ such that

$$\sigma_i \text{ is globally Lipschitz, } \sigma_i \text{ is increasing and } \sigma_i(0) = 0,\ i = 1, 2, \dots, n. \tag{2.1}$$

With the above notation, we introduce the quasilinear hyperbolic problem

$$\begin{gathered} u'' - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \big[\sigma_i\big(\frac{\partial u}{\partial x_i}\big) + \frac{\partial u'}{\partial x_i}\big] = 0 \quad \text{in } \Omega \times (0, \infty), \\ u = 0 \text{ in } \Gamma_0 \times (0, \infty), \\ \sum_{i=1}^{n} \big[\sigma_i\big(\frac{\partial u}{\partial x_i}\big) + \frac{\partial u'}{\partial x_i}\big]\nu_i + u'' = 0 \quad \text{on } \Gamma_1 \times (0, \infty), \\ u(0) = u^0, \quad u'(0) = u^1 \quad \text{in } \Omega. \end{gathered} \tag{2.2}$$

Here, $u' = \frac{\partial u}{\partial t}$. We obtain the following results.

**Theorem 2.1.** *Assume hypotheses* (2.1) *hold and*

$$u^0, u^1 \in H^1_0(\Omega) \cap H^2(\Omega) \quad \text{with } \frac{\partial u^0}{\partial \nu} = \frac{\partial u^1}{\partial \nu} = 0 \quad \text{on } \Gamma_1. \tag{2.3}$$

*Then, there exists an unique function $u$ with*

$$\begin{gathered} u \in L^\infty_{\rm loc}(0, \infty; H^1_{\Gamma_0}(\Omega)), \\ u' \in L^\infty(0, \infty, L^2(\Omega)) \cap L^2(0, \infty; H^1_{\Gamma_0}(\Omega)), \\ u'' \in L^\infty(0, \infty; L^2(\Omega)) \cap L^2(0, \infty; H^1_{\Gamma_0}(\Omega)), \\ u'' \in L^\infty(0, \infty; L^2(\Gamma_1)), \end{gathered} \tag{2.4}$$

*such that $u$ satisfies the equations*

$$u'' - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \big[\sigma_i\big(\frac{\partial u}{\partial x_i}\big) + \frac{\partial u'}{\partial x_i}\big] = 0 \quad \text{in } L^2_{\rm loc}(0, \infty; H^1_{\Gamma_0}(\Omega)), \tag{2.5}$$

$$\sum_{i=1}^{n} \big[\sigma_i\big(\frac{\partial u}{\partial x_i}\big) + \frac{\partial u'}{\partial x_i}\big]\nu_i + u'' = 0 \quad \text{in } L^2_{\rm loc}(0, \infty; H^{1/2}_{\Gamma_0}(\Omega)) \tag{2.6}$$

*and the initial conditions*

$$u(0) = u^0, \quad u'(0) = u^1. \tag{2.7}$$

Let $\widehat{\sigma}_i(s) = \int_0^s \sigma_i(\tau)d\tau$, $i = 1, 2, \dots, n$. The energy functional for (2.2) is

$$E(t) = \frac{1}{2}|u'(t)|^2 + \sum_{i=1}^{n} \int_\Omega \widehat{\sigma}_i\big(\frac{\partial u}{\partial x_i}\big)dx + \frac{1}{2}|u'(t)|^2_{L^2(\Gamma_1)}, \quad t \geq 0.$$

To state the estimates on the decay of $E(t)$, we introduced some notation and consider one more hypothesis on $\sigma_i$. We set the notation

$$\begin{gathered} |v|^2 \leq a_1\|v\|^2, \quad \forall v \in H^1_{\Gamma_0}(\Omega), \\ |v|^2_{L^2(\Gamma_1)} \leq a_2\|v\|^2, \quad \forall v \in H^1_{\Gamma_0}(\Omega), \end{gathered} \tag{2.8}$$

in which $a_1$ and $a_2$ are positive constants. We assume that there exist positive constants $b_i$ $(i = 1, 2, \dots, n)$ such that

$$s^2 \leq b_i\widehat{\sigma}_i(s), \quad \forall s \in \mathbb{R},\ i = 1, 2, \dots, n. \tag{2.9}$$

Consider the constants

$$b = \max\{b_1, \dots, b_n\}, \quad d = \frac{1}{2}b(a_1 + 1 + a_2), \tag{2.10}$$

$$\varepsilon_0 = \min\{\frac{1}{2}, \frac{1}{2d}\}, \quad \varepsilon_1 = \min\{\frac{1}{3a_1}, \frac{1}{3a_2}\}, \tag{2.11}$$

$$\eta = \min\{\varepsilon_0, \varepsilon_1\} \tag{2.12}$$

**Theorem 2.2.** *Let $u$ be the solution obtained in Theorem 2.1. Assume that* (2.9) *is satisfied. Then*

$$E(t) \leq 3E(0) \exp\big(-\frac{2}{3}\eta t\big), \quad \forall t \geq 0. \tag{2.13}$$

To prove Theorem 2.1, we need some previous results.

## 3. Results

We denote by $k_i$ the Lipschitz constants of $\sigma_i$ $(i = 1, 2, \dots, n)$ and by $k = \max\{k_i; i = 1, 2, \dots, n\}$. In rest of this article we use the notation.

$$\langle Au, v\rangle = \sum_{i=1}^{n} \int_\Omega \sigma_i\big(\frac{\partial u}{\partial x_i}\big)\frac{\partial v}{\partial x_i}dx, \quad u, v \in H^1_{\Gamma_0}(\Omega).$$

**Proposition 3.1.** *We have*

- (i) $A : H^1_{\Gamma_0}(\Omega) \to H^{-1}_{\Gamma_0}(\Omega)$;
- (ii) *$A$ maps bounded sets of $H^1_{\Gamma_0}(\Omega)$ into bounded sets of $H^{-1}_{\Gamma_0}(\Omega)$;*
- (iii) *$A$ is monotone;*
- (iv) *$A$ is hemicontinuous.*

*Proof.* We have

$$|\langle Au, v\rangle| \leq \sum_{i=1}^{n} k_i \int_\Omega \big|\frac{\partial u}{\partial x_i}\big|\big|\frac{\partial v}{\partial x_i}\big|dx \leq k\|u\|\|v\|.$$

Thus, $Au \in H^{-1}_{\Gamma_0}(\Omega)$ and

$$\|Au\|_{H^{-1}_{\Gamma_0}(\Omega)} \leq k\|u\|, \quad \forall u \in H^1_{\Gamma_0}(\Omega).$$

This inequality proves (i) and (ii). Item (iii) follows from the fact that each $\sigma_i$ is an increasing function. Item (iv) is proved by using the continuity of each $\sigma_i$ and the Lebesgue Dominated Convergence Theorem. □

The following result is concerned with the trace of non-smooth functions. Consider the Hilbert space

$$E(\Omega) = \{f = (f_1, \dots, f_n) \in (L^2(\Omega))^n : \operatorname{div} f \in L^2(\Omega)\}$$

provided with the scalar product

$$(f, g)_{E(\Omega)} = \sum_{i=1}^{n} (f_i, g_i) + (\operatorname{div} f, \operatorname{div} g).$$

Note that $(\mathcal{D}(\overline{\Omega}))^n$ is dense in $E(\Omega)$ (cf. Temam [17, Theorem 1.1, p.6]). Take $f \in (\mathcal{D}(\overline{\Omega}))^n$ and $z \in H^1_{\Gamma_0}(\Omega)$. Then

$$(\operatorname{div} f, z) = -\sum_{i=1}^{n} \big(f_i, \frac{\partial z}{\partial x_i}\big) + \int_{\Gamma_1} \Big(\sum_{i=1}^{n} f_i \nu_i\Big) z d\Gamma,$$

in which $\nu(x) = (\nu_1(x), \dots, \nu_n(x))$ is the unit outward normal at $x \in \Gamma_1$. The above motivates the following result.

**Proposition 3.2.** *The map*

$$E(\Omega) \to H^{-1/2}(\Gamma_1), \quad f \mapsto \gamma_\nu f = f \cdot \nu$$

*is continuous. Also we have*

$$\langle \gamma_\nu f, z\rangle_{X' \times X} = \langle f \cdot \nu, z\rangle_{X' \times X} = (\operatorname{div} f, z) + \sum_{i=1}^{n} \big(f_i, \frac{\partial z}{\partial x_i}\big)$$

*for all $z \in (\mathcal{D}(\overline{\Omega}))^n$ and all $z \in H^1_{\Gamma_0}(\Omega)$. Here $X = H^{1/2}(\Gamma_1)$.*

*Proof.* Consider $f \in (\mathcal{D}(\overline{\Omega}))^n$ and $z \in H^{1/2}(\Gamma_1)$. By the trace Theorem there exists $w \in H^1_{\Gamma_0}(\Omega)$ such that $\gamma_0 w = z$ and

$$\|w\| \le C\|z\|_{H^{1/2}(\Gamma_1)}, \tag{3.1}$$

in which $C$ is a positive constant independent of $w$ and $z$. We have

$$\begin{aligned} |\langle \gamma_\nu f, z\rangle_{X' \times X}| &\le |(\operatorname{div} f, w)| + \sum_{i=1}^{n} \big|\big(f_i, \frac{\partial w}{\partial x_i}\big)\big| \\ &\le C_1 |\operatorname{div} f| \|w\| + \Big(\sum_{i=1}^{n} |f_i|^2\Big)^{1/2} \|w\| \\ &\le (C_1 + 1)\|f\|_{E(\Omega)} \|w\|. \end{aligned}$$

This inequality and (3.1) provide $\gamma_\nu f \in H^{-1/2}(\Gamma_1)$ and

$$\|\gamma_\nu f\|_{H^{-1/2}(\Gamma_1)} \le C_2 \|f\|_{E(\Omega)},$$

where $C_2 > 0$ is a constant independent of $f \in E(\Omega)$. The proposition follows by the denseness of $(\mathcal{D}(\overline{\Omega}))^n$ in $E(\Omega)$. □

## 4. Proof of Theorem 2.1

We used the Faedo-Galerkin method with a special basis of $H^1_{\Gamma_0}(\Omega)$. Consider a basis $\{w_1, w_2, \dots, \}$ of $H^1_{\Gamma_0}(\Omega)$ such that $u^0, u^1 \in [w_1, w_2]$ where $[w_1, w_2]$ is the subspace generated by $w_1$ and $w_2$. Let $u_m$ be an approximate solution of Problem (2.2), that is,

$$u_m = \sum_{i=1}^{m} g_{jm}(t) w_j$$

and $u_m$ be solution of the system

$$\begin{gathered}(u''_m, w) + \sum_{i=1}^{n} \Big(\sigma_i\big(\frac{\partial u_m}{\partial x_i}\big), \frac{\partial w}{\partial x_i}\Big) + ((u'_m, w)) + (u''_m, w)_{L^2(\Gamma_1)} = 0, \\ \forall w \in V_m = [w_1, w_2, \dots, w_m], \\ u_m(0) = u^0, \quad u'_m(0) = u^1\end{gathered} \tag{4.1}$$

**First estimate.** Setting $w = u'_m$ in $(4.1)_1$, we obtain

$$\frac{1}{2}\frac{d}{dt}|u'_m|^2 + \sum_{i=1}^{n} \frac{d}{dt}\int_\Omega \widehat{\sigma}_i\big(\frac{\partial u_m}{\partial x_i}\big)dx + \|u'_m\|^2 + \frac{1}{2}\frac{d}{dt}|u'_m|^2_{L^2(\Gamma_1)} = 0.$$

Integrating on $[0, t]$, $0 < t < t_m$, we obtain

$$\begin{aligned}&\frac{1}{2}|u'_m(t)|^2 + \sum_{i=1}^{n}\int_\Omega \widehat{\sigma}_i\big(\frac{\partial u_m(t)}{\partial x_i}\big)dx + \int_0^t \|u'_m(\tau)\|^2 d\tau + \frac{1}{2}|u'_m(t)|^2_{L^2(\Gamma_1)} \\ &= \frac{1}{2}|u^1|^2 + \sum_{i=1}^{n}\int_\Omega \widehat{\sigma}_i\big(\frac{\partial u^0}{\partial x_i}\big)dx + \frac{1}{2}|u^1|^2_{L^2(\Gamma_1)}.\end{aligned} \tag{4.2}$$

**Remark 4.1.** We have

$$|\widehat{\sigma}_i(s)| \le k_i \frac{s^2}{2}, \quad \forall s \in \mathbb{R},\ i = 1, 2, , \dots, n.$$

Therefore,

$$\int_\Omega \widehat{\sigma}_i\big(\frac{\partial u^0}{x_i}\big)dx \le \frac{k_i}{2}\int_\Omega \big(\frac{\partial u^0}{x_i}\big)dx, \quad i = 1, 2, \dots, n.$$

Taking into account Remark 4.1 in (4.2), we obtain

$$\begin{aligned}&\frac{1}{2}|u'_m(t)|^2 + \sum_{i=1}^{n}\int_\Omega \widehat{\sigma}_i\Big(\frac{\partial u_m(t)}{\partial x_i}\Big)dx + \int_0^t \|u'_m(\tau)\|^2 d\tau + \frac{1}{2}|u'_m(t)|^2_{L^2(\Gamma_1)} \\ &\le C, \quad \forall m,\ \forall t \in [0, \infty).\end{aligned} \tag{4.3}$$

We denote by $C > 0$ the various constants independent of $m$ and $t \in [0, \infty)$.

**Second estimate.** Differentiate the approximate equation $(4.1)_1$ with respect to $t$ then set $w = u''_m$. We obtain

$$\frac{1}{2}\frac{d}{dt}|u'_m|^2 + \sum_{i=1}^{n}\Big(\sigma'_i\big(\frac{\partial u_m}{\partial x_i}\big)\frac{\partial u'_m}{\partial x_i}, \frac{\partial u''_m}{\partial x_i}\Big) + \|u''_m\|^2 + \frac{1}{2}\frac{d}{dt}|u''_m|^2_{L^2(\Gamma_1)} = 0. \tag{4.4}$$

We have

$$\Big|\int_\Omega \sigma'_i\big(\frac{\partial u_m}{\partial x_i}\big)\frac{\partial u'_m}{\partial x_i}, \frac{\partial u''_m}{\partial x_i}dx\Big| \le k_i\Big|\frac{\partial u'_m}{\partial x_i}\Big|\Big|\frac{\partial u''_m}{\partial x_i}\Big| \le \frac{1}{2}k^2\Big|\frac{\partial u'_m}{\partial x_i}\Big|^2 + \frac{1}{2}\Big|\frac{\partial u''_m}{\partial x_i}\Big|^2.$$

Thus

$$\Big|\int_\Omega \sigma_i'\big(\frac{\partial u_m}{\partial x_i}\big)\frac{\partial u_m'}{\partial x_i}, \frac{\partial u_m''}{\partial x_i}dx\Big| \le \frac{1}{2}k^2\|u_m'\|^2 + \frac{1}{2}\|u_m''\|^2, \quad \forall m,\ \forall t\in[0,\infty).$$

Combining this inequality with (4.4), then integrating on $[0,t]$ and using estimate (4.3), we obtain

$$\begin{aligned}
&\frac{1}{2}|u_m''(t)|^2 + \frac{1}{2}\int_0^t \|u_m''(\tau)\|^2 d\tau + \frac{1}{2}|u_m''(t)|^2_{L^2(\Gamma_1)} \\
&\le \frac{1}{2}k^2 C + \frac{1}{2}|u_m''(0)|^2 + \frac{1}{2}|u_m''(0)|^2_{L^2(\Gamma_1)}, \quad \forall m,\ \forall t\in[0,\infty).
\end{aligned} \tag{4.5}$$

Next, we estimate the two last terms of the second member of (4.5).

**Third estimate.** Make $t = 0$ in the approximate equation $(4.1)_1$ and then set $w = u_m''(0)$. We find

$$\begin{aligned}
&|u_m''(0)|^2 + |u_m''(0)|^2_{L^2(\Gamma_1)} \\
&= -\sum_{i=1}^n \Big(\sigma_i\big(\frac{\partial u^0}{\partial x_i}\big), \frac{\partial u_m''(0)}{\partial x_i}\Big) - \sum_{i=1}^n \big(\frac{\partial u^1}{\partial x_i}, \frac{\partial u_m''(0)}{\partial x_i}\big).
\end{aligned} \tag{4.6}$$

Since $u^0\in H_0^1(\Omega)\cap H^2(\Omega)$, we have $\frac{\partial u^0}{\partial x_i} = \nu_i\frac{\partial u^0}{\partial \nu}$ on $\Gamma_1$. Also from (2.3), we have $\frac{\partial u^0}{\partial \nu} = 0$ on $\Gamma_1$. Then $\frac{\partial u^0}{\partial x_i} = 0$ on $\Gamma_1$ and therefore $\sigma_i\left(\frac{\partial u^0}{\partial \nu}\right) = 0$ on $\Gamma_1$. Thus by Gauss' Theorem

$$\Big(\sigma_i\big(\frac{\partial u^0}{\partial x_i}\big), \frac{\partial u_m''(0)}{\partial x_i}\Big) = -\Big(\sigma_i'\big(\frac{\partial u^0}{\partial x_i}\big)\frac{\partial^2 u^0}{\partial x_i^2}, u_m''(0)\Big).$$

This implies

$$\Big|\sum_{i=1}^n \Big(\sigma_i\big(\frac{\partial u^0}{\partial x_i}\big), \frac{\partial u_m''(0)}{\partial x_i}\Big)\Big| \le k|\triangle u^0||u_m''(0)|. \tag{4.7}$$

In a similar way, we obtain

$$\Big|\sum_{i=1}^n \big(\frac{\partial u^1}{\partial x_i}, \frac{\partial u_m''(0)}{\partial x_i}\big)\Big| \le |\triangle u^1||u_m''(0)|. \tag{4.8}$$

Taking into account (4.7) and (4.8) in (4.6), we obtain

$$|u_m''(0)|^2 + |u_m''(0)|^2_{L^2(\Gamma_1)} \le C, \quad \forall m.$$

This inequality and (4.5) provide

$$\frac{1}{2}|u_m''(t)|^2 + \frac{1}{2}\int_0^t \|u_m''(\tau)\|^2 d\tau + \frac{1}{2}|u_m''(t)|^2_{L^2(\Gamma_1)} \le C, \quad \forall m,\ \forall t\in[0,\infty). \tag{4.9}$$

By estimate (4.3) and the equality $u_m(t) = \int_0^t u_m'(\tau)d\tau + u^0$, we obtain that

$$(u_m) \text{ is bounded in } L^\infty_{\rm loc}(0,\infty; H^1_{\Gamma_0}(\Omega)).$$

This estimate, Proposition 3.1 and part (ii) imply

$$(Au_m) \text{ is bounded in } L^\infty_{\rm loc}(0,\infty; H^{-1}_{\Gamma_0}(\Omega)). \tag{4.10}$$

Estimates (4.3), (4.9)-(4.10) provide a subsequence of $(u_m)$, still denoted by $(u_m)$, and a function $u$ such that

$$\begin{gathered}
u_m \to u \text{ weak star in } L^\infty_{\rm loc}(0,\infty; H^1_{\Gamma_0}(\Omega)),\\
Au_m \to \chi \text{ weak star in } L^\infty_{\rm loc}(0,\infty; H^{-1}_{\Gamma_0}(\Omega)),\\
u'_m \to u' \text{ weak star in } L^\infty(0,\infty; L^2(\Omega)),\\
u'_m \to u' \text{ weak in } L^2(0,\infty; H^1_{\Gamma_0}(\Omega)),\\
u''_m \to u'' \text{ weak star in } L^\infty(0,\infty; L^2(\Omega)),\\
u''_m \to u'' \text{ weak in } L^2(0,\infty; H^1_{\Gamma_0}(\Omega)),\\
u'_m \to u' \text{ weak star in } L^\infty(0,\infty; L^2(\Gamma_1)),\\
u''_m \to u'' \text{ weak star in } L^\infty(0,\infty; L^2(\Gamma_1)).
\end{gathered} \tag{4.11}$$

The above convergences allow us to pass the limit in the approximate equation $(4.1)_1$ and obtain

$$\int_0^\infty (u'',z)dt + \int_0^\infty \langle \chi, z\rangle dt + \int_0^\infty ((u',z))dt + \int_0^\infty (u'',z)_{L^2(\Gamma_1)}dt = 0, \tag{4.12}$$

for all $z \in L^2_{\rm loc}(0,\infty; H^1_{\Gamma_0}(\Omega))$, $z$ with compact support.

**Convergence of $(Au_m)$.** In this part, we use the method of the monotone operator (cf. J.L. Lions [9] and Medeiros and Pereira [15]). Fix an arbitrary $T>0$. As $A$ is monotone, we have

$$\int_0^T \langle Av - Au_m, v - u_m\rangle dt \ge 0, \quad \forall v \in L^1(0,T; H^1_{\Gamma_0}(\Omega)).$$

Then by convergence (4.11), we find that

$$\int_0^T \langle Av, v-u\rangle dt - \int_0^T \langle \chi, v\rangle dt + \limsup \int_0^T \langle Au_m, u_m\rangle dt \ge 0. \tag{4.13}$$

By the approximate equation $(4.1)_1$, we obtain

$$\begin{aligned}
&\int_0^T \langle Au_m, u_m\rangle dt\\
&= -(u'_m(T), u_m(T)) + (u^1,u^0) + \int_0^T |u'_m|^2 dt - \frac{1}{2}\|u_m(T)\|^2 + \frac{1}{2}\|u^0\|^2\\
&\quad - (u'_m(T), u_m(T))_{L^2(\Gamma_1)} + (u^1,u^0)_{L^2(\Gamma_1)} + \int_0^T |u'_m|^2_{L^2(\Gamma_1)} = 0.
\end{aligned} \tag{4.14}$$

Now we will find the limit of first and third term of the second member of the last equality. By convergences $(4.11)_1$, $(4.11)_3$, the compact embedding of $H^1_{\Gamma_0}(\Omega)$ in $L^2(\Omega)$ and the Aubin-Lions Compactness Theorem, we have

$$u_m(T) \to u(T) \quad \text{in } L^2(\Omega).$$

Note that convergences $(4.11)_3$ and $(4.11)_5$ provide

$$u'_m(T) \to u'(T) \quad \text{weak in } L^2(\Omega).$$

Convergences $(4.11)_4$ and $(4.11)_5$ and the compactness embedding of $H^1_{\Gamma_0}(\Omega)$ in $L^2(\Omega)$ imply

$$u'_m \to u' \quad \text{in } L^2(0,T; L^2(\Omega)).$$

The above three convergences provide

$$-(u_m'(T),u_m(T))+\int_0^T |u_m'(t)|^2dt \to -(u'(T),u(T))+\int_0^T |u'(t)|^2dt. \tag{4.15}$$

On the other hand, convergences $(4.11)_1$ and $(4.11)_3$ imply

$$u_m(T)\to u(T) \quad \text{weak in } H^1_{\Gamma_0}(\Omega).$$

Thus

$$\limsup\big(-\frac{1}{2}\|u_m(T)\|^2\big)\le -\frac{1}{2}\|u(T)\|^2. \tag{4.16}$$

By convergences $(4.11)_1$, $(4.11)_4$ and noting that the embedding of $H^{1/2}(\Gamma_1)$ in $L^2(\Gamma_1)$ is compact, we obtain

$$u_m(T)\to u(T) \quad \text{in } L^2(\Gamma_1).$$

Also $(4.11)_7$ and $(4.11)_8$ imply

$$u_m'(T)\to u'(T) \quad \text{weak in } L^2(\Gamma_1)$$

and $(4.11)_4$, $(4.11)_8$ imply

$$u_m'\to u' \quad \text{in } L^2(0,T;L^2(\Gamma_1)).$$

The las two convergences provide

$$\begin{aligned} &-(u_m'(T),u_m(T))_{L^2(\Gamma_1)}+\int_0^T |u_m'(t)|^2_{L^2(\Gamma_1)}dt\\ &\to -(u'(T),u(T))_{L^2(\Gamma_1)}+\int_0^T |u'(t)|^2_{L^2(\Gamma_1)}dt. \end{aligned} \tag{4.17}$$

From (4.14), (4.15), (4.16) and (4.17) we obtain

$$\begin{aligned} &\limsup\int_0^T \langle Au_m,u_m\rangle dt\\ &\le -(u'(T),u(T))+(u^1,u^0)+\int_0^T |u'(t)|^2dt-\frac{1}{2}\|u(T)\|^2\\ &\quad+\frac{1}{2}\|u^0\|^2-(u'(T),u(T))_{L^2(\Gamma_1)}+(u^1,u^0)_{L^2(\Gamma_1)}+\int_0^T |u'(t)|^2_{L^2(\Gamma_1)}dt. \end{aligned} \tag{4.18}$$

Make $z=u1_{(0,T)}$ in (4.12), where $1_{(0,T)}$ is the characteristic function of the interval $(0,T)$. We obtain

$$\begin{aligned} \int_0^T\langle\chi,u\rangle dt=&-(u'(T),u(T))+(u^1,u^0)+\int_0^T |u'(t)|^2dt-\frac{1}{2}\|u(T)\|^2+\frac{1}{2}\|u^0\|^2\\ &-(u'(T),u(T))_{L^2(\Gamma_1)}+(u^1,u^0)_{L^2(\Gamma_1)}+\int_0^T |u'(t)|^2_{L^2(\Gamma_1)}dt. \end{aligned}$$

Comparing this equality with (4.18), we derive

$$\limsup\int_0^T\langle Au_m,u_m\rangle dt\le\int_0^T\langle\chi,u\rangle dt.$$

Taking into account the last inequality in (4.13), we find

$$\int_0^T\langle Av,v-u\rangle dt-\int_0^T\langle\chi,v-u\rangle dt\ge 0,\quad \forall v\in L^1(0,T;H^1_{\Gamma_0}(\Omega)).$$

This inequality and the hemicontinuity of $A$ provide

$$\chi = Au \quad \text{in } L^\infty(0,T;H^{-1}_{\Gamma_0}(\Omega)).$$

By diagonalization process and noting that $T > 0$ was arbitrary, this equality implies

$$\chi = Au \quad \text{in } L^\infty_{\rm loc}(0,\infty;H^{-1}_{\Gamma_0}(\Omega)).$$

Thus equation (4.12) becomes

$$\int_0^\infty (u'',z)dt + \int_0^\infty \langle Au,z\rangle dt + \int_0^\infty ((u'',z))dt + \int_0^\infty (u'',z)_{L^2(\Gamma_1)}dt = 0, \quad (4.19)$$

for all $z \in L^2_{\rm loc}(0,\infty;H^1_{\Gamma_0}(\Omega))$, $z$ with compact support.

Taking $z \in \mathcal{D}(\Omega\times(0,\infty))$ in (4.19) and noting that $u''$ belongs to $L^2(0,\infty;H^1_{\Gamma_0})$, we obtain equation (2.5).

Consider $f = (f_1, f_2, \dots, f_n)$, where

$$f_i = \sigma_i\big(\frac{\partial u}{\partial x_i}\big) + \frac{\partial u'}{\partial x_i}, \quad i = 1,2,\dots,n.$$

Then by (2.5) we obtain $f \in [L^2_{\rm loc}(0,\infty;L^2(\Omega))]^n$ and $\operatorname{div} f \in L^2_{\rm loc}(0,\infty;L^2(\Omega))$. Therefore by Proposition 3.2, we find $\gamma_\nu f \in L^2_{\rm loc}(0,\infty;H^{-1/2}(\Gamma_1))$.

Multiply both sides of (2.5) by $z$, $z \in L^2_{\rm loc}(0,\infty;H^1_{\Gamma_0}(\Omega))$ of compact support, and then integrate. We obtain

$$\int_0^\infty (u'',z)dt + \int_0^\infty \langle Au,z\rangle dt + \int_0^\infty ((u'',z))dt - \int_0^\infty \langle \gamma_\nu f,\gamma_0 z\rangle dt = 0.$$

On the other hand, equation (4.19) implies

$$\int_0^\infty (u'',z)dt + \int_0^\infty \langle Au,z\rangle dt + \int_0^\infty ((u'',z))dt + \int_0^\infty (u'',z)_{L^2(\Gamma_1)}dt = 0.$$

Comparing the last two equations, we obtain

$$\gamma_\nu f + u'' = 0 \quad \text{in } L^2_{\rm loc}(0,\infty;L^2(\Gamma_1)).$$

Then the regularity of $u''$ given by $(4.11)_6$, allows us to obtain equation (2.6).

Convergence (4.11) say us that $u$ belong to class (2.4). The verification of the initial conditions (2.7) follows by convergences (4.11). Thus the proof of the existence of solutions is concluded.

**Uniqueness.** Let $u$ and $v$ be in the class (2.4) that satisfy (2.5)-(2.7). Consider $w = u - v$. Introduce the notation

$$B_i u = \sigma_i\big(\frac{\partial u}{\partial x_i}\big) + \frac{\partial u'}{\partial x_i}, \quad i = 1,2,\dots,n.$$

For short notation, we write $\sum$ instead of $\sum_{i=1}^n$. By equation (2.5), we obtain

$$(w'',w') - \Big(\sum \frac{\partial}{\partial x_i} B_i u - \sum \frac{\partial}{\partial x_i} B_i v, w'\Big) = 0$$

Then, by Proposition 3.2 and (2.6), we have

$$(w'',w') + \sum \Big(\big[\sigma_i\big(\frac{\partial u}{\partial x_i}\big) - \sigma_i\big(\frac{\partial v}{\partial x_i}\big)\big], \frac{\partial w'}{\partial x_i}\Big) + \|w'\|^2 + \int_{\Gamma_1} w''w' d\Gamma = 0,$$

that is,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|w'|^2 + \|w'\|^2 + \frac{1}{2}\frac{d}{dt}|w'|^2_{L^2(\Gamma_1)} \\ &= -\sum \Big[\Big(\sigma_i(\frac{\partial u}{\partial x_i}) - \sigma_i(\frac{\partial v}{\partial x_i}), \frac{\partial w'}{\partial x_i}\Big)\Big]. \end{aligned} \tag{4.20}$$

Modifying the last term of this expression, we have

$$\sum \Big[\Big(\sigma_i(\frac{\partial u}{\partial x_i}) - \sigma_i(\frac{\partial v}{\partial x_i}), \frac{\partial w'}{\partial x_i}\Big)\Big] \le k \sum |\frac{\partial w}{\partial x_i}||\frac{\partial w'}{\partial x_i}| \le k\|w\|\|w'\|.$$

Fix an arbitrary real number $T > 0$. Taking into account the last inequality in (4.20) and then integrating on $[0,s], 0 < s \le T$, we obtain

$$\frac{1}{2}|w'(s)|^2 + \int_0^s \|w'(\tau)\|^2 d\tau + \frac{1}{2}|w'(s)|^2_{L^2(\Gamma_1)} \le k \int_0^s \|w(\tau)\|\|w'(\tau)\| d\tau. \tag{4.21}$$

From the equality $w(\tau) = \int_0^\tau w'(\sigma)d\sigma$, we derive

$$\|w(\tau)\|^2 \le \tau \int_0^\tau \|w'(\sigma)\|^2 d\sigma.$$

Thus by using this inequality and Cauchy-Schwarz inequality in (4.21), we derive

$$\frac{1}{2}|w'|^2 + \int_0^s \|w'(\tau)\|^2 d\tau + \frac{1}{2}|w'|^2_{L^2(\Gamma_1)} \le ks \int_0^s \|w'(\tau)\|^2 d\tau.$$

Choose $0 < s_0 \le T$ such that $ks_0 \le 1$. Then the last inequality implies

$$\frac{1}{2}|w'(s)|^2 + \frac{1}{2}|w'(s)|^2_{L^2(\Gamma_1)} \le 0, \quad \text{for } 0 \le s \le s_0.$$

Thus,

$$w(s) = 0, \quad w'(s) = 0, \quad \forall s \in [0, s_0].$$

We apply the above arguments to the interval $[s_0, T]$. Since $s_0$ does not depend on $T$, we obtain

$$w(s) = 0, \quad w'(s) = 0, \quad \forall s \in [s_0, 2s_0].$$

After a finite number of steps, we prove that $w(t) = 0$ for all $t \in [0, T]$. As $T > 0$ was arbitrary, we conclude that $u = v$ on $[0, \infty)$.

## 5. Proof of Theorem 2.2

Let $u$ be the solution given by Theorem 2.1. Multiplying both sides of equation (2.5) by $u'$, we obtain

$$\frac{d}{dt}\Big[\frac{1}{2}|u'(t)|^2 + \sum_{i=1}^n \int_\Omega \widehat{\sigma}_i(\frac{\partial u}{\partial x_i})dx + \frac{1}{2}|u'(t)|^2_{L^2(\Gamma_1)}\Big] = -\|u'(t)\|^2;$$

that is,

$$\frac{d}{dt}E(t) = -\|u'(t)\|^2. \tag{5.1}$$

Also multiply both sides of equation (2.5) by $u$. We find

$$\begin{aligned} &\frac{d}{dt}\Big[(u'(t), u(t)) + \frac{1}{2}\|u(t)\|^2 + (u'(t), u(t))_{L^2(\Gamma_1)}\Big] \\ &= |u'(t)|^2 - \sum_{i=1}^n \int_\Omega \sigma_i(\frac{\partial u(t)}{\partial x_i})\frac{\partial u(t)}{\partial x_i} dx + |u'(t)|^2_{L^2(\Gamma_1)}; \end{aligned}$$

that is,

$$\frac{d}{dt}\rho(t) = |u'(t)|^2 - \sum_{i=1}^{n}\int_\Omega \sigma_i\big(\frac{\partial u(t)}{\partial x_i}\big)\frac{\partial u(t)}{\partial x_i}dx + |u'(t)|^2_{L^2(\Gamma_1)}, \tag{5.2}$$

where

$$\rho(t) = (u'(t), u(t)) + \frac{1}{2}\|u(t)\|^2 + (u'(t), u(t))_{L^2(\Gamma_1)}, \quad t \ge 0.$$

Consider $\varepsilon > 0$. We introduce the perturbed energy

$$E_\varepsilon(t) = E(t) + \varepsilon\rho(t), \quad t \ge 0.$$

**Relation between $E_\varepsilon(t)$ and $E(t)$.** We have

$$|E_\varepsilon(t) - E(t)| = \varepsilon|\rho(t)|. \tag{5.3}$$

We obtain

$$|\rho(t)| \le \frac{1}{2}|u'(t)|^2 + \frac{1}{2}(a_1 + 1 + a_2)\|u(t)\|^2 + \frac{1}{2}|u(t)|^2_{L^2(\Gamma_1)}, \quad t \ge 0,$$

where $a_1$ and $a_2$ were introduced in (2.8). Then by hypothesis (2.9), we have

$$|\rho(t)| \le \frac{1}{2}|u'(t)|^2 + d\sum_{i=1}^{n}\int_\Omega \widehat{\sigma}_i\big(\frac{\partial u(t)}{\partial x_i}\big)dx + \frac{1}{2}|u(t)|^2_{L^2(\Gamma_1)}.$$

Consider $\varepsilon_0 = \min\{\frac{1}{2}, \frac{1}{2d}\}$. Then

$$\varepsilon|\rho(t)| \le \frac{1}{2}E(t), \quad \forall 0 < \varepsilon \le \varepsilon_0. \tag{5.4}$$

From (5.3) and (5.4) it follows that

$$\frac{1}{2}E(t) \le E_\varepsilon(t) \le \frac{3}{2}E(t), \quad \forall t \ge 0,\ \forall 0 < \varepsilon \le \varepsilon_0. \tag{5.5}$$

**Boundedness of $E'_\varepsilon(t)$.** From (5.1) and (5.2), we obtain

$$\begin{aligned} E'_\varepsilon(t) = -\|u'(t)\|^2 + \varepsilon\Big[|u'(t)|^2 - \sum_{i=1}^{n}\int_\Omega \sigma_i\big(\frac{\partial u(t)}{\partial x_i}\big)\frac{\partial u(t)}{\partial x_i}\,dx \\ + |u'(t)|^2_{L^2(\Gamma_1)}\Big]. \end{aligned} \tag{5.6}$$

By (2.8) we deduce that

$$-\|u'(t)\|^2 \le -\frac{1}{2a_1}|u'(t)|^2 - \frac{1}{2a_2}|u'(t)|^2_{L^2(\Gamma_1)}. \tag{5.7}$$

Since $\sigma_i$ is an increasing continuous function, we have

$$\widehat{\sigma}_i(s) \le s\sigma_i(s), \quad \forall s \in \mathbb{R}.$$

Thus

$$-\sum\int_\Omega \sigma_i\big(\frac{\partial u(t)}{\partial x_i}\big)\frac{\partial u(t)}{\partial x_i}dx \le -\sum_{i=1}^{n}\int_\Omega \widehat{\sigma}_i\big(\frac{\partial u(t)}{\partial x_i}\big)dx. \tag{5.8}$$

Taking into account (5.7) and (5.8) in (5.6), we have

$$E'_\varepsilon(t) \le -\big(\frac{1}{2a_1} - \varepsilon\big)|u'(t)|^2 - \varepsilon\sum_{i=1}^{n}\int_\Omega \widehat{\sigma}_i\big(\frac{\partial u(t)}{\partial x_i}\big)dx - \big(\frac{1}{2a_2} - \varepsilon\big)|u'(t)|^2_{L^2(\Gamma_1)}.$$

Take $\varepsilon_1 = \min\{\frac{1}{3a_1}, \frac{1}{3a_2}\}$. Then the above inequality implies

$$E_\varepsilon'(t) \le -\frac{\varepsilon}{2}|u'(t)|^2 - \varepsilon\sum_{i=1}^n \int_\Omega \widehat{\sigma_i}\big(\frac{\partial u(t)}{\partial x_i}\big)dx - \frac{\varepsilon}{2}|u'(t)|^2_{L^2(\Gamma_1)};$$

that is,

$$E_\varepsilon'(t) \le -\varepsilon E(t), \quad \text{for } 0 < \varepsilon \le \varepsilon_1. \tag{5.9}$$

Consider $\eta > 0$ defined in (2.12). Then by (5.9) and (5.5), we obtain

$$E_\eta'(t) \le -\frac{2\eta}{3}E_\eta(t),$$

and therefore

$$E_\eta(t) \le E_\eta(0)\exp\big(-\frac{2}{3}\eta t\big).$$

This inequality and (5.5) provide inequality (2.13).

## 6. Examples

In what follows we will give examples of functions that satisfy the hypotheses considered in Section 1. Consider real numbers $p$ and $L_i$ with $p \ge 1$ and $L_i > 1$. The function

$$\sigma_i(s) = \begin{cases} L_i^p s, & s > L_i \\ |s|^p s, & -L_i \le s \le L_i \\ L_i^p s, & s < -L_i. \end{cases}$$

satisfies hypothesis (2.1). The function

$$\sigma_i(s) = \begin{cases} L_i^p s, & s > L_i \\ |s|^p s, & 1 < s \le L_i \\ s, & -1 \le s \le 1 \\ |s|^p s, & -L_i \le s < -1 \\ L_i^p s, & s < -L_i \end{cases}$$

satisfies hypotheses (2.1) and (2.9).

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