Electronic Journal of Differential Equations, Vol. 2020 (2020), No. 102, pp. 1–25.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

OPTIMAL TIME DECAY RATES FOR THE COMPRESSIBLE NAVIER-STOKES SYSTEM WITH AND WITHOUT YUKAWA-TYPE POTENTIAL

QING CHEN, GUOCHUN WU, YINGHUI ZHANG, LAN ZOU

Communicated by Hongjie Dong

Abstract. We consider the time decay rates of smooth solutions to the Cauchy problem for the compressible Navier-Stokes system with and without a Yukawa-type potential. We prove the existence and uniqueness of global solutions by the standard energy method under small initial data assumptions. Furthermore, if the initial data belong to $L^1(\mathbb{R}^3)$, we establish the optimal time decay rates of the solution as well as its higher-order spatial derivatives. In particular, we obtain the optimal decay rates of the highest-order spatial derivatives of the velocity. Finally, we derive the lower bound time decay rates for the solution and its spacial derivatives.

1. Introduction

We consider the Cauchy problem of the compressible Navier-Stokes system with and without the Yukawa-type potential term in the whole space $\mathbb{R}^3$,

$$
\begin{gathered}
\rho_t + \operatorname{div}(\rho u) = 0, \\
(\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) + \gamma \rho \nabla \psi = \mu \Delta u + (\mu + \nu) \nabla \operatorname{div} u, \\
-\Delta \psi + \psi = \rho - 1, \\
(\rho, u)|_{t=0} = (\rho_0, u_0) \to (1, 0) \quad \text{as } |x| \to \infty.
\end{gathered} \tag{1.1}
$$

Here $\rho = \rho(t,x)$, $u = u(t,x)$, $P = P(t,x)$ and $\psi(t,x)$ represent the density, the velocity vector field of the fluid, the pressure and the potential force exerted in the fluid respectively, at time $t \geq 0$ and position $x \in \mathbb{R}^3$. The Lamé coefficients μ and ν satisfy $\mu > 0$ and $\frac{2}{3}\mu + \nu > 0$. And the constant $\gamma \in \mathbb{R}$ may be arbitrary and it is essential on its sign. When $\gamma = 0$, (1.1) reduces to compressible Navier-Stokes system, which describes the motion of a barotropic viscous compressible flow. When $\gamma \neq 0$, (1.1) becomes the compressible Navier-Stokes equations with a Yukawa-potential, which is a simplified hydrodynamical model describing the nuclear matter [3, 8]. In this paper, we consider the compressible Navier-Stokes system with and without a Yukawa-potential. For technical consideration, we assume that the constant $\gamma \geq 0$ and the pressure P is some smooth function depending only on ρ and $P'(\rho) > 0$.

2010 *Mathematics Subject Classification.* 35Q30, 76N15.
Key words and phrases. Compressible flow; energy method; optimal decay rates.

Submitted February 2, 2020. Published September 29, 2020.

There are many important investigations on large time behavior of the solutions to the compressible Navier-Stokes system in multi-dimensional space, see [5, 10, 11, 12, 18, 20, 21, 23, 28, 27] and references therein. Matsumura and Nishida [20, 21] first proved the existence of the small global solutions in $H^3(\mathbb{R}^3)$ to compressible Navier-Stokes equations, and particularly, for the initial perturbation small in $L^1(\mathbb{R}^3)\cap H^3(\mathbb{R}^3)$, by employing time-decay properties of the linear system, the authors in [20] obtained the decay rate of the solution in L^2-norm:

$$\|(\rho-1,u)\|_{L^2(\mathbb{R}^3)}\leq C(1+t)^{-3/4},$$

which is the same as for the heat equation with initial data in $L^1(\mathbb{R}^3)$. Later, for small initial disturbance in $H^l(\mathbb{R}^d)\cap W^{l,1}(\mathbb{R}^d)$ with $l\geq 4$, Ponce [23] gave the optimal L^p $(p\geq 2)$ decay rates of the solutions and their first and second order derivatives. For the initial perturbation in $H^3(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)$ with $1\leq p<\frac{6}{5}$, Duan, Liu, Ukai and Yang [6] proved the optimal convergence rates of the solution in L^q-norm with $2\leq q\leq 6$ and its first order derivative in L^2-norm. For the small data (ρ_0-1,m_0) with the momentum $m=\rho u$ in $H^l(\mathbb{R}^3)\cap \dot{B}^{-s}_{1,\infty}(\mathbb{R}^3)$ with $l\geq 4$ and $0\leq s\leq 1$, Li and Zhang [17] studied the Cauchy problem for compressible Navier-Stokes system (1.1) with $\gamma=0$ and obtained the following decay rate

$$\|(\rho-1,m)\|_{L^2(\mathbb{R}^3)}\leq C(1+t)^{-(\frac{3}{4}+\frac{s}{2})}.$$

Moreover, they established the lower bound of the time-decay rate for the global solution. For the small initial perturbation in $\|(\rho_0-1,u_0)\|_{H^l(\mathbb{R}^3)}$ with $l\geq 3$ and the data bounded in $\dot{H}^{-s}(\mathbb{R}^3)$ with $s\in[0,\frac{3}{2})$, instead of resorting to the decay properties of the linear system, Guo and Wang [10] developed a general energy method and obtained the optimal decay rates of the higher-order spatial derivatives of solutions

$$\|\nabla^k(\rho-1,u)\|_{L^2(\mathbb{R}^3)}\leq C(1+t)^{-\frac{k+s}{2}},\quad 1\leq k\leq l-1.$$

Recently, Danchin and Xu [5] studied the L^p decay rates of the global solutions in the critical L^p framework and Xin and Xu [27] improved the result by removing some low frequencies conditions. Under the discontinuous initial data assumption, Hu and Wu [13] established the optimal convergence rates of the solutions with low regularity in L^p-norm with $2\leq p\leq\infty$ and of the first order derivative of the velocity in L^2-norm.

Compared with the compressible Navier-Stokes equations, there are few results on the system (1.1) with $\gamma\neq 0$. For instance, Chikami [3] studied the existence and uniqueness of the solution in the critical space, and they also developed a blow-up criterion of the solution. In this paper, we first establish the global existence of the smooth solutions for the system (1.1), and then we will continue to address the optimal decay rates for the solutions. In particular, we can derive the optimal decay rate on the highest-order derivatives of the velocity.

The system (2.1) can be rewritten as

$$\begin{gathered}U_t=DU+\mathcal{N},\\ U|_{t=0}=U_0,\end{gathered}$$

with the solution $U=(\rho-1,u)$ and the matrix-valued differential operator D has the form

$$D=\begin{pmatrix}0 & -\operatorname{div}\\ -P'(1)\nabla-\gamma\nabla(1-\Delta)^{-1} & \mu\Delta+(\mu+\nu)\nabla\operatorname{div}\end{pmatrix}.$$

Thus the solution can be expressed as

$$U(t) = E(t)U_0 + \int_0^t E(t-\tau)\mathcal{N}(U(\tau))d\tau,$$

where $E(t) = e^{tD}$ is the solution semigroup. Due to the estimates on the semigroup and the energy estimates on the solution to the nonlinear problem (cf. [6, 17, 20]), it is difficult to show that the optimal decay rate of the high-order derivatives of the solution since the nonlinear term involves the derivatives. To improve the known results, motivated by [2, 4], we introduce Hodge decomposition to analyze the system (1.1). Hence the second equation of (1.1) can be divided into two systems. One is a mere heat equation on the "incompressible part", whose decay rate is exponential; another one is a mixed system, which seems more complicated because of the nonlinear term. Fortunately, we find that the solution semigroup of the mixed system also keep good properties (the detail can be seen in the proof of Proposition 3.3). As a result, we can eventually derive the optimal decay rate of the highest order derivatives of the velocity.

Notation. In this paper, ∇^ℓ with an integer $\ell \geq 0$ stands for the usual any spatial derivatives of order ℓ. We use $L^p(\mathbb{R}^3)$ with $1 \leq p \leq \infty$ to denote the usual L^p spaces with norm $\|\cdot\|_{L^p}$, and $H^s(\mathbb{R}^3)$ to denote the usual Sobolev spaces with norm $\|\cdot\|_{H^s}$. Furthermore, we use $\dot{H}^s(\mathbb{R}^3)$ to denote the homogenous Sobolev spaces with norm $\|\cdot\|_{\dot{H}^s}$ defined as

$$\|f\|_{\dot{H}^s} =: \|\Lambda^s f\|_{L^2} = \||\xi|^s \hat{f}\|_{L^2}$$

with $s \in \mathbb{R}$ and here Λ^s is a Riesz potential operator of order s. We will employ the notation $a \lesssim b$ to mean that $a \leq Cb$ for a universal constant $C > 0$ that only depends on the parameters coming from the problem. And $C_i (i = 1, 2, 3, 4)$ will also denote some positive constants depending only on the given problems.

Our main results are stated in the following theorems.

Theorem 1.1. *Assume that $\|(\rho_0 - 1, u_0)\|_{H^l}$ with an integer $l \geq 3$ is sufficiently small. Then there exists a unique global solution $(\rho(t,x), u(t,x), \psi(t,x))$ to the initial value problem (1.1) such that*

$$\begin{aligned} &\|(\rho-1,u)(t)\|_{H^l}^2 + \|\psi(t)\|_{H^{l+2}}^2 \\ &+ \int_0^t (\|\nabla\rho(\tau)\|_{H^{l-1}}^2 + \|\nabla u(\tau)\|_{H^l}^2 + \|\nabla\psi(\tau)\|_{H^{l+1}}^2) d\tau \\ &\lesssim \|(\rho_0 - 1, u_0)\|_{H^l}^2. \end{aligned} \tag{1.2}$$

If further $\|(\rho_0 - 1, u_0)\|_{L^1} < +\infty$, then for $k = 0, 1, \dots, l-1$,

$$\|\nabla^k(\rho-1)(t)\|_{L^2} + \|\nabla^k\psi(t)\|_{H^2} \lesssim (1+t)^{-(\frac{3}{4}+\frac{k}{2})}, \tag{1.3}$$

and for $k = 0, 1, \dots, l$,

$$\|\nabla^k u(t)\|_{L^2} \lesssim (1+t)^{-(\frac{3}{4}+\frac{k}{2})}. \tag{1.4}$$

Remark 1.2. The results in Theorem 1.1 indicate that the optimal time decay rates are same for the solutions of the Navier-Stokes system and what with a Yukawa-type potential.

It is worth noting that the optimal time decay rates of the highest-order spatial derivatives of the velocity are obtained. By comparison, the optimal time decay rates of the compressible Navier-Stokes equations and Navier-Stokes-Poisson equations, see [6, 7, 10, 15, 16, 26] and the references therein for instance, can be obtained but except the highest-order one. This is because of the decomposition on the system.

To obtain the optimal time decay rates of the higher-order derivatives of the solution, we can represent the spatial derivatives of the solutions to the equation $U_t = BU + G$ with the initial data $U|_{t=0} = U_0$ which follows from the Duhamel principle as (cf. [25])

$$\nabla^k U = \nabla^k S(t)U_0 + \int_0^{t/2} \nabla^k S(t-\tau)G(\tau)d\tau + \int_{t/2}^t \nabla^r S(t-\tau)\nabla^{k-r}G(\tau)d\tau, \tag{1.5}$$

where $S(t) := e^{tB}$ is the solution semigroup and $0 \le r \le k$.

Note that the decay rates for the solutions and their derivatives above are optimal. Indeed, we shall establish the lower bound of the time decay rates for the global solution.

Theorem 1.3. *Besides the assumptions of Theorem 1.1, assume that the Fourier transform* $(\mathfrak{F}[\rho_0 - 1], \mathfrak{F}[m_0])$ *with* $m_0 = \rho_0 u_0$ *satisfies* $|\mathfrak{F}[\rho_0 - 1](\xi)| > c_0 K_0$, *and* $\mathfrak{F}[m_0](\xi) = 0$ *for* $0 \le |\xi| \ll 1$ *with a positive constant* c_0 *and* $K_0 = \|(\rho_0 - 1, u_0)\|_{L^1\cap H^l}$. *Then, the global solution* (ρ, u, ψ) *given by Theorem 1.1 satisfies for* $t \ge t_0$ *with* $t_0 > 0$ *a sufficiently large time such that for* $k = 0, \dots, l-1$,

$$\begin{aligned} c_1 K_0 (1+t)^{-(\frac{3}{4}+\frac{k}{2})} &\le \min\left\{\|\nabla^k(\rho-1)(t)\|_{L^2}, \|\nabla^k u(t)\|_{L^2}, \|\nabla^k \psi(t)\|_{H^2}\right\} \\ &\le C(1+t)^{-(\frac{3}{4}+\frac{k}{2})}, \end{aligned} \tag{1.6}$$

and

$$c_1 K_0 (1+t)^{-(\frac{3}{4}+\frac{k}{2})} \le \|\nabla^l u(t)\|_{L^2} \le C(1+t)^{-(\frac{3}{4}+\frac{k}{2})}, \tag{1.7}$$

where c_1 *is a positive constant independent of time.*

Remark 1.4. Compared to the lower bound of the time-decay rate for the solution obtained in [17], we can also get the lower-time-decay-rate for the high-order derivatives of the solution as well as the highest-order derivatives of the velocity.

The rest of this article is organized as follows. In Section 2, we reformulate the problem and state the equivalent theorem and propositions. In Section 3, we use the decomposition of the momentum to analyze the linearized system and establish the linear L^2 decay estimates. In Section 4, we prove the global existence Proposition 2.2. In Section 5, we prove the optimal time decay rates Proposition 2.3 and the lower time decay rates Proposition 2.5 respectively.

2. Reformulated system

Denoting $\varrho = \rho - 1$, then we rewrite (1.1) in a perturbation form as

$$\begin{gathered} \varrho_t + \operatorname{div} u = N_1, \\ u_t + P'(1)\nabla\varrho + \gamma\nabla\psi - \mu\Delta u - (\mu+\nu)\nabla \operatorname{div} u = N_2, \\ -\Delta\psi + \psi = \varrho, \\ (\varrho, u)|_{t=0} = (\varrho_0, u_0) = (\rho_0 - 1, u_0), \end{gathered} \tag{2.1}$$

where the nonlinear terms are

$$N_1 = -\operatorname{div}(\varrho u), \tag{2.2}$$

$$N_2 = -u \cdot \nabla u - \big(\frac{P'(\rho)}{\rho} - P'(1)\big)\nabla\varrho - \frac{\mu}{\rho}\varrho\Delta u - \frac{\mu+\nu}{\rho}\varrho\nabla \operatorname{div} u. \tag{2.3}$$

For any $T > 0$, we define the solution space by

$$\begin{aligned} X(0,T) = \Big\{&(\varrho,u,\psi) : \varrho \in C^0(0,T;H^l(\mathbb{R}^3)) \cap C^1(0,T;H^{l-1}(\mathbb{R}^2)),\\ &u \in C^0(0,T;H^l(\mathbb{R}^3)) \cap C^1(0,T;H^{l-2}(\mathbb{R}^2)),\\ &\psi \in C^0(0,T;H^{l+2}(\mathbb{R}^3)) \cap C^1(0,T;H^{l+1}(\mathbb{R}^2)),\\ &\nabla\varrho \in L^2(0,T;H^{l-1}(\mathbb{R}^3)), \nabla u \in L^2(0,T;H^l(\mathbb{R}^3)),\\ &\nabla\psi \in L^2(0,T;H^{l+1}(\mathbb{R}^3))\Big\}, \end{aligned} \tag{2.4}$$

and the solution norm by

$$\begin{aligned} \chi^2(T) = \sup_{0\le t\le T}\Big\{&\|(\varrho,u)(t)\|_{H^l}^2 + \|\psi\|_{H^{l+2}}^2\\ &+ \int_0^T (\|\nabla\varrho(t)\|_{H^{l-1}}^2 + \|\nabla u(t)\|_{H^l}^2 + \|\nabla\psi(t)\|_{H^{l+1}}^2)dt\Big\}. \end{aligned} \tag{2.5}$$

We now state a local existence theorem for the system (2.1), which can be established by a standard contraction mapping argument; we may refer to [15].

Proposition 2.1 (local existence)**.** *Let $(\varrho_0, v_0) \in H^l(\mathbb{R}^3)$ for an integer $l \ge 3$ and*

$$\inf_{x\in\mathbb{R}^3}\{\varrho_0 + 1\} > 0. \tag{2.6}$$

Then there exists a positive constant T_0 depending on $\chi(0)$ such that the problem (2.1) *has a unique solution $(\varrho,u,\psi) \in X(0,T_0)$ satisfying*

$$\inf_{x\in\mathbb{R}^3, 0\le t\le T_0}\{\varrho(t,x)+1\} > 0 \quad \text{and} \quad \chi(T_0) \le 2\chi(0). \tag{2.7}$$

It is easy to check that the global existence part of Theorem 1.1 is equivalent to the following proposition.

Proposition 2.2 (Global existence)**.** *Assume $\|(\varrho_0,u_0)\|_l$ with an integer $l \ge 3$ is sufficiently small. Then there exists a unique global solution $(\rho(t,x), u(t,x), \psi(t,x))$ to the initial value problem* (2.1) *such that*

$$\begin{aligned} &\|(\varrho,u)(t)\|_{H^l}^2 + \|\psi(t)\|_{H^{l+2}}^2\\ &+ \int_0^t (\|\nabla\varrho(\tau)\|_{H^{l-1}}^2 + \|\nabla u(\tau)\|_{H^l}^2 + \|\nabla\psi(\tau)\|_{H^{l+1}}^2)d\tau \le C\|(\varrho_0,u_0)\|_l^2. \end{aligned} \tag{2.8}$$

To obtain the lower time decay rate for the system (1.1), we consider the following linearized system, which is equivalent to (1.1):

$$\begin{gathered} \varrho_t + \operatorname{div} m = 0,\\ m_t + P'(1)\nabla\varrho + \gamma\nabla\psi - \mu\Delta m - (\mu+\nu)\nabla \operatorname{div} m = N,\\ -\Delta\psi + \psi = \varrho,\\ (\varrho,m)|_{t=0} = (\varrho_0,m_0) = (\rho_0 - 1, \rho_0 u_0), \end{gathered} \tag{2.9}$$

where

$$\begin{aligned} N =: \operatorname{div} F =: \operatorname{div} \Big(& (-P(1+\varrho) + P(1) + P'(1)\varrho) I_3 \\ & + \gamma \nabla\psi \otimes \nabla\psi - \frac{\gamma}{2}(|\psi|^2 + |\nabla\psi|^2) I_3 - \frac{m \otimes m}{1+\varrho} \\ & - \mu \nabla\big(\frac{\varrho m}{1+\varrho}\big) - (\mu+\nu) \operatorname{div}\big(\frac{\varrho m}{1+\varrho}\big) I_3 \Big). \end{aligned} \tag{2.10}$$

For simplicity, we will also use the system (2.9) to analyze the upper decay rate for the solutions in the following proposition.

Proposition 2.3 (Optimal time-decay-rate)**.** *Under the assumptions in Proposition 2.2, and that* $\|(\varrho_0, u_0)\|_{L^1} < +\infty$*, for* $k = 0, \dots, l-1$*, we have*

$$\|\nabla^k \varrho(t)\|_{L^2} + \|\nabla^k \psi(t)\|_{H^2} \lesssim (1+t)^{-(\frac{3}{4}+\frac{k}{2})}, \tag{2.11}$$

$$\|\nabla^l \varrho(t)\|_{L^2} + \|\nabla^{l+2} \psi(t)\|_{L^2} \lesssim (1+t)^{-(\frac{3}{4}+\frac{l-1}{2})}, \tag{2.12}$$

and for $k = 0, \dots, l$*, we have*

$$\|\nabla^k m(t)\|_{L^2} \lesssim (1+t)^{-(\frac{3}{4}+\frac{k}{2})}. \tag{2.13}$$

Remark 2.4. Regarding Proposition 2.3 we have the following observations.

- $\|(\varrho_0, u_0)\|_{L^1} < +\infty$ and $\|\rho_0\|_{L^\infty} < +\infty$ imply $\|m_0\|_{L^1} < +\infty$ (cf. [9]).
- By (2.11) and (2.13), we can check that under the assumptions of Proposition 2.3, (1.4) holds for $k = 0, \dots, l$.

Proposition 2.5 (lower-time decay rate)**.** *Assume that the conditions in Propositions 2.2 and 2.3 hold,* $|\mathfrak{F}[\varrho_0](\xi)| > c_0 K_0$*, and* $\mathfrak{F}[m_0](\xi) = 0$ *for* $0 \le |\xi| \ll 1$*. Then for* $t \ge t_0$ *with* $t_0 > 0$ *a sufficiently large time and for* $k = 0, \dots, l-1$*, the global solution* (ρ, m, ψ) *given by Proposition 2.2 satisfies*

$$c_2 K_0 (1+t)^{-(\frac{3}{4}+\frac{k}{2})} \le \min\left\{\|\nabla^k(\varrho, m)(t)\|_{L^2}, \|\psi(t)\|_{H^2}\right\} \le C(1+t)^{-(\frac{3}{4}+\frac{k}{2})}, \tag{2.14}$$

$$c_2 K_0 (1+t)^{-\left(\frac{3}{4}+\frac{l}{2}\right)} \le \|\nabla^l m(t)\|_{L^2} \le C(1+t)^{-\left(\frac{3}{4}+\frac{l}{2}\right)}, \tag{2.15}$$

where c_2 *is a positive constant independent of time.*

By using the estimates on the upper decay rates of the solution, we can conclude from (2.14)–(2.15) that (1.7) holds for $k = 0, \dots, l$.

3. Spectral analysis and linear L^2 estimates

In this section, we focus on the decay rate of the solution to the linear system

$$\begin{gathered} \varrho_t + \operatorname{div} m = 0, \\ m_t + \left(P'(1)\nabla + \gamma\nabla(1-\Delta)^{-1}\right)\varrho - \mu\Delta m - (\mu+\nu)\nabla \operatorname{div} m = 0, \\ (\varrho, m)|_{t=0} = (\varrho_0, m_0) = (\rho_0 - 1, \rho_0 u_0). \end{gathered} \tag{3.1}$$

Motivated by [2], we decompose the momentum m to analyze the above system (3.1) similarly as Hodge decomposition of the vector field, the system (3.1) can be transformed into two systems. One is a mere heat equation on the "incompressible part", and another one has distinct eigenvalues. Let $n = \Lambda^{-1} \operatorname{div} m$ and $M =$

$\Lambda^{-1}\operatorname{curl} m$ (with $\operatorname{curl} z = (\partial_{x_2} z^3 - \partial_{x_3} z^2, \partial_{x_3} z^1 - \partial_{x_1} z^3, \partial_{x_1} z^2 - \partial_{x_2} z^1)^t$), then we can rewrite (2.9) as follows

$$\begin{gathered} \varrho_t + \Lambda n = 0, \\ n_t - \Lambda(P'(1) + \gamma(1-\Delta)^{-1})\varrho - (2\mu+\nu)\Delta n = 0, \\ M_t - \mu\Delta M = 0, \\ (\varrho, n, M)|_{t=0} = (\varrho_0, n_0, M_0) = (\varrho_0, \Lambda^{-1}\operatorname{div} m_0, \Lambda^{-1}\operatorname{curl} m_0). \end{gathered} \tag{3.2}$$

Indeed, as the definition of n and M, and relation

$$m = -\Lambda^{-1}\nabla n - \Lambda^{-1}\operatorname{curl} M \tag{3.3}$$

involve pseudo-differential operators of degree zero, the estimates in the space $H^l(\mathbb{R}^3)$ for the original function m can be derived from n and M.

Now, we study the time decay rates of the solutions for the system (3.1). It follows immediately that the convergence rate of M is exponential in any norm. Hence, it suffices to consider the system

$$\begin{gathered} \varrho_t = -\Lambda n, \\ n_t = \Lambda\left(P'(1) + \gamma(1-\Delta)^{-1}\right)\varrho + (2\mu+\nu)\Delta n, \\ (\varrho, n)|_{t=0} = (\varrho_0, n_0). \end{gathered} \tag{3.4}$$

In terms of the semigroup theory, by denoting $V = (\varrho, n)^t$, we may express (3.4) as,

$$\begin{gathered} V_t = BV, \\ V|_{t=0} = V_0 \end{gathered} \tag{3.5}$$

with

$$B = \begin{pmatrix} 0 & -\Lambda \\ \Lambda\left(P'(1) + \gamma(1-\Delta)^{-1}\right) & (2\mu+\nu)\Delta \end{pmatrix}$$

Applying the Fourier transform to the system (3.5), we have

$$\begin{gathered} \widehat{V}_t = A(\xi)\widehat{V}, \\ \widehat{V}|_{t=0} = \widehat{V}_0, \end{gathered} \tag{3.6}$$

where $\widehat{V}(t,\xi) = \mathscr{F}V(t,x)$, $\xi = (\xi_1, \xi_2, \xi_3)^t$ and $A(\xi)$ is defined by

$$A(\xi) = \begin{pmatrix} 0 & -|\xi| \\ \left(P'(1) + \frac{\gamma}{1+|\xi|^2}\right)|\xi| & -(2\mu+\nu)|\xi|^2 \end{pmatrix}. \tag{3.7}$$

The eigenvalues of the matrix $A(\xi)$ are computed from the determinant

$$\det(A(\xi) - \lambda I) = \lambda^2 + (2\mu+\nu)|\xi|^2\lambda + \Big(P'(1) + \frac{\gamma}{1+|\xi|^2}\Big)|\xi|^2 = 0, \tag{3.8}$$

which implies the eigenvalues of the matrix A can be expressed as

$$\lambda_\pm(|\xi|) = -\big(\mu + \frac{\nu}{2}\big)|\xi|^2 \pm \sqrt{\big(\mu+\frac{\nu}{2}\big)^2|\xi|^4 - \big(P'(1) + \frac{\gamma}{1+|\xi|^2}\big)|\xi|^2}. \tag{3.9}$$

The semigroup $S(t) = e^{tA}$ can be decomposed into

$$e^{tA(\xi)} = e^{\lambda_+ t}P_+(\xi) + e^{\lambda_- t}P_-(\xi), \tag{3.10}$$

where the projector $P_\pm(\xi)$ is

$$P_\pm(\xi) = \frac{A(\xi) - \lambda_\mp I}{\lambda_\pm - \lambda_\mp}. \tag{3.11}$$

To estimate the semigroup e^{tA} in L^2 frame, we analyze the asymptotical expansions of $\lambda_\pm$, $P_\pm$ and $e^{tA(\xi)}$ for both lower and higher frequencies. From [21], we have the following lemma by careful computation.

Lemma 3.1. *(a) For $|\xi| \ll 1$, the spectral has the Taylor series expansion*

$$\lambda_\pm = -\big(\mu + \frac{\nu}{2}\big)|\xi|^2 + O(|\xi|^4) \pm i\big((P'(1)+\gamma)^{1/2}|\xi| + O(|\xi|^3)\big). \tag{3.12}$$

(b) For $|\xi| \gg 1$, the spectral has the Laurent expansion

$$\begin{aligned} \lambda_+ &= -\frac{P'(1)}{2\mu+\nu} + O(|\xi|^{-2}), \\ \lambda_- &= -(2\mu+\nu)|\xi|^2 + \frac{P'(1)}{2\mu+\nu} + O(|\xi|^{-2}). \end{aligned} \tag{3.13}$$

By (3.10)–(3.11) and Lemma 3.1, we obtain the following estimates for the solution $\widehat{V}(t,\xi)$ to the system (3.6).

Lemma 3.2. *(a) For $|\xi| \ll 1$, we have*

$$|\widehat{\varrho}|, |\widehat{n}| \lesssim e^{-(\mu+\frac{\nu}{2})|\xi|^2 t}(|\widehat{\varrho}_0| + |\widehat{n}_0|), \tag{3.14}$$

(b) For $|\xi| \gg 1$, we have

$$|\widehat{\varrho}| \lesssim e^{-Rt}\left(|\widehat{\varrho}_0| + |\xi|^{-1}|\widehat{n}_0|\right), \tag{3.15}$$

$$|\widehat{n}| \lesssim |\xi|^{-1}e^{-Rt}|\widehat{\varrho}_0| + \Big(e^{-(\mu+\frac{\nu}{2})|\xi|^2 t} + |\xi|^{-2}e^{-Rt}\Big)|\widehat{n}_0| \tag{3.16}$$

for some positive constant R.

Proof. By the formula (3.10)–(3.11), we can calculate the semigroup S as follows.

$$\begin{aligned} &S(t,\xi) \\ &= (S_{ij}(t,\xi))_{2\times 2} \\ &= \begin{pmatrix} g_1(\lambda_+,\lambda_-) & -|\xi|g_2(\lambda_+,\lambda_-) \\ |\xi|\big(P'(1)+\frac{\gamma}{1+|\xi|^2}\big)g_2(\lambda_+,\lambda_-) & g_1(\lambda_+,\lambda_-) - (2\mu+\nu)|\xi|^2 g_2(\lambda_+,\lambda_-) \end{pmatrix}, \end{aligned} \tag{3.17}$$

where

$$g_1(\lambda_+,\lambda_-) = \frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-}, \quad g_2(\lambda_+,\lambda_-) = \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}.$$

From the expansions (3.12) and (3.13) of $\lambda_\pm$, we can estimate $g_i(\lambda_+,\lambda_-)$ ($i = 1,2$) as follows

$$\begin{aligned} &g_1(\lambda_+,\lambda_-) \\ &= \begin{cases} e^{-(\mu+\frac{\nu}{2})|\xi|^2}\Big(\frac{(\mu+\frac{\nu}{2})|\xi|^2}{(P'(1)+\gamma)^{1/2}|\xi|+O(|\xi|^3)}\sin\Big(\big((P'(1)+\gamma)^{1/2}|\xi| \\ \quad +O(|\xi|^3)\big)t\Big) + \cos\Big(\Big((P'(1)+\gamma)^{1/2}|\xi| + O(|\xi|^3)\Big)t\Big)\Big), \\ \quad \text{if } |\xi| \ll 1, \\ \frac{O(1)e^{(-(2\mu+\nu)|\xi|^2+O(1))t} + \big((2\mu+\nu)|\xi|^2+O(1)\big)e^{(-\frac{P'(1)}{2\mu+\nu}+O(|\xi|^{-2}))t}}{(2\mu+\nu)|\xi|^2+O(1)}, \\ \quad \text{if } |\xi| \gg 1, \end{cases} \end{aligned} \tag{3.18}$$

and

$$g_2(\lambda_+,\lambda_-)=\begin{cases}\frac{\sin\Big(\big((P'(1)+\gamma)^{1/2}|\xi|+O(|\xi|^3)\big)t\Big)}{(P'(1)+\gamma)^{1/2}|\xi|+O(|\xi|^3)}e^{-(\mu+\frac{\nu}{2})|\xi|^2}, & |\xi|\ll 1,\\ \frac{e^{\big(-\frac{P'(1)}{2\mu+\nu}+O(|\xi|^{-2})\big)t}-e^{\big(-(2\mu+\nu)|\xi|^2+O(1)\big)t}}{(2\mu+\nu)|\xi|^2+O(1)}, & |\xi|\gg 1.\end{cases}\tag{3.19}$$

Hence we conclude that

$$|g_1(\lambda_+,\lambda_-)|\lesssim\begin{cases}e^{-(\mu+\frac{\nu}{2})|\xi|^2t}, & |\xi|\ll 1,\\ e^{-Rt}, & |\xi|\gg 1\end{cases}\tag{3.20}$$

for a positive constant R, and

$$|g_2(\lambda_+,\lambda_-)|\lesssim\begin{cases}|\xi|^{-1}e^{-(\mu+\frac{\nu}{2})|\xi|^2t}, & |\xi|\ll 1,\\ |\xi|^{-2}e^{-Rt}, & |\xi|\gg 1.\end{cases}\tag{3.21}$$

Moreover, by delicate calculations, we have the estimate

$$\begin{aligned}&|g_1(\lambda_+,\lambda_-)-(2\mu+\nu)|\xi|^2g_2(\lambda_+,\lambda_-)|\\&\lesssim\begin{cases}e^{-(\mu+\frac{\nu}{2})|\xi|^2t}, & |\xi|\ll 1,\\ e^{-(\mu+\frac{\nu}{2})|\xi|^2t}+|\xi|^{-2}e^{-Rt}, & |\xi|\gg 1.\end{cases}\end{aligned}\tag{3.22}$$

Now we represent the solution of (3.6) as

$$\widehat{V}(t,\xi)=e^{tA}\widehat{V}_0.\tag{3.23}$$

Therefore, by plugging (3.20)–(3.22) into the expression (3.17) of $S(t)$, we obtain (3.14)–(3.16). □

As in [21], we obtain the decay rates for the solution (ϱ,n,M) of the linear system (3.2) as follows.

Proposition 3.3. *Assume that $(\varrho_0,m_0)\in H^l\cap L^1$. Let $n=\Lambda^{-1}\operatorname{div}m$ and $M=\Lambda^{-1}\operatorname{curl}m$. Then the solution (ϱ,n,M) of the linear system* (3.2) *satisfies*

(a)

$$\|\varrho\|_{L^2}^2\lesssim(1+t)^{-3/2}\|(\varrho_0,n_0)\|_{L^1}^2+e^{-2Rt}\|(\varrho_0,n_0)\|_{L^2}^2,\tag{3.24}$$

and for $1\le k\le l$,

$$\begin{aligned}\|\nabla^k\varrho\|_{L^2}^2\lesssim{}&(1+t)^{-(\frac{3}{2}+k)}\|(\varrho_0,n_0)\|_{L^1}^2\\&+e^{-2Rt}\Big(\|\nabla^k\varrho_0\|_{L^2}^2+\|\nabla^{k-1}n_0\|_{L^2}^2\Big).\end{aligned}\tag{3.25}$$

(b) *For $k=0,1$,*

$$\|\nabla^kn\|_{L^2}^2\lesssim(1+t)^{-\left(\frac{3}{2}+k\right)}\|(\varrho_0,n_0)\|_{L^1}^2+e^{-2Rt}\|(\varrho_0,n_0)\|_{L^2}^2,\tag{3.26}$$

and for $2\le k\le l$,

$$\|\nabla^kn\|_{L^2}^2\lesssim(1+t)^{-(\frac{3}{2}+k)}\|(\varrho_0,n_0)\|_{L^1}^2+e^{-2Rt}\Big(\|\nabla^{k-1}\varrho_0\|_{L^2}^2+\|\nabla^{k-2}n_0\|_{L^2}^2\Big),\tag{3.27}$$

and for $0\le k\le l$,

$$\|\nabla^kM\|_{L^2}^2\lesssim(1+t)^{-(\frac{3}{2}+k)}\|M_0\|_{L^1}^2.\tag{3.28}$$

Proof. We only prove (3.27). By Lemma 3.2, Plancherel theorem and Hausdorff-Young's inequality, from (3.14) and (3.16) we have that for each $2 \le k \le l$ and for some $\eta > 0$,

$$\begin{aligned}
&\|\nabla^k n\|_{L^2}^2 = \||\xi|^k \widehat{n}\|_{L^2}^2 \\
&\lesssim \int_{|\xi|<\eta} |\xi|^{2k} e^{-(2\mu+\nu)|\xi|^2 t}(|\widehat{\varrho}_0|^2 + |\widehat{n}_0|^2) d\xi + \int_{|\xi|\ge\eta} |\xi|^{2k} e^{-(2\mu+\nu)|\xi|^2 t} |\widehat{n}_0|^2 d\xi \\
&\quad + \int_{|\xi|\ge\eta} e^{-2Rt}(|\xi|^{k-1}|\widehat{\varrho}_0| + |\xi|^{k-2}|\widehat{n}_0|)^2 d\xi \\
&\lesssim \left(\|\widehat{\varrho}_0\|_{L^\infty}^2 + \|\widehat{n}_0\|_{L^\infty}^2\right) \int_{|\xi|<\infty} |\xi|^{2k} e^{-(2\mu+\nu)|\xi|^2 t} d\xi \\
&\quad + e^{-2Rt} \int_{|\xi|\ge\eta} (|\xi|^{k-1}|\widehat{\varrho}_0| + |\xi|^{k-2}|\widehat{n}_0|)^2 d\xi \\
&\lesssim (1+t)^{-(\frac{3}{2}+k)} \|(\varrho_0, n_0)\|_{L^1}^2 + e^{-2Rt}\left(\|\nabla^{k-1}\varrho_0\|_{L^2}^2 + \|\nabla^{k-2} n_0\|_{L^2}^2\right).
\end{aligned} \tag{3.29}$$

□

Finally, to obtain the lower decay rates for the solution, we also need the following decay rates for the solution (ϱ, n, M) of the linear system (3.2).

Proposition 3.4. *Assume that $\varrho_0 \in H^l \cap \dot{H}^{-s}$ and $m_0 \in L^2 \cap \dot{H}^{-s}$. Let $n = \Lambda^{-1} \operatorname{div} m$ and $M = \Lambda^{-1} \operatorname{curl} m$. Then the solution (ϱ, n, M) of the linear system* (3.2) *satisfies*

(a)

$$\|\varrho\|_{L^2}^2 \lesssim (1+t)^{-s} \|(\varrho_0, n_0)\|_{\dot{H}^{-s}}^2 + e^{-2Rt} \|(\varrho_0, n_0)\|_{L^2}^2, \tag{3.30}$$

$$\|\nabla\varrho\|_{L^2}^2 \lesssim (1+t)^{-(1+s)} \|(\varrho_0, n_0)\|_{\dot{H}^{-s}}^2 + e^{-2Rt}\left(\|\nabla\varrho_0\|_{L^2}^2 + \|n_0\|_{L^2}\right). \tag{3.31}$$

(b) *For $k = 0, 1$,*

$$\|\nabla^k n\|_{L^2}^2 \lesssim (1+t)^{-(k+s)} \|(\varrho_0, n_0)\|_{\dot{H}^{-s}}^2 + e^{-2Rt} \|(\varrho_0, n_0)\|_{L^2}^2, \tag{3.32}$$

$$\|\nabla^k M\|_{L^2}^2 \lesssim (1+t)^{-(k+s)} \|M_0\|_{\dot{H}^{-s}}^2. \tag{3.33}$$

See [1] for a proof of this proposition, or use an argument similar to the one in Proposition 3.3.

4. Energy estimates

To prove Proposition 2.2, by the standard continuity argument, it suffices to derive the following a priori energy estimates.

Proposition 4.1 (a priori estimate)**.** *Let $(\varrho_0, u_0) \in H^l(\mathbb{R}^3)$ with an integer $l \ge 3$. Suppose that* (2.1) *has a solution $(\varrho, u, \psi) \in X(0,T)$, where T is a positive constant. Then there exists a small constant $\delta > 0$, independent of T, such that if*

$$\sup_{0\le t\le T} \left\{\|(\varrho, u)(t)\|_{H^l} + \|\psi(t)\|_{H^{l+2}}^2\right\} \le \delta, \tag{4.1}$$

then for any $t \in [0, T]$,

$$\begin{aligned}
&\|(\varrho, u)(t)\|_{H^l}^2 + \|\psi(t)\|_{H^{l+2}}^2 + \int_0^t (\|\nabla\varrho(\tau)\|_{H^{l-1}}^2 + \|\nabla u(\tau)\|_{H^l}^2 + \|\nabla\psi(\tau)\|_{H^{l+1}}^2) d\tau \\
&\lesssim \|(\varrho_0, u_0)\|_{H^l}^2.
\end{aligned} \tag{4.2}$$

First from $(2.1)_3$, we can easily deduce the following lemma and we omit the proof.

Lemma 4.2. *Under the assumption of Proposition 4.1, for $k = 0, \dots, l$, it holds*

$$\|\nabla^k \psi\|_{H^2} \approx \|\nabla^k \varrho\|_{L^2}. \tag{4.3}$$

Next we derive an energy estimates for (ϱ, u).

Lemma 4.3. *Under the assumption of Proposition 4.1, there exists a positive constant C_1, such that*

$$\begin{aligned} &\frac{d}{dt}\Big\{\frac{P'(1)}{2}\|\varrho(t)\|_{L^2}^2 + \frac{1}{2}\|u\|_{L^2}^2 + \frac{\gamma}{2}\|\psi\|_{H^1}^2 + \frac{\mu}{4C_1}\langle\nabla\varrho, u\rangle\Big\} \\ &+ \frac{\mu}{2}\|\nabla u(t)\|_{L^2}^2 + \frac{\mu P'(1)}{16C_1}\|\nabla\varrho(t)\|_{L^2}^2 + \frac{\mu\gamma}{4C_1}\|\nabla\psi\|_{H^1}^2 \\ &\le C\|\nabla^l u(t)\|_{L^2}^2. \end{aligned} \tag{4.4}$$

Proof. From (4.1) and the Sobolev inequality, we obtain upper and lower bounds

$$\frac{1}{2} \le \rho \le \frac{3}{2}. \tag{4.5}$$

Multiplying $(2.1)_1$ and$(2.1)_2$ with $P'(1)\varrho$ and u respectively, and then integrating the resulting equalities over $\mathbb{R}^3$, one has

$$\begin{aligned} &\frac{d}{dt}\Big\{\frac{P'(1)}{2}\|\varrho(t)\|_{L^2}^2 + \frac{1}{2}\|u\|_{L^2}^2\Big\} + \langle\gamma\nabla\psi, u\rangle + \mu\|\nabla u\|_{L^2}^2 + (\mu+\nu)\|\operatorname{div} u\|_{L^2}^2 \\ &= \langle N_1, P'(1)\varrho\rangle + \langle N_2, u\rangle. \end{aligned} \tag{4.6}$$

The terms in (4.6) can be estimated as follows. By using $(2.1)_1$, $(2.1)_3$, and integration by parts, we have

$$\begin{aligned} \langle\gamma\nabla\psi, u\rangle &= \langle\gamma\psi, -\operatorname{div} u\rangle \\ &= \langle\gamma\psi, \varrho_t + \operatorname{div}(\varrho u)\rangle \\ &= \langle\gamma\psi, (-\Delta\psi+\psi)_t\rangle + \langle\gamma\psi, \varrho\operatorname{div} u + \nabla\varrho\cdot u\rangle \\ &\ge \frac{\gamma}{2}\frac{d}{dt}\|\psi\|_{H^1}^2 - C\|\psi\|_{L^3}\left(\|\varrho\|_{L^6}\|\operatorname{div} u\|_{L^2} + \|\nabla\varrho\|_{L^2}\|u\|_{L^6}\right) \\ &\ge \frac{\gamma}{2}\frac{d}{dt}\|\psi\|_{H^1}^2 - C\delta\left(\|\nabla\varrho\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\right), \end{aligned} \tag{4.7}$$

where the a priori assumption (4.1), Hölder's inequality, Cauchy's inequality and the Sobolev embedding theorem are used. In addition, by (4.5), we have

$$\langle N_1, \varrho\rangle = \langle -\operatorname{div}(\varrho u), \varrho\rangle \le C\delta\left(\|\nabla\varrho\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\right), \tag{4.8}$$

$$
\begin{aligned}
&\langle N_2, u\rangle \\
&= \langle -u\cdot\nabla u, u\rangle + \Big\langle -\big(\frac{P'(\rho)}{\rho} - P'(1)\big)\nabla\varrho, u\Big\rangle + \Big\langle \mu\nabla\big(\frac{\varrho}{\rho}\big)\cdot\nabla u, u\Big\rangle \\
&\quad + \Big\langle \frac{\mu}{\rho}\varrho\nabla u, \nabla u\Big\rangle + \Big\langle (\mu+\nu)\nabla\big(\frac{\varrho}{\rho}\big)\operatorname{div} u, u\Big\rangle + \Big\langle \frac{\mu+\nu}{\rho}\varrho\operatorname{div} u, \operatorname{div} u\Big\rangle \\
&\le C\Big(\|u\|_{L^3}\|\nabla u\|_{L^2}\|u\|_{L^6} + \|\varrho\|_{L^3}\|\nabla\varrho\|_{L^2}\|u\|_{L^6} \\
&\quad + \|u\|_{L^\infty}\|\nabla\varrho\|_{L^2}\|\nabla u\|_{L^2} + \|\varrho\|_{L^\infty}\|\nabla u\|_{L^2}^2\Big) \\
&\le C\delta(\|\nabla\varrho\|_{L^2}^2 + \|\nabla u\|_{L^2}^2).
\end{aligned}
\tag{4.9}
$$

Plugging (4.7)–(4.9) into (4.6) and using Cauchy's inequality and the smallness of δ, we can deduce that

$$
\begin{aligned}
&\frac{d}{dt}\Big\{\frac{P'(1)}{2}\|\varrho(t)\|_{L^2}^2 + \frac{1}{2}\|u\|_{L^2}^2 + \frac{\gamma}{2}\|\psi\|_{H^1}^2\Big\} + \frac{3\mu}{4}\|\nabla u\|_{L^2}^2 + (\mu+\nu)\|\operatorname{div} u\|_{L^2}^2 \\
&\le C\delta\|\nabla\varrho\|_{L^2}^2.
\end{aligned}
\tag{4.10}
$$

Next we shall deal with the L^2-norm of $\nabla\varrho$. By taking $\langle\nabla(2.1)_1, u\rangle + \langle(2.1)_2, \nabla\varrho\rangle$, we have

$$
\begin{aligned}
&\frac{d}{dt}\langle\nabla\varrho, u\rangle + P'(1)\|\nabla\varrho(t)\|_{L^2}^2 + \langle\gamma\nabla\psi, \nabla\varrho\rangle \\
&= \langle\nabla(-\operatorname{div} u + N_1), u\rangle + \langle\mu\Delta u + (\mu+\nu)\nabla\operatorname{div} u + N_2, \nabla\varrho\rangle.
\end{aligned}
\tag{4.11}
$$

By using $(2.1)_3$, we obtain the following estimates:

$$
\langle\gamma\nabla\psi, \nabla\varrho\rangle = \langle\gamma\nabla\psi, \nabla(-\Delta\psi + \psi)\rangle = \gamma\|\nabla\psi\|_{H^1}^2,
\tag{4.12}
$$

$$
\begin{aligned}
\langle\nabla(-\operatorname{div} u + N_1), u\rangle &= \langle\operatorname{div} u + \operatorname{div}(\varrho u), \operatorname{div} u\rangle \\
&\le \|\operatorname{div} u\|_{L^2}^2 + \|(\varrho, u)\|_{L^\infty}\|\nabla(\varrho, u)\|_{L^2}\|\nabla u\|_{L^2} \\
&\le C\|\nabla u\|_{L^2}^2 + C\delta\|\nabla\varrho\|_{L^2}^2,
\end{aligned}
\tag{4.13}
$$

and

$$
\begin{aligned}
&\langle\mu\Delta u + (\mu+\nu)\nabla\operatorname{div} u + N_2, \nabla\varrho\rangle \\
&\le \frac{P'(1)}{4}\|\nabla\rho\|_{L^2}^2 + C\|\nabla^2 u\|_{L^2}^2 + C\|(\varrho, u)\|_{L^\infty}\|\left(\nabla\varrho, \nabla u, \nabla^2 u\right)\|_{L^2}\|\nabla\varrho\|_{L^2} \\
&\le \frac{P'(1)}{4}\|\nabla\rho\|_{L^2}^2 + C\left(\|\nabla u\|_{L^2}^2 + \|\nabla^l u\|_{L^2}^2\right) + C\delta\left(\|\nabla\varrho\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\right).
\end{aligned}
\tag{4.14}
$$

Thus plugging (4.12)–(4.14) into (4.11) yields

$$
\frac{d}{dt}\langle\nabla\varrho, u\rangle + \frac{P'(1)}{2}\|\nabla\varrho(t)\|_{L^2}^2 + \gamma\|\nabla\psi\|_{H^1}^2 \le C_1\|\nabla u\|_{L^2}^2 + C\|\nabla^l u\|_{L^2}^2,
\tag{4.15}
$$

where C_1 is some positive number. Then the estimate (4.4) follows by taking the addition of (4.10) and $\mu/(4C_1)$ times (4.15) and using the smallness of δ. $\square$

Lemma 4.4. *Under the assumption of Proposition 4.1, there exist two positive constants C_2 and C_3 such that*

$$\begin{aligned}&\frac{d}{dt}\Big\{\frac{P'(1)}{2}\|\nabla^{l-1}\varrho\|_{H^1}^2+\frac{1}{2}\|\nabla^{l-1}u\|_{H^1}^2+\frac{\gamma}{2}\|\left(\nabla^{l-1}\psi,\nabla^l\psi\right)\|_{H^1}^2\\&+\frac{\mu}{4C_2}\langle\nabla^l\varrho,\nabla^{l-1}u\rangle\Big\}+\frac{\mu}{2}\|\nabla^l u(t)\|_{H^1}^2+\frac{\mu P'(1)}{16C_2}\|\nabla^l\varrho(t)\|_{L^2}^2\\&+\frac{\mu\gamma}{4C_2}\|\nabla^l\psi\|_{H^1}^2\le 0.\end{aligned}\tag{4.16}$$

Proof. By taking derivatives with $k=l-1$ or l, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big\{P'(1)\|\nabla^k\varrho\|_{L^2}^2+\|\nabla^k u\|_{L^2}^2\Big\}+\mu\|\nabla^{k+1}u(t)\|_{L^2}^2+(\mu+\nu)\|\nabla^k\operatorname{div}u\|_{L^2}^2\\&+\langle\gamma\nabla^k\nabla\psi,\nabla^k u\rangle\\&=\langle P'(1)\nabla^k N_1,\nabla^k\varrho\rangle+\langle\nabla^k N_2,\nabla^k u\rangle.\end{aligned}\tag{4.17}$$

By using $(2.1)_1$, $(2.1)_3$, and Lemmas 4.2, 6.1–Lemma 6.4, we can estimate the terms in (4.17) as follows.

$$\begin{aligned}\langle\gamma\nabla^k\nabla\psi,\nabla^k u\rangle&=\langle\gamma\nabla^k\psi,\nabla^k(-\operatorname{div}u)\rangle\\&=\langle\gamma\nabla^k\psi,\nabla^k(\varrho_t+\operatorname{div}(\varrho u))\rangle\\&=\langle\gamma\nabla^k\psi,\nabla^k(-\Delta\psi+\psi)_t\rangle+\langle\gamma\nabla^k\psi,\nabla^k\operatorname{div}(\varrho u)\rangle\\&=\frac{\gamma}{2}\frac{d}{dt}\|\nabla^k\psi\|_{H^1}^2-\langle\gamma\nabla^{k+1}\psi,\nabla^k(\varrho u)\rangle.\end{aligned}\tag{4.18}$$

By

$$\begin{aligned}\left|\langle\gamma\nabla^l\psi,\nabla^{l-1}(\varrho u)\rangle\right|&\lesssim\|\nabla^l\psi\|_{L^2}\left(\|\nabla^{l-1}\varrho\|_{L^6}\|u\|_{L^3}+\|\varrho\|_{L^3}\|\nabla^{l-1}u\|_{L^6}\right)\\&\lesssim\|(\varrho,u)\|_{H^1}\|\nabla^l\psi\|_{L^2}\|\nabla^l(\varrho,u)\|_{L^2}\\&\lesssim\delta\|\nabla^l(\varrho,u)\|_{L^2}^2,\end{aligned}\tag{4.19}$$

and

$$\begin{aligned}|\langle\gamma\nabla^{l+1}\psi,\nabla^l(\varrho u)\rangle|&\lesssim\|\nabla^{l+1}\psi\|_{L^2}\left(\|\varrho\|_{L^3}\|\nabla^l u\|_{L^6}+\|\nabla^l\varrho\|_{L^2}\|u\|_{L^\infty}\right)\\&\lesssim\delta\Big(\|\nabla^l\varrho\|_{L^2}^2+\|\nabla^{l+1}u\|_{L^2}^2\Big),\end{aligned}\tag{4.20}$$

we can conclude that for $k=l-1,l$,

$$\langle\gamma\nabla^k\nabla\psi,\nabla^k u\rangle\ge\frac{\gamma}{2}\frac{d}{dt}\|\nabla^k\psi\|_{H^1}^2-C\delta\left(\|\nabla^l\varrho\|_{L^2}^2+\|\nabla^{k+1}u\|_{L^2}^2\right).\tag{4.21}$$

Similarly we have

$$\begin{aligned}\langle\nabla^k N_1,\nabla^k\varrho\rangle&=\langle\nabla^k(-\nabla\varrho\cdot u-\varrho\operatorname{div}u),\nabla^k\varrho\rangle\\&=\int_{\mathbb{R}^3}(\operatorname{div}u)\frac{|\nabla^k\varrho|^2}{2}dx-\langle[\nabla^k,u]\cdot\nabla\rho,\nabla^k\rho\rangle\\&\quad+\|\nabla^k(\varrho\operatorname{div}u)\|_{L^2}\|\nabla^k\varrho\|_{L^2},\end{aligned}\tag{4.22}$$

where the commutator $[\nabla^k, f]g$ is defined in (6.6). This together with

$$
\begin{aligned}
&\int_{\mathbb{R}^3} (\operatorname{div} u)\frac{|\nabla^{l-1}\varrho|^2}{2}dx - \langle [\nabla^{l-1}, u]\cdot\nabla\rho, \nabla^{l-1}\rho\rangle \\
&+ \|\nabla^{l-1}(\varrho \operatorname{div} u)\|_{L^2}\|\nabla^{l-1}\varrho\|_{L^2} \\
&\lesssim \left(\|\nabla(\varrho,u)\|_{L^{3/2}}\|\nabla^{l-1}(\varrho,u)\|_{L^6} + \|\varrho\|_{L^3}\|\nabla^l u\|_{L^2}\right)\|\nabla^{l-1}\varrho\|_{L^6} \\
&\lesssim \delta\|\nabla^l(\varrho,u)\|_{L^2}^2,
\end{aligned} \tag{4.23}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^3} (\operatorname{div} u)\frac{|\nabla^{l}\varrho|^2}{2}dx - \langle [\nabla^{l}, u]\cdot\nabla\rho, \nabla^{l}\rho\rangle + \|\nabla^{l}(\varrho \operatorname{div} u)\|_{L^2}\|\nabla^{l}\varrho\|_{L^2} \\
&\lesssim \left(\|\nabla u\|_{L^\infty}\|\nabla^l\varrho\|_{L^2} + \|\nabla\varrho\|_{L^3}\|\nabla^l u\|_{L^6} + \|\varrho\|_{L^\infty}\|\nabla^{l+1}u\|_{L^2}\right)\|\nabla^l\varrho\|_{L^2} \\
&\lesssim \delta\Big(\|\nabla^l\varrho\|_{L^2}^2 + \|\nabla^{l+1}u\|_{L^2}^2\Big)
\end{aligned} \tag{4.24}
$$

implies that for $k = l-1, l$,

$$
\langle \nabla^k N_1, \nabla^k\varrho\rangle \lesssim \delta\Big(\|\nabla^l\varrho\|_{L^2}^2 + \|\nabla^{k+1}u\|_{L^2}^2\Big). \tag{4.25}
$$

Furthermore, we have from (2.3) that for $k = l-1$,

$$
\begin{aligned}
&\langle \nabla^{l-1}N_2, \nabla^{l-1}u\rangle \\
&= \langle \nabla^{l-1}\big(-u\cdot\nabla u\big), \nabla^{l-1}u\rangle + \big\langle \nabla^{l-1}\Big(-\big(\frac{P'(\rho)}{\rho} - P'(1)\big)\nabla\varrho\Big), \nabla^{l-1}u\big\rangle \\
&\quad + \big\langle \nabla^{l-1}\big(\frac{\mu}{\rho}\varrho\nabla u\big), \nabla^l u\big\rangle + \big\langle \nabla^{l-1}\Big(\nabla\big(\frac{\mu}{\rho}\varrho\big)\cdot\nabla u\Big), \nabla^{l-1}u\big\rangle \\
&\quad + \big\langle \nabla^{l-1}\big(\frac{\mu+\nu}{\rho}\varrho \operatorname{div} u\big), \nabla^{l-1}\operatorname{div} u\big\rangle + \big\langle \nabla^{l-1}\Big(\nabla\big(\frac{\mu+\nu}{\rho}\varrho\big)\operatorname{div} u\Big), \nabla^{l-1}u\big\rangle \\
&\lesssim \Big(\|(\varrho,u)\|_{L^3}\|\nabla^l(\varrho,u)\|_{L^2} + \|\nabla^{l-1}\big(\frac{P'(\rho)}{\rho} \\
&\quad - P'(1), u\big)\|_{L^6}\|\nabla(\varrho,u)\|_{L^{3/2}}\Big)\|\nabla^{l-1}u\|_{L^6} \\
&\quad + \Big(\|\varrho\|_{L^\infty}\|\nabla^l u\|_{L^2} + \|\nabla^{l-1}\big(\frac{\varrho}{\rho}\big)\|_{L^6}\|\nabla u\|_{L^3}\Big)\|\nabla^l u\|_{L^2} \\
&\quad + \Big(\|\nabla\varrho\|_{L^3}\|\nabla^l u\|_{L^2} + \|\nabla^l\big(\frac{\varrho}{\rho}\big)\|_{L^2}\|\nabla u\|_{L^3}\Big)\|\nabla^{l-1}u\|_{L^6} \\
&\lesssim \delta\Big(\|\nabla^l\varrho\|_{L^2}^2 + \|\nabla^l u\|_{L^2}^2\Big) + \delta\Big(\|\nabla^l\big(\frac{P'(\rho)}{\rho} - P'(1)\big)\|_{L^2} \\
&\quad + \|\nabla^l\left(\frac{\varrho}{\rho}\right)\|_{L^2}\Big)\|\nabla^l u\|_{L^2} \\
&\lesssim \delta\Big(\|\nabla^l\varrho\|_{L^2}^2 + \|\nabla^l u\|_{L^2}^2\Big),
\end{aligned} \tag{4.26}
$$

where (6.9) is used, and for $k=l$,

$$\begin{aligned}
&\langle \nabla^l N_2, \nabla^l u\rangle \\
&= \langle \nabla^l(-u\cdot\nabla u), \nabla^l u\rangle + \Big\langle \nabla^{l-1}\Big(\big(\frac{P'(\rho)}{\rho} - P'(1)\big)\nabla\varrho\Big), \nabla^l \operatorname{div} u\Big\rangle \\
&\quad + \Big\langle \nabla^{l-1}\big(\frac{\mu}{\rho}\varrho\Delta u\big), \nabla^l \operatorname{div} u\Big\rangle + \Big\langle \nabla^{l-1}\big(\frac{\mu+\nu}{\rho}\varrho\nabla \operatorname{div} u\big), \nabla^l \operatorname{div} u\Big\rangle \\
&\lesssim \Big(\|u\|_{L^3}\|\nabla^{l+1}u\|_{L^2} + \|\nabla^l u\|_{L^6}\|\nabla u\|_{L^{3/2}}\Big)\|\nabla^l u\|_{L^6} \\
&\quad + \Big(\|\varrho\|_{L^\infty}\|\nabla^l\varrho\|_{L^2} + \|\nabla^{l-1}\Big(\frac{P'(\rho)}{\rho} - P'(1)\Big)\|_{L^6}\|\nabla\varrho\|_{L^3} \\
&\quad + \|\varrho\|_{L^\infty}\|\nabla^{l+1}u\|_{L^2} + \|\nabla^{l-1}\big(\frac{\varrho}{\rho}\big)\|_{L^6}\|\nabla^2 u\|_{L^3}\Big)\|\nabla^{l+1}u\|_{L^2} \\
&\lesssim \|(\varrho,u)\|_{H^3}\|\left(\nabla^l\varrho, \nabla^{l+1}u\right)\|_{L^2}\|\nabla^{l+1}u\|_{L^2} \\
&\lesssim \delta\left(\|\nabla^l\varrho\|_{L^2}^2 + \|\nabla^{l+1}u\|_{L^2}^2\right).
\end{aligned} \tag{4.27}$$

Therefore, from (4.26) and (4.27) we conclude that for $k=l-1,l$,

$$\langle \nabla^k N_2, \nabla^k u\rangle \lesssim \delta\left(\|\nabla^l\varrho\|_{L^2}^2 + \|\nabla^{k+1}u\|_{L^2}^2\right). \tag{4.28}$$

Hence plugging (4.21), (4.25) and (4.28) into (4.17) yields that for $k=l-1,l$,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\{P'(1)\|\nabla^k\varrho\|_{L^2}^2 + \|\nabla^k u\|_{L^2}^2 + \gamma\|\nabla^k\psi\|_{H^1}^2\right\} + \frac{3\mu}{4}\|\nabla^{k+1}u(t)\|_{L^2}^2 \\
&\le C\delta\|\nabla^l\varrho\|_{L^2}^2.
\end{aligned} \tag{4.29}$$

Here the smallness of δ is used again.

Now we turn to estimate the L^2-norm of $\nabla^l\varrho$. By taking the derivative, we have

$$\begin{aligned}
&\frac{d}{dt}\langle \nabla^l\varrho, \nabla^{l-1}u\rangle + P'(1)\|\nabla^l\varrho(t)\|_{L^2}^2 + \langle \gamma\nabla^l\psi, \nabla^l\varrho\rangle \\
&= \langle \nabla^l(-\operatorname{div} u + N_1), \nabla^{l-1}u\rangle + \langle \nabla^{l-1}\left(\mu\Delta u + (\mu+\nu)\nabla\operatorname{div} u + N_2\right), \nabla^l\varrho\rangle.
\end{aligned} \tag{4.30}$$

By using $(2.1)_3$ and Lemmas 6.1–6.4, we obtain the following estimates.

$$\langle \gamma\nabla^l\psi, \nabla^l\varrho\rangle = \langle \gamma\nabla^l\psi, \nabla^l(-\Delta\psi + \psi)\rangle = \gamma\|\nabla^l\psi\|_{H^1}^2, \tag{4.31}$$

$$\begin{aligned}
&\langle \nabla^l(-\operatorname{div} u + N_1), \nabla^{l-1}u\rangle \\
&= \langle \nabla^{l-1}(\operatorname{div} u + \operatorname{div}(\varrho u)), \nabla^{l-1}\operatorname{div} u\rangle \\
&\le \|\nabla^{l-1}\operatorname{div} u\|_{L^2}^2 + \|(\varrho,u)\|_{L^\infty}\|\nabla^l(\varrho,u)\|_{L^2}\|\nabla^l u\|_{L^2} \\
&\le C\|\nabla^l u\|_{L^2}^2 + C\delta\|\nabla^l\varrho\|_{L^2}^2,
\end{aligned} \tag{4.32}$$

and as in the proof of (4.26) and (4.27),

$$\begin{aligned}
&\langle \nabla^{l-1}\left(\mu\Delta u + (\mu+\nu)\nabla\operatorname{div} u + N_2\right), \nabla^l\varrho\rangle \\
&\le \frac{P'(1)}{4}\|\nabla^l\varrho\|_{L^2}^2 + C\|\nabla^{l+1}u\|_{L^2}^2 \\
&\quad + C\|(\varrho,u)\|_{H^3}\|\left(\nabla^l\varrho, \nabla^l u, \nabla^{l+1}u\right)\|_{L^2}\|\nabla^l\varrho\|_{L^2} \\
&\le \frac{3P'(1)}{8}\|\nabla^l\varrho\|_{L^2}^2 + C\|\nabla^l u\|_{H^1}^2.
\end{aligned} \tag{4.33}$$

Thus plugging (4.31)–(4.33) into (4.30) yields

$$\frac{d}{dt}\langle\nabla^l\varrho,\nabla^{l-1}u\rangle+\frac{P'(1)}{2}\|\nabla^l\varrho(t)\|_{L^2}^2+\gamma\|\nabla^l\psi\|_{H^1}^2\le C_2\|\nabla^l u\|_{H^1}^2, \tag{4.34}$$

where we take $C_2>\mu/(4\sqrt{P'(1)})$.

Therefore, (4.16) can be deduced by adding (4.29) and $\mu/(4C_2)$ times (4.34), and using the smallness of δ. □

Now we are in a position to prove Proposition 4.1.

Proof of Proposition 4.1. Choosing a sufficiently small positive number η_1 and then taking η_1 times (4.4) plus (4.16), we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big\{\frac{\eta_1P'(1)}{2}\|\varrho(t)\|_{L^2}^2+\frac{\eta_1}{2}\|u\|_{L^2}^2+\frac{\eta_1\gamma}{2}\|\psi\|_{H^1}^2+\frac{\mu\eta_1}{4C_1}\langle\nabla\varrho,u\rangle\\
&+\frac{P'(1)}{2}\|\nabla^{l-1}\varrho\|_{H^1}^2+\frac12\|\nabla^{l-1}u\|_{H^1}^2+\frac{\gamma}{2}\|(\nabla^{l-1}\psi,\nabla^l\psi)\|_{H^1}^2\\
&+\frac{\mu}{4C_2}\langle\nabla^l\varrho,\nabla^{l-1}u\rangle\Big\}+\frac{\eta_1\mu}{2}\|\nabla u(t)\|_{L^2}^2+\frac{\eta_1\mu P'(1)}{16C_1}\|\nabla\varrho(t)\|_{L^2}^2\\
&+\frac{\eta_1\mu\gamma}{4C_1}\|\nabla\psi\|_{H^1}^2+\frac{\mu}{4}\|\nabla^l u(t)\|_{H^1}^2+\frac{\mu P'(1)}{16C_2}\|\nabla^l\varrho(t)\|_{L^2}^2+\frac{\mu\gamma}{4C_2}\|\nabla^l\psi\|_{H^1}^2\\
&\le 0.
\end{aligned}\tag{4.35}$$

By integrating (4.35) with respect to t, and using Cauchy's inequality, the smallness of δ and η_1, the fact that $C_2>\frac{\mu}{4\sqrt{P'(1)}}$, and Sobolev interpolation inequality, we can finally deduce the a priori estimate (4.2). □

5. Decay estimates

In this section, we shall prove Proposition 2.3 and Proposition 2.5, which imply the optimal decay rate for the solution. To this end, we define

$$H(t)=\sup_{0\le\tau\le t}\sum_{0\le k\le l-1}(1+\tau)^{\frac32+k}\|\nabla^k(\varrho,m)(\tau)\|_{L^2}^2. \tag{5.1}$$

For technical considerations, we do not intend to estimate the optimal time decay rates of the highest-order derivatives of the density and the velocity at first. However, we will show that by energy estimates, $\|\nabla^l(\varrho,u)\|_{L^2}$ can be controlled by $\|(\nabla^{l-1}\varrho,\nabla^{l-1}u)\|_{L^2}$.

First we shall introduce the following useful inequality.

Lemma 5.1 ([6, 7]). *Assume $r_1>1$, $r_2\in[0,r_1]$, then we have*

$$\int_0^t(1+t-\tau)^{-r_1}(1+\tau)^{-r_2}d\tau\le C(r_1,r_2)(1+t)^{-r_2}. \tag{5.2}$$

Lemma 5.2. *Under the assumption of Proposition 2.3, it holds that*

$$\|\nabla^l(\varrho,m)\|_{L^2}^2\lesssim(1+t)^{-(\frac12+l)}\big\{\|(\varrho_0,u_0)\|_{H^l}^2+H(t)\big\}. \tag{5.3}$$

Proof. We define

$$\begin{aligned}
L(t)=&\frac{P'(1)}{2}\|\nabla^{l-1}\varrho\|_{H^1}^2+\frac12\|\nabla^{l-1}u\|_{H^1}^2+\frac{\gamma}{2}\|\left(\nabla^{l-1}\psi,\nabla^l\psi\right)\|_{H^1}^2\\
&+\frac{\mu}{4C_2}\langle\nabla^l\varrho,\nabla^{l-1}u\rangle.
\end{aligned}\tag{5.4}$$

Since by $C_2 > \mu/(4P'(1))$ from (4.5), it is easy to verify that

$$L(t) \approx \|\nabla^{l-1}(\varrho, u)\|_{H^1}^2 \approx \|\nabla^{l-1}(\varrho, m)\|_{H^1}^2. \tag{5.5}$$

Then by taking (4.16) plus $C_3\|\nabla^{l-1}(\varrho, u)\|_{L^2}^2$ with some large number $C_3 > 0$, one arrive at

$$\frac{d}{dt}L(t) + C_4 L((t) \le C_3\|\nabla^{l-1}(\varrho, u)\|_{L^2}^2 \tag{5.6}$$

for some constant $C_4 > 0$. Hence by Gronwall's inequality, the definition (5.1) of $H(t)$ and Lemma 5.1, we have that

$$\begin{aligned} L(t) &\le e^{-C_4 t}L(0) + C\int_0^t e^{-C_4(t-\tau)}\|\nabla^{l-1}(\varrho, u)(\tau)\|_{L^2}^2 d\tau \\ &\le e^{-C_4 t}L(0) + C\int_0^t e^{-C_4(t-\tau)}(1+\tau)^{-(\frac{1}{2}+l)}H(t)d\tau \\ &\lesssim (1+t)^{-(\frac{1}{2}+l)}\{L(0) + H(t)\}, \end{aligned} \tag{5.7}$$

where the monotonicity of $H(t)$ is used. Combining (5.7) with (5.5) yields

$$\|\nabla^l(\varrho, m)\|_{L^2}^2 \approx \|\nabla^l(\varrho, u)\|_{L^2}^2 \lesssim L(t) \lesssim (1+t)^{-(l+\frac{1}{2})}\{L(0) + H(t)\},$$

which completes the proof of Lemma 5.2. □

Next we estimate the decay rate of the solution (ϱ, m).

Lemma 5.3. *Under the assumption of Proposition 2.3, it holds that*

$$\|(\varrho, m)\|_{L^2} \lesssim (1+t)^{-3/4}\Big(\|(\varrho_0, n_0)\|_{L^1\cap L^2} + \delta\sqrt{H(t)}\Big). \tag{5.8}$$

Proof. By (3.24), (3.26), (3.28) and the Duhamel principle, we have

$$\begin{aligned} &\|(\varrho, n, M)\|_{L^2} \\ &= \|(\widehat{\varrho}, \widehat{n}, \widehat{M})\|_{L^2} \\ &\lesssim (1+t)^{-3/4}\left(\|(\varrho_0, n_0, M_0)\|_{L^1} + \|(\varrho_0, n_0)\|_{L^2}\right) \\ &\quad + \int_0^t (1+t-\tau)^{-3/4}\left(\|\left(\Lambda^{-1}\operatorname{div} N, \Lambda^{-1}\operatorname{curl} N\right)\|_{L^1} + \|\Lambda^{-1}\operatorname{div} N\|_{L^2}\right)d\tau. \end{aligned} \tag{5.9}$$

We define

$$\begin{aligned} F_1 = &\big(-P(1+\varrho) + P(1) + P'(1)\varrho\big)I_3 + \gamma\nabla\psi\otimes\nabla\psi \\ &- \frac{\gamma}{2}\big(|\psi|^2 + |\nabla\psi|^2\big)I_3 - \frac{m\otimes m}{1+\varrho}, \end{aligned} \tag{5.10}$$

$$F_2 = \frac{\varrho m}{1+\varrho}. \tag{5.11}$$

Then from the definition (2.10) of N, we have that $|F| \lesssim |F_1| + |\nabla F_2|$. Moreover, F_1 and F_2 can be treated as the product of smooth functions depending on ϱ, ψ, $\nabla\psi$ and/or m. Therefore, we can estimate the terms in (5.9) in the following:

$$\begin{aligned} \|\left(\Lambda^{-1}\operatorname{div} N, \Lambda^{-1}\operatorname{curl} N\right)\|_{L^1} &\lesssim \|\nabla F\|_{L^1} \\ &\lesssim \|\nabla F_1\|_{L^1} + \|\nabla^2 F_2\|_{L^1} \\ &\lesssim \|(\varrho, \nabla\varrho, \psi, \nabla\psi, m)\|_{L^2}\|(\nabla\varrho, \nabla\psi, \nabla m)\|_{H^1} \\ &\lesssim \delta(1+t)^{-5/4}\sqrt{H(t)}, \end{aligned} \tag{5.12}$$

where (5.1) and Lemma 6.2 are used, and similarly,

$$\begin{aligned}\|\Lambda^{-1}\operatorname{div} N\|_{L^2} &\approx \|\operatorname{div} F\|_{L^2}\\ &\lesssim \|\nabla F_1\|_{L^2}+\|\nabla^2 F_2\|_{L^2}\\ &\lesssim \|(\varrho,\nabla\varrho,\psi,\nabla\psi,m)\|_{L^\infty}\|(\nabla\varrho,\nabla\psi,\nabla m)\|_{H^1}\\ &\lesssim \delta(1+t)^{-5/4}\sqrt{H(t)}.\end{aligned} \tag{5.13}$$

Plugging (5.12)–(5.13) into (5.9) yields

$$\begin{aligned}\|(\varrho,m)\|_{L^2} &\le \|(\widehat{\varrho},\widehat{n},\widehat{M})\|_{L^2}\\ &\lesssim (1+t)^{-3/4}\left(\|(\varrho_0,n_0,M_0)\|_{L^1}+\|(\varrho_0,n_0)\|_{L^2}\right)\\ &\quad+\int_0^t (1+t-\tau)^{-3/4}\delta(1+\tau)^{-5/4}\sqrt{H(t)}d\tau\\ &\lesssim (1+t)^{-3/4}\left(\|(\varrho_0,m_0)\|_{L^1\cap L^2}+\delta\sqrt{H(t)}\right).\end{aligned} \tag{5.14}$$

□

Now we estimate the decay of $\|\nabla^{l-1}(\varrho,m)\|_{L^2}$.

Lemma 5.4. *Under the assumption of Proposition 2.3, it holds*

$$\|\nabla^{l-1}(\varrho,m)\|_{L^2}\lesssim (1+t)^{-(\frac14+\frac l2)}\left(\|(\varrho_0,m_0)\|_{L^1\cap H^{l-1}}+\delta\sqrt{H(t)}\right). \tag{5.15}$$

Proof. By using the Duhamel principal (1.5) with $k=l-1$ and $r=1$, we deduce from Proposition 3.3 that

$$\begin{aligned}\|\nabla^{l-1}\varrho\|_{L^2} &\lesssim (1+t)^{-(\frac14+\frac l2)}\left(\|(\varrho_0,n_0)\|_{L^1}+\|(\varrho_0,n_0)\|_{H^{l-1}}\right)\\ &\quad+\int_0^{t/2}(1+t-\tau)^{-(\frac14+\frac l2)}\|\Lambda^{-1}\operatorname{div} N(\tau)\|_{L^1}d\tau\\ &\quad+\int_{t/2}^t (1+t-\tau)^{-5/4}\|\nabla^{l-2}\Lambda^{-1}\operatorname{div} N(\tau)\|_{L^1}d\tau\\ &\quad+\int_0^t e^{-R(t-\tau)}\|\nabla^{l-2}\Lambda^{-1}\operatorname{div} N(\tau)\|_{L^2}d\tau.\end{aligned} \tag{5.16}$$

By Lemmaa 5.2, 6.2 and 6.4, we have

$$\begin{aligned}\|\nabla^{l-2}\Lambda^{-1}\operatorname{div} N\|_{L^1} &\lesssim \|\nabla^{l-1}F_1\|_{L^1}+\|\nabla^l F_2\|_{L^1}\\ &\lesssim \|(\varrho,\psi,\nabla\psi,m)\|_{L^2}\|\nabla^{l-1}(\varrho,\psi,m)\|_{H^1}\\ &\lesssim \delta(1+t)^{-(\frac14+\frac l2)}\left(\|\varrho_0,u_0)\|_{H^l}+\sqrt{H(t)}\right),\end{aligned} \tag{5.17}$$

and

$$\begin{aligned}\|\nabla^{l-2}\Lambda^{-1}\operatorname{div} N\|_{L^2} &\lesssim \|\nabla^{l-1}F_1\|_{L^2}+\|\nabla^l F_2\|_{L^2}\\ &\lesssim \|(\varrho,\psi,\nabla\psi,m)\|_{L^\infty}\|\nabla^{l-1}(\varrho,\psi,m)\|_{H^1}\\ &\lesssim \delta(1+t)^{-(\frac14+\frac l2)}\left(\|\varrho_0,u_0)\|_{H^l}+\sqrt{H(t)}\right).\end{aligned} \tag{5.18}$$

Then, by substituting the estimates (5.12), (5.17) and (5.18) into (5.16), we obtain

$$
\begin{aligned}
&\|\nabla^{l-1}\varrho\|_{L^2}\\
&\lesssim (1+t)^{-(\frac{1}{4}+\frac{l}{2})}\|(\varrho_0,n_0)\|_{L^1\cap H^{l-1}}\\
&\quad+\int_0^{t/2}(1+t-\tau)^{-(\frac{1}{4}+\frac{l}{2})}\delta(1+\tau)^{-5/4}\sqrt{H(t)}d\tau\\
&\quad+\int_{t/2}^{t}(1+t-\tau)^{-5/4}\delta(1+\tau)^{-(\frac{1}{4}+\frac{l}{2})}\Big(\|(\varrho_0,u_0)\|_{H^l}+\sqrt{H(t)}\Big)d\tau\\
&\quad+\int_0^{t}e^{-R(t-\tau)}\delta(1+\tau)^{-(\frac{1}{4}+\frac{l}{2})}\Big(\|(\varrho_0,u_0)\|_{H^l}+\sqrt{H(t)}\Big)d\tau\\
&\lesssim (1+t)^{-(\frac{1}{4}+\frac{l}{2})}\Big(\|(\varrho_0,m_0)\|_{L^1\cap H^l}+\delta\sqrt{H(t)}\Big).
\end{aligned}
\tag{5.19}
$$

Similarly as in (5.16), we have the estimate

$$
\begin{aligned}
\|\nabla^{l-1}n\|_{L^2}\lesssim\ &(1+t)^{-(\frac{1}{4}+\frac{l}{2})}\left(\|(\varrho_0,n_0)\|_{L^1}+\|(\varrho_0,n_0)\|_{H^{l-1}}\right)\\
&+\int_0^{t/2}(1+t-\tau)^{-(\frac{1}{4}+\frac{l}{2})}\|\Lambda^{-1}\operatorname{div}N(\tau)\|_{L^1}d\tau\\
&+\int_{t/2}^{t}(1+t-\tau)^{-5/4}\|\nabla^{l-2}\Lambda^{-1}\operatorname{div}N(\tau)\|_{L^1}d\tau\\
&+\int_0^{t}e^{-R(t-\tau)}\|\nabla^{l-3}\Lambda^{-1}\operatorname{div}N(\tau)\|_{L^2}d\tau.
\end{aligned}
\tag{5.20}
$$

Here we only estimate the last term in (5.20) as follows.

$$
\begin{aligned}
&\|\nabla^{l-3}\Lambda^{-1}\operatorname{div}N\|_{L^2}\\
&\lesssim \|\nabla^{l-2}F_1\|_{L^2}+\|\nabla^{l-1}F_2\|_{L^2}\\
&\lesssim \|(\varrho,\psi,\nabla\psi,m)\|_{L^3}\|\nabla^{l-2}(\varrho,\psi,\nabla\psi,m)\|_{L^6}\\
&\quad+\|\nabla^{l-3}(P'(1+\varrho)-P'(1))\|_{L^6}\|\nabla\varrho\|_{L^3}+\|(\varrho,m)\|_{L^\infty}\|\nabla^{l-1}\big(\frac{\varrho}{1+\varrho},m\big)\|_{L^2}\\
&\lesssim \|(\varrho,m)\|_{H^2}\|\nabla^{l-1}(\varrho,m)\|_{L^2}+\|\nabla^{l-2}\varrho\|_{L^2}\|\nabla\varrho\|_{L^3}\\
&\lesssim \Big(\|(\varrho,m)\|_{H^2}+\|\varrho\|_{L^2}^{\frac{1}{l-1}}\|\varrho\|_{L^{\frac{6(l-2)}{2l-5}}}^{\frac{l-2}{l-1}}\Big)\|\nabla^{l-1}(\varrho,m)\|_{L^2}\\
&\lesssim \delta(1+t)^{-(\frac{1}{4}+\frac{l}{2})}\sqrt{H(t)}.
\end{aligned}
\tag{5.21}
$$

Hence plugging (5.12), (5.17) and (5.21) into (5.20) leads to

$$
\|\nabla^{l-1}n\|_{L^2}\lesssim (1+t)^{-(\frac{1}{4}+\frac{l}{2})}\Big(\|(\varrho_0,m_0)\|_{L^1\cap H^l}+\delta\sqrt{H(t)}\Big).\tag{5.22}
$$

Similarly we have

$$
\|\nabla^{l-1}M\|_{L^2}\lesssim (1+t)^{-(\frac{1}{4}+\frac{l}{2})}\Big(\|(\varrho_0,m_0)\|_{L^1\cap H^l}+\delta\sqrt{H(t)}\Big).\tag{5.23}
$$

This combining with (5.19) and (5.22) yields (5.8). □

By the definition (5.1) of $H(t)$ and using the smallness of δ, we have from Lemmas 5.3–5.4 and Sobolev interpolation inequality that

$$
H(t)\le CK_0^2.\tag{5.24}
$$

Moreover, by (5.24) and Lemma 5.2, we can obtain

$$\|\nabla^l(\varrho,m)\|_{L^2}\lesssim(1+t)^{-(\frac14+\frac l2)}. \tag{5.25}$$

Finally, we only need to obtain the optimal estimate on the highest-order derivatives of m to complete the proof of Proposition 2.3.

Lemma 5.5. *Under the assumption of Proposition 2.3, it holds*

$$\|\nabla^l m\|_{L^2}\le C(1+t)^{-(\frac34+\frac l2)}. \tag{5.26}$$

Proof. By using the Duhamel principal (1.5) with $k=l$ and $r=2$, we deduce from Proposition 3.3 that

$$\begin{aligned}\|\nabla^l n\|_{L^2}&\lesssim(1+t)^{-(\frac34+\frac l2)}\{\|(\varrho_0,n_0)\|_{L^1}+\|(\varrho_0,n_0)\|_{H^l}\}\\&\quad+\int_0^{t/2}(1+t-\tau)^{-(\frac34+\frac l2)}\|\Lambda^{-1}\operatorname{div}N(\tau)\|_{L^1}d\tau\\&\quad+\int_{t/2}^t(1+t-\tau)^{-\frac74}\|\nabla^{l-2}\Lambda^{-1}\operatorname{div}N(\tau)\|_{L^1}d\tau\\&\quad+\int_0^t e^{-R(t-\tau)}\|\nabla^{l-2}\Lambda^{-1}\operatorname{div}N(\tau)\|_{L^2}d\tau.\end{aligned} \tag{5.27}$$

Using (5.17) and (5.18), we have

$$\|\nabla^{l-2}\Lambda^{-1}\operatorname{div}N\|_{L^1}\lesssim\|(\varrho,m)\|_{L^2}\|\nabla^{l-1}(\varrho,m)\|_{H^1}\lesssim K_0^2(1+t)^{-(1+\frac l2)}, \tag{5.28}$$

$$\|\nabla^{l-2}\Lambda^{-1}\operatorname{div}N\|_{L^2}\lesssim\|(\varrho,m)\|_{H^2}\|\nabla^{l-1}(\varrho,m)\|_{H^1}\lesssim K_0^2(1+t)^{-(1+\frac l2)}. \tag{5.29}$$

Plugging (5.12), (5.28) and (5.29) into (5.27), one arrives at

$$\begin{aligned}&\|\nabla^l n\|_{L^2}\\&\lesssim K_0(1+t)^{-(\frac34+\frac l2)}+\int_0^{t/2}(1+t-\tau)^{-(\frac34+\frac l2)}\delta(1+\tau)^{-5/4}\sqrt{H(t)}d\tau\\&\quad+K_0^2\int_{t/2}^t(1+t-\tau)^{-\frac74}(1+\tau)^{-(1+\frac l2)}d\tau\\&\quad+K_0^2\int_0^t e^{-R(t-\tau)}(1+\tau)^{-(1+\frac l2)}d\tau\\&\lesssim(K_0+K_0^2)(1+t)^{-(\frac34+\frac l2)}.\end{aligned} \tag{5.30}$$

Similarly we deduce the same decay rate on $\nabla^l M$ and obtain (5.26). □

By combining the definition of $H(t)$, the inequality (5.24), and Lemma 5.5, we complete the proof of Proposition 2.3.

Now we estimate the lower boundedness of the decay rate for the solution to complete the proof of Proposition 2.5. First from (3.18) and (3.19), for $|\xi|\ll1$ we have

$$\begin{aligned}g_1(\lambda_+,\lambda_-)&\ge\cos\Big(\Big((P'(1)+\gamma)^{1/2}|\xi|+O(|\xi|^3)\Big)t\Big)e^{-(\mu+\frac\nu2)|\xi|^2t}\\&\quad-C|\xi|e^{-(\mu+\frac\nu2)|\xi|^2t},\end{aligned} \tag{5.31}$$

and

$$g_2(\lambda_+,\lambda_-)\sim C|\xi|^{-1}\sin\Big(\Big(\sqrt{P'(1)+\gamma}|\xi|+O(|\xi|^3)\Big)t\Big)e^{-(\mu+\frac\nu2)|\xi|^2t}, \tag{5.32}$$

Hence by using the Duhamel principal (1.5) and the condition that $\widehat{m}_0(\xi) = 0$ for $0 \le |\xi| \ll 1$, we deduce from (3.30) with $s = 2$ that

$$\begin{aligned}\|\varrho\|_{L^2}^2 = \|\widehat{\varrho}\|_{L^2}^2 \ge &\int_{|\xi|<\eta} |g_1(\lambda_+,\lambda_-)|^2 |\widehat{\varrho}_0|^2 d\xi \\ &- C\int_0^t \Big((1+t-\tau)^{-2}\|\Lambda^{-2}\Lambda^{-1}\operatorname{div} N(\tau)\|_{L^2}^2 \\ &+ e^{-2R(t-\tau)}\|\Lambda^{-1}\operatorname{div} N(\tau)\|_{L^2}^2\Big)d\tau.\end{aligned} \tag{5.33}$$

In the spirit of [17], the first term in the right hand side of (5.33) can be estimated as

$$\begin{aligned}&\int_{|\xi|<\eta} |g_1(\lambda_+,\lambda_-)|^2 |\widehat{\varrho}_0|^2 d\xi \\ &\ge \int_{|\xi|<\eta} \cos^2\Big(\big(\sqrt{P'(1)+\gamma}|\xi| + O(|\xi|^3)\big)t\Big) e^{-(2\mu+\nu)|\xi|^2 t} |\widehat{\varrho}_0|^2 d\xi \\ &\quad - C\int_{|\xi|<\eta} |\xi|^2 e^{-(2\mu+\nu)|\xi|^2 t}|\widehat{\varrho}_0|^2 d\xi \\ &\ge \int_{|\xi|<\eta} \cos^2\Big(\Big(\sqrt{P'(1)+\gamma}|\xi|\Big)t\Big) e^{-(2\mu+\nu)|\xi|^2 t}|\widehat{\varrho}_0|^2 d\xi \\ &\quad - C\int_{|\xi|<\eta} \big(|\xi|^3 t\big)^2 e^{-(2\mu+\nu)|\xi|^2 t}|\widehat{\varrho}_0|^2 d\xi - C\int_{|\xi|<\eta} |\xi|^2 e^{-(2\mu+\nu)|\xi|^2 t}|\widehat{\varrho}_0|^2 d\xi \\ &\ge Cc_0^2K_0^2 t^{-3/2}\int_{r\le\eta\sqrt{t}} \cos^2\Big(\sqrt{P'(1)+\gamma}\, r\sqrt{t}\Big) e^{-(2\mu+\nu)r^2} r^2 dr \\ &\quad - C(1+t)^{-5/2}\|\varrho_0\|_{L^1}^2 \\ &\ge Cc_0^2K_0^2 t^{-3/2} - CK_0^2(1+t)^{-5/2},\end{aligned} \tag{5.34}$$

where $|\widehat{\varrho_0}| > c_0K_0$ and $t \ge t_0$ for some sufficiently large $t_0 > 0$.

Furthermore, by using the estimates in (5.17), we have

$$\begin{aligned}\|\Lambda^{-2}\Lambda^{-1}\operatorname{div} N\|_{L^2}^2 &\lesssim \|\Lambda^{-1}F_1\|_{L^2}^2 + \|F_2\|_{L^2}^2 \\ &\lesssim \|F_1\|_{L^{6/5}}^2 + \|F_2\|_{L^2}^2 \\ &\lesssim \|(\varrho, m)\|_{H^1}^4 \lesssim (1+t)^{-3},\end{aligned} \tag{5.35}$$

where (6.10) is used, and obviously

$$\|\Lambda^{-1}\operatorname{div} N\|_{L^2}^2 \le C(1+t)^{-3}. \tag{5.36}$$

Plugging (5.34)–(5.36) into (5.33), and taking t_0 large enough, one arrives at

$$\|\varrho\|_{L^2}^2 \ge Cc_0^2K_0^2 t^{-3/2}. \tag{5.37}$$

As in (5.33), from (1.5) and (3.32) we have

$$
\begin{aligned}
\|n\|_{L^2}^2 &= \|\widehat{n}\|_{L^2}^2 \\
&\geq \int_{|\xi|<\eta} \Big||\xi|\big(P'(1)+\frac{\gamma}{1+|\xi|^2}\big)g_2(\lambda_+,\lambda_-)\Big|^2|\widehat{\varrho}_0|^2d\xi \\
&\quad - C\int_0^t \Big((1+t-\tau)^{-2}\|\Lambda^{-2}\Lambda^{-1}\operatorname{div} N\|_{L^2}^2 \\
&\quad + e^{-2R(t-\tau)}\|\Lambda^{-1}\operatorname{div} N\|_{L^2}^2\Big)d\tau \\
&\geq Cc_0^2K_0^2t^{-3/2} - C(1+t)^{-2},
\end{aligned} \tag{5.38}
$$

which gives

$$
\|n\|_{L^2}^2 \geq Cc_0^2K_0^2t^{-3/2} \tag{5.39}
$$

with $t \geq t_0$. Moreover, by using the condition $\widehat{m}_0(\xi)=0$ for $0\leq|\xi|\ll 1$ again we have that

$$
\begin{aligned}
\|M\|_{L^2}^2 &= \|\widehat{M}\|_{L^2}^2 \\
&\leq \int_{|\xi|\geq\eta} e^{-(2\mu+\nu)|\xi|^2t}|\widehat{M}_0|^2d\xi \\
&\quad + C\int_0^t (1+t-\tau)^{-2}\|\Lambda^{-2}\Lambda^{-1}\operatorname{curl} N\|_{L^2}^2d\tau \\
&\leq \int_{|\xi|\geq\eta} e^{-(2\mu+\nu)\eta^2t}|\widehat{M}_0|^2d\xi + C(1+t)^{-2} \\
&\leq C(1+t)^{-2},
\end{aligned} \tag{5.40}
$$

Thus we conclude that for $t\geq t_0$,

$$
\|m\|_{L^2}^2 \geq \|n\|_{L^2}^2 - \|M\|_{L^2}^2 \geq CK_0^2t^{-3/2}. \tag{5.41}
$$

Next we shall estimate the lower boundedness on the first derivatives of the solution. As in (5.33), we have from (1.5) and (3.31) that

$$
\begin{aligned}
\|\nabla\varrho\|_{L^2}^2 &\geq \int_{|\xi|<\eta} |g_1(\lambda_+,\lambda_-)|^2|\xi|^2|\widehat{\varrho}_0|^2d\xi \\
&\quad - C\int_0^t (1+t-\tau)^{-3}\left(\|\Lambda^{-2}\Lambda^{-1}\operatorname{div} N\|_{L^2}^2 + \|\Lambda^{-1}\operatorname{div} N\|_{L^2}^2\right)d\tau \\
&\geq CK_0^2t^{-5/2} - C(1+t)^{-3} \\
&\geq CK_0^2t^{-5/2}
\end{aligned} \tag{5.42}
$$

with some sufficiently large positive constant t_0. Similarly, we have

$$
\|\nabla m\|_{L^2}^2 \geq CK_0^2t^{-5/2}. \tag{5.43}
$$

Moreover, by the Sobolev interpolation inequality, we can deduce that for $2\leq k\leq l-1$,

$$
\|\nabla\varrho\|_{L^2} \lesssim \|\varrho\|_{L^2}^{\frac{k-1}{k}}\|\nabla^k\varrho\|_{L^2}^{1/k} \lesssim (1+t)^{-\frac{3(k-1)}{4k}}\|\nabla^k\varrho\|_{L^2}^{1/k}, \tag{5.44}
$$

which implies that for $2\leq k\leq l-1$,

$$
\|\nabla^k\varrho\|_{L^2} \geq (1+t)^{\frac{3(k-1)}{4}}\|\nabla\varrho\|_{L^2}^k \geq t^{\frac{3(k-1)}{4}}t^{-\frac{5}{4}k} = t^{-\left(\frac{3}{4}+\frac{k}{2}\right)}. \tag{5.45}
$$

Similarly, for $2 \le k \le l$, we have

$$\|\nabla^l m\|_{L^2} \ge t^{-\left(\frac{3}{4}+\frac{k}{2}\right)}, \tag{5.46}$$

which complets the proof of Proposition 2.5.

6. Appendix: Analytic tools

We will extensively use the Sobolev interpolation of the Gagliardo-Nirenberg inequality.

Lemma 6.1. *Let $0 \le i, j \le k$, then we have*

$$\|\nabla^i f\|_{L^p} \lesssim \|\nabla^j f\|_{L^q}^{1-a} \|\nabla^k f\|_{L^r}^a, \tag{6.1}$$

where a satisfies

$$\frac{i}{3} - \frac{1}{p} = \big(\frac{j}{3} - \frac{1}{q}\big)(1-a) + \big(\frac{k}{3} - \frac{1}{r}\big)a. \tag{6.2}$$

Especially, when $p = q = r = 2$, we have

$$\|\nabla^i f\|_{L^2} \lesssim \|\nabla^j f\|_{L^2}^{\frac{k-i}{k-j}} \|\nabla^k f\|_{L^2}^{\frac{i-j}{k-j}}. \tag{6.3}$$

The above lemma is a special case of [22, theorem on p. 125]. To estimate the commutator and the product of two functions, we shall recall the following estimate.

Lemma 6.2 ([2, 14])**.** *For $k \ge 0$, we have*

(i)

$$\|\Lambda^k (gh)\|_{L^{p_0}} \lesssim \|g\|_{L^{p_1}} \|\Lambda^k h\|_{L^{p_2}} + \|\Lambda^k g\|_{L^{p_3}} \|h\|_{L^{p_4}}, \tag{6.4}$$

where $p_0, p_2, p_3 \in (1, \infty)$ and

$$\frac{1}{p_0} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

(ii)

$$\|\nabla^k (gh)\|_{L^1} \lesssim \|g\|_{L^2} \|\nabla^k h\|_{L^2} + \|\nabla^k g\|_{L^2} \|h\|_{L^2}, \tag{6.5}$$

Thus we can easily deduce from Lemma 6.2 the following commutator estimate, or one can refer to [19, pp. 98, Lemma 3.4].

Lemma 6.3 ([19])**.** *Let f and g be smooth functions belonging to $H^k \cap L^\infty$ for any integer $k \ge 1$ and define the commutator*

$$[\nabla^k, f]g = \nabla^k (fg) - f \nabla^k g. \tag{6.6}$$

Then we have

$$\|[\nabla^k, f]g\|_{L^2} \lesssim \|\nabla f\|_{L^\infty} \|\nabla^{k-1} g\|_{L^2} + \|\nabla^k f\|_{L^2} \|g\|_{L^\infty}. \tag{6.7}$$

Next, to estimate the L^2-norm of the spatial derivatives of some smooth function $F(f)$, we shall recall the following estimate.

Lemma 6.4 ([1, 2])**.** *Let $F(f)$ be a smooth function of f with bounded derivatives of any order and f belong to H^k for any integer $k \ge 3$, then we have*

$$\begin{aligned} &\|\nabla^k (F(f))\|_{L^2} \\ &\lesssim \sup_{0 \le i \le k} \|F^{(i)}(f)\|_{L^\infty} \Big(\sum_{j=2}^{k} \|f\|_{L^2}^{j-1-\frac{3(j-1)}{2k}} \|\nabla^k f\|_{L^2}^{1+\frac{3(j-1)}{2k}} + \|\nabla^k f\|_{L^2} \Big). \end{aligned} \tag{6.8}$$

Moreover, if f has lower and upper bounds, and $\|f\|_k \le 1$, we have

$$\|\nabla^k(F(f))\|_{L^2} \lesssim \|\nabla^k f\|_{L^2}. \tag{6.9}$$

Lemma 6.5 ([9, 24])**.** *Let $0 < s < 3$, $1 < p < q < \infty$, $\frac{1}{q} + \frac{s}{3} = \frac{1}{p}$, then*

$$\|\Lambda^{-s} f\|_{L^q} \lesssim \|f\|_{L^p}. \tag{6.10}$$

In particular, for $-1 < s < 3/2$, we have

$$\|\Lambda^{-s} f\|_{L^2} \lesssim \|f\|_{L^1}^{\frac{2+2s}{5}} \|\nabla f\|_{L^2}^{\frac{3-2s}{5}}. \tag{6.11}$$

Proof. (6.10) can be seen in [24, p. 119, Theorem 1], and [9, P. 10, Remark 1.2.2]. (6.11) can be proved using (6.10) and (6.1). □

Acknowledgments. Qing Chen was supported by Natural Science Foundation of Fujian Province (No. 2018J01430). Guochun Wu was supported by National Natural Science Foundation of China (No. 11701193, 11671086), by the Natural Science Foundation of Fujian Province (No. 2018J05005, 2017J01562), by the Program for Innovative Research Team in Science and Technology in Fujian Province University Quanzhou High-Level Talents Support Plan (No. 2017ZT012). Yinghui Zhang was supported by the Guangxi Natural Science Foundation (No. 2019JJG110003, 2019AC20214), and by the National Natural Science Foundation of China (No. 11771150, 11571280, 11301172 and 11226170.)

References

[1] Q. Chen; *Optimal time decay rates for the compressible Navier-Stokes system*, preprint.
[2] Q. Chen, Z. Tan; *Time decay of solutions to the compressible euler equations with damping*, Kinetic and Related Models, 7(4) (2014), 605–619.
[3] N. Chikami; *The blow-up criterion for the compressible Navier-Stokes system with a Yukawa-potential in the critical Besov space*, Differ. Integral Equ., 27 (2014), 801–820.
[4] R. Danchin; *Global existence in critical spaces for compressible Navier-Stokes equations*, Invent. math., 141 (2000), 579–614.
[5] R. Danchin, J. Xu; *Optimal time-decay estimates for the compressible Navier-Stokes equations in the critical L^p framework*, Arch. Rational Mech. Anal., 224 (2017), 53–90.
[6] R. J. Duan, H. X. Liu, S. Ukai, T. Yang; *Optimal L^p-L^q convergence rate for the compressible Navier-Stokes equations with potential force*, J. Differential Equations, 238 (2007), 220–233.
[7] R. J. Duan, S. Ukai, T. Yang, H. J. Zhao; *Optimal convergence rate for the compressible Navier-Stokes equations with potential force*, Math. Mod. Meth. Appl. Sci., 17 (2007), 737–758.
[8] B. Ducomet; *Simplified models of quantum fluids in nuclear physics*, Proceedings of Partial Differential Equations and Applications (Olomouc, 1999), Math. Bohem., 126(2) (2001), 323–336.
[9] L. Grafakos; *Modern fourier analysis*, Springer, 2009.
[10] Y. Guo, Y. J. Wang; *Decay of dissipative equations and negative Sobolev spaces*, Comm. PDE, 37 (2012), 2165–2208.
[11] D. Hoff, K. Zumbrun; *Multi-dimensional diffusion waves for the Navier-Stokes equations of compressible flow*, Indiana U. Math. J., 44 (1995), 603–676.
[12] D. Hoff, K. Zumbrun; *Pointwise decay estimates for multidimensional Navier-Stokes diffusion waves*, Z. Angew. Math. Phys., 48 (1997), 597–614.
[13] X. P. Hu, G. C. Wu; *Optimal rates of decay for solutions to the isentropic compressible Navier-Stokes equations with discontinuous initial data*, J. Differential Equations, 269 (2020), 8132–8172.
[14] N. Ju; *Existence and uniqueness of the solution to the dissipative 2D quasi-geostrophic equations in the Sobolev space*, Comm. Math. Phys., 251 (2) (2004), 365–376.
[15] S. Kawashima; *System of a Hyperbolic-Parabolic Composite Type, with Applications to the Equations of Magnetohydrodynamics*, thesis, Kyoto University, Kyoto, 1983.

[16] H. Li, A. Matsumura, G. Zhang; *Optimal decay rate of the compressible Navier-Stokes-Poisson system in* $\mathbb{R}^3$, Arch. Ration. Mech. Anal., 196 (2010) 681-713.
[17] H. L. Li, T. Zhang; *Large time behavior of isentropic compressible Navier-Stokes system in* $\mathbb{R}^3$, Math. Meth. Appl. Sci. 34 (2011), 670–682.
[18] T. P. Liu, W. K. Wang; *The pointwise estimates of diffusion waves for the Navier-Stokes equations in odd multi-dimensions*, Comm. Math. Phys., 196 (1998), 145–173.
[19] A. J. Majda, A. L. Bertozzi; *Vorticity and incompressible flow*, Cambridge University Press, Cambridge, 2002.
[20] A. Matsumura, T. Nishida; *The initial value problem for the equations of motion of compressible viscous and heat-conductive fluids*, Proc. Japan Acad. Ser. A, 55 (1979), 337–342.
[21] A. Matsumura, T. Nishida; *The initial value problem for the equations of motion of viscous and heat-conductive gases*, J Math Kyoto Univ, 20 (1980), 67–104.
[22] L. Nirenberg; *On elliptic partial differential equations*, Ann. Scuola Norm. Sup. Pisa, 13 (1959), 115–162.
[23] G. Ponce; *Global existence of small solution to a class of nonlinear evolution equations*, Nonlinear Anal., 9 (1985), 339–418.
[24] E. M. Stein; *Singular Integrals and Differentiability Properties of Functions*, Princeton, NJ: Princeton University Press, 1970.
[25] S. Ukai, T. Yang; *The Boltzmann equation in the space* $L^2 \cap L^\infty_\beta$*: global and time-periodic solutions*, Anal. Appl., 4 (2006), 263–310.
[26] Y. Wang; *Decay of the Navier-Stokes-Poisson equations*, J. Differential Equations, 253(2012) 273-297.
[27] Z. P. Xin, J. Xu; *Optimal decay for the compressible Navier-Stokes equations without additional smallness assumptions*, https://arxiv.org/abs/1812.11714.
[28] Y. Zheng; *Global smooth solutions to the adiabatic gas dynamics system with dissipation terms*, Chinese Ann. Math. Ser. A, 17 (1996), 155–162.

Qing Chen
School of Applied Mathematics, Xiamen University of Technology, Xiamen, Fujian 361024, China
Email address: chenqing@xmut.edu.cn

Guochun Wu
Fujian Province University Key Laboratory of Computational Science, School of Mathematical Sciences, Huaqiao University, Quanzhou 362021, China
Email address: guochunwu@126.com

Yinghui Zhang (corresponding author)
School of Mathematics and Statistics, Guangxi Normal University, Guilin, Guangxi 541004, China
Email address: yinghuizhang@mailbox.gxnu.edu.cn

Lan Zou
Fujian Province University Key Laboratory of Computational Science, School of Mathematical Sciences, Huaqiao University, Quanzhou 362021, China
Email address: zlyoung@163.com