

SCHRÖDINGER-POISSON SYSTEMS WITH SINGULAR POTENTIAL AND CRITICAL EXPONENT

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ABSTRACT. In this article we study the Schrödinger-Poisson system

$$\begin{aligned} -\Delta u + V(|x|)u + \lambda\phi u &= f(u), \quad x \in \mathbb{R}^3, \\ -\Delta\phi &= u^2, \quad x \in \mathbb{R}^3, \end{aligned}$$

where V is a singular potential with the parameter α and the nonlinearity f satisfies critical growth. By applying a generalized version of Lions-type theorem and the Nehari manifold theory, we establish the existence of the nonnegative ground state solution when $\lambda = 0$. By the perturbation method, we obtain a nontrivial solution to above system when $\lambda \neq 0$.

1. INTRODUCTION

We consider the Schrödinger-Poisson system

$$\begin{aligned} -\Delta u + V(|x|)u + \lambda\phi u &= f(u), \quad x \in \mathbb{R}^3, \\ -\Delta\phi &= u^2, \quad x \in \mathbb{R}^3, \end{aligned} \tag{1.1}$$

where V is a singular potential with the parameter α and satisfies the following conditions:

- (A1) There exist $B \geq A > 0$ such that $A/t^\alpha \leq V(t) \leq B/t^\alpha$ for almost all $t > 0$.
- (A2) $V \in L^1(a, b)$ for some (a, b) with $b > a > 0$.

The simplest function satisfying the above assumptions is $V(x) = 1/|x|^\alpha$. This is the so-called external Coulomb potential for Helium, see [27]. Coulomb potential arises in many scientific areas such as quantum mechanics, nuclear physics, molecular physics and quantum cosmology. For more details on the Coulomb potential, we refer to [2, 20] and on the physical phenomena of system (1.1), we refer to [10, 14].

System (1.1) was initially introduced as a model describing waves interacting with its own electrostatic field in quantum mechanics [9], and is related to

$$\begin{aligned} i\frac{\partial\psi}{\partial t} &= -\Delta\psi + W(|x|)\psi + \lambda\phi\psi - f(\psi), \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ -\Delta\phi &= \psi^2, \quad x \in \mathbb{R}^3, \end{aligned} \tag{1.2}$$

where the functions u and ϕ represent the wave functions associated with the particle and the electric potential, respectively. $W(|x|)$ denotes an external potential,

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and the nonlinearity $f(u)$ represents the interaction among particles or an external nonlinear perturbation. It is well-known that the standing waves $\psi(t, x) = e^{i\omega t}u(x)$ is a solution of system (1.2), if and only if the real valued function $u(x)$ solves system (1.1) with $V(|x|) = W(|x|) + \lambda$.

In the past decades, system (1.1) has attracted considerable attention in the community of mathematical physics. In particular, the existence and nonexistence of ground state solutions, nodal solutions and multiplicity of solutions have been extensively studied [3, 17, 19, 24, 25] and qualitative properties such as regularity, symmetry, uniqueness and decay of nontrivial solutions to system (1.1) can be seen in [16, 18, 29, 33] etc. For example, Ruiz [30] considered the existence and multiplicity of positive solutions to system (1.1) with the suitable parameter λ and $f(u) = u^p$ for $p \in (1, 5)$. The nonexistence results were also obtained for $p \leq 3$ and $p \geq 6$. Azzollini-Pomponio [4] obtained a positive ground state solution of system (1.1) with $\lambda = 1$ and $f(u) = u^p$ for $p \in (2, 5)$. Ambrosetti-Ruiz [1] extended the results described in [30] and proved that system (1.1) admits infinitely many solutions when $p \in (2, 5)$ and $\lambda > 0$. Some multiplicity results for system (1.1) were also established with the proper range of the parameter λ and $p \in (1, 2)$ or $p = 2$, respectively.

Recently, there have been a number of results of system (1.1) under various assumptions on the potential V . When V is a sign-changing potential, Batista-Furtado [8] obtained a nonnegative solution and a sign-changing solution for the Schrödinger-Poisson systems by employing the Nehari manifold theory and variational methods. When V vanishes at infinity, Bonheure-Di Cosmo-Mercuri [11] investigated the existence and concentration phenomena of solutions to a class of Schrödinger-Poisson system, under the following conditions:

- (A3) $V \in C(\mathbb{R}^3, \mathbb{R})$ and $\inf_{x \in \mathbb{R}^3} V(x) \geq V_0 > 0$.
- (A4) There exists $z > 0$ such that the set $\{x \in \mathbb{R}^3 | V(x) < z\}$ is nonempty and has finite measure.
- (A5) $\Omega = \text{int}V^{-1}(0)$ is nonempty and has smooth boundary with $\bar{\Omega} = V^{-1}(0)$.

Bartsch-Wang [7] considered a nonlinear Schrödinger equation, where λV is called the steep potential well. Jiang-Zhou [23] presented the existence of nontrivial solutions to system (1.1) with $f(u) = |u|^{p-1}u$ for $p \in (1, 5)$.

He-Zou [21] studied the semiclassical solutions of the Schrödinger-Poisson system

$$\begin{aligned} -\varepsilon^2 \Delta u + V(x)u + \phi u &= f(u) + |u|^4 u, & x \in \mathbb{R}^3, \\ -\varepsilon^2 \Delta \phi &= u^2, & x \in \mathbb{R}^3, \\ u &\in H^1(\mathbb{R}^3), \quad u > 0, & x \in \mathbb{R}^3, \end{aligned} \tag{1.3}$$

where the potential $V(x)$ satisfies:

- (A6) There is constant $V_0 > 0$ such that $V_0 := \inf_{x \in \mathbb{R}^3} V(x)$.
- (A7) There is a bounded open set $\Omega \subset \mathbb{R}^3$ such that $V_0 < \min_{\partial\Omega} V(x)$ and $M = \{x \in \Omega | V(x) = V_0\} \neq \emptyset$.

Under the above conditions and some suitable hypotheses for $f(u)$, the existence and the concentration results of system (1.3) were presented by using the generalized Nehari manifold theory, penalization techniques and Ljusternik-Schnirelmann theory.

In this article, we are interested in the case when V satisfies (A1)-(A2), which is different from the above mentioned cases. Let us briefly recall some related results

on the Schrödinger equation. Su-Wang-Willem [31] considered the Schrödinger equation

$$\begin{aligned} -\Delta u + V(|x|)u &= Q(|x|)f(u), \quad u > 0, \quad x \in \mathbb{R}^N, \\ |u(x)| &\rightarrow 0, \quad \text{as } |x| \rightarrow \infty, \end{aligned} \quad (1.4)$$

where $N \geq 2$, and V and f satisfy the following assumptions:

(A8) $V \in C((0, \infty), \mathbb{R})$, $V(t) \geq 0$, and there exist a and a_0 such that

$$\liminf_{t \rightarrow 0} \frac{V(t)}{t^{a_0}} > 0, \quad \liminf_{t \rightarrow \infty} \frac{V(t)}{t^a} > 0.$$

(A9) $Q \in C((0, \infty), \mathbb{R})$, $Q(t) > 0$, and there exist b and b_0 such that

$$\limsup_{t \rightarrow 0} \frac{Q(t)}{t^{b_0}} < \infty, \quad \limsup_{t \rightarrow \infty} \frac{Q(t)}{t^b} < \infty.$$

(A10) $f \in C(\mathbb{R}, \mathbb{R})$, $f(0) = 0$, there exists $C > 0$ such that

$$f(u) \leq C(|t|^{p_1-1} + |t|^{p_2-1}), \quad t \in \mathbb{R},$$

where p_1 and p_2 satisfy one of the following conditions

$$\begin{aligned} 2_\alpha^* < p_1 \leq p_2 < 2^*, \quad \alpha \in (0, 2), \\ 2^* < p_1 \leq p_2 < \infty, \quad \alpha \in (2, 2N - 2), \\ 2^* < p_1 \leq p_2 < \infty, \quad \alpha \in [2N - 2, \infty). \end{aligned}$$

(A11) There exists $\mu > 2$ such that $\mu F(t) \leq t f(t)$ for $t \in \mathbb{R}$, where $F(t) := \int_0^t f(s) ds$.

(A12) $F(t) > 0$, for $t \in \mathbb{R}$.

(A13) f is odd.

Based on the improved Strauss radial lemmas, Su-Wang-Willem [31] established some radial inequalities. As an application of these radial inequalities, they obtained some continuous and compact embeddings as follows.

Proposition 1.1 ([31]). *Assume that $N \geq 3$ and conditions (A1), (A2) hold. Then the following continuous embeddings hold:*

$$\begin{aligned} W_{\text{rad}}^{1,2}(\mathbb{R}^N, V) &\hookrightarrow L^s(\mathbb{R}^N), \quad s \in [2_\alpha^*, 2^*], \quad \alpha \in (0, 2), \\ W_{\text{rad}}^{1,2}(\mathbb{R}^N, V) &\hookrightarrow L^s(\mathbb{R}^N), \quad s \in [2^*, 2_\alpha^*], \quad \alpha \in (2, 2N - 2), \\ W_{\text{rad}}^{1,2}(\mathbb{R}^N, V) &\hookrightarrow L^s(\mathbb{R}^N), \quad s \in [2^*, \infty), \quad \alpha \in [2N - 2, \infty). \end{aligned}$$

Furthermore, the embeddings are compact if $s \neq 2_\alpha^*$ or $s \neq 2^*$, where $2_\alpha^* = 2 + \frac{4\alpha}{2N-2-\alpha}$.

By Proposition 1.1 and the variational methods, the existence and multiplicity of positive radial solutions to equation (1.3) were also established.

Li-Su-Zhao [26] studied the Schrödinger-Poisson system:

$$\begin{aligned} -\Delta u + V(|x|)u + \phi u &= \lambda Q(|x|)f(u), \quad x \in \mathbb{R}^3, \\ -\Delta \phi &= u^2, \quad x \in \mathbb{R}^3, \end{aligned} \quad (1.5)$$

and obtained the existence and multiplicity of nontrivial radial solutions to system (1.5) under the following conditions:

(A14) $V \in C(\mathbb{R}^3, \mathbb{R}^+)$ and there exists $\bar{a} \in \mathbb{R}$ such that

$$\liminf_{t \rightarrow \infty} \frac{V(t)}{t^{\bar{a}}} > 0.$$

(A15) $Q \in C(\mathbb{R}^3, \mathbb{R}^+)$ and there exists $\bar{b} \in \mathbb{R}$ such that

$$\limsup_{t \rightarrow \infty} \frac{Q(t)}{t^{\bar{b}}} < \infty.$$

Moreover, the nonlinearity f satisfies:

(A16) $f \in C((-\delta, \delta), \mathbb{R})$ for some $\delta > 0$ and there exists $q_1 \in (4, 6)$ such that

$$\lim_{|t| \rightarrow 0} \frac{F(t)}{|t|^{q_1}} = +\infty,$$

where $F(t) = \int_0^t f(\xi) d\xi$.

(A17) There exists $q_2 \in (4, 6)$ with $q_2 < q_1$ such that

$$\lim_{|t| \rightarrow 0} \frac{f(t)t}{|t|^{q_2}} = 0.$$

(A18) There exist $\beta \in (4, 6)$ and $\delta > 0$ such that

$$0 < \beta F(t) \leq tf(t), \quad 0 < |t| \leq \delta.$$

For more related results about elliptic equations satisfying conditions (A1) and (A2), we refer to [5, 6, 13] and the references therein.

Here it is natural for us to ask:

Does system (1.1) with f satisfying critical growth admit nontrivial solutions?

To the best of our knowledge, there is no answer to the above question in the existing literature. Our purpose of this paper is to make an effort in providing an affirmative answer to this question. To this end, we consider the Shrödinger-Poisson system

$$\begin{aligned} -\Delta u + V(|x|)u + \lambda \phi u &= |u|^{2_\alpha^* - 2}u + \beta|u|^{q-2}u + |u|^4u, \quad x \in \mathbb{R}^3, \\ -\Delta \phi &= u^2, \quad x \in \mathbb{R}^3, \end{aligned} \quad (1.6)$$

where $q \in (2_\alpha^*, 6)$, $2_\alpha^* = 2 + \frac{4\alpha}{4-\alpha}$, and the nonlinearity f contains the embedding top and bottom indices.

By the Lax-Milgram theorem, for $u \in W_{\text{rad}}^{1,2}(\mathbb{R}^3, V)$, there exists a unique solution $\phi_u \in D^{1,2}(\mathbb{R}^3)$ satisfying $-\Delta \phi_u = u^2$, which can be represented by

$$\phi_u = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{|u(y)|^2}{|x-y|} dy.$$

Substituting ϕ_u into system (1.6), we can reduce this system to a single equation:

$$-\Delta u + V(|x|)u + \lambda \phi_u u = |u|^{2_\alpha^* - 2}u + \beta|u|^{q-2}u + |u|^4u, \quad x \in \mathbb{R}^3. \quad (1.7)$$

Let us state our result on the nonnegative ground state solution of system (1.6).

Theorem 1.2. *Assume that $\alpha \in (0, 2)$, $q \in (2_\alpha^*, 6)$, $\lambda = 0$ and conditions (A1), (A2) hold. Then there exists $\tilde{\beta} > 0$ such that for $\beta \in (\tilde{\beta}, +\infty)$ system (1.6) possesses a nonnegative ground state solution.*

Note that the effect of $|u|^{q-2}u$ can be regarded as a perturbation, which is used to lower the energy and to ensure that the (PS) sequences obtained at the mountain pass level of system (1.6) is non-vanishing.

To prove Theorem 1.2, we face the following two major difficulties: (1) In view of $\alpha \in (0, 2)$ and Proposition 1.1, we know that the following embedding

$$W_{\text{rad}}^{1,2}(\mathbb{R}^3, V) \hookrightarrow L^s(\mathbb{R}^3), \quad s \in (2_\alpha^*, 6),$$

is compact. In [26, 31], the compactness is guaranteed by assuming the nonlinearity f satisfies conditions (A10) or (A16). However, the methods used in [26, 31] are not applicable for our case due to the presence of the embedding top and bottom indices and the lack of compactness of the following embeddings

$$W_{\text{rad}}^{1,2}(\mathbb{R}^3, V) \hookrightarrow L^{2_\alpha^*}(\mathbb{R}^3) \quad \text{and} \quad W_{\text{rad}}^{1,2}(\mathbb{R}^3, V) \hookrightarrow L^6(\mathbb{R}^3).$$

(2) For $\alpha \in (0, 2)$, we know $2_\alpha^* \neq 6$, which means that system (1.6) contains two different kinds of critical embedding indices. Obviously, this case is more difficult than the single critical case.

Following [15, 22], we can extend the existence result of Theorem 1.2 from $\lambda = 0$ to $\lambda \neq 0$. Our second result can be summarized as follows.

Theorem 1.3. *Assume that $\alpha \in (0, 4/11)$, $\beta \in (\tilde{\beta}, +\infty)$, $q \in (2_\alpha^*, 6)$ and conditions (A1), (A2) hold, where $\tilde{\beta}$ is taken as in Theorem 1.2. Then there exists $\lambda_0 > 0$ small enough such that for any $\lambda \in (0, \lambda_0)$ system (1.6) possesses a nontrivial solution.*

Compared with the proof of Theorem 1.2, there is an extra difficulty in proving Theorem 1.3. For $\alpha \in (0, \frac{4}{11})$, we have $2_\alpha^* < 4$. Then if $\lambda \neq 0$, it is not easy for us to guarantee the boundedness of the (PS) sequences.

To prove Theorem 1.3, following [15, 22], we define the energy functional corresponding to equation (1.7) by

$$\begin{aligned} J_\lambda(u) &= \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V(|x|)|u|^2) dx + \frac{\lambda}{4} \int_{\mathbb{R}^3} \phi_u |u|^2 dx - \frac{1}{2_\alpha^*} \int_{\mathbb{R}^3} |u|^{2_\alpha^*} dx \\ &\quad - \frac{\beta}{q} \int_{\mathbb{R}^3} |u|^q dx - \frac{1}{6} \int_{\mathbb{R}^3} |u|^6 dx. \end{aligned}$$

For $\lambda > 0$ small enough, we view the functional J_λ as a perturbation of the functional J_0 :

$$J_\lambda(u) = J_0(u) + P_\lambda(u),$$

where

$$\begin{aligned} J_0(u) &= \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V(|x|)|u|^2) dx - \frac{1}{2_\alpha^*} \int_{\mathbb{R}^3} |u|^{2_\alpha^*} dx \\ &\quad - \frac{\beta}{q} \int_{\mathbb{R}^3} |u|^q dx - \frac{1}{6} \int_{\mathbb{R}^3} |u|^6 dx, \\ P_\lambda(u) &= \frac{\lambda}{4} \int_{\mathbb{R}^3} \phi_u |u|^2 dx. \end{aligned}$$

Let Ω be the set of ground state critical points of J_0 . The perturbation method mainly includes the following two aspects:

- (i) The mountain pass type critical point of J_0 is a ground state solution.
- (ii) The set Ω is compact in $W_{\text{rad}}^{1,2}(\mathbb{R}^3, V)$.

If the two conditions above hold, then for $\lambda > 0$ small enough there exists a (PS) sequence of J_λ near the set Ω .

The remainder of this paper is organized as follows. In Section 2, two technical lemmas are presented. In Sections 3 and 4, we prove Theorems 1.2 and 1.3, respectively.

2. PRELIMINARY RESULTS

Throughout this paper, we use symbols C, C_i ($i = 1, 2, \dots$) to denote different positive constants which may change from line to line.

Let $C_0^\infty(\mathbb{R}^3)$ be the collection of smooth functions with the compact support. Let $D^{1,2}(\mathbb{R}^3)$ be the completion of $C_0^\infty(\mathbb{R}^3)$ with the semi-norm

$$\|u\|_{D^{1,2}(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^{1/2}.$$

We denote by $D_{\text{rad}}^{1,2}(\mathbb{R}^3)$ the space of radial functions in $D^{1,2}(\mathbb{R}^3)$ and define

$$\begin{aligned} W^{1,2}(\mathbb{R}^3, V) &:= \{u \in D^{1,2}(\mathbb{R}^3) : \|u\|_{L^2(\mathbb{R}^3, V)}^2 = \int_{\mathbb{R}^3} V(|x|)|u|^2 dx < \infty\} \\ &= D^{1,2}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3, V) \end{aligned}$$

with the norm

$$\|u\|_{W^{1,2}(\mathbb{R}^3, V)}^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 dx + V(|x|)|u|^2) dx.$$

Let $W_{\text{rad}}^{1,2}(\mathbb{R}^3, V) := D_{\text{rad}}^{1,2}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3, V)$ denote the radial subspace of $W^{1,2}(\mathbb{R}^3, V)$.

Lemma 2.1 ([28], Hardy-Littlewood-Sobolev inequality). *Let $s, t > 1$ and $\theta \in (0, 3)$ with $\frac{1}{s} + \frac{1}{t} = 1 + \frac{\theta}{3}$. Then there exists $C(\theta, s, t) > 0$ such that for any $u \in L^s(\mathbb{R}^3)$ and $v \in L^t(\mathbb{R}^3)$ it holds*

$$\left| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u(x)v(y)}{|x-y|^{3-\theta}} dx dy \right| \leq C_{\theta, s, t} \|u\|_{L^s(\mathbb{R}^3)} \|v\|_{L^t(\mathbb{R}^3)}.$$

If $s = t = \frac{6}{3+\theta}$, then

$$C(\theta, s, t) = \pi^{\frac{3-\theta}{2}} \frac{\Gamma(\frac{\theta}{2})}{\Gamma(\frac{3+\theta}{2})} \left[\frac{\Gamma(\frac{3}{2})}{\Gamma(3)} \right]^{-\theta/3}.$$

The following two inequalities play an important role in the estimation of the mountain pass energy.

$$\mathcal{S}_\alpha \left(\int_{\mathbb{R}^3} |u|^{2_\alpha^*} dx \right)^{2/2_\alpha^*} \leq \|u\|_{W_{\text{rad}}^{1,2}(\mathbb{R}^3, V)}^2, \quad u \in W_{\text{rad}}^{1,2}(\mathbb{R}^3, V), \quad (2.1)$$

$$\mathcal{S} \left(\int_{\mathbb{R}^3} |u|^6 dx \right)^{1/3} \leq \int_{\mathbb{R}^3} |\nabla u|^2 dx, \quad u \in W_{\text{rad}}^{1,2}(\mathbb{R}^3, V), \quad (2.2)$$

where \mathcal{S} and \mathcal{S}_α are the embedding constants with

$$\mathcal{S} = \inf_{u \in D^{1,2}(\mathbb{R}^3) \setminus \{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^3} |u|^6 dx \right)^{1/3}}.$$

The following lemma states some properties of ϕ_u .

Lemma 2.2. *Assume that $\alpha \in (0, \frac{4}{11})$ and conditions (A1), (A2) hold. For any $u \in W_{\text{rad}}^{1,2}(\mathbb{R}^3, V)$, the following statements hold:*

- (i) $\phi_u : W_{\text{rad}}^{1,2}(\mathbb{R}^3, V) \rightarrow D^{1,2}(\mathbb{R}^3)$ is continuous and maps bounded sets into bounded sets.
- (ii) $\int_{\mathbb{R}^3} \phi_u |u|^2 dx \leq C \|u\|_{W^{1,2}(\mathbb{R}^3, V)}^4$.
- (iii) If $u_n \rightharpoonup u$ in $W_{\text{rad}}^{1,2}(\mathbb{R}^3, V)$, then, up to a subsequence, $\phi_{u_n} \rightharpoonup \phi_u$ in $D^{1,2}(\mathbb{R}^3)$.
- (iv) If $u_n \rightharpoonup u$ in $W_{\text{rad}}^{1,2}(\mathbb{R}^3, V)$ and $u_n \rightarrow u$ a.e. in \mathbb{R}^3 , then, as $n \rightarrow +\infty$, we have

$$\int_{\mathbb{R}^3} \phi_{u_n - u} |u_n - u|^2 dx = \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^2 dx - \int_{\mathbb{R}^3} \phi_u |u|^2 dx.$$

The proof of the above is similar to that of [30, Lemma 2.1], so we omit it here.

3. PROOF OF THEOREM 1.2

In this section, we shall prove Theorem 1.2 by applying a generalized version of Lions-type theorem and the Nehari manifold theory. When $\lambda = 0$, system (1.6) is reduced to

$$-\Delta u + V(|x|)u = |u|^{2^*_\alpha - 2}u + \beta |u|^{q-2}u + |u|^4u, \quad x \in \mathbb{R}^3. \quad (3.1)$$

The energy functional associated with equation (3.1) can be defined as

$$J_0(u) = \frac{1}{2} \|u\|_{W^{1,2}(\mathbb{R}^3, V)}^2 - \frac{1}{2^*_\alpha} \int_{\mathbb{R}^3} |u|^{2^*_\alpha} dx - \frac{\beta}{q} \int_{\mathbb{R}^3} |u|^q dx - \frac{1}{6} \int_{\mathbb{R}^3} |u|^6 dx.$$

It is standard to show that J_0 is well-defined on $W_{\text{rad}}^{1,2}(\mathbb{R}^3, V)$ and belongs to $C^1(W_{\text{rad}}^{1,2}(\mathbb{R}^3, V), \mathbb{R})$. Moreover, for any $u, \varphi \in W_{\text{rad}}^{1,2}(\mathbb{R}^3, V)$, we have

$$\begin{aligned} \langle J'_0(u), \varphi \rangle &= \int_{\mathbb{R}^3} (\nabla u \nabla \varphi + V(|x|)u\varphi) dx - \int_{\mathbb{R}^3} |u|^{2^*_\alpha - 2} u \varphi dx \\ &\quad - \beta \int_{\mathbb{R}^3} |u|^{q-2} u \varphi dx - \int_{\mathbb{R}^3} |u|^4 u \varphi dx. \end{aligned}$$

Lemma 3.1. *Assume that all conditions described in Theorem 1.2 hold. Then the following statements hold.*

- (i) The functional J_0 possesses the mountain pass geometry.
- (ii) For any $u \in W_{\text{rad}}^{1,2}(\mathbb{R}^3, V) \setminus \{0\}$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}$ and $J_0(t_u u) = \max_{t>0} J_0(tu)$, where

$$\mathcal{N} = \{u \in W_{\text{rad}}^{1,2}(\mathbb{R}^3, V) \setminus \{0\} : \langle J'_0(u), u \rangle = 0\}.$$

- (iii) $c_0 = \bar{c}_0 = \bar{c}_0 > 0$, where

$$\begin{aligned} c_0 &= \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_0(\gamma(t)), \quad \bar{c}_0 = \inf_{u \in \mathcal{N}} J_0(u), \\ \bar{c}_0 &= \inf_{u \in W_{\text{rad}}^{1,2}(\mathbb{R}^3, V) \setminus \{0\}} \max_{t>0} J_0(tu), \\ \Gamma &= \{\gamma \in C([0,1], W_{\text{rad}}^{1,2}(\mathbb{R}^3, V)) : \gamma(0) = 0, J_0(\gamma(1)) < 0\}. \end{aligned}$$

We will present the proof of this lemma in the appendix.

3.1. Estimation of c_0 . The main feature of the functional J_0 is that it satisfies the local compactness condition. We now give an estimation of c_0 .

Lemma 3.2. *Assume that all conditions described in Theorem 1.2 hold. Then we have*

$$0 < c_0 < c_0^* := \min \left\{ \frac{1}{3} \mathcal{S}^{3/2}, \left(\frac{1}{2} - \frac{1}{2_\alpha^*} \right) \mathcal{S}_\alpha^{\frac{2_\alpha^*}{2_\alpha^* - 2}} \right\}.$$

Proof. We choose

$$\|v\|_{W^{1,2}(\mathbb{R}^3, V)}^2 = 1, \quad \int_{\mathbb{R}^3} |v|^q dx > 0, \quad \lim_{t \rightarrow +\infty} J_0(tv) = -\infty.$$

Then

$$\sup_{t \geq 0} J_0(tv) = J_0(t_{v,\beta}v)$$

for some $t_{v,\beta} > 0$. Hence, $t_{v,\beta} > 0$ satisfies

$$t_{v,\beta}^2 \|v\|_{W^{1,2}(\mathbb{R}^3, V)}^2 = t_{v,\beta}^{2_\alpha^*} \int_{\mathbb{R}^3} |v|^{2_\alpha^*} dx + \beta t_{v,\beta}^q \int_{\mathbb{R}^3} |v|^q dx + t_{v,\beta}^6 \int_{\mathbb{R}^3} |v|^6 dx \quad (3.2)$$

and

$$t_{v,\beta}^2 \|v\|_{W^{1,2}(\mathbb{R}^3, V)}^2 \geq t_{v,\beta}^6 \int_{\mathbb{R}^3} |v|^6 dx.$$

This implies that $\{t_{v,\beta}\}$ is bounded.

We claim that $t_{v,\beta} \rightarrow 0$ as $\beta \rightarrow +\infty$. Argue by contradiction, suppose that there exist $t_0 > 0$ and a sequence $\{\beta_n\}$ with $\beta_n \rightarrow +\infty$ as $n \rightarrow +\infty$, such that $t_{v,\beta_n} \rightarrow t_0$ as $n \rightarrow +\infty$. Then, we have

$$\beta_n t_{v,\beta_n}^q \int_{\mathbb{R}^3} |v|^q dx \rightarrow +\infty, \quad \text{as } n \rightarrow +\infty.$$

Substituting this into (3.2) yields

$$t_{v,\beta}^2 \|v\|_{W^{1,2}(\mathbb{R}^3, V)}^2 = +\infty,$$

which leads to a contradiction. That is, $t_{v,\beta} \rightarrow 0$ as $\beta \rightarrow +\infty$, and

$$\lim_{\beta \rightarrow +\infty} \sup_{t \geq 0} J_0(tv) = \lim_{\beta \rightarrow +\infty} J_0(t_{v,\beta}v) = 0.$$

So there exists $0 < \tilde{\beta} < +\infty$ such that for any $\beta > \tilde{\beta}$,

$$\sup_{t \geq 0} J_0(tv) < \min \left\{ \frac{1}{3} \mathcal{S}^{3/2}, \left(\frac{1}{2} - \frac{1}{2_\alpha^*} \right) \mathcal{S}_\alpha^{\frac{2_\alpha^*}{2_\alpha^* - 2}} \right\}.$$

If we take $e = Tv$ with $T > 0$ large enough such that $J_0(e) < 0$, then

$$c_0 \leq \max_{t \in [0,1]} J_0(\gamma(t)),$$

where $\gamma(t) = tTv$. Thus, $c_0 \leq \sup_{t \geq 0} J_0(tv) < c_0^*$. \square

3.2. **Non-vanishing of the $(PS)_{c_0}$ sequence.**

Lemma 3.3. *Assume that all conditions described in Theorem 1.2 hold. Let $\{u_n\}$ be a bounded $(PS)_{c_0}$ sequence of J_0 with $0 < c_0 < c^*$. Then*

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{R}^3} |u_n|^{2^*_\alpha} dx > 0 \quad \text{and} \quad \lim_{n \rightarrow +\infty} \int_{\mathbb{R}^3} |u_n|^6 dx > 0.$$

Proof. Let $\{u_n\}$ be a $(PS)_{c_0}$ sequence of J_0 . We first show $\lim_{n \rightarrow +\infty} \int_{\mathbb{R}^3} |u_n|^{2^*_\alpha} dx > 0$. Otherwise, we suppose that

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{R}^3} |u_n|^{2^*_\alpha} dx = 0. \tag{3.3}$$

It follows from (3.3) and Hölder’s inequality that

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{R}^3} |u_n|^q dx \leq C \lim_{n \rightarrow +\infty} \left(\int_{\mathbb{R}^3} |u_n|^{2^*_\alpha} dx \right)^{\frac{6-q}{6-2^*_\alpha}} \left(\int_{\mathbb{R}^3} |u_n|^6 dx \right)^{\frac{q-2^*_\alpha}{6-2^*_\alpha}} = 0. \tag{3.4}$$

By using (3.3)-(3.4) and in view of definition of the $(PS)_{c_0}$ sequence, we can deduce

$$c_0 + o_n(1) = \frac{1}{2} \|u_n\|_{W^{1,2}(\mathbb{R}^3, V)}^2 - \frac{1}{6} \int_{\mathbb{R}^3} |u_n|^6 dx, \tag{3.5}$$

$$o_n(1) = \|u_n\|_{W^{1,2}(\mathbb{R}^3, V)}^2 - \int_{\mathbb{R}^3} |u_n|^6 dx. \tag{3.6}$$

Using (3.5)-(3.6) we have

$$c_0 + o_n(1) = \frac{1}{3} \|u_n\|_{W^{1,2}(\mathbb{R}^3, V)}^2 \geq \frac{1}{3} \int_{\mathbb{R}^3} |\nabla u_n|^2 dx. \tag{3.7}$$

It follows from (2.2) and (3.6) that

$$\int_{\mathbb{R}^3} |\nabla u_n|^2 dx \geq \mathcal{S} \left(\int_{\mathbb{R}^3} |u_n|^6 dx \right)^{1/3} \geq \mathcal{S} \left(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx \right)^{1/3},$$

which implies

$$\int_{\mathbb{R}^3} |\nabla u_n|^2 dx \geq \mathcal{S}^{3/2}. \tag{3.8}$$

Combining (3.7) and (3.8) leads to

$$c_0 \geq \frac{1}{3} \mathcal{S}^{3/2},$$

which yields a contradiction with $c_0 < c_0^*$. Therefore, $\lim_{n \rightarrow +\infty} \int_{\mathbb{R}^3} |u_n|^{2^*_\alpha} dx > 0$.

To prove $\lim_{n \rightarrow +\infty} \int_{\mathbb{R}^3} |u_n|^6 dx > 0$, we suppose that

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{R}^3} |u_n|^6 dx = 0. \tag{3.9}$$

According to (3.4), (3.9) and the definition of the $(PS)_{c_0}$ sequence, it holds

$$c_0 + o_n(1) = \frac{1}{2} \|u_n\|_{W^{1,2}(\mathbb{R}^3, V)}^2 - \frac{1}{2^*_\alpha} \int_{\mathbb{R}^3} |u_n|^{2^*_\alpha} dx, \tag{3.10}$$

$$o_n(1) = \|u_n\|_{W^{1,2}(\mathbb{R}^3, V)}^2 - \int_{\mathbb{R}^3} |u_n|^{2^*_\alpha} dx. \tag{3.11}$$

Using (3.10) and (3.11), we have

$$c_0 + o_n(1) = \left(\frac{1}{2} - \frac{1}{2^*_\alpha} \right) \|u_n\|_{W^{1,2}(\mathbb{R}^3, V)}^2. \tag{3.12}$$

Taking into account (2.1) and (3.11), we obtain

$$\|u_n\|_{W^{1,2}(\mathbb{R}^3, V)}^2 \geq \mathcal{S}_\alpha \left(\int_{\mathbb{R}^3} |u_n|^{2_\alpha^*} dx \right)^{2/2_\alpha^*} = \mathcal{S}_\alpha \|u_n\|_{W^{1,2}(\mathbb{R}^3, V)}^{\frac{4}{2_\alpha^*}},$$

which gives

$$\|u_n\|_{W^{1,2}(\mathbb{R}^3, V)}^2 \geq \mathcal{S}_\alpha^{\frac{2_\alpha^*}{2_\alpha^*-2}}. \quad (3.13)$$

It follows from (3.12) and (3.13) that

$$c_0 \geq \left(\frac{1}{2} - \frac{1}{2_\alpha^*} \right) \mathcal{S}_\alpha^{\frac{2_\alpha^*}{2_\alpha^*-2}},$$

which yields another contradiction with $c_0 < c_0^*$. Consequently, we have

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{R}^3} |u_n|^6 dx > 0.$$

□

3.3. Existence of ground state solution.

Theorem 3.4 ([32]). *Assume that $\alpha \in (0, 2)$ and conditions (A1), (A2) hold. Let $\{u_n\} \subset W_{\text{rad}}^{1,2}(\mathbb{R}^3, V)$ be any bounded sequence satisfying*

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{R}^3} |u_n|^{2_\alpha^*} dx > 0 \quad \text{and} \quad \lim_{n \rightarrow +\infty} \int_{\mathbb{R}^3} |u_n|^6 dx > 0.$$

Then the sequence $\{u_n\}$ converges weakly and a.e. to $u \neq 0$ in $L_{\text{loc}}^2(\mathbb{R}^3)$.

Proof of Theorem 1.2. Let $\{u_n\}$ be a $(PS)_{c_0}$ sequence of J_0 . Then we have

$$c_0 + o_n(1) = \frac{1}{2} \|u_n\|_{W^{1,2}(\mathbb{R}^3, V)}^2 - \frac{1}{2_\alpha^*} \int_{\mathbb{R}^3} |u_n|^{2_\alpha^*} dx - \frac{1}{q} \int_{\mathbb{R}^3} |u_n|^q dx - \frac{1}{6} \int_{\mathbb{R}^3} |u_n|^6 dx$$

and

$$o_n(1) = \|u_n\|_{W^{1,2}(\mathbb{R}^3, V)}^2 - \int_{\mathbb{R}^3} |u_n|^{2_\alpha^*} dx - \int_{\mathbb{R}^3} |u_n|^q dx - \int_{\mathbb{R}^3} |u_n|^6 dx.$$

Combining the two equalities above we have

$$\begin{aligned} c_0 + o_n(1) &= \left(\frac{1}{2} - \frac{1}{2_\alpha^*} \right) \|u_n\|_{W^{1,2}(\mathbb{R}^3, V)}^2 + \left(\frac{1}{2_\alpha^*} - \frac{1}{q} \right) \int_{\mathbb{R}^3} |u_n|^q dx \\ &\quad + \left(\frac{1}{2_\alpha^*} - \frac{1}{6} \right) \int_{\mathbb{R}^3} |u_n|^6 dx, \end{aligned}$$

which implies that $\{u_n\}$ is bounded in $W_{\text{rad}}^{1,2}(\mathbb{R}^3, V)$. According to Lemma 3.3 and Theorem 3.4, we can see that $\{u_n\}$ converges weakly and a.e. to $u_0 \neq 0$ in $L_{\text{loc}}^2(\mathbb{R}^3)$. From $u_n \rightharpoonup u_0$ in $W_{\text{rad}}^{1,2}(\mathbb{R}^3, V)$ and $\lim_{n \rightarrow +\infty} \langle J'_0(u_n), \varphi \rangle = o_n(1)$, we deduce

$$\langle J'_0(u_0), \varphi \rangle = o_n(1).$$

Since $u_0 \neq 0$, we obtain $u_0 \in \mathcal{N}$. By Lemma 2.2 and the Brézis-Lieb lemma [12], we have

$$\begin{aligned} \bar{c}_0 &\leq J_0(u_0) \\ &= J_0(u_0) - \frac{1}{2_\alpha^*} \langle J'_0(u_0), u_0 \rangle \\ &= \left(\frac{1}{2} - \frac{1}{2_\alpha^*} \right) \|u_0\|_{W^{1,2}(\mathbb{R}^3, V)}^2 + \left(\frac{1}{2_\alpha^*} - \frac{1}{q} \right) \int_{\mathbb{R}^3} |u_0|^q dx + \left(\frac{1}{2_\alpha^*} - \frac{1}{6} \right) \int_{\mathbb{R}^3} |u_0|^6 dx \end{aligned}$$

$$\begin{aligned} &\leq \lim_{n \rightarrow +\infty} \left[\left(\frac{1}{2} - \frac{1}{2^*_\alpha} \right) \|u_n\|_{W^{1,2}(\mathbb{R}^3, V)}^2 + \left(\frac{1}{2^*_\alpha} - \frac{1}{q} \right) \int_{\mathbb{R}^3} |u_n|^q dx \right. \\ &\quad \left. + \left(\frac{1}{2^*_\alpha} - \frac{1}{6} \right) \int_{\mathbb{R}^3} |u_n|^6 dx \right] \\ &= \lim_{n \rightarrow +\infty} \left[J_0(u_n) - \frac{1}{2^*_\alpha} \langle J'_0(u_n), u_n \rangle \right] \\ &= \lim_{n \rightarrow +\infty} J_0(u_n) \\ &= \bar{c}_0, \end{aligned}$$

which implies $J_0(u_0) = \bar{c}_0$. Then, it is easy to see that

$$\begin{aligned} &\lim_{n \rightarrow +\infty} \left[\left(\frac{1}{2} - \frac{1}{2^*_\alpha} \right) \|u_n - u_0\|_{W^{1,2}(\mathbb{R}^3, V)}^2 + \left(\frac{1}{2^*_\alpha} - \frac{1}{q} \right) \int_{\mathbb{R}^3} |u_n - u_0|^q dx \right. \\ &\quad \left. + \left(\frac{1}{2^*_\alpha} - \frac{1}{6} \right) \int_{\mathbb{R}^3} |u_n - u_0|^6 dx \right] = 0, \end{aligned}$$

which implies

$$\lim_{n \rightarrow +\infty} \|u_n - u_0\|_{W^{1,2}(\mathbb{R}^3, V)}^2 = 0.$$

Thus, we have $u_n \rightarrow u_0$ in $W^{1,2}_{\text{rad}}(\mathbb{R}^3, V)$. Moreover, we can choose $u_0 \geq 0$. That is, $u_0 \in W^{1,2}_{\text{rad}}(\mathbb{R}^3, V)$ is a nonnegative ground state solution of system (1.6) with $\lambda = 0$. □

4. PROOF OF THEOREM 1.3

In this section, we apply the perturbation method to prove the existence of nontrivial solutions to system (1.6) with $\lambda \neq 0$. The associated energy functional with system (1.6) can be defined as

$$\begin{aligned} J_\lambda(u) &= \frac{1}{2} \|u\|_{W^{1,2}(\mathbb{R}^3, V)}^2 + \frac{\lambda}{4} \int_{\mathbb{R}^3} \phi_u |u|^2 dx - \frac{1}{2^*_\alpha} \int_{\mathbb{R}^3} |u|^{2^*_\alpha} dx \\ &\quad - \frac{\beta}{q} \int_{\mathbb{R}^3} |u|^q dx - \frac{1}{6} \int_{\mathbb{R}^3} |u|^6 dx. \end{aligned}$$

From Lemma 2.1, it is easy to show that $J_\lambda \in C^1(W^{1,2}_{\text{rad}}(\mathbb{R}^3, V), \mathbb{R})$ and

$$\begin{aligned} \langle J'_\lambda(u), \varphi \rangle &= \int_{\mathbb{R}^3} (\nabla u \nabla \varphi + V(|x|)u\varphi) dx + \lambda \int_{\mathbb{R}^3} \phi_u u\varphi dx - \int_{\mathbb{R}^3} |u|^{2^*_\alpha - 2} u\varphi dx \\ &\quad - \beta \int_{\mathbb{R}^3} |u|^{q-2} u\varphi dx - \int_{\mathbb{R}^3} |u|^4 u\varphi dx \end{aligned}$$

for $u, \varphi \in W^{1,2}_{\text{rad}}(\mathbb{R}^3, V)$.

Lemma 4.1. *Assume that all conditions described in Theorem 1.3 hold. The functional J_0 satisfies the following properties.*

- (i) *There exist $\rho, \delta > 0$ such that if $\|u\|_{W^{1,2}(\mathbb{R}^3, V)} = \rho$, then $J_0(u) \geq \delta$ and there exists $v_0 \in W^{1,2}_{\text{rad}}(\mathbb{R}^3, V)$ such that $\|v_0\|_{W^{1,2}(\mathbb{R}^3, V)} > \rho$ and $J_0(v_0) < 0$.*
- (ii) *There exists a critical point u_0 of J_0 such that*

$$J_0(u_0) = c_0 := \min_{\gamma \in \Gamma_0} \max_{t \in [0,1]} J_0(\gamma(t)),$$

where

$$\Gamma_0 = \{ \gamma \in C([0, 1], W^{1,2}_{\text{rad}}(\mathbb{R}^3, V)) : \gamma(0) = 0, \gamma(1) = v_0 \}.$$

- (iii) For any $u \in W_{\text{rad}}^{1,2}(\mathbb{R}^3, V) \setminus \{0\}$, we have $c_0 = \inf\{J_0(u) : J'_0(u) = 0\}$.
- (iv) There exists a path $\gamma_0(t) \in \Gamma_0$ passing through u_0 at $t = t_0$ and satisfies

$$J_0(u_0) > J_0(\gamma_0(t)), \quad t \neq t_0.$$

- (v) The set

$$\Omega := \{u \in W_{\text{rad}}^{1,2}(\mathbb{R}^3, V) : J_0(u) = c_0, J'_0(u) = 0\}$$

is compact in $W_{\text{rad}}^{1,2}(\mathbb{R}^3, V)$.

Proof. Since the proofs of (i)–(iii) are closely similar to those in Lemma 3.1, we only present the proof of (iv) and (v). Let u_0 be a critical point of J_0 and $v_0 = Tu_0$ with $T > 0$ large enough such that $J_0(v_0) < 0$. Then $\gamma_0(t) \in C([0, 1], W_{\text{rad}}^{1,2}(\mathbb{R}^3, V))$ can be defined by

$$\gamma_0(t) = tv_0 = tTu_0.$$

By taking $t_0 = 1/T$, we can see that (iv) is true. Analogous to the proof of Theorem 1.2, the weak convergence of the critical point sequence can be upgraded into the strong convergence. That is, (v) is also true. \square

Following [15, 22], we define a modified mountain pass energy level of J_λ as

$$c_\lambda := \min_{\gamma \in \Gamma_M} \max_{t \in [0,1]} J_\lambda(\gamma(t)),$$

where

$$\Gamma_M = \{\gamma \in \Gamma_0 : \sup_{t \in [0,1]} \|\gamma(t)\|_{W^{1,2}(\mathbb{R}^3, V)} \leq M\},$$

$$M = 2 \max \left\{ \sup_{u \in \Omega} \|u\|_{W^{1,2}(\mathbb{R}^3, V)}, \sup_{t \in [0,1]} \|\gamma(t)\|_{W^{1,2}(\mathbb{R}^3, V)} \right\}.$$

Clearly, taking a suitable choice of M , we have $\gamma_0 \in \Gamma_M$. Then

$$c_0 = \min_{\gamma \in \Gamma_M} \max_{t \in [0,1]} J_0(\gamma(t)).$$

Taking into account that $\Gamma_M \subsetneq \Gamma_0$, the standard mountain pass theorem cannot be applied. So we have to show that c_λ is a critical value.

Lemma 4.2. *Let $\lambda > 0$. Then $\lim_{\lambda \rightarrow 0} c_\lambda = c_0$.*

Proof. On the one hand, for $\lambda > 0$, it is easy to see $c_\lambda \geq c_0$. On the other hand, in view of Lemmas 2.2 and 4.1, we can deduce

$$\begin{aligned} \lim_{\lambda \rightarrow 0} c_\lambda &= \lim_{\lambda \rightarrow 0} J_\lambda(\gamma_0(t)) \\ &\leq J_0(\gamma_0(t)) + \lim_{\lambda \rightarrow 0} \frac{\lambda}{4} \int_{\mathbb{R}^3} \phi_{u_0} |u_0|^2 dx \\ &= J_0(u_0) + o(1) \\ &= c_0 + o(1). \end{aligned}$$

\square

For a $d > 0$ we define

$$B_d(u) := \{v \in W_{\text{rad}}^{1,2}(\mathbb{R}^3, V) : \|u - v\|_{W^{1,2}(\mathbb{R}^3, V)} \leq d\},$$

and for any $X \subset W_{\text{rad}}^{1,2}(\mathbb{R}^3, V)$ we set

$$X^d := \cup_{u \in X} B_d(u).$$

Lemma 4.3. *Let $d_1 = \sqrt{3c_0}$ and $d \in (0, d_1)$. Then for any $u \in \Omega^d$, we have $u \neq 0$.*

Proof. For each $v \in \Omega^d$, we obtain

$$c_0 = \frac{1}{3} \|v\|_{W^{1,2}(\mathbb{R}^3, V)}^2 + \left(\frac{1}{6} - \frac{1}{2_\alpha^*}\right) \int_{\mathbb{R}^3} |v|^{2_\alpha^*} dx + \beta \left(\frac{1}{6} - \frac{1}{q}\right) \int_{\mathbb{R}^3} |v|^q dx,$$

which gives

$$\|v\|_{W^{1,2}(\mathbb{R}^3, V)} \geq d_1.$$

In view of $u \in \Omega^d$, we know that there exists some $v \in \Omega^d$ such that

$$\|u - v\|_{W^{1,2}(\mathbb{R}^3, V)} \leq d < d_1.$$

Thus, we obtain

$$\|u\|_{W^{1,2}(\mathbb{R}^3, V)} \geq \|v\|_{W^{1,2}(\mathbb{R}^3, V)} - \|u - v\|_{W^{1,2}(\mathbb{R}^3, V)} \geq d_1 - d > 0.$$

□

Lemma 4.4. *Suppose that $d > 0$ is a fixed number and $\{u_n\} \subset \Omega^d$. Then, up to a subsequence, it holds $u_n \rightharpoonup \tilde{u} \in \Omega^{2d}$.*

Proof. By the definition of Ω^d , there exists a sequence $\{v_n\} \subset \Omega$ such that $\{u_n\} \subset B_d(v_n)$. According to Lemma 4.1 (v), we can assume that there exists $v \in \Omega$ such that $v_n \rightarrow v$ in $W_{\text{rad}}^{1,2}(\mathbb{R}^3, V)$. Thus, we obtain

$$\|u_n - v\|_{W^{1,2}(\mathbb{R}^3, V)} \leq \|u_n - v_n\|_{W^{1,2}(\mathbb{R}^3, V)} + \|v_n - v\|_{W^{1,2}(\mathbb{R}^3, V)} \leq 2d,$$

which implies $\{u_n\} \subset B_{2d}(v)$ for $n > 0$ large enough. Moreover, $\{u_n\}$ is bounded and, up to a subsequence, there exists \tilde{u} such that $u_n \rightharpoonup \tilde{u}$ in $W_{\text{rad}}^{1,2}(\mathbb{R}^3, V)$. Since $B_{2d}(v)$ is weakly closed in $W_{\text{rad}}^{1,2}(\mathbb{R}^3, V)$, we arrive at $\tilde{u} \in B_{2d}(v) \subset \Omega^{2d}$. □

Lemma 4.5. *Let $d \in (0, d_1)$. Assume that there exist a sequence $\{\lambda_n\} \rightarrow 0$ and $\{u_n\} \subset \Omega^d$ satisfying*

$$\lim_{n \rightarrow +\infty} J_{\lambda_n}(u_n) \leq c_0 \quad \text{and} \quad \lim_{n \rightarrow +\infty} J'_{\lambda_n}(u_n) = 0.$$

Then there exists a $\tilde{u} \in \Omega$ such that, up to a subsequence, $\{u_n\}$ converges strongly to \tilde{u} .

Proof. Note that $\lim_{n \rightarrow +\infty} J'_{\lambda_n}(u_n) = 0$ and $\{u_n\}$ is bounded. From Lemma 4.4, up to a subsequence, there exists $u_n \rightharpoonup \tilde{u} \in \Omega^{2d}$. For any $\varphi \in W_{\text{rad}}^{1,2}(\mathbb{R}^3, V)$, it follows that

$$\begin{aligned} \langle J'_0(\tilde{u}), \varphi \rangle &= \int_{\mathbb{R}^3} (\nabla \tilde{u} \nabla \varphi + V(|x|)\tilde{u}\varphi) dx - \int_{\mathbb{R}^3} |\tilde{u}|^{2_\alpha^*-2} \tilde{u}\varphi dx \\ &\quad - \beta \int_{\mathbb{R}^3} |\tilde{u}|^{q-2} \tilde{u}\varphi dx - \int_{\mathbb{R}^3} |\tilde{u}|^4 \tilde{u}\varphi dx \\ &= \lim_{n \rightarrow +\infty} \left[\int_{\mathbb{R}^3} (\nabla u_n \nabla \varphi + V(|x|)u_n\varphi) dx - \int_{\mathbb{R}^3} |u_n|^{2_\alpha^*-2} u_n\varphi dx \right. \\ &\quad \left. - \beta \int_{\mathbb{R}^3} |u_n|^{q-2} u_n\varphi dx - \int_{\mathbb{R}^3} |u_n|^4 u_n\varphi dx \right] \\ &= \lim_{n \rightarrow +\infty} \left[\langle J'_{\lambda_n}(u_n), \varphi \rangle - \frac{\lambda_n}{4} \int_{\mathbb{R}^3} \phi_{u_n} u_n\varphi dx \right] = 0. \end{aligned}$$

So $J'_0(\tilde{u}) = 0$. From $\lim_{n \rightarrow +\infty} J'_{\lambda_n}(u_n) = 0$ and $\{\lambda_n\} \rightarrow 0$ as $n \rightarrow +\infty$, it follows that

$$\lim_{n \rightarrow +\infty} \langle J'_0(u_n), \varphi \rangle = \lim_{n \rightarrow +\infty} \left[\langle J'_{\lambda_n}(u_n), \varphi \rangle - \frac{\lambda_n}{4} \int_{\mathbb{R}^3} \phi_{u_n} u_n \varphi dx \right] = 0.$$

On the other hand,

$$\begin{aligned} c_0 &\geq \lim_{n \rightarrow +\infty} J_{\lambda_n}(u_n) \\ &= \lim_{n \rightarrow +\infty} \left[J_0(u_n) + \frac{\lambda_n}{4} \int_{\mathbb{R}^3} \phi_{u_n} u_n \varphi dx \right] \\ &= \lim_{n \rightarrow +\infty} J_0(u_n). \end{aligned} \tag{4.1}$$

Therefore, $\{u_n\}$ is a $(PS)_m$ sequence of J_0 , where $m := \lim_{n \rightarrow +\infty} J_0(u_n)$. Since $u_n \rightarrow \tilde{u}$, up to a subsequence, we can deduce

$$\begin{aligned} J_0(\tilde{u}) &= J_0(\tilde{u}) - \frac{1}{2^*_\alpha} \langle J'_0(\tilde{u}), \tilde{u} \rangle \\ &= \left(\frac{1}{2} - \frac{1}{2^*_\alpha} \right) \|\tilde{u}\|_{W^{1,2}(\mathbb{R}^3, V)}^2 + \beta \left(\frac{1}{2^*_\alpha} - \frac{1}{q} \right) \int_{\mathbb{R}^3} |\tilde{u}|^q dx + \left(\frac{1}{2^*_\alpha} - \frac{1}{6} \right) \int_{\mathbb{R}^3} |\tilde{u}|^6 dx \\ &\leq \liminf_{n \rightarrow +\infty} \left[\left(\frac{1}{2} - \frac{1}{2^*_\alpha} \right) \|u_n\|_{W^{1,2}(\mathbb{R}^3, V)}^2 + \beta \left(\frac{1}{2^*_\alpha} - \frac{1}{q} \right) \int_{\mathbb{R}^3} |u_n|^q dx \right. \\ &\quad \left. + \left(\frac{1}{2^*_\alpha} - \frac{1}{6} \right) \int_{\mathbb{R}^3} |u_n|^6 dx \right] \\ &= \liminf_{n \rightarrow +\infty} \left[J_0(u_n) - \frac{1}{2^*_\alpha} \langle J'_0(u_n), u_n \rangle \right] = m. \end{aligned}$$

In view of Lemma 4.1 (iii), we have $m \geq J_0(\tilde{u}) \geq c_0$. Moreover, combining the above inequality and (4.1), we obtain $J_0(\tilde{u}) = c_0 = m$, which implies $\tilde{u} \in \Omega$. \square

Let

$$m_\lambda := \max_{t \in [0,1]} J_\lambda(\gamma_0(t)).$$

Then $c_\lambda \leq m_\lambda$. It is easy to see that

$$\lim_{\lambda \rightarrow 0} m_\lambda \leq c_0.$$

From Lemma 4.2 it follows that

$$\lim_{\lambda \rightarrow 0} c_\lambda = \lim_{\lambda \rightarrow 0} m_\lambda = c_0.$$

We define

$$J^{m_\lambda}_\lambda = \{u \in W^{1,2}_{\text{rad}}(\mathbb{R}^3, V) : J_\lambda(u) \leq m_\lambda\}.$$

Lemma 4.6. *For every $d_2, d_3 > 0$ satisfying $d_3 < d_2 < d_1$, there exist $\delta > 0$ and $\lambda_0 > 0$ dependent on d_2, d_3 such that for any $\lambda \in (0, \lambda_0)$, we have*

$$\|J'_\lambda(u)\|_{W^{-1,2}(\mathbb{R}^3, V)} \geq \delta, \quad u \in J^{m_\lambda}_\lambda \cap (\Omega^{d_2} \setminus \Omega^{d_3}).$$

Proof. We argue by contradiction. For every $d_2, d_3 > 0$ satisfying $d_3 < d_2 < d_1$, we suppose that there exist sequences $\{\lambda_n\}$ with $\lim_{n \rightarrow +\infty} \lambda_n = 0$ and $\{u_n\} \subset J^{m_\lambda}_\lambda \cap (\Omega^{d_2} \setminus \Omega^{d_3})$ satisfying

$$\lim_{n \rightarrow +\infty} J_{\lambda_n}(u_n) \leq c_0 \quad \text{and} \quad \lim_{n \rightarrow +\infty} J'_{\lambda_n}(u_n) = 0.$$

By Lemma 4.5, there exists $\tilde{u} \in \Omega$ such that $u_n \rightarrow \tilde{u}$ in $W_{\text{rad}}^{1,2}(\mathbb{R}^3, V)$, as $n \rightarrow +\infty$. Passing the limit as $n \rightarrow +\infty$, we have $\text{dist}(u_n, \tilde{u}) = 0$. It leads to a contradiction with $\{u_n\} \not\subset \Omega^{d_3}$. \square

Lemma 4.7. *For $d > 0$, there exists $\delta > 0$ such that if $\lambda > 0$ is small, then*

$$t \in [0, 1] \quad \text{and} \quad J_\lambda(\gamma_0(t)) \geq c_\lambda - \delta \implies \gamma_0(t) \in \Omega^d.$$

The proof of the above lemma is similar to that of [22, Proposition 4], so we omit it here.

Lemma 4.8 ([15]). *For any $d \in (0, d_1)$, there exist a number $\lambda_0 > 0$ and a sequence $\{u_n\} \subset J_\lambda^{m_\lambda} \cap \Omega^d$ such that for all $\lambda \in (0, \lambda_0)$, we have*

$$J'_\lambda(u_n) \rightarrow 0, \quad \text{as } n \rightarrow +\infty.$$

Proof of Theorem 1.3. Taking $d \in (0, d_1)$, it follows from Lemma 4.8 that there exists some small $\lambda_0 > 0$ such that for any fixed $\lambda \in (0, \lambda_0)$, there exists a $(PS)_{m_\lambda}$ sequence $\{u_n^\lambda\} \subset \Omega^{\frac{d}{2}}$. It is easy to see that $\{u_n^\lambda\}$ is bounded in $W_{\text{rad}}^{1,2}(\mathbb{R}^3, V)$. According to Lemma 4.4, up to a subsequence, there exists $\tilde{u}^\lambda \in \Omega^d$ such that $u_n^\lambda \rightharpoonup \tilde{u}^\lambda$. Hence, we obtain $J'_\lambda(\tilde{u}^\lambda) = 0$. By the choice of a proper d , we can see that $\tilde{u}^\lambda \neq 0$. Consequently, \tilde{u}^λ is a nontrivial solution of system (1.6), when $\lambda \neq 0$. \square

APPENDIX

Proof of Lemma 3.1. (i) It suffices to show that J_0 satisfies the mountain pass geometry. By the mountain pass theorem, we can obtain a $(PS)_{c_0}$ sequence of J_0 .

(ii) For $t > 0$, let

$$g(t) = J_0(tu) = \frac{t^2}{2} \|u\|_{W^{1,2}(\mathbb{R}^3, V)}^2 - \frac{t^{2^*_\alpha}}{2^*_\alpha} \int_{\mathbb{R}^3} |u|^{2^*_\alpha} dx - \frac{t^q}{q} \int_{\mathbb{R}^3} |u|^q dx - \frac{t^6}{6} \int_{\mathbb{R}^3} |u|^6 dx.$$

For $t > 0$ small enough, it follows from Proposition 1.1 that

$$g(t) \geq \frac{t^2}{2} \|u\|_{W^{1,2}(\mathbb{R}^3, V)}^2 - C_1 t^{2^*_\alpha} \|u\|_{W^{1,2}(\mathbb{R}^3, V)}^{2^*_\alpha} - C_2 t^q \|u\|_{W^{1,2}(\mathbb{R}^3, V)}^q - C_3 t^6 \|u\|_{W^{1,2}(\mathbb{R}^3, V)}^6.$$

Clearly, we have $g(t) > 0$ for $t > 0$ small enough. Furthermore, it is easy to see that $J_0(tu) \rightarrow -\infty$, as $t \rightarrow +\infty$. Thus, $g(t)$ has a maximum at $t = t_u > 0$, and we further have $g'(t_u) = 0$ and $t_u u \in \mathcal{N}$.

Next, we show that t_u is unique. On the contrary, we suppose that there exist $0 < t_u < \tilde{t}_u$ such that $\tilde{t}_u u, t_u u \in \mathcal{N}$. Then we have

$$\begin{aligned} & \left(\frac{1}{\tilde{t}_u^{2^*_\alpha - 2}} - \frac{1}{t_u^{2^*_\alpha - 2}} \right) \|u\|_{W^{1,2}(\mathbb{R}^3, V)}^2 \\ &= (\tilde{t}_u^{q - 2^*_\alpha} - t_u^{q - 2^*_\alpha}) \int_{\mathbb{R}^3} |u|^q dx + (\tilde{t}_u^{6 - 2^*_\alpha} - t_u^{6 - 2^*_\alpha}) \int_{\mathbb{R}^3} |u|^6 dx, \end{aligned}$$

which is impossible because $0 < t_u < \tilde{t}_u$ and $q \in (2^*_\alpha, 6)$.

(iii) Using (ii), we have $\bar{c}_0 = \bar{c}_0$. Choose $\bar{t} > 0$ large enough such that

$$J_0(\bar{t}u) < 0.$$

Define a path $\tilde{\gamma} : [0, 1] \rightarrow W_{\text{rad}}^{1,2}(\mathbb{R}^3, V)$ by $\tilde{\gamma}(t) = \bar{t}tu$, so we have $\tilde{\gamma} \in \Gamma$. Thus, we have $c \leq \bar{c}$. On the other hand, let $h(t) := \langle J'_0(\gamma(t)), \gamma(t) \rangle$, where $\gamma \in \Gamma$. Then we obtain $h(t) > 0$ for $t > 0$ small enough. Set $\gamma(1) = e$. Then we derive

$$\begin{aligned} & J_0(e) - \frac{1}{2_\alpha^*} \langle J'_0(e), e \rangle \\ &= \left(\frac{1}{2} - \frac{1}{2_\alpha^*} \right) \|e\|_{W^{1,2}(\mathbb{R}^3, V)}^2 + \left(\frac{1}{2_\alpha^*} - \frac{1}{q} \right) \int_{\mathbb{R}^3} |e|^q dx + \left(\frac{1}{2_\alpha^*} - \frac{1}{6} \right) \int_{\mathbb{R}^3} |e|^6 dx > 0, \end{aligned}$$

which leads to

$$\langle J'_0(e), e \rangle < 2_\alpha^* J_0(e) < 0.$$

This indicates that there exists $\bar{t} \in (0, 1)$ such that $\langle J'_0(\gamma(\bar{t})), \gamma(\bar{t}) \rangle = 0$, i.e. $\gamma(\bar{t}) \in \mathcal{N}$. Hence, we obtain $\bar{c}_0 \leq c_0$. \square

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