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QUASILINEARIZATION METHOD FOR CAUSAL TERMINAL VALUE PROBLEMS INVOLVING RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVES

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Abstract. In this work, we construct new definitions for a causal terminal value problem involving Riemann-Liouville fractional derivatives, and study the unique solution by combining techniques from generalized quasilinearization.

1. Introduction

It has been shown that causal differential equations [2, 7, 8, 14, 21, 22, 30, 31, 32] provide excellent models for real world problems [7] and in a variety of disciplines. This is the main advantage of causal differential equations in comparison with the traditional models [12]. There has also been a growing interest to study causal dynamic systems [7, 14]. The theory of terminal value problems [1, 3, 10, 23, 25, 27, 31, 32] for ordinary differential equations is more complicated than that of initial value problems of ordinary differential equations, and it is such an interesting theory to study. The study of a terminal value problem for ordinary differential equations using the method of lower and upper solutions can be found in [12]. The information is given at the end point of the interval and one has to work backwards to find the initial value at which the solution must start in order to reach the prescribed value at the end point of the interval. This problem becomes more interesting in the case of a fractional differential equation where it closely resembles a boundary value problem, in the sense that the initial value is inherently involved in the definition of the differential operator, and the terminal value provides the condition at the right end point of the interval.

The study of differential equations with causal operators has rapidly developed in recent years; see for example [7, 14]. The term for causal operators was adopted from the engineering literature, and the theory these operators have is the powerful quality of unifying the fractional order differential equations [4, 33], ordinary differential equations [7], integro-differential equations [26], differential equations with finite or infinite delay [9], Volterra integral equations [26], and neutral functional equations [7, 14, 22]. Especially, they are very common equations for modeling

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problems in mechanical engineering, physical engineering, electric and electronics engineering [11, 19, 20, 24]. Moreover, causality is a basic concept in physical sciences to describe the process of cause and effect in a particular situation.

The most important application of the quasilinearization method [3, 5, 6, 13, 14, 15, 17, 23, 28, 29, 30, 31, 32, 33, 34] in fractional causal differential equations has been to obtain a sequence of lower and upper bounds which are the solutions of linear fractional causal differential equations, that converge quadratically. As a result, the method has been popular in applied areas. However, the convexity assumption that is demanded by the method of quasilinearization has been a stumbling block for further development of the theory. Recently, this method has been generalized, refined and extended in several directions so as to be applicable to a much larger class of nonlinear problems by not demanding convexity and concavity property. Moreover, other possibilities that have been explored make the method of generalized quasilinearization universally useful in applications [16].

In a fractional causal terminal value problem (2.2) is used to obtain upper and lower sequences in terms of the solutions of a linear fractional causal terminal value problem, and bound the solutions of a given nonlinear fractional causal terminal value problem. Moreover, we have also shown that these sequences converge to the unique solution of the nonlinear equation uniformly and quadratically.

2. Preliminaries

In this section, we state some fundamental definitions and useful theorems used for proving the main result. Let $E = C[J, X]$ where J is an appropriate time interval, X represents either finite or infinite dimensional space, depending on the requirement of the context, so that E is a function space.

An operator $Q : E \to E$ is said to be a causal operator if, for each couple of elements x, y in E such that $x(s) = y(s)$ for $0 \leq t_0 \leq s \leq t$, the equality $(Qx)(s) = (Qy)(s)$ holds for $0 \leq t_0 \leq s \leq t$, $t < T$, T is a given number.

If E is a space of measurable functions on $[t_0, T)$ for $t_0 \geq 0$, then the definition needs a slight modification, requiring the property to be valid almost everywhere on $[t_0, T]$. One can point out that for causal operators, a notation identical with what is encountered for a general equation with a memory can be stated as follows. A representation of the form

$$x(t) = (Qx)(t)$$

where for each $t \in [t_0, T)$. The functional $(Qx)(t)$ on E which takes values in X, for each t, while the whole family of functionals, $t \in [t_0, T)$, define the operator from $E = C([t_0, T), X)$ to itself.

For illustration, let us take $E = C[[t_0, T), \mathbb{R}^n]$ as the underlying space. Let $\{Q_n\}$ be a sequence of causal operators on E such that

$$\lim_{n\to\infty} (Q_n x)(t) = (Qx)(t) \tag{2.1}$$

for each $(t, x) \in [t_0, T) \times E$. The question is whether we can infer that the limit $Q : E \to E$ is also a causal operator. The answer is yes because the causality of $\{Q_n\}$ implies

$$(Q_n x)(s) = (Q_n y)(s), \quad s \in E[t_0, T).$$

If we let $n \to \infty$ on both sides, in the above relation and use (2.1) for each fixed $s \in [t_0, T)$, we obtain the causality of Q.

The Riemann-Liouville Fractional Causal Terminal Value Problem (FCTVP) is defined as follows,

$$D^q u(t) = (Qu)(t), \quad u(T) = u^T = u(t)(T-t)^{1-q}\big|_{t=T} \tag{2.2}$$

where $0 < q < 1$ and the terminal value T and the solution $u(T, t_0, u_0) = u^T$. The corresponding Volterra fractional integral equation is given by

$$u(t) = u^T(t) + \frac{1}{\Gamma(q)} \int_t^T (t-\tau)^{q-1} (Qu)(\tau) d\tau \tag{2.3}$$

where $u^T(t) = \frac{u^T (T-t)^{q-1}}{\Gamma(q)}$ and $\Gamma(q)$ is the standard Gamma function.

Let $p = 1 - q$ and

$$C_p([t_0, T], \mathbb{R}) = \{u : u \in C([t_0, T], \mathbb{R}) \text{ and } (T-t)^p u(t) \in C([t_0, T], \mathbb{R})\}$$

consider the fractional terminal value problem (FTVP)

$$D^q u(t) = f\ (t, u(t)), \quad u(T) = u^{\scriptscriptstyle T} = u(t)(T-t)^{1-q}\big|_{t=T} \tag{2.4}$$

where $f \in C[[t_0, T] \times \mathbb{R}, \mathbb{R}]$ and $u^T(t) = \frac{u^T (T-t)^{q-1}}{\Gamma(q)}$. In fact, the terminal condition $u(T) = u^{\scriptscriptstyle T}$ and $u(t)$ is a solution of (2.4).

Definition 2.1. A function $f : (t_0, T] \to \mathbb{R}$ is Hölder continuous if there are nonnegative real constants C, α such that $|f(x) - f(y)| \le C|x - y|^\alpha$ for all $x, y \in (t_0, T]$.

Lemma 2.2. *Let $m \in C_p[[t_0, T], \mathbb{R}]$ be locally Hölder continuous with exponent $\lambda > q$, and for any $t_1 \in (t_0, T]$ we have that on $(t_1, T]$: $m(t_1) = 0$, $m(t) \le 0$ and $m(t)(T-t)^{1-q}\big|_{t=T} \le 0$ for $t_0 \le t \le t_1$. Then*

$$D^q m(t_1) \le 0. \tag{2.5}$$

Proof. By definition of the Riemann-Liouville fractional derivative is

$$D^q m(t) = \frac{1}{\Gamma(p)} \frac{d}{dt} \int_t^T (s-t)^{p-1} m(s) ds.$$

Let $H(t) = \int_t^T (s-t)^{p-1} m(s) ds$. For small $h > 0$, consider

$$\begin{aligned}
&H(t_1 + h) - H(t_1) \\
&= \int_{t_1+h}^T (s - t_1 - h)^{p-1} m(s) ds - \int_{t_1}^T (s - t_1)^{p-1} m(s) ds \\
&= \int_{t_1+h}^T [(s - t_1 - h)^{p-1} - (s - t_1)^{p-1}] m(s) ds - \int_{t_1}^{t_1+h} (s - t_1)^{p-1} m(s) ds \\
&= I_1 - I_2
\end{aligned}$$

Since $[(s - t_1 - h)^{p-1} - (s - t_1)^{p-1}] > 0$ for $t_1 \le s \le T$ and $m(s) \le 0$ by hypothesis one has $I_1 \le 0$. This leads to

$$H(t_1 + h) - H(t_1) = -\int_{t_1}^{t_1+h} (s - t_1)^{p-1} m(s) ds = -I_2.$$

Since $m(t)$ is locally Hölder continuous there exists a $k(t_1) > 0$ such that for $t_1 - h \le s \le t_1 + h$,

$$-k(t_1)(s - t_1)^\lambda \le m(s) - m(t_1) \le k(t_1)(s - t_1)^\lambda$$

where $0 < \lambda < 1$ is such that $\lambda > q$. By Hölder continuity and from the fact that $m(t_1) = 0$ we obtain

$$\begin{aligned}\int_{t_1}^{t_1+h} (s-t_1)^{p-1} m(s) ds &\geq \int_{t_1}^{t_1+h} (s-t_1)^{p-1} [m(t_1) - k(t_1)(s-t_1)^{\lambda}] ds \\ &= -k(t_1) \int_{t_1}^{t_1+h} (s-t_1)^{p-1+\lambda} ds.\end{aligned}$$

Thus

$$-I_2 = \int_{t_1}^{t_1+h} (s-t_1)^{p-1} m(s) ds \leq k(t_1) \int_{t_1}^{t_1+h} (s-t_1)^{p-1+\lambda} ds = k(t_1) \frac{h^{p+\lambda}}{p+\lambda}.$$

Hence

$$H(t_1+h) - H(t_1) - k(t_1) \frac{h^{p+\lambda}}{p+\lambda} \leq 0$$

for sufficiently small $h > 0$. Letting $h \to 0$, we obtain $\frac{d}{dt} H(t_1) \leq 0$, which implies that $D^q m(t_1) \leq 0$ and the proof is complete. □

Lemma 2.3. *Let $\{u_\epsilon(t)\}$ be a family of continuous functions on $[t_0, T]$, for $\epsilon > 0$, such that*

$$\begin{gathered} D^q u_\epsilon(t) = f(t, u_\epsilon(t)), \\ u_\epsilon^T = u_\epsilon(t)(T-t)^{1-q}\big|_{t=T}, \quad |f(t, u_\epsilon(t))| \leq M \quad \text{for } t_0 \leq t \leq T. \end{gathered}$$

Then the family of functions $\{u_\epsilon(t)\}$ is equicontinuous on $[t_0, T]$.

The proof of the above lemma can be found in [18].

Definition 2.4. Function $v, w \in C_p[[t_0, T], \mathbb{R}]$ are said to be lower and the upper solutions of (2.2) if v and w satisfy the differential inequalities, respectively,

$$\begin{gathered} D^q v(t) \geq (Qv)(t), \quad v(T) \leq u^T \\ D^q w(t) \leq (Qw)(t), \quad w(T) \geq u^T \end{gathered}$$

where the causal operator $Q \in E = C(\mathbb{R}_+, \mathbb{R})$, $Q : E \to E$ is continuous.

Definition 2.5. The causal operator $Q : E \to E$ is said to be semi nondecreasing in t for each x if

$$(Qx)(t_1) = (Qy)(t_1) \quad \text{and} \quad (Qx)(t) \leq (Qy)(t),\ 0 \leq t < t_1 < T,\ T \in \mathbb{R}_+$$

for

$$x(t_1) = y(t_1), \quad x(t) < y(t), \quad 0 \leq t < t_1 < T,\ T\ \in \mathbb{R}_+.$$

Definition 2.6. Let the causal operator $Q \in C(\mathbb{R}_+, \mathbb{R})$. At $x \in E$,

$$(Q(x+h))(t) = (Qx)(t) + L(x,h)(t) + \|h\| \eta(x,h)(t)$$

where $\lim_{\|h\| \to 0} \|\eta(x,h)(t)\| = 0$ and $L(x, \cdot)(t)$ is a linear operator. $L(x,h)(t)$ is said to be Fréchet derivative of Q at x with the increment h for the remainder $\eta(x,h)(t)$.

Theorem 2.7. *Assume that $(Qu)(t) \in C[\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}]$, where the causal operator $Q \in E = C(\mathbb{R}_+, \mathbb{R})$, $Q : E \to E$ is continuous. In addition to $v, w \in C_p[[t_0, T], \mathbb{R}]$ be with continuous exponent $\lambda > q$, such that*

(i) *$D^q v(t) \geq (Qv)(t)$;*
(ii) *$D^q w(t) \leq (Qw)(t)$;*

(iii) *$(Qu)(t)$ is nondecreasing in u for each t, $t_0 \leq t \leq T$ with one of the inequalities (i) or (ii) being strict.*

Then $v(T) < w(T)$, where $v(T) = v^T = v(t)(T-t)^{1-q}\big|_{t=T} \leq v(t_0)$ and $w(T) = w^T = w(t)(T-t)^{1-q}\big|_{t=T} \geq w(t_0)$, implies $v(t) < w(t)$, $t \in [t_0, T]$.

Proof. Assume that one of the inequalities is strict; let (i) be strict and then set $m(t) = v(t) - w(t)$. If the conclusion of the theorem is not true, then there exists $t_1 \in (t_0, T]$ such that $m(t_1) = 0$, $m(t) \leq 0$ for $t_0 \leq t \leq t_1$.

Consider the case when $t_1 \in (t_0, T]$, then $m(t_1) = 0$, $m(t) \leq 0$ on (t_0, t_1). By using Lemma 2.2, we obtain to be $D^q m(t_1) \leq 0$. Thus

$$\begin{gathered}(Qv)(t_1) < D^q v(t_1) \leq D^q w(t_1) \leq (Qw)(t_1),\\ (Qv)(t_1) < (Qw)(t_1)\end{gathered}$$

which is a contradiction. Therefore $v(t) < w(t)$.

We set, for the nonstrict inequality

$$\widetilde{v}(t) = v(t) - \epsilon[(T-t)^{q-1} E_{q,q}[-2L(t-t_0)^q]]$$

for $\epsilon, L > 0$, where $E_{q,q}$ is the Mittag-Leffler function that define as $E_{q,q}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma((k+1)q)}$, $q > 0$. This implies that

$$\widetilde{v}(t)(T-t)^{1-q}\big|_{t=T} = \widetilde{v}^T = v(t)(T-t)^{1-q}\big|_{t=T} - \epsilon g(t)(T-t)^{1-q}\big|_{t=T}.$$

So that $\widetilde{v}^T = v^T - \epsilon g^T$. Then $\widetilde{v}(t) < v(t)$ for $t \in [t_0, T]$ and $\widetilde{v}(T) < v(T)$. Thus, it follows from (i) and the fact that $(Qu)(t)$ is nondecreasing, that

$$\begin{aligned}D^q \widetilde{v}(t) &= D^q v(t) - \epsilon D^q g(t) \geq (Qv)(t) + 2\epsilon L g(t)\\ &\geq (Q\widetilde{v})(t) + 2\epsilon L g(t) > (Q\widetilde{v})(t).\end{aligned}$$

It follows by the earlier argument that $\widetilde{v}(t) < w(t)$. Finally, letting $\varepsilon \to 0$, we have $v(t) \leq w(t)$. The proof is complete. □

Theorem 2.8. *Assume that $v, w \in C_p[[t_0, T], \mathbb{R}]$ such that $v(t) \leq w(t)$, $t \in [t_0, T]$ and $Q : \Omega \to \mathbb{R}$ is the continuous causal operator where $\Omega = [(t, u) : v(t) \leq u \leq w(t)]$. Suppose further that*

(i) *$D^q v(t) \geq (Qv)(t)$;*
(ii) *$D^q w(t) \leq (Qw)(t)$;*
(iii) *$(Qu)(t) \leq \lambda(t)$ on Ω such that $\lambda \in L^1[\mathbb{R}_+, \mathbb{R}]$.*

Then (2.2) has a solution which satisfies $v(t) \leq u(t) \leq w(t)$ on $[t_0, T]$ provided that $v(T) \leq u(T) \leq w(T)$ for some $t_0 \geq 0$.

Proof. Consider $P : [t_0, T] \times \mathbb{R} \to \mathbb{R}$ defined by

$$(Pu)(t) = \max\{v(t), \min\{u, w(t)\}\}. \tag{2.6}$$

Then Q is a continuous causal operators and by the assumption (iii), we have $(Qu)(t) \leq \lambda(t)$. So that $Q(t, (Pu)(t))$ defines a continuous extension of Q to $[t_0, T] \times \mathbb{R}$ which is also bounded. Therefore, the FCTVP

$$D^q u = Q(t, (Pu)(t)), \quad u(T) = u^T \tag{2.7}$$

has a solution $u(t)$ on $[t_0, T]$. We show $v(t) \leq u(t) \leq w(t)$ for $t \in [t_0, T]$ and therefore $u(t)$ is a solution of (2.2).

For $\epsilon, L > 0$, consider

$$\begin{aligned} \widetilde{v}(t)(T-t)^{1-q}\big|_{t=T} &= v(t)(T-t)^{1-q}\big|_{t=T} - \epsilon g(t)(T-t)^{1-q}\big|_{t=T} \\ \widetilde{w}(t)(T-t)^{1-q}\big|_{t=T} &= w(t)(T-t)^{1-q}\big|_{t=T} + \epsilon g(t)(T-t)^{1-q}\big|_{t=T} \end{aligned} \tag{2.8}$$

where $g(t) = (T-t)^{q-1}E_{q,q}[-2L(t-t_0)^q]$.

Then $\widetilde{w}(t) > w(t)$, $\widetilde{v}(t) < v(t)$ and $\widetilde{v}(T) < u(T) < \widetilde{w}(T)$. We claim that $\widetilde{v}(t) < u(t) < \widetilde{w}(t)$ on $[t_0, T]$. Suppose that it is not true and thus there exists $t_1 \in [t_0, T]$ such that $u(t_1) = \widetilde{w}(t_1)$ and $\widetilde{v}(t) < u(t) < \widetilde{w}(t)$, $t_0 \le t \le t_1$.

Then $u(t_1) > w(t_1)$ and hence $(Pu)(t_1) = w(t_1)$. Also $v(t_1) \le (Pu)(t_1) \le w(t_1)$. Setting $m(t) = u(t) - \widetilde{w}(t)$, we have $m(t_1) = 0$ and $m(t) \le 0$, $t_0 \le t \le t_1$. Hence by Lemma 2.2, we obtain $D^q m(t_1) \le 0$ that yields

$$\begin{aligned} Q(t_1, (Pu)(t_1)) &= D^q u(t_1) \le D^q \widetilde{w}(t_1) = D^q w(t_1) - 2\epsilon L g(t_1) \\ &\le Q(t_1, w_{t_1}) - 2\epsilon L g(t_1) = Q(t_1, (Pu)(t_1)) - 2\epsilon L g(t_1) \\ &< Q(t_1, (Pu)(t_1)) \end{aligned}$$

which is a contradiction. Then, we have $u(t) < \widetilde{w}(t)$ on $[t_0, T]$ provided that $u(T) \le w(T)$ for some $t_0 \ge 0$. Similarly, the other case $\widetilde{v}(t) < u(t)$ for $t_0 \le t \le T$ can be proved.

Consequently, combining the proved results, we have $\widetilde{v}(t) < u(t) < \widetilde{w}(t)$ on $t \in [t_0, T]$. Letting $\epsilon \to 0$, we obtain $v(t) \le u(t) \le w(t)$, on $[0, T]$. The proof is complete. □

3. Quasilinearization Method

In this section, we extend the generalized quasilinearization method for nonlinear terminal value problems in [3]. We prove the main theorem that gives several conditions to apply the method of quasilinearization to the nonlinear causal terminal value problem involving Riemann-Liouville fractional derivatives.

Theorem 3.1. *Assume that Q, $\Phi : C[\mathbb{R}_+, \mathbb{R}] \to C[\mathbb{R}_+, \mathbb{R}]$ are continuous causal operator such that $(Qu)(t)$, $(\Phi u)(t) \in C[\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}]$ and*

- (M1) *$|(Qu)(t)| \le \lambda(t)|u(t)|$ on $\Omega = [(t,u) \in [t_0, T] \times C^q[[t_0, T], \mathbb{R}] : v(t) \le u \le w(t)]$, where $\lambda \in L^1[0, \infty)$;*
- (M2) *$v, w \in C^q[[t_0, T], \mathbb{R}]$ are the lower and upper solutions of (2.2) such that $v(t) \le w(t)$, $t \in [t_0, T]$;*
- (M3) *$v_0, w_0 \in C^q[[t_0, T], \mathbb{R}]$ with $v_0(t) \le w_0(t)$ on $[t_0, T]$, $v_0(T)$, $w_0(T)$ exist and*
  - (a) *$D^q v_0(t) \ge (Qv_0)(t)$, $v_0(T) \le u^T$ for $t \in [t_0, T]$;*
  - (b) *$D^q w_0(t) \le (Qw_0)(t)$, $w_0(T) \ge u^T$ for $t \in [t_0, T]$;*
- (M4) *$Q, \Phi \in C^q[\mathbb{R}_+, \mathbb{R}]$ and for $(t, u) \in \Omega$ the Fréchet derivatives $(Q_u u)(t)$, $(\Phi_u u)(t)$, $(Q_{uu} u)(t)$ and $(\Phi_{uu} u)(t)$ exists and are continuous on $[0, \infty)$ such that $(Q_u u)(t) \le B$, $(Q_{uu} u)(t) +$ $(\Phi_{uu} u)(t) \le 0$ for some function Φ with $|(\Phi u)(t)| \le \lambda_1(t)|u(t)|$, $|(\Phi_u u)(t)| \le F$ and $(Q_{uu} u)(t) \ge 0$, $(\Phi_{uu} u)(t) \le 0$ on $\mathbb{R}_+ \times \mathbb{R}$, where $B, F, \lambda_1 \in L^1[0, \infty)$.*

Then there exist the monotone sequences $\{v_n\}$ and $\{w_n\}$ which converge uniformly to the unique solution $u(t) = u^T(t) + \frac{1}{\Gamma(q)} \int_t^T (t-\tau)^{q-1} (Qu)(\tau) d\tau$ that satisfy $u(T, t_0, u_0) = u^T$ of (2.2) on $[t_0, T]$. Moreover, the convergence is quadratic.

Proof. Let us initially define a continuous causal operator $\Psi : C[\mathbb{R}_+,\mathbb{R}] \to C[\mathbb{R}_+,\mathbb{R}]$ and $(\Psi u)(t) \in C[\mathbb{R}_+ \times \mathbb{R},\mathbb{R}]$, such that

$$(\Psi u)(t) = (Qu)(t) + (\Phi u)(t). \tag{3.1}$$

In view of (M4), we see that $(\Psi_{uu}u)(t) \le 0$, and $|(\Psi u)(t)| \le (\lambda(t)+\lambda_1(t))|u(t)| = P|u(t)|$, where $P = (\lambda(t) + \lambda_1(t)) \in L^1[0,\infty)$. Also, $|(\Psi_u u)(t)| \le B + F = P_1 \in L^1[0,\infty)$. Using the generalized mean value theorem and (3.1), we have

$$(Qu)(t) \le (\Psi\alpha)(t) + (\Psi_u\alpha)(t)(u-\alpha) - (\Phi u)(t)$$

where $u,\alpha \in C^q[[t_0,T],\mathbb{R}]$ such that $\alpha(t) \le u(t)$, $t \in [t_0,T]$. Get

$$(Gu\alpha)(t) = (\Psi\alpha)(t) + (\Psi_u\alpha)(t)(u-\alpha) - (\Phi u)(t) \tag{3.2}$$

and observe that

$$\begin{gathered}(Gu\alpha)(t) \ge (Qu)(t)\\ (Guu)(t) = (Qu)(t).\end{gathered} \tag{3.3}$$

Further, in view of the nonincreasing property of $(\Phi_u u)(t)$, we obtain

$$(G_u u\alpha)(t) = (\Psi_u\alpha)(t) - (\Phi_u u)(t) \ge (\Psi_u\alpha)(t) - (\Phi_u\alpha)(t) \ge (Q_u\alpha)(t) \ge 0.$$

Thus, $(Gu\alpha)(t)$ is nondecreasing in u for each fixed $(t,\alpha) \in [t_0,T] \times C^q[[t_0,T],\mathbb{R}]$. Further,

$$(Gu\alpha)(t) = (\Psi\alpha)(t) + (\Psi_u u)(t)(u-\alpha) - (\Phi u)(t),$$

which, together with (M1), (M4) and (3.2) implies that

$$(Gu\alpha)(t) = P|\alpha| + B(|u| + |\alpha|) + \lambda_1|u| = P_2(t)|\alpha| + P_3(t)|u| = (H|u|)(t), \tag{3.4}$$

where $P_2 = P + B,\ P_3 = \lambda_1 + B \in L^1[0,\infty)$. Now, using the mean value theorem and the nonincreasing nature of $(\Psi_u u)(t)$, we obtain

$$\begin{aligned}(Gu\alpha_1)(t) - (Gu\alpha_2)(t) &\le (\Psi_u\mu_1)(t)(\alpha_1-\alpha_2) + (\Psi_u\alpha_2)(t)(\alpha_2-\alpha_1)\\ &= (\Psi_{uu}\mu_2)(t)(\mu_1-\alpha_2)(\alpha_1-\alpha_2) \le 0\end{aligned} \tag{3.5}$$

where $\alpha_2 \le \mu_2 \le \mu_1 \le \alpha_1$. Expression (3.5) implies that $(Gu\alpha)(t)$ is nonincreasing in α for each fixed $(t,u) \in [t_0,T] \times C^q[[t_0,T],\mathbb{R}]$. Set $v = \beta_0$ and consider the FCTVP

$$D^q u(t) = (Gu\beta_0)(t), \quad u(T) = \gamma^T \tag{3.6}$$

Because of expression (3.4), the problem (3.6) has a unique solution $\beta_1(t)$ on $[a,\infty)$, $a > 0$ satisfying $u_1(T) = u^T$. Also, in view of $(M2)$ and (3.3), we have

$$\begin{gathered}D^q\beta_0 \ge (Q\beta_0)(t) = (G\beta_0\beta_0)(t), \quad \beta_0(T) \le \gamma^T,\\ D^q w(t) \le (Qw)(t) \le (Gw\beta_0)(t), \quad w(T) \ge \gamma^T\end{gathered}$$

which imply

$$v(t) \le u_1(t) \le w(t) \quad \text{for some } a \ge 0.$$

Next, we consider the FCTVP

$$D^q u(t) = (Gu\beta_1)(t), \quad u(T) = \gamma^T \tag{3.7}$$

As above, we can show that (3.7) has a unique solution $\beta_2(t)$ satisfying $\beta_2(T) = \gamma^T$. Using (3.3) and the nonincreasing property of $(Gu\alpha)(t)$ in α, we have

$$D^q\beta_1(t) = (G\beta_1\beta_0)(t) \ge (G\beta_1\beta_1)(t), \quad \beta_1(T) = \gamma^T$$

which implies that $\beta_1(t)$ is a lower solution of (3.7) and

$$D^q w(t) \le (Qw)(t) \le (Gw\beta_1)(t), \quad \beta(T) \ge \gamma^T$$

implies that $w(t)$ is an upper solution of (3.7). Further, $\beta_1(T) \leq \beta_2(T) \leq w(T)$. Again, by Theorem 2.8, we obtain

$$\beta_1(t) \leq \beta_2(t) \leq w(t), \quad t \in [a, T)$$

for some $a \geq 0$. Continuing this process successively, we obtain

$$v \leq \beta_1 \leq \beta_2 \leq \beta_3 \leq \cdots \leq \beta_{n-1} \leq \beta_n \leq w \text{ on } [t_0, T]$$

where the elements of the monotone sequence $\{\beta_n\}$ are the solutions of the problem

$$D^q u(t) = (Gu\beta_{(n-1)})(t), \; u(T) = \gamma^T.$$

Since the sequence $\{\beta_n\}$ is monotone, it follows that it has a pointwise limit $\beta(t)$. To show that $\beta(t)$ is in fact a solution of (2.2), we observe that β_n is a solution of the linear FCTVP

$$D^q u(t) = (G\beta_n\beta_{(n-1)})(t) = F_n(t), \quad \beta_n(T) = \gamma^T \tag{3.8}$$

where

$$F_n(t) = (\Psi\beta_{(n-1)})(t) + (\Psi_u\beta_{(n-1)})(t)(\beta_n - \beta_{n-1}) - (\Phi\beta_n)(t).$$

Since G is continuous on $\mathbb{R}_+$, therefore, in view of (3.4), it follows that for each $n \in \mathbb{N}$, the sequence $\{F_n(t)\}$ is a sequence of continuous functions and is bounded by $(H\beta_n)(t) \in L^1[0, \infty)$. Consequently, $\int_t^\infty F_n(s)ds < \infty$. Now, taking the limits both side as $n \to \infty$, we have

$$\lim_{n\to\infty} F_n(t) = \lim_{n\to\infty} (G\beta_n\beta_{(n-1)})(t) = (Q\beta)(t).$$

Now, by using the Lebesque dominated convergence theorem, we obtain

$$\lim_{n\to\infty} \int_t^\infty F_n(s)ds = \int_t^\infty (Qu)(s)ds$$

which implies $\int_t^\infty (Qu)(s)ds < \infty$. Now, the solution of (3.8) is

$$\beta_n(t) = \gamma^T - \int_t^\infty F_n(s)ds$$

which, by taking the limit $n \to \infty$, yields

$$\beta(t) = \gamma^T - \int_t^\infty (Qu)(s)ds.$$

This shows that $\beta(t)$ is solution of the (2.2).

To prove the quadratic convergence of $\{\alpha_n\}$ and $\{\beta_n\}$ to the unique solution, we consider

$$\sigma_n(t) = \beta(t) - \beta_n(t), \quad n = 1, 2, 3, \ldots$$

Observe that $\sigma_n(t) \geq 0$ and $\sigma_n(\infty) = 0$. Here we use the mean value theorem and assumption (M4), to obtain

$$
\begin{aligned}
&D^q\sigma_{n+1}(t)\\
&= D^q\beta(t) - D^q\beta_{n+1}(t)\\
&= (Q\beta)(t) - [(\Psi\beta_n)(t) + (\Psi_u\beta_n)(t)(\beta_{n+1} - \beta_n) - (\Phi\beta_{(n+1)})(t)]\\
&= (\Psi_u\beta_n)(t)(\beta - \beta_n) + (\Psi_{uu}\xi)(t)\frac{(\beta - \beta_n)^2}{2!}\\
&\quad - (\Psi_u\beta_n)(t)(\beta_{n+1} - \beta_n) - ((\Phi\beta)(t) - (\Phi\beta_{(n+1)})(t))\\
&= (\Psi_u\beta_n)(t)(\beta - \beta_{n+1}) + (\Psi_{uu}\xi)(t)\frac{(\beta - \beta_n)^2}{2!} - (\Psi_u\xi_1)(t)(\beta - \beta_{n+1})\\
&\geq (Q_u\beta_n)(t)\sigma_{n+1}(t) + (\Psi_{uu}\zeta_1)(t)\frac{(\sigma_n(t))^2}{2!}\\
&\geq -B(t)\sigma_{n+1}(t) - \frac{D^qP(t)}{2}(\sigma_n(t))^2,
\end{aligned}
\tag{3.9}
$$

$\sigma_{n+1}(\infty) = 0$, where $\beta_n \leq \zeta \leq \beta$. From (3.9) and using the definition of lower solution and Theorem 2.8, we have $\sigma_{n+1}(t) \leq r(t)$ for some $t \geq a > 0$, where

$$
r(t) = \exp\Big(\int_t^\infty B(s)ds\Big)\Big[\int_t^\infty \frac{D^qP(s)}{2}(\sigma_n(s))^2 \exp\Big(-\int_t^\infty B(l)dl\Big)ds\Big],
$$

which is a unique solution of the nonhomogeneous linear problem

$$
D^qr(t) = -B(t)r(t) - \frac{D^qP(t)}{2}(\sigma_n(t))^2, \quad \beta(\infty) = 0.
$$

Thus,

$$
\sigma_{n+1}(t) \leq \exp\Big(\int_t^\infty B(s)ds\Big)\Big[\int_t^\infty \frac{D^qP(s)}{2}(\sigma_n(s))^2 \exp(-\int_t^\infty B(l)dl)ds\Big].
$$

Hence,

$$
\begin{aligned}
|\sigma_{n+1}(t)| &\leq |\exp(\int_t^\infty B(s)ds)|\Big|\int_t^\infty \frac{D^qP(s)}{2}(\sigma_n(s))^2 \exp(-\int_t^\infty B(l)dl)ds\Big|\\
&\leq K|\sigma_n(s)|^2T = A|\sigma_n(s)|^2,
\end{aligned}
$$

where $|\exp(\int_t^\infty B(s)ds)| \leq K$,

$$
|\int_t^\infty \frac{D^qP(s)}{2}(\sigma_n(s))^2 \exp(-\int_t^\infty B(l)dl)ds\Big| \leq 2T
$$

and $A = KT$. This establishes the quadratic convergence and therefore completes the proof. □

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Addendum posted by the editor on April 2, 2020

A reader sent us the comments below. I asked the authors for corrections several times, via email, but they did not reply; so I post the comments as received.

The paper studies what is usually called the right-sided Riemann-Liouville fractional derivative with a terminal condition. The terminal condition is not written correctly in the paper it should be $\lim_{t\to T-} u(t)(T-t)^{1-q} = u^T$, but not $u(T) = u^T$ which should be deleted everywhere it occurs.

By a simple change of variable this problem is actually equivalent to a Cauchy problem (initial value problem) for the usually studied left-sided Riemann-Liouville fractional derivative, which is why this terminal problem is not studied in the textbooks.

The fractional integral as defined in Section 2.1 of the well-known text [K-S-T] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and applications of fractional differential equations. North-Holland Mathematics Studies 204, Elsevier Science: B.V. Amsterdam, 2006, is, with some notation of [K-S-T],

$$I^q_{T-}f(t) = \frac{1}{\Gamma(q)}\int_t^T (\tau-t)^{q-1}f(\tau)d\tau,$$

and the corresponding fractional derivative is

$$D^q_{T-}f(t) = -\frac{1}{\Gamma(1-q)}\frac{d}{dt}\int_t^T (\tau-t)^{-q}f(\tau)d\tau.$$

In (2.3) there is a typo, the term $(t-\tau)^{-q}$ is usually not well defined for $\tau > t$ and it seems there should be no $\Gamma(q)$ in definition of $u^T(t)$.

The space C_p has a typo in the definition, it should be $u \in C([t_0, T), \mathbb{R})$.

Lemma 2.2, the minus sign in the fractional derivative is omitted at the start of the proof, so that result must be reconsidered.

Lemma 2.3 Unclear: if u_ϵ is continuous then $u^T_\epsilon = 0$. It is using result from [18] so is using the equivalence with the Cauchy problem mentioned above.

Theorem 3.1, Either there is a typo in (M4) or the claim on the line after (3.1) is not clear.

End of addendum.

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