

## BOUND STATES OF THE DISCRETE SCHRÖDINGER EQUATION WITH COMPACTLY SUPPORTED POTENTIALS

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ABSTRACT. The discrete Schrödinger operator is considered on the half-line lattice  $n \in \{1, 2, 3, \dots\}$  with the Dirichlet boundary condition at  $n = 0$ . It is assumed that the potential belongs to class  $\mathcal{A}_b$ , i.e. it is real valued, vanishes when  $n > b$  with  $b$  being a fixed positive integer, and is nonzero at  $n = b$ . The proof is provided to show that the corresponding number of bound states,  $N$ , must satisfy the inequalities  $0 \leq N \leq b$ . It is shown that for each fixed nonnegative integer  $k$  in the set  $\{0, 1, 2, \dots, b\}$ , there exist infinitely many potentials in class  $\mathcal{A}_b$  for which the corresponding Schrödinger operator has exactly  $k$  bound states. Some auxiliary results are presented to relate the number of bound states to the number of real resonances associated with the corresponding Schrödinger operator. The theory presented is illustrated with some explicit examples.

### 1. INTRODUCTION

We consider the discrete Schrödinger equation on the half-line lattice, i.e. the difference equation

$$-\psi_{n+1} + 2\psi_n - \psi_{n-1} + V_n\psi_n = \lambda\psi_n, \quad n \geq 1, \quad (1.1)$$

with the Dirichlet boundary condition

$$\psi_0 = 0. \quad (1.2)$$

Here, the discrete independent variable  $n$  takes positive integer values, the boundary of the half-line lattice corresponds to  $n = 0$ ,  $V_n$  denotes the value of the potential  $V$  at the lattice location  $n$ ,  $\lambda$  is the spectral parameter, and  $\psi_n$  denotes the value of the wave function at the location  $n$ . We assume that the potential is real valued, i.e.

$$V_n = V_n^*, \quad n \geq 1,$$

where the asterisk is used for complex conjugation.

There are various physical models [6] governed by the discrete Schrödinger equation on a half-line lattice, and such models describe the quantum mechanical behavior of a particle of energy  $\lambda$  in a semi-infinite crystal where the particle experiences at each lattice point the force associated with the potential value  $V_n$ .

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In this article we assume that the potential belongs to class  $\mathcal{A}_b$ , which is defined below as in [2].

**Definition 1.1.** The potential  $V$  appearing in (1.1) belongs to the class  $\mathcal{A}_b$  if the  $V_n$ -values are real and the support of the potential  $V$  is confined to the finite set  $\{1, 2, \dots, b\}$  for some positive integer  $b$ , i.e.  $V_n = 0$  for  $n > b$  and  $V_b \neq 0$ .

We refer to a potential  $V$  in class  $\mathcal{A}_b$  as a compactly-supported potential, and we see that  $b$  in the definition of  $\mathcal{A}_b$  refers to the smallest positive integer beyond which the potential vanishes. In class  $\mathcal{A}_b$ , it is possible to have  $V_n = 0$  for some or all  $n$ -values with  $1 \leq n < b$  but we must have  $V_b \neq 0$ . Let us remark that the trivial potential where  $V_n = 0$  for all  $n \geq 1$  can either be included in class  $\mathcal{A}_b$  by letting  $b$  also take the value  $b = 0$  or that trivial potential can be studied separately.

When the potential  $V$  belongs to class  $\mathcal{A}_b$  the discrete Schrödinger operator corresponding to (1.1) with the Dirichlet boundary condition (1.2) is a selfadjoint operator acting on square-summable functions on the half-line lattice and its spectrum is well understood [1, 2, 3, 4, 5]. The corresponding spectrum has two parts, where the first part is the continuous spectrum  $\lambda \in [0, 4]$ , and the second part is the discrete spectrum consisting of a finite number  $\lambda$ -values in the set  $\lambda \in (-\infty, 0) \cup (4, +\infty)$ . Each  $\lambda$ -value in the interval  $(0, 4)$  corresponds to a scattering state, each  $\lambda$  in the discrete spectrum corresponds to a bound state, and the values  $\lambda = 0$  and  $\lambda = 4$  correspond to the edges of the continuous spectrum.

We denote the number of discrete eigenvalues by  $N$ . In this paper we prove that, when the potential  $V$  belongs to class  $\mathcal{A}_b$  for some fixed positive integer  $b$ , the value of  $N$  is restricted to the set  $\{0, 1, \dots, b\}$ , where every value in the set including  $N = 0$  and  $N = b$  is always attained by an infinite number of potentials in class  $\mathcal{A}_b$ . Thus, in our paper we prove two main results for potentials in class  $\mathcal{A}_b$ . The first is that  $0 \leq N \leq b$  for any potential in class  $\mathcal{A}_b$  having  $N$  bound states. The second is that for every integer  $k$  in the set  $\{0, 1, \dots, b\}$ , there exists at least one potential in class  $\mathcal{A}_b$  for which there are exactly  $k$  bound states, and in fact there are infinitely many potentials in  $\mathcal{A}_b$  for which there are exactly  $k$  bound states.

While proving the two aforementioned main results, we also obtain some upper bounds on the number of bound states located in each of the intervals  $\lambda \in (-\infty, 0)$  and  $\lambda \in (4, +\infty)$ , where the corresponding upper bounds are related to the number of certain real resonances associated with the Schrödinger operator for (1.1) and (1.2).

Our paper is organized as follows. In Section 2 we briefly present the preliminaries needed to prove our two main results. This involves the introduction of the Jost solution  $f_n$  to (1.1), the Jost function  $f_0$  associated with (1.1) and (1.2), the alternate spectral parameter  $z$  related to  $\lambda$  as in (2.1), and the real and complex resonances associated with (1.1) and (1.2). In Section 3 we prove our two main results; namely, we have  $0 \leq N \leq b$  for every potential in class  $\mathcal{A}_b$  and that  $\mathcal{A}_b$  contains infinitely many potentials with  $k$  bound states for every  $k \in \{0, 1, \dots, b\}$ . We also obtain certain upper bounds on  $N_-$  and on  $N_+$ , where  $N_-$  denotes the number of bound states when  $\lambda \in (4, +\infty)$  and  $N_+$  denotes the number of bound states when  $\lambda \in (-\infty, 0)$ . Finally, in Section 4 we present various explicit examples illustrating the theory presented in Section 3. In particular, we illustrate the inequalities  $0 \leq N \leq b$  and the attainment of  $N$  being equal to any integer between zero and  $b$  in class  $\mathcal{A}_b$ .

## 2. PRELIMINARIES

For the analysis of bound states of the discrete Schrödinger operator corresponding to (1.1) and (1.2), it is useful to use the parameter  $z$  related to the spectral parameter  $\lambda$  as

$$z = 1 - \frac{\lambda}{2} + \frac{1}{2}\sqrt{\lambda(\lambda - 4)}, \quad (2.1)$$

where the square root denotes the principal branch of the complex square-root function. Under the transformation  $\lambda \mapsto z$  specified in (2.1), the extended real  $\lambda$ -axis is mapped in a one-to-one manner onto the boundary of the upper half of the unit disk in the complex- $z$  plane. In particular, the interval  $\lambda \in (-\infty, 0)$  is mapped to the interval  $z \in (0, 1)$ , the interval  $\lambda \in (0, 4)$  is mapped to the upper semicircle  $z = e^{i\theta}$  with  $0 < \theta < \pi$ , and the interval  $\lambda \in (4, +\infty)$  is mapped to the interval  $z \in (-1, 0)$ . One can invert (2.1) to obtain

$$\lambda = 2 - z - \frac{1}{z}. \quad (2.2)$$

Using this expression we can transform (1.1) into

$$\psi_{n+1} + \psi_{n-1} = (z + z^{-1} + V_n)\psi_n, \quad n \geq 1. \quad (2.3)$$

A particular solution to (2.3), whose values are denoted by  $f_n$ , with the asymptotics

$$f_n = z^n[1 + o(1)], \quad n \rightarrow \infty, \quad (2.4)$$

is usually called the Jost solution. We occasionally also use  $f_n(z)$  to denote  $f_n$ . The value  $f_0$ , i.e. the value of  $f_n$  evaluated at  $n = 0$ , is known as the Jost function. When the potential  $V$  belongs to class  $\mathcal{A}_b$ , as seen from (2.3) and (2.4) we have

$$\begin{aligned} f_n &= z^n, \quad n \geq b, \\ f_{b-1} &= -f_{b+1} + (z + z^{-1} + V_b)f_b, \\ f_{b-2} &= -f_b + (z + z^{-1} + V_{b-1})f_{b-1}, \\ f_{b-3} &= -f_{b-1} + (z + z^{-1} + V_{b-2})f_{b-2}, \\ &\vdots \\ f_1 &= -f_3 + (z + z^{-1} + V_2)f_2, \\ f_0 &= -f_2 + (z + z^{-1} + V_1)f_1. \end{aligned} \quad (2.5)$$

A bound-state solution corresponds to a square-summable solution to (1.1) satisfying the Dirichlet boundary condition (1.2).

In the following theorem we summarize some basic facts needed later on.

**Theorem 2.1.** *Assume that the potential  $V$  belongs to class  $\mathcal{A}_b$  specified in Definition 1.1. We then have the following:*

(a) *The Jost function  $f_0(z)$  associated with (1.1) is a polynomial in  $z$  of degree  $2b - 1$  and it has the form*

$$f_0(z) = 1 + K_{01}z + K_{02}z^2 + \cdots + K_{0(2b-1)}z^{2b-1}, \quad (2.6)$$

where each double-indexed coefficient  $K_{0j}$  is real valued and  $K_{0(2b-1)} = V_b$ .

(b) *Each coefficient  $K_{0j}$  appearing in (2.6) is a polynomial in the multivariable  $(V_1, V_2, \dots, V_b)$ . In each term in the coefficient  $K_{0j}$ , each  $V_n$ -value for  $n = 1, \dots, b$  appears either to the first power or does not appear at all. The Jost function  $f_0(z)$ ,*

viewed as a polynomial in  $(V_1, V_2, \dots, V_b)$  contains a single monomial with degree  $b$ , and that monomial is given by  $(V_1 V_2 \cdots V_b) z^b$ . Any other term in  $f_0(z)$  has a degree of  $b - 1$  or less. Thus, the Jost function  $f_0(z)$  has the unique decomposition

$$f_0(z) = F(z) + G(z), \quad (2.7)$$

where we have defined

$$F(z) := \left( \prod_{j=1}^b V_j \right) z^b, \quad (2.8)$$

$$G(z) := f_0(z) - \left( \prod_{j=1}^b V_j \right) z^b. \quad (2.9)$$

**(c)** For each  $b \geq 1$ , there are exactly  $2b - 1$  zeros of  $f_0(z)$  in the complex- $z$  plane. Such zeros occur either when  $z \in \mathbf{R} \setminus \{0\}$  or they occur as complex-conjugate pairs symmetrically located with respect to the real axis  $\mathbf{R}$  in the complex- $z$  plane. The point  $z = 0$  does not correspond to a zero of  $f_0(z)$ .

**(d)** The zeros of  $f_0(z)$  occurring in  $z \in (-1, 0) \cup (0, 1)$  are each simple, and each such zero corresponds to a bound state of the discrete Schrödinger operator associated with (1.1) and (1.2). We use  $N_-$  to denote the number of zeros of  $f_0(z)$  in the interval  $z \in (-1, 0)$ , use  $N_+$  to denote the number of zeros of  $f_0(z)$  in the interval  $z \in (0, 1)$ , and use  $N$  to denote the number of zeros of  $f_0(z)$  when  $z \in (-1, 0) \cup (0, 1)$ . Thus, we have

$$N = N_- + N_+.$$

We remark that, as seen from (2.2),  $N_-$  coincides with the number of bound states when  $\lambda \in (4, +\infty)$  and  $N_+$  coincides with the number of bound states when  $\lambda \in (-\infty, 0)$ .

**(e)** The Jost function  $f_0(z)$  may have a simple zero at  $z = -1$  and may have a simple zero at  $z = 1$ . Such zeros do not correspond to bound states for the discrete Schrödinger operator. Let us use  $\mu_-$  to denote the number of zeros of  $f_0(z)$  at  $z = -1$  and use  $\mu_+$  to denote the number of zeros of  $f_0(z)$  at  $z = 1$ . Hence, we have

$$\mu_- = \begin{cases} 1, & f_0(-1) = 0, \\ 0, & f_0(-1) \neq 0, \end{cases} \quad \mu_+ = \begin{cases} 1, & f_0(1) = 0, \\ 0, & f_0(1) \neq 0. \end{cases}$$

**(f)** The zeros of  $f_0(z)$  when  $z \in (-\infty, -1)$  are not necessarily simple. Similarly, the zeros of  $f_0(z)$  when  $z \in (1, +\infty)$  are not necessarily simple. We refer to the zeros of  $f_0(z)$  when  $z \in (-\infty, -1) \cup (1, +\infty)$  as real resonances for the discrete Schrödinger operator.

**(g)** The nonreal zeros of  $f_0(z)$  cannot occur inside or on the unit circle  $|z| = 1$ . Such zeros, if they exist, are not necessarily simple and they occur as complex-conjugate pairs located outside the unit circle in the complex- $z$  plane.

**(h)** We have

$$2b - 1 = Z(-\infty, -1] + Z(-1, 0) + Z(0, 1) + Z[1, +\infty) + 2Z_c, \quad (2.10)$$

where the nonnegative integer  $Z(-\infty, 1]$  is the number of zeros of  $f_0(z)$  in the interval  $z \in (-\infty, -1]$ , the nonnegative integer  $Z(-1, 0)$  is the number of zeros of  $f_0(z)$  in the interval  $z \in (-1, 0)$ , the nonnegative integer  $Z(0, 1)$  is the number of zeros of  $f_0(z)$  in the interval  $z \in (0, 1)$ , the nonnegative integer  $Z[1, +\infty)$  is the

number of zeros of  $f_0(z)$  in the interval  $z \in [1, +\infty)$ , and the nonnegative integer  $Z_c$  is the number of zeros of  $f_0(z)$  located in the interior of the upper-half complex- $z$  plane.

*Proof.* For the results stated in (a), (c), (d), (e) we refer the reader to [1, Theorems 2.2, 2.4, 2.5]. The proof of (b) is directly obtained from (2.5) as follows. From [2, (2.50)] we see that each coefficient  $K_{0j}$  in (2.6) is a polynomial in  $(V_1, V_2, \dots, V_b)$  and that  $f_0(z)$  has the form

$$f_0(z) = 1 + (V_1 + \dots + V_b)z + \dots + V_b(V_1 + \dots + V_{b-1})z^{2b-2} + V_b z^{2b-1}. \quad (2.11)$$

In each line of (2.5) expressing  $f_{n-1}$  for  $n = 1, \dots, b$ , only one single potential value, i.e.  $V_n$  appears. Thus, in expressing  $f_0(z)$  as in (2.11), each term in  $f_0(z)$  contains  $V_n$  either to the first power or to the zeroth power. Let us view each  $f_n$  as a polynomial in the multivariable  $(V_1, \dots, V_b)$ . From (2.5) we observe that the highest-degree term in  $f_{b-1}$  is the single monomial given by  $V_b f_b$  or equivalently by  $V_b z^b$ ; the highest-degree term in  $f_{b-2}$  is the term  $V_{b-1} f_{b-1}$  or equivalently  $V_{b-1} V_b z^b$ ; the highest-degree term in  $f_{b-3}$  is the term  $V_{b-2} f_{b-2}$ . Continuing in this manner, from (2.5) we see that the highest-degree term in  $f_0$  is the single monomial given by  $(V_1 V_2 \dots V_b) z^b$ . Thus, the proof of (b) is complete. For (f) we refer the reader to (4.14) and (4.15) in Example 4.3 where we illustrate a double zero of  $f_0(z)$  when  $z \in (-\infty, -1)$  and refer to (4.12) and (4.13) in Example 4.3 for a double zero in  $z \in (1, +\infty)$ . For (g) we refer the reader to [1, Theorem 2.4] and refer to (4.19) and (4.20) in Example 4.3 where we illustrate a double complex zero. Finally, the proof of (h) is obtained as follows. By (c) we know that  $f_0(z)$  is a polynomial in  $z$  of degree  $2b - 1$  and that it has exactly  $2b - 1$  zeros in the complex- $z$  plane. By (c) we also know that the zeros of  $f_0(z)$  off the real axis must occur in complex conjugate pairs and hence the number of such zeros can be represented as  $2Z_c$ , where  $Z_c$  is the number of zeros of  $f_0(z)$  in  $z \in \mathbb{C}^+$ , where  $\mathbb{C}^+$  denotes the open upper-half complex- $z$  plane. From (c) we also know that  $f_0(z)$  cannot have a zero at  $z = 0$ . Thus, (2.10) holds, which expresses the total number of zeros of  $f_0(z)$  in terms of the number of zeros at various different locations in the complex- $z$  plane.  $\square$

By Theorem 2.1(d) we know that each zero of  $f_0(z)$  when  $z \in (-1, 0) \cup (0, 1)$  corresponds to a bound state of the discrete Schrödinger operator associated with (1.1) and (1.2). It is known [2] that for each bound-state zero, there exists a positive constant, known as the Marchenko norming constant, and when the potential belongs to class  $\mathcal{A}_b$ , the Marchenko norming constant  $c_k$  corresponding to the zero  $\alpha_k$  of  $f_0(z)$  with  $\alpha_k \in (-1, 0) \cup (0, 1)$  is related to the Jost function  $f_0(z)$  as

$$c_k^2 = \text{Res} \left[ \frac{f_0\left(\frac{1}{z}\right)}{z f_0(z)}, \alpha_k \right], \quad (2.12)$$

where the notation  $\text{Res}[h(z), \alpha]$  is used to denote the residue of the function  $h(z)$  at  $z = \alpha$ . The expression given in (2.12) follows from [2, (2.45), (2.49) and (3.19)].

In the next proposition, we express the right-hand side of (2.12) in terms of all the zeros of  $f_0(z)$ . In our paper we use the convention that if the lower limit is greater than the upper limit in a product, then that product is understood to be absent or equivalently its value is equal to one.

**Proposition 2.2.** *Assume that the potential  $V$  appearing in (1.1) belongs to class  $\mathcal{A}_b$  specified in Definition 1.1. Then, the Marchenko norming constant  $c_k$  appearing*

in (2.12) satisfies

$$c_k^2 = \frac{1}{\alpha_k^{2b}} \frac{\prod_{j=1}^{2b-1} (1 - \alpha_k \alpha_j)}{\prod_{j \neq k} (\alpha_k - \alpha_j)}, \quad (2.13)$$

where  $\{\alpha_j\}_{j=1}^{2b-1}$  is the set of all zeros of  $f_0(z)$  and the product appearing in the denominator of (2.13) is over all  $j = 1, \dots, 2b-1$  except  $j = k$ .

*Proof.* From Theorem 2.1(a) we know that  $f_0(z)$  is a polynomial in  $z$  of degree  $2b-1$ , and hence it can be written in terms of its zeros as

$$f_0(z) = \left(1 - \frac{z}{\alpha_1}\right) \left(1 - \frac{z}{\alpha_2}\right) \cdots \left(1 - \frac{z}{\alpha_{2b-1}}\right),$$

where by Theorem 2.1(d) we know that those  $\alpha_k$  values confined to  $(-1, 0) \cup (0, 1)$  are simple. We can write  $f_0(z)$  in terms of its zeros also as

$$f_0(z) = V_b (z - \alpha_1) (z - \alpha_2) \cdots (z - \alpha_{2b-1}), \quad (2.14)$$

where we have used the fact that the coefficient of  $z^{2b-1}$  in the expression (2.11) is equal to  $V_b$ . From [2, (2.46), (2.49), (3.19), and (3.22)], we have

$$c_k^2 = \frac{f_0\left(\frac{1}{\alpha_k}\right)}{\alpha_k \dot{f}_0(\alpha_k)}, \quad (2.15)$$

where  $\dot{f}_0(\alpha_k)$  is  $df_0/dz$  evaluated at  $z = \alpha_k$ . From (2.14) we obtain

$$f_0\left(\frac{1}{\alpha_k}\right) = V_b \prod_{j=1}^{2b-1} \left(\frac{1}{\alpha_k} - \alpha_j\right), \quad (2.16)$$

$$\dot{f}_0(\alpha_k) = V_b \prod_{j \neq k} (\alpha_k - \alpha_j). \quad (2.17)$$

Using (2.16) and (2.17) in (2.15), we obtain

$$c_k^2 = \frac{\prod_{j=1}^{2b-1} \left(\frac{1}{\alpha_k} - \alpha_j\right)}{\alpha_k \prod_{j \neq k} (\alpha_k - \alpha_j)},$$

or equivalently

$$c_k^2 = \frac{1}{\alpha_k} \frac{1}{\alpha_k^{2b-1}} \frac{\prod_{j=1}^{2b-1} (1 - \alpha_k \alpha_j)}{\prod_{j \neq k} (\alpha_k - \alpha_j)},$$

which yields (2.13).  $\square$

### 3. MAIN RESULTS

To prove our two main results, i.e. the number of bound states  $N$  for any potential in class  $\mathcal{A}_b$  must satisfy  $0 \leq N \leq b$  and that every integer  $k$  in the set  $\{0, 1, \dots, b\}$  is equal to the number of bound states for at least one potential, and in fact for infinitely many potentials, in class  $\mathcal{A}_b$ , we first need to prove some auxiliary results.

In reference to the nonzero integers appearing in (2.10), let us define

$$p := Z(-\infty, -1], \quad (3.1)$$

$$q := Z(-\infty, -1] + Z(-1, 0), \quad (3.2)$$

$$r := Z(-\infty, -1] + Z(-1, 0) + Z(0, 1), \quad (3.3)$$

$$s := Z(-\infty, -1] + Z(-1, 0) + Z(0, 1) + Z[1, +\infty). \quad (3.4)$$

The integers  $p, q, r, s$  help us to order the real zeros of  $f_0(z)$  in an increasing order and they indicate the location of those zeros as

$$\alpha_1, \alpha_2, \dots, \alpha_p \in (-\infty, -1], \quad (3.5)$$

$$\alpha_{p+1}, \alpha_{p+2}, \dots, \alpha_q \in (-1, 0), \quad (3.6)$$

$$\alpha_{q+1}, \alpha_{q+2}, \dots, \alpha_r \in (0, 1), \quad (3.7)$$

$$\alpha_{r+1}, \alpha_{r+2}, \dots, \alpha_s \in [1, +\infty), \quad (3.8)$$

so that  $\alpha_{s+1}, \alpha_{s+2}, \dots, \alpha_{2b-1}$  correspond to the nonreal zeros of  $f_0(z)$ . We know from Theorem 2.1(g) that those nonreal zeros must occur in complex-conjugate pairs and hence we have

$$2b - 1 - s = 2Z_c, \quad (3.9)$$

where  $Z_c$  is the nonnegative integer appearing in (2.10).

Let us denote the signum function by

$$\operatorname{sgn}(x) := \begin{cases} 1, & x > 0, \\ -1, & x < 0. \end{cases}$$

In the following proposition we consider the sign of the product appearing in the numerator of (2.13).

**Proposition 3.1.** *Assume that the potential  $V$  appearing in (1.1) belongs to class  $\mathcal{A}_b$ , and further assume that there exists at least one bound state. Let  $\{\alpha_j\}_{j=1}^{2b-1}$  denote the set of zeros of  $f_0(z)$  ordered as indicated in (3.5)-(3.9). Then, for the bound state occurring at  $z = \alpha_k$  we have*

$$\operatorname{sgn} \left( \prod_{j=1}^{2b-1} (1 - \alpha_k \alpha_j) \right) = \begin{cases} \operatorname{sgn} \left( \prod_{j=1}^p (1 - \alpha_k \alpha_j) \right), & \alpha_k \in (-1, 0), \\ \operatorname{sgn} \left( \prod_{j=r+1}^s (1 - \alpha_k \alpha_j) \right), & \alpha_k \in (0, 1), \end{cases} \quad (3.10)$$

where  $p, r$ , and  $s$  are the integers defined in (3.1), (3.3), and (3.4), respectively.

*Proof.* Let us omit  $(1 - \alpha_k \alpha_j)$  from the argument of the product and e.g. write  $\prod_1^p$  to denote  $\prod_{j=1}^p (1 - \alpha_k \alpha_j)$ . We have

$$\prod_1^{2b-1} = \left( \prod_1^p \right) \left( \prod_{p+1}^q \right) \left( \prod_{q+1}^r \right) \left( \prod_{r+1}^s \right) \left( \prod_{s+1}^{2b-1} \right). \quad (3.11)$$

With the help of (3.6) and (3.7) we see that each of  $\prod_{p+1}^q$  and  $\prod_{q+1}^r$  is positive because  $|\alpha_j| < 1$  when  $p+1 \leq j \leq r$ . By Theorem 2.1(g), the zeros  $\alpha_j$  for  $s+1 \leq j \leq 2b-1$  occur in complex-conjugate pairs and hence the quantity  $\prod_{s+1}^{2b-1}$  appearing in (3.11) consists of the products of the form  $|1 - \alpha_k \alpha_j|^2$  when the  $\alpha_j$ -values consist of all nonreal zeros of  $f_0(z)$  in the upper-half complex- $z$  plane. Since  $\alpha_k$  is real and such  $\alpha_j$ -values are nonreal, we conclude that  $\prod_{s+1}^{2b-1}$  is positive. Let us now consider the sign of  $\prod_1^p$  appearing in (3.11). As seen from (3.5) and (3.7), for  $1 \leq j \leq p$  we have  $1 - \alpha_k \alpha_j > 0$  if  $\alpha_k \in (0, 1)$ . Let us also consider the sign of  $\prod_{r+1}^s$  appearing in (3.11). As seen from (3.6) and (3.8), for  $r+1 \leq j \leq s$  we have  $1 - \alpha_k \alpha_j > 0$  if  $\alpha_k \in (-1, 0)$ . Thus, from (3.11) we directly conclude (3.10).  $\square$

In the following proposition, we consider the sign of the product appearing in the denominator of (2.13).

**Proposition 3.2.** *Assume that the potential  $V$  appearing in (1.1) belongs to class  $\mathcal{A}_b$ , and further assume that there exists at least one bound state. Let  $\{\alpha_j\}_{j=1}^{2b-1}$  denote the set of zeros of  $f_0(z)$  ordered as indicated in (3.5)-(3.9). Then, for the bound state occurring at  $z = \alpha_k$  we have*

$$\operatorname{sgn} \left( \prod_{j \neq k} (\alpha_k - \alpha_j) \right) = (-1)^{k-1}, \quad k = p+1, \dots, r, \quad (3.12)$$

where  $p$  and  $r$  are the respective integers appearing in (3.1) and (3.3), the product  $\prod_{j \neq k}$  is over all  $j$ -values with  $1 \leq j \leq 2b-1$  except  $j = k$ , and we recall that  $\operatorname{sgn}$  denotes the signum function.

*Proof.* Let us drop  $(\alpha_k - \alpha_j)$  from the argument of the product and use  $\prod_{j \neq k}$  to denote the product on the left-hand side of (3.12). We have

$$\prod_{j \neq k} = \left( \prod_1^{k-1} \right) \left( \prod_{k+1}^s \right) \left( \prod_{s+1}^{2b-1} \right), \quad (3.13)$$

where  $\prod_1^{k-1}$  denotes  $\prod_{j=1}^{k-1} (\alpha_k - \alpha_j)$ , etc. From (3.5)-(3.7) we see that  $\prod_1^{k-1}$  is positive because  $\alpha_k > \alpha_j$  when  $1 \leq j < k$ . The sign of  $\prod_{k+1}^s$  is the same as the sign of  $(-1)^{s-k}$  because each factor  $(\alpha_k - \alpha_j)$  is negative as we have  $\alpha_k < \alpha_j$  for  $k+1 \leq j \leq s$ . Furthermore, the quantity  $\prod_{s+1}^{2b-1}$  is positive because it consists of the products  $(\alpha_k - \alpha_j)(\alpha_k - \alpha_j^*)$ , which is equivalent to  $|\alpha_k - \alpha_j|^2$ , when the  $\alpha_j$ -values are on the upper-half complex- $z$  plane, as indicated in Theorem 2.1(g). Since  $\alpha_k$  is real and such  $\alpha_j$ -values are nonreal, each factor  $|\alpha_k - \alpha_j|^2$  is positive and hence  $\prod_{s+1}^{2b-1}$  is positive. Thus, from (3.13) we conclude that the sign of its right-hand side is the same as the sign of  $(-1)^{s-k}$ . On the other hand, from (2.10) and (3.9) we know that  $s$  is an odd positive integer. Thus the sign of  $(-1)^{s-k}$  is the same as the sign of  $(-1)^{k-1}$ , which establishes (3.12).  $\square$

We recall from Theorem 2.1(f) that the real zeros of  $f_0(z)$  when  $z \in (-\infty, -1) \cup (1, +\infty)$  correspond to the real resonances, whereas from Theorem 2.1(d) we know that the zeros when  $z \in (-1, 0) \cup (0, 1)$  correspond to the bound states. In the following proposition we investigate the relationship between the number of real resonances and the number of bound states for a potential in class  $\mathcal{A}_b$ .

**Proposition 3.3.** *Assume that the potential  $V$  appearing in (1.1) belongs to class  $\mathcal{A}_b$ , and further assume that there exists at least one bound state. Let  $\{\alpha_j\}_{j=1}^{2b-1}$  denote the set of zeros of  $f_0(z)$  ordered as indicated in (3.5)-(3.9). As in Theorem 2.1(h), let  $Z(-\infty, -1]$ ,  $Z(-1, 0)$ ,  $Z(0, 1)$ , and  $Z[1, +\infty)$  denote the number of zeros of  $f_0(z)$  in the intervals  $(-\infty, -1]$ ,  $(-1, 0)$ ,  $(0, 1)$ , and  $[1, +\infty)$ , respectively. We then have the following:*

(a) *If  $Z(-1, 0) = 0$  then*

$$Z(-\infty, -1] = Z(-1, 0) - 1 + \varepsilon_-, \quad (3.14)$$

where  $\varepsilon_-$  is a positive integer with  $\varepsilon_- \geq 1$ .

(b) *If  $Z(-1, 0) \geq 1$  then*

$$Z(-\infty, -1] = Z(-1, 0) - 1 + \varepsilon_-, \quad (3.15)$$

where  $\varepsilon_-$  is a nonnegative integer with  $\varepsilon_- \geq 0$ .

(c) If  $Z(0, 1) = 0$  then

$$Z[1, +\infty) = Z(0, 1) - 1 + \varepsilon_+, \quad (3.16)$$

where  $\varepsilon_+$  is a positive integer with  $\varepsilon_+ \geq 1$ .

(d) If  $Z(0, 1) \geq 1$  then

$$Z[1, +\infty) = Z(0, 1) - 1 + \varepsilon_+, \quad (3.17)$$

where  $\varepsilon_+$  is a nonnegative integer with  $\varepsilon_+ \geq 0$ .

(e) The sum of nonnegative integers  $\varepsilon_-$  and  $\varepsilon_+$  appearing in (a)-(d) must satisfy  $\varepsilon_- + \varepsilon_+ \geq 1$ .

(f) We have

$$Z(-\infty, -1] \geq \begin{cases} 0, & Z(-1, 0) = 0, \\ Z(-1, 0) - 1, & Z(-1, 0) \geq 1, \end{cases} \quad (3.18)$$

$$Z[1, \infty) \geq \begin{cases} 0, & Z(0, 1) = 0, \\ Z(0, 1) - 1, & Z(0, 1) \geq 1. \end{cases} \quad (3.19)$$

*Proof.* Note that (3.14) and (3.16) automatically follow from the facts that the integers  $Z(-\infty, -1]$  and  $Z[1, +\infty)$  must both be nonnegative as they represent the number of zeros of  $f_0(z)$  in the appropriate intervals. So, it is enough to prove (3.15) and (3.17) only. If  $Z(-1, 0) \geq 1$  then there exists at least one  $\alpha_k$ -value in the interval  $z \in (-1, 0)$ , which corresponds to a bound state. Let  $c_k$  be the corresponding Marchenko norming constant. From (2.13), (3.12), and the first line of (3.10) we obtain

$$\operatorname{sgn}(c_k^2) = (-1)^{k-1} \operatorname{sgn} \left( \prod_{j=1}^p (1 - \alpha_k \alpha_j) \right), \quad (3.20)$$

where  $p$  is the integer defined in (3.1). Being a norming constant, we have  $c_k > 0$  and hence from (3.20) we conclude that

$$\operatorname{sgn} \left( \prod_{j=1}^p (1 - \alpha_k \alpha_j) \right) = (-1)^{k-1}, \quad k = p+1, \dots, q, \quad (3.21)$$

where  $q$  is the integer appearing in (3.2) and (3.6). Defining

$$P_-(z) := \prod_{j=1}^p (1 - \alpha_j z), \quad (3.22)$$

let us investigate the sign of  $P_-(z)$  when  $z$  takes values in the interval  $z \in (-1, 0)$ . From (3.21) and (3.22) we know that

$$\operatorname{sgn}(P_-(\alpha_k)) = (-1)^{k-1}, \quad k = p+1, \dots, q. \quad (3.23)$$

Thus, from (3.23) we conclude that in the interval  $z \in (-1, 0)$ , the polynomial  $P_-(z)$  defined in (3.22) changes sign at least  $(q-p-1)$  times. This indicates that the polynomial  $P_-(z)$  must have degree no less than  $q-p-1$ . From (3.1) and (3.2) we have  $q-p = Z(-1, 0)$ , and from (3.22) we know that the degree of  $P_-(z)$  is the same as  $p$ , which is equal to  $Z(-\infty, -1]$  as seen from (3.1). Thus, we have proved that

$$Z(-\infty, -1] \geq Z(-1, 0) - 1, \quad (3.24)$$

when  $Z(-1, 0) \geq 1$ . We can then write (3.24) as (3.15), and hence the proof of (b) is complete. In a similar way, let us prove (d). If  $Z(0, 1) \geq 1$  then there exists at least one  $\alpha_k$ -value in the interval  $z \in (0, 1)$ , which corresponds to a bound state. Let  $c_k$  be the corresponding Marchenko norming constant. From (2.13), (3.12), and the second line of (3.10), we obtain

$$\operatorname{sgn}(c_k^2) = (-1)^{k-1} \operatorname{sgn} \left( \prod_{j=r+1}^s (1 - \alpha_k \alpha_j) \right), \quad (3.25)$$

where  $r$  and  $s$  are the integers appearing in (3.3), (3.4), (3.7), and (3.8). The norming constant  $c_k$  is positive and hence from (3.25) we conclude that

$$\operatorname{sgn} \left( \prod_{j=r+1}^s (1 - \alpha_k \alpha_j) \right) = (-1)^{k-1}, \quad k = q + 1, \dots, r, \quad (3.26)$$

where we recall that  $q$  is the integer appearing in (3.2), (3.6), and (3.7). Letting

$$P_+(z) := \prod_{j=r+1}^s (1 - \alpha_j z), \quad (3.27)$$

we notice that the degree of  $P_+(z)$  is  $s - r$ , which is equal to  $Z[1, +\infty)$  as seen from (3.3) and (3.4). Let us investigate the sign of  $P_+(z)$  when  $z$  takes values in the interval  $z \in (0, 1)$ . From (3.26) and (3.27) we see that

$$\operatorname{sgn}(P_+(\alpha_k)) = (-1)^{k-1}, \quad k = q + 1, \dots, r. \quad (3.28)$$

Thus, from (3.28) we conclude that in the interval  $z \in (0, 1)$  the polynomial  $P_+(z)$  changes sign at least  $(r - q - 1)$  times. Hence, the degree of  $P_+(z)$  cannot be less than  $(r - q - 1)$ . From (3.2) and (3.3) we see that  $r - q = Z(0, 1)$ , and we have already seen that the degree of  $P_+(z)$  is equal to  $Z[1, +\infty)$ . Thus, we have proved that

$$Z[1, +\infty) \geq Z(0, 1) - 1, \quad (3.29)$$

when  $Z(0, 1) \geq 1$ . We can write (3.29) in the form given in (3.17), and hence the proof of (d) is complete. The proof of (e) is obtained as follows. Using (3.14)-(3.17) in (2.10) we obtain

$$2b - 1 = 2[Z(-1, 0) + Z(0, 1) - 1 + Z_c] + \varepsilon_- + \varepsilon_+,$$

where the left-hand side is an odd integer and we have  $\varepsilon_- + \varepsilon_+ \geq 0$  on the right-hand side. Thus, (e) must hold. The result in (f) directly follows from (a)-(d).  $\square$

To prepare the proof of one of our two main theorems, namely that for any potential in class  $\mathcal{A}_b$  having  $N$  bound states we must have  $0 \leq N \leq b$ , we first prove that for any  $b \geq 1$  there exist infinitely many nontrivial potentials in class  $\mathcal{A}_b$  for which  $N = 0$  and also there exist infinitely many potentials in class  $\mathcal{A}_b$  for which  $N = b$ . We then prove our other main theorem, namely, that for each  $k$  in the set  $\{0, 1, \dots, b\}$  there are infinitely many potentials in class  $\mathcal{A}_b$  having exactly  $k$  bound states. We remark that in the limiting case where  $b = 0$  the potential class  $\mathcal{A}_b$  contains only the trivial potential where  $V_n \equiv 0$ , for which we already know that  $N = 0$ . Thus, our main result  $0 \leq N \leq b$  automatically and trivially holds also when  $b = 0$ .

In the next theorem we prove that for any fixed  $b \geq 1$  the class  $\mathcal{A}_b$  contains infinitely many potentials with  $N = 0$ . We recall that for each potential in class  $\mathcal{A}_b$  we have  $V_b \neq 0$ .

**Theorem 3.4.** *For any fixed  $b$  with  $b \geq 1$ , the class  $\mathcal{A}_b$  specified in Definition 1.1 contains infinitely many potentials having no bound states.*

*Proof.* From Theorem 2.1(b) we already know that the Jost function  $f_0(z)$  is a polynomial in the multivariable  $(V_1, V_2, \dots, V_b)$ . In fact,  $f_0(z)$  considered as such a polynomial consists of terms where the degree of each term is between 0 and  $b$ . In Theorem 2.1(b) we have seen that the term with the largest degree is the unique monomial given by  $(V_1 \cdots V_b)z^b$ . In fact,  $f_0(z)$  as a polynomial in  $(V_1, \dots, V_b)$  contains a single monomial term with zero degree and that term is the term given by 1 in  $f_0(z)$ . We already know that the zero potential  $V_n \equiv 0$  corresponds to  $N = 0$  with the corresponding Jost function being  $f_0 \equiv 1$ . By using a small perturbation on the trivial potential appropriately, we can get a nontrivial potential with  $N = 0$ . This can be seen as follows. In (2.6) every term in  $f_0(z) - 1$ , viewed as a polynomial in  $(V_1, \dots, V_b)$ , has degree at least one. Thus, we can choose each  $|V_j|$  small enough and with  $V_b \neq 0$  so that the corresponding  $K_{0j}$  appearing in (2.6) satisfies

$$|K_{0j}| < \frac{1}{2b}, \quad 1 \leq j \leq 2b - 1. \quad (3.30)$$

By Theorem 2.1(a) we know that each  $K_{0j}$  is real and hence (3.30) implies that when  $z \in (-1, 1)$  we obtain

$$-\frac{1}{2b} < K_{0j}z < \frac{1}{2b}, \quad 1 \leq j \leq 2b - 1. \quad (3.31)$$

From (2.6) we have the standard inequality

$$|f_0(z)| \geq |1 - |K_{01}z + K_{0,2}z^2 + \cdots + K_{0(2b-1)}z^{2b-1}||. \quad (3.32)$$

Using (3.31) in (3.32), for  $z \in (-1, 1)$  we obtain

$$|f_0(z)| \geq 1 - (2b - 1)\frac{1}{2b},$$

or equivalently

$$|f_0(z)| \geq \frac{1}{2b},$$

implying that there cannot be any  $z$ -value in the interval  $z \in (-1, 1)$  for which  $f_0(z) = 0$ . Thus, for such potentials we must have  $N = 0$ .  $\square$

In the next theorem we prove that for any fixed  $b \geq 1$ , the class  $\mathcal{A}_b$  contains infinitely many potentials for which the number of bound states is exactly  $b$ .

**Theorem 3.5.** *For any fixed  $b$  with  $b \geq 1$ , the class  $\mathcal{A}_b$  specified in Definition 1.1 contains infinitely many potentials having exactly  $b$  bound states.*

*Proof.* From Theorem 2.1(b) we know that we can write the Jost function  $f_0(z)$  as in (2.7) as the sum of  $F(z)$  and  $G(z)$  where  $F(z)$  is the monomial in  $(V_1, \dots, V_b)$  defined in (2.8) and  $G(z)$  is the polynomial in  $(V_1, \dots, V_b)$  of degree no larger than  $b - 1$  given in (2.9). Let us choose our potential  $V$  in class  $\mathcal{A}_b$  such that  $|V_b| > 1$  and

$$|V_j| = |V_b|, \quad j = 1, \dots, b - 1. \quad (3.33)$$

Let us estimate the corresponding  $|F(z)|$  and  $|G(z)|$  on the unit circle  $|z| = 1$ . Using (3.33) in (2.8) we obtain

$$|F(z)|\Big|_{|z|=1} = |V_b|^b. \quad (3.34)$$

On the other hand, using (3.33) in (2.9) we obtain

$$|G(z)|\Big|_{|z|=1} \leq c|V_b|^{b-1}, \quad (3.35)$$

for some positive constant  $c$ , and this can be seen as follows. Note that  $G(z)$  can be viewed as a finite sum of monomials in  $(V_1, \dots, V_b)$  of degree  $b-1$  or less multiplied with some nonnegative integer power of  $|z|$ . Thus, (2.9) implies that each such monomial is bounded by a constant multiple of  $|V_b|^{b-1}$ , and since there are only a finite number of such monomials, there exists a positive constant  $c$  depending on  $b$  for which (3.35) holds. We can choose  $|V_b|$  large enough so that  $|V_b| > c$ , and hence from (3.34) and (3.35) we obtain

$$|G(z)|\Big|_{|z|=1} < |F(z)|\Big|_{|z|=1}. \quad (3.36)$$

Using (3.36) in the decomposition (2.7), we can apply Rouché's theorem of complex variables and conclude that the number of zeros of  $f_0(z)$  inside the unit circle  $|z| = 1$  coincides with the number of zeros of  $F(z)$  inside that unit circle. By Theorem 2.1(d), the number of zeros of  $f_0(z)$  inside the unit circle is equal to  $N$ , the number of bound states. On the other hand, using (3.33) in (2.8) we have

$$|F(z)| = |V_b|^b |z|^b,$$

and hence  $F(z)$  has exactly  $b$  zeros inside the unit circle, and in fact all such zeros occur at  $z = 0$ . Thus, we have proved that there exists at least one potential with  $V_b \neq 0$  in class  $\mathcal{A}_b$  for which  $N = b$ . In fact, since we can choose  $V_b$  in an infinite number of ways such that (3.33) and  $|V_b| > c$  are satisfied, it follows that there are indeed an infinite number of potentials in class  $\mathcal{A}_b$  for which we have  $N = b$ .  $\square$

The next theorem shows that for each integer  $k$  in the set  $\{0, 1, \dots, b\}$  there are an infinite number of potentials in class  $\mathcal{A}_b$  having exactly  $k$  bound states. We recall that  $V_b \neq 0$  for each potential in  $\mathcal{A}_b$ .

**Theorem 3.6.** *For any fixed  $b$  with  $b \geq 1$  and for each nonnegative integer  $k$  in the set  $\{0, 1, \dots, b\}$ , there correspond infinitely many potentials in class  $\mathcal{A}_b$  having exactly  $k$  bound states.*

*Proof.* The proof for  $k = 0$  is given in Theorem 3.4 and the proof for  $k = b$  is given in Theorem 3.5. So, it is enough to provide the proof for  $k$  in the set  $\{1, \dots, b-1\}$ . In the proof of Theorem 3.5, the potentials explicitly constructed in class  $\mathcal{A}_b$  with  $b$  bound states and with  $V_b \neq 0$  are all generic, i.e. the corresponding Jost functions  $f_0(z)$  do not vanish at  $z = \pm 1$ . This can be seen from (2.7) and (3.36) by noting that on the unit circle  $|z| = 1$  we have

$$|f_0(z)|\Big|_{|z|=1} = |F(z) + G(z)|\Big|_{|z|=1} \geq |F(z)|\Big|_{|z|=1} - |G(z)|\Big|_{|z|=1} > 0,$$

and hence  $f_0(z)$  cannot vanish on the unit circle and in particular it cannot vanish at  $z = \pm 1$ . Thus, for each integer  $k$  in the set  $\{1, \dots, b-1\}$  we conclude that there is at least one generic potential  $V$  with exactly  $k$  bound states in class  $\mathcal{A}_k$  with  $V_k \neq 0$  and  $V_n = 0$  for  $n > k$ . Let us assume that the corresponding bound-state zeros of the Jost function  $f_0(z)$  occur at  $z = z_l$  with  $l = 1, \dots, k$ . We know from Theorem 2.1(d) that each such  $z_l$  is simple and belongs to the set  $z \in (-1, 0) \cup (0, 1)$ .

Let us continuously perturb the potential  $V$  to  $\tilde{V}$  in such a way that  $\tilde{V}_n = V_n$  for  $n \leq k$ ,  $\tilde{V}_n = \epsilon$  for  $k < n \leq b$ , and  $\tilde{V}_n = 0$  for  $n > b$ , where  $\epsilon$  is a nonzero real parameter. The perturbed potential  $\tilde{V}$  belongs to class  $\mathcal{A}_b$ , and let us use  $\tilde{f}_0(z)$  for the corresponding Jost function. By Theorem 2.1(b) we know that as we perturb  $V$  to  $\tilde{V}$  continuously, the coefficients of  $\tilde{f}_0(z)$  change continuously, and hence the zeros of  $\tilde{f}_0(z)$  also change continuously. We claim that when  $\epsilon$  is small the number of zeros of  $\tilde{f}_0(z)$  in  $z \in (-1, 0) \cup (0, 1)$  must be equal to  $k$ . This can be seen as follows. Because of Theorem 2.1(g), as we introduce the perturbation none of  $z_l$ -values can abruptly change to nonreal values, and because of Theorem 2.1(c) none of those  $z_l$ -values can change to zero. Thus, the only way to change the number of bound states during the continuous perturbation would be for a zero of  $f_0(z)$  to move into or out of  $z \in (-1, 0) \cup (0, 1)$  through  $z = -1$  or  $z = 1$ . By choosing  $\epsilon$  small enough, we know that we must have  $\tilde{f}_0(\pm 1) \neq 0$  because we have  $f_0(\pm 1) \neq 0$ . Thus, we have shown that for small enough  $\epsilon$  we have  $\tilde{V}$  belonging to class  $\mathcal{A}_b$  with  $\tilde{V}_b \neq 0$  and the corresponding perturbed operator has exactly  $k$  bound states. Let us remark that, as we continuously perturb the potential  $V$  to  $\tilde{V}$ , the degree of  $\tilde{f}_0(z)$  abruptly changes from  $2k - 1$  to  $2b - 1$ , which is the degree of  $\tilde{f}_0(z)$ . However, the additional zeros of the Jost function enter the complex- $z$  plane from  $z = \infty$  and hence they do not increase the number of bound states. Thus, our proof is complete.  $\square$

Having proved one of our main results in Theorem 3.6, in the next theorem we prove our other main result.

**Theorem 3.7.** *Assume that the potential  $V$  appearing in (1.1) belongs to class  $\mathcal{A}_b$  for some positive integer  $b$ . Then, the number of bound states, denoted by  $N$ , for the corresponding discrete Schrödinger operator associated with (1.1) and (1.2) satisfies the inequalities*

$$0 \leq N \leq b. \quad (3.37)$$

We remark that the result trivially holds also when  $b = 0$ .

*Proof.* Let us first indicate that (3.37) holds when  $b = 0$  because in that case  $\mathcal{A}_b$  consists of the trivial potential  $V_n \equiv 0$  for which we already know [2] that  $N = 0$ . By Theorem 2.1(d) we have

$$Z(-1, 0) + Z(0, 1) = N, \quad (3.38)$$

where we recall that  $Z(-1, 0)$  and  $Z(0, 1)$  denote the number of zeros of the Jost function  $f_0(z)$  in the respective intervals  $z \in (-1, 0)$  and  $z \in (0, 1)$ . With the help of (3.38), from (3.18) and (3.19) we conclude that

$$Z(-\infty, -1] + Z[1, +\infty) \geq \begin{cases} 0, & N = 0, \\ N - 1, & N = 1, \\ N - 2, & N \geq 2. \end{cases} \quad (3.39)$$

Using (3.38) and (3.39) in (2.10), we obtain

$$2b - 1 \geq N + 2Z_c + \begin{cases} 0, & N = 0, \\ N - 1, & N = 1, \\ N - 2, & N \geq 2. \end{cases} \quad (3.40)$$

Since  $Z_c \geq 0$ , from (3.40) we obtain

$$2b - 1 \geq \begin{cases} N, & N = 0, \\ 2N - 1, & N = 1, \\ 2N - 2, & N \geq 2, \end{cases}$$

or equivalently

$$2b - 1 \geq \begin{cases} 0, & N = 0, \\ 1, & N = 1, \\ 2N - 2, & N \geq 2. \end{cases} \quad (3.41)$$

From (3.41) we see that

$$b \geq \begin{cases} 1/2, & N = 0, \\ 1, & N = 1, \\ N - \frac{1}{2}, & N \geq 2. \end{cases} \quad (3.42)$$

From (3.42) we infer that

$$N \leq \begin{cases} b, & N = 0, \\ b, & N = 1, \\ b + \frac{1}{2}, & N \geq 2. \end{cases} \quad (3.43)$$

From the third line of (3.43) we conclude that any potential in class  $\mathcal{A}_b$  satisfies  $N < b$  or  $N \leq b$ . By Theorem 3.5 we know that  $N \leq b$  holds for an infinite number of potentials in class  $\mathcal{A}_b$ . Hence we must have  $N \leq b$  for any potential in class  $\mathcal{A}_b$ .  $\square$

Let us comment on our result given in (3.37) for the discrete Schrödinger operator where the discrete spectrum is a discrete subset of  $\lambda \in (-\infty, 0) \cup (4, +\infty)$ , and let us contrast it with a result by Simon for the continuous Schrödinger operator where the discrete spectrum is a discrete subset of  $\lambda \in (-\infty, 0)$ . For this, we write our discrete Schrödinger operator as  $H_0 + V$ , where  $H_0$  is the free discrete Schrödinger operator with the Dirichlet boundary condition having no bound states and  $V$  is the multiplication operator by the potential  $V$  which has at most  $b$  nonzero eigenvalues (viewed as an infinite diagonal matrix with only  $b$  nonzero elements). Our result in (3.37) that the operator  $H_0 + V$  has at most  $b$  bound states is an analog of Simon's result given in Lemma 5.3 of [8] as well as in (125) appearing in the proof of [7, Theorem XIII.80, p. 274].

#### 4. SOME EXPLICIT EXAMPLES

In this section we illustrate the results presented in Section 3 with some explicit examples. In our first example we consider all the possibilities for the bound states for potentials in class  $\mathcal{A}_b$  with  $b = 1$ .

**Example 4.1.** Consider the potentials in class  $\mathcal{A}_b$  with  $b = 1$ , where we have  $V_n = 0$  for  $n \geq 2$  and  $V_1 \neq 0$ . From (2.11) we know that the corresponding Jost function is given by

$$f_0 = 1 + V_1 z. \quad (4.1)$$

According to Theorem 3.7 we must have  $0 \leq N \leq 1$  satisfied by the number of bound states, and by Theorem 3.6 we must have  $N = 0$  for an infinite number of  $V_1$ -values and we must also have  $N = 1$  for an infinite number of  $V_1$ -values. From

(4.1) we see that the only zero of  $f_0(z)$  occurs at  $z = \alpha_1$  with  $\alpha_1 = -1/V_1$ . Thus, when  $0 < |V_1| < 1$  we have  $|\alpha_1| > 1$  and hence  $\alpha_1 \notin (-1, 0) \cup (0, 1)$ , indicating that  $N = 0$ . On the other hand, when  $|V_1| > 1$ , we have  $0 < |\alpha_1| < 1$  and hence  $\alpha_1 \in (-1, 0) \cup (0, 1)$ , indicating that  $N = 1$ . Let us also remark that the inequalities given in (3.18) and (3.19) hold when  $N = 0$  and also when  $N = 1$ .

In the next example, we explore all the possibilities for the bound states in class  $\mathcal{A}_b$  with  $b = 2$ .

**Example 4.2.** Consider the potential class  $\mathcal{A}_b$  with  $b = 2$ , and hence we have  $V_n = 0$  for  $n > 2$  and  $V_2 \neq 0$ . From (2.11) we obtain the corresponding Jost function as

$$f_0(z) = 1 + (V_1 + V_2)z + V_1V_2z^2 + V_2z^3. \quad (4.2)$$

In this case  $f_0(z)$  has three zeros  $\alpha_1, \alpha_2, \alpha_3$  and we have

$$f_0(z) = \left(1 - \frac{z}{\alpha_1}\right) \left(1 - \frac{z}{\alpha_2}\right) \left(1 - \frac{z}{\alpha_3}\right). \quad (4.3)$$

Equating the corresponding coefficients in (4.2) and (4.3) we obtain

$$\begin{cases} V_1 + V_2 = -\frac{1}{\alpha_1} - \frac{1}{\alpha_2} - \frac{1}{\alpha_3}, \\ V_1V_2 = \frac{1}{\alpha_1\alpha_2} + \frac{1}{\alpha_1\alpha_3} + \frac{1}{\alpha_2\alpha_3}, \\ V_2 = -\frac{1}{\alpha_1\alpha_2\alpha_3}. \end{cases} \quad (4.4)$$

The nonlinear algebraic equations given in (4.4) impose certain restrictions on the locations of  $\alpha_1, \alpha_2, \alpha_3$  in the complex- $z$  plane when  $V_1$  and  $V_2$  take real values. We can view  $V_1$  and  $V_2$  as the two solutions to the quadratic equation

$$x^2 - (V_1 + V_2)x + V_1V_2 = 0, \quad (4.5)$$

where the solutions are given by

$$x = \frac{V_1 + V_2 \pm \sqrt{(V_1 + V_2)^2 - 4V_1V_2}}{2}. \quad (4.6)$$

Thus,  $V_2$  must be equal to one of the two quantities on the right-hand side of (4.6). Using the first two lines of (4.4) in (4.6) we obtain

$$x = \frac{-\left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3}\right) \pm \sqrt{\left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3}\right)^2 - 4\left(\frac{1}{\alpha_1\alpha_2} + \frac{1}{\alpha_1\alpha_3} + \frac{1}{\alpha_2\alpha_3}\right)}}{2}. \quad (4.7)$$

Comparing (4.7) with the third line of (4.4) we see that we must have

$$\begin{aligned} & -\frac{2}{\alpha_1\alpha_2\alpha_3} + \left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3}\right) \\ &= \sqrt{\left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3}\right)^2 - 4\left(\frac{1}{\alpha_1\alpha_2} + \frac{1}{\alpha_1\alpha_3} + \frac{1}{\alpha_2\alpha_3}\right)}, \end{aligned} \quad (4.8)$$

or

$$\begin{aligned} & -\frac{2}{\alpha_1\alpha_2\alpha_3} + \left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3}\right) \\ &= -\sqrt{\left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3}\right)^2 - 4\left(\frac{1}{\alpha_1\alpha_2} + \frac{1}{\alpha_1\alpha_3} + \frac{1}{\alpha_2\alpha_3}\right)}. \end{aligned} \quad (4.9)$$

In order for (4.8) or (4.9) to hold, we must have

$$-(\alpha_1\alpha_2\alpha_3)(\alpha_1 + \alpha_2 + \alpha_3) + (\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) = 1, \quad (4.10)$$

which is obtained by squaring both sides of (4.8) and (4.9) and then simplifying the resulting expressions. We can write (4.10) in the equivalent form as

$$\alpha_1\alpha_2(1 - \alpha_3^2) + \alpha_1\alpha_3(1 - \alpha_2^2) + \alpha_2\alpha_3(1 - \alpha_1^2) = 1. \quad (4.11)$$

The conditions (4.8) and (4.9) impose various restrictions on  $\alpha_1, \alpha_2, \alpha_3$ . For example, we cannot have  $\alpha_1, \alpha_2, \alpha_3$  all located in the interval  $[1, +\infty)$ . Otherwise, the left-hand side of (4.11) would be negative. Similarly, we cannot have  $\alpha_1, \alpha_2, \alpha_3$  all located in the interval  $(-\infty, -1]$ . Otherwise, the left-hand side of (4.10) would again be negative. On the other hand, for example, we have a double real resonance and a bound state with

$$\alpha_1 = \alpha_2 = 2, \quad \alpha_3 = -\frac{3}{2} + \sqrt{3} = 0.23205\bar{1}, \quad (4.12)$$

which correspond to

$$V_1 = -\frac{5}{2} - \sqrt{3} = -4.23205\bar{5}, \quad V_2 = -\frac{1}{2} - \frac{1}{\sqrt{3}} = -1.07731\bar{5}. \quad (4.13)$$

Here, we use an overline on a digit to indicate a round off. For example, we have a double real resonance and another real resonance with

$$\alpha_1 = \alpha_2 = 2, \quad \alpha_3 = -\frac{3}{2} - \sqrt{3} = -3.23205\bar{1},$$

with the corresponding potential values given by

$$V_1 = -\frac{5}{2} + \sqrt{3} = -0.76795\bar{5}, \quad V_2 = -\frac{1}{2} + \frac{1}{\sqrt{3}} = 0.07735\bar{5}.$$

We also get a double real resonance and another real resonance with

$$\alpha_1 = \alpha_2 = -2, \quad \alpha_3 = \frac{3}{2} + \sqrt{3} = 3.23205\bar{5}, \quad (4.14)$$

which correspond to

$$V_1 = \frac{5}{2} - \sqrt{3} = 0.76795\bar{5}, \quad V_2 = \frac{1}{2} - \frac{1}{\sqrt{3}} = -0.07735\bar{5}. \quad (4.15)$$

The restriction (3.18) indicates that if we have two bound states with  $\alpha_1$  and  $\alpha_2$  both being in the interval  $(-1, 0)$ , then we must have  $\alpha_3 \leq -1$ . However, in that case the restriction (4.11) rules out the possibility  $\alpha_3 = -1$  and dictates that we must have  $\alpha_3 \leq -1.56105\bar{5}$ . Similarly, the restriction (3.19) indicates that if we have two bound states with  $\alpha_1$  and  $\alpha_2$  both being in the interval  $(0, 1)$ , then we must have a real resonance with  $\alpha_3 \geq 1.56105\bar{5}$ . Since nonreal resonances must occur in complex-conjugate pairs it is impossible to have a complex resonance and two bound states. We can have two bound states and one real resonance by choosing  $\alpha_1, \alpha_2, \alpha_3$  appropriately so that the corresponding  $V_1$  and  $V_2$  are real valued. For example, for

$$V_1 = -\sqrt{5} = -2.23607\bar{7}, \quad V_2 = \frac{4}{\sqrt{5}} = 1.78885\bar{5},$$

we obtain

$$\alpha_1 = \frac{1}{2}, \quad \alpha_2 = -\frac{1}{2}, \quad \alpha_3 = \sqrt{5}.$$

Choosing

$$V_1 = \sqrt{5}, \quad V_2 = -\frac{4}{\sqrt{5}} = -1.7888\bar{5},$$

we obtain

$$\alpha_1 = \frac{1}{2}, \quad \alpha_2 = -\frac{1}{2}, \quad \alpha_3 = -\sqrt{5}.$$

Choosing

$$V_1 = \frac{-13 + \sqrt{22}}{3} = -2.7698\bar{6}, \quad V_2 = -4 - 4\sqrt{\frac{2}{11}} = -5.7056\bar{1},$$

we obtain

$$\alpha_1 = \frac{1}{6}, \quad \alpha_2 = \frac{1}{2}, \quad \alpha_3 = \frac{11 - \sqrt{22}}{3} = 2.1031\bar{9},$$

and choosing

$$V_1 = \frac{13 - \sqrt{22}}{3} = 2.7698\bar{6}, \quad V_2 = 4 + 4\sqrt{\frac{2}{11}} = 5.7056\bar{1},$$

we obtain

$$\alpha_1 = -\frac{1}{6}, \quad \alpha_2 = -\frac{1}{2}, \quad \alpha_3 = \frac{-11 + \sqrt{22}}{3} = -2.1031\bar{9}.$$

In the following example, we illustrate some of the possibilities for the number of bound states and resonances for potentials in class  $\mathcal{A}_b$  with  $b = 3$ .

**Example 4.3.** Consider the potential class  $\mathcal{A}_b$  with  $b = 3$ , and hence  $V_n = 0$  for  $n > 3$  and  $V_3 \neq 0$ . From (2.11) we see that the corresponding Jost function  $f_0(z)$  is expressed in terms of  $V_1, V_2, V_3$  as

$$\begin{aligned} f_0(z) = & 1 + (V_1 + V_2 + V_3)z + [V_1V_2 + (V_1 + V_2)V_3]z^2 \\ & + [V_2 + V_3(1 + V_1V_2)]z^3 + V_3(V_1 + V_2)z^4 + V_3z^5. \end{aligned} \quad (4.16)$$

In terms of the zeros  $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$  of  $f_0(z)$  we have the representation

$$f_0(z) = \left(1 - \frac{z}{\alpha_1}\right) \left(1 - \frac{z}{\alpha_2}\right) \left(1 - \frac{z}{\alpha_3}\right) \left(1 - \frac{z}{\alpha_4}\right) \left(1 - \frac{z}{\alpha_5}\right). \quad (4.17)$$

By equating the corresponding coefficients in (4.16) and (4.17) we express  $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$  in terms of  $V_1, V_2, V_3$  as a nonlinear system of five equations given by

$$\left\{ \begin{array}{l} V_1 + V_2 + V_3 = -\left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} + \frac{1}{\alpha_4} + \frac{1}{\alpha_5}\right), \\ V_1V_2 + (V_1 + V_2)V_3 = \frac{1}{\alpha_1\alpha_2} + \frac{1}{\alpha_1\alpha_3} + \cdots + \frac{1}{\alpha_4\alpha_5}, \\ V_2 + V_3(1 + V_1V_2) = -\left(\frac{1}{\alpha_1\alpha_2\alpha_3} + \frac{1}{\alpha_1\alpha_2\alpha_4} + \cdots + \frac{1}{\alpha_3\alpha_4\alpha_5}\right), \\ (V_1 + V_2)V_3 = \frac{1}{\alpha_1\alpha_2\alpha_3\alpha_4} + \frac{1}{\alpha_1\alpha_2\alpha_3\alpha_5} + \cdots + \frac{1}{\alpha_2\alpha_3\alpha_4\alpha_5}, \\ V_3 = -\frac{1}{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5}. \end{array} \right. \quad (4.18)$$

Notice that if (4.18) has a solution, then we can change the sign of each of  $V_1, V_2, V_3$  and  $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$  and get another solution. The nonlinear relations given in (4.18) put certain restrictions on the locations of  $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$  on the complex- $z$  plane in order to have  $V_1, V_2, V_3$  as real-valued constants. The system in

(4.18) can be solved to express  $V_1, V_2, V_3, \alpha_5$  in terms of  $\alpha_1, \alpha_2, \alpha_3, \alpha_4$  by solving the fifth line in (4.18) for  $V_3$ , then solving the fourth line for  $V_2$ , then solving the first line for  $\alpha_5$ , and solving the second line for  $V_1$ . Then, we can use the resulting expressions for  $V_1, V_2, V_3, \alpha_5$  in the third line of (4.18) to get the consistency. We then obtain a consistency equation involving  $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ . By assigning various allowable values for  $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ , we can then produce some explicit examples. For example, by choosing

$$\alpha_2 = \alpha_1, \quad \alpha_3 = \alpha_4 = \alpha_1^*,$$

we can obtain the existence of a double nonreal resonance with  $\alpha_1 = -0.3196\bar{8} + 2i$  corresponding to

$$\alpha_5 = -0.60017\bar{2}, \quad V_1 = 1.1327\bar{9}, \quad V_2 = 0.74610\bar{6}, \quad V_3 = 0.099012\bar{9},$$

and we observe that this case has exactly one bound state at  $z = \alpha_5$ . We obtain another example with

$$\alpha_1 = 1.161\bar{3} + i, \quad \alpha_2 = \alpha_1, \quad \alpha_3 = \alpha_4 = \alpha_1^*, \quad \alpha_5 = 0.2779\bar{7}, \quad (4.19)$$

$$V_1 = -1.8911\bar{4}, \quad V_2 = -3.0320\bar{2}, \quad V_3 = -0.652\bar{2}, \quad (4.20)$$

which indicates that we have one bound state at  $z = \alpha_5$ , a double complex resonance at  $z = \alpha_1$ , and a double complex resonance at  $z = \alpha_1^*$ .

In the final example below we present a specific example where  $N = b$  is attained in class  $\mathcal{A}_b$ .

**Example 4.4.** For a fixed positive value of  $b$ , let us choose the potential appearing in (1.1) as

$$V_n = \begin{cases} (-1)^n 2, & 1 \leq n \leq b, \\ 0, & n > b, \end{cases} \quad (4.21)$$

so that it belongs to class  $\mathcal{A}_b$ . Using (2.11) we then get the Jost function  $f_0(z)$  explicitly expressed as a polynomial in  $z$  of degree  $2b - 1$ . Using the symbolic computing system Mathematica, we evaluate the zeros of  $f_0(z)$  numerically and observe that e.g. for each  $b = 1, 2, \dots, 110$  the resulting  $f_0(z)$  has exactly  $b$  real zeros in  $z \in (-1, 0) \cup (0, 1)$ . This numerically confirms the result presented in Theorem 3.5. We remark that as  $b$  increases some of the zeros of  $f_0(z)$  start getting closer to  $z = \pm 1$ . In that case, one needs to increase the accuracy of the numerical program used in Mathematica to evaluate the zeros of a polynomial function to avoid any discrepancies. If one replaces the value of 2 in the first line of (4.21) with a larger value, then the zeros of  $f_0(z)$  in the set  $(-1, 0) \cup (0, 1)$  move away from  $z = \pm 1$  and hence it becomes easier to confirm  $N = b$  for large  $b$ -values during the numerical evaluation of the zeros of  $f_0(z)$ .

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