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NONLINEAR FREDHOLM EQUATIONS IN MODULAR FUNCTION SPACES

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Abstract. We investigate the existence of solutions in modular function spaces of the Fredholm integral equation

$$\Phi(\theta) = g(\theta) + \int_0^1 f(\theta, \sigma, \Phi(\sigma))\, d\sigma,$$

where $\Phi(\theta), g(\theta) \in L_\rho, \theta \in [0,1], f : [0,1] \times [0,1] \times L_\rho \to \mathbb{R}$. An application in the variable exponent Lebesgue spaces is derived under minimal assumptions on the problem data.

1. Introduction

The purpose of this work is to study the existence of the solutions of Fredholm equations within the general theory of functional analysis in modular function spaces L_ρ [8, 9, 4]. Several authors study the case of integral equations in the space of all ρ-continuous functions from $[0,1]$ into L_ρ using the Banach contraction principle or the Brouwer fix point theorem for continuous functions, see the references [2, 6, 7, 15]. Taleb and Hanebaly [15] considered the Banach contraction mapping principle in order to solve integral equations in modular function spaces taking into account the Δ_2-type condition. Hajji and Hanebaly [6, 7] use the argument in [15]. The theory of modular function spaces has gained attention with the publication of the book by Diening et al. [4] on variable exponent spaces. We work within the general theory of the spaces of all ρ-continuous functions from $[0,1]$ into L_ρ, denoted by $\mathcal{C}_\rho([0,1], L_\rho)$.

The aim of this article is to investigate the existence of solutions of Fredholm equations in $\mathcal{C}_\rho([0,1], L_\rho)$. Preliminaries on modular function spaces are presented in Section 2. In Section 3, we present our main results concerning the existence of the solutions in Theorem 3.5. In Section 4, we give an application of our main result in the variable exponent spaces $L^{p(\cdot)}$.

2. Preliminaries

Let us consider the Fredholm integral equation

$$\Phi(\theta) = g(\theta) + \int_0^1 f(\theta, \sigma, \Phi(\sigma))\, d\sigma. \tag{2.1}$$

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In general, this equation need not have a solution. See [1].

For basic definitions and properties of modular function spaces the reader is referred to [8, 4]. We denote by $\mathcal{M}_\infty$ the set of Lebesgue measurable functions $\phi : [0,1] \to [0,\infty]$.

Definition 2.1 ([8])**.** We say that $\rho : \mathcal{M}_\infty \to [0,\infty]$ is a convex regular modular function if the following conditions hold:

(1) $\rho(\phi) = 0$ if and only if $\phi = 0$;
(2) $\rho(\alpha\phi) = \rho(\phi)$, if $|\alpha| = 1$;
(3) $\rho(\alpha\phi + (1-\alpha)\psi) \le \alpha\rho(\phi) + (1-\alpha)\rho(\psi)$, for any $\alpha \in [0,1]$,

where $\phi, \psi \in \mathcal{M}_\infty$.

We say that a set $A \subset [0,1]$ is ρ-null if $\rho(\mathbf{1}_A) = 0$, where $\mathbf{1}_A$ denote the characteristic function of the set A, its clear that ρ-null sets are the same as Lebesgue-null sets, by condition (1) of Definition 2.1. We say that a property holds ρ-almost everywhere (ρ-a.e.) if the exceptional set is ρ-null. Define

$$X = \{\phi \in \mathcal{M}_\infty : |\phi(\theta)| < \infty \ \rho\text{-a.e.}\}.$$

The modular function space $L_\rho([0,1])$, or briefly L_ρ, is defined as

$$L_\rho = \{\phi \in X : \lim_{\lambda \to 0} \rho(\lambda\phi) = 0\}.$$

Throughout this paper, the Luxemburg norm in L_ρ is defined as:

$$\|\phi\|_\rho = \inf\big\{t > 0; \rho\big(\frac{1}{t}\phi\big) \le 1\big\}.$$

Definition 2.2 ([8])**.** Let ρ be a convex regular modular function.

(a) $\{\phi_n\}$ is said to be ρ-convergent to ϕ if $\lim_{n\to+\infty} \rho(\phi_n - \phi) = 0$.
(b) $\{\phi_n\}$ is said to be ρ-Cauchy if $\lim_{n,m\to+\infty} \rho(\phi_n - \phi_m) = 0$.
(c) $B \subset L_\rho$ is said to be ρ-closed if for any sequence $\{\phi_n\}$ in B which ρ-converges to ϕ, we have $\phi \in B$. We will denote by $\overline{B}^\rho$ the intersection of all ρ-closed subsets which contain B. $\overline{B}^\rho$ will be called the ρ-closure of B.
(d) $B \subset L_\rho$ is said to be ρ-bounded if
$$\operatorname{diam}_\rho(B) = \sup\{\rho(\phi - \psi); \phi \in B, \psi \in B\} < \infty.$$
(e) $B \subset L_\rho$ is called ρ-compact if for any $\{\Phi_n\}$ in B, there exists a subsequence $\{\Phi_{n_k}\}$ such that $\{\Phi_{n_k}\}$ is ρ-convergent to $\Phi \in B$.
(f) $B \subset L_\rho$ is said to be ρ-relatively compact if its ρ-closure is ρ-compact.

In the following theorem we recall some of the needed properties of modular spaces.

Theorem 2.3 ([8, 9])**.** *Let ρ be a convex regular modular function. The following properties hold:*

(i) *if $\lim_{n\to+\infty} \rho(\lambda\phi_n) = 0$, for some $\lambda > 0$, then there exists a subsequence $\{\phi_{k_n}\}$ which converges ρ-a.e. to 0;*
(ii) *(Fatou property) if $\{\phi_n\}$ converges ρ-a.e. to ϕ, then we have*
$$\rho(\phi) \le \liminf_{n\to+\infty} \rho(\phi_n).$$

Definition 2.4. We say that ρ satisfies the Δ_2-type condition if there exists $K > 0$ such that $\rho(2\phi) \le K\ \rho(\phi)$, for any $\phi \in L_\rho$.

This property is crucial when studying modular function spaces. Note that if ρ satisfies the Δ_2-type condition, then

$$\omega_\rho(2) = \inf\{K : \rho(2\phi) \le K\ \rho(\phi) \text{ where } \phi \in L_\rho\} < \infty.$$

In this case, we have

$$\rho(\phi+\psi) = \rho\Big(2\frac{\phi+\psi}{2}\Big) \le \omega_\rho(2)\rho\Big(\frac{\phi+\psi}{2}\Big) \le \frac{\omega_\rho(2)}{2}\Big(\rho(\phi)+\rho(\psi)\Big), \tag{2.2}$$

for any $\phi, \psi \in L_\rho$. This property is satisfied in b-metric spaces, see for example [11]. Since ρ fails the triangle inequality, then the ρ-convergence may not imply the ρ-Cauchy behavior. But if ρ satisfies the Δ_2-type condition, then the ρ-convergence does imply the ρ-Cauchy behavior.

As a consequence to Theorem 2.3, and without using the Δ_2-type condition of ρ, we have the following proposition:

Proposition 2.5 ([8]). *Let ρ be a convex regular modular function. Then L_ρ is complete with respect to the ρ-convergence, i.e., any ρ-Cauchy sequence is ρ-convergent. Moreover, the ρ-balls*

$$B_\rho(\phi, r) = \{\psi \in L_\rho;\ \rho(\phi-\psi) \le r\},$$

are ρ-closed, for any $\phi \in L_\rho$ and $r \ge 0$.

Let us come back to the question of the relationship between convex regular modular functions and norm convergence in modular function spaces L_ρ.

Proposition 2.6 ([8]). *Let ρ be a convex regular modular function. The ρ-convergence with respect to the function modular ρ is equivalent to the $\|\cdot\|_\rho$-convergence with respect to the Luxemburg norm $\|\cdot\|_\rho$ in L_ρ if and only if ρ satisfies the Δ_2-type condition.*

3. Solvability of nonlinear Fredholm equations

In this section we discuss the solvability of the Fredholm integral equation in modular function spaces L_ρ,

$$\Phi(\theta) = g(\theta) + \int_0^1 f(\theta, \sigma, \Phi(\sigma))\, d\sigma, \tag{3.1}$$

where $\Phi(\theta), g(\theta) \in L_\rho$, $\theta \in [0,1]$ and $f : [0,1] \times [0,1] \times L_\rho \to \mathbb{R}$. First, we need to recall the definition of ρ-continuity [7, 15] and fix some notation.

Definition 3.1. Let ρ be a convex regular modular function, and $\Phi : [0,1] \to L_\rho$. Then Φ is said to be ρ-continuous at $\theta_0 \in [0,1]$, if $\rho(\Phi(\theta) - \Phi(\theta_0)) \to 0$ as $\theta \to \theta_0$. A function $\Phi : [0,1] \to L_\rho$ is said to be ρ-continuous if it is ρ-continuous at every point of $[0,1]$.

If ρ satisfies the Δ_2-type condition, then any ρ-continuous function in L_ρ is also $\|\cdot\|_\rho$-continuous function in L_ρ. We denote by $\mathcal{C}_\rho([0,1], L_\rho)$ the space of all ρ-continuous functions from $[0,1]$ into L_ρ. Define $\rho_{\mathcal{C}_\rho} : \mathcal{C}_\rho([0,1], L_\rho) \to [0,\infty]$, by

$$\rho_{\mathcal{C}_\rho}(\Phi) = \sup_{\theta \in [0,1]} \rho(\Phi(\theta)), \tag{3.2}$$

for any $\Phi \in \mathcal{C}_\rho([0,1], L_\rho)$. For any nonempty subset $B \subset L_\rho$, we denote by $\mathcal{C}_\rho([0,1], B)$ the set of functions $\Phi \in \mathcal{C}_\rho([0,1], L_\rho)$ such that $\Phi([0,1]) \subset B$. The following Lemma will be useful throughout our work.

Lemma 3.2 ([15]). *Let ρ be a convex regular modular function. Assume that ρ satisfies the Δ_2-type condition, and $B \subset L_\rho$ is a nonempty ρ-closed convex subset of L_ρ. Then:*

(a) *$\rho_{\mathcal{C}_\rho}$ is convex modular satisfying the Fatou property and the Δ_2-type condition with:*
$$\omega_{\rho_{\mathcal{C}_\rho}}(2) \leq \omega_\rho(2).$$
(b) *$\mathcal{C}_\rho([0,1], L_\rho)$ is $\rho_{\mathcal{C}_\rho}$-complete.*
(c) *$\mathcal{C}_\rho([0,1], B)$ is a $\rho_{\mathcal{C}_\rho}$-closed, convex subset of $\mathcal{C}_\rho([0,1], L_\rho)$.*

Now, we can define the Luxemburg norm in $\mathcal{C}_\rho([0,1], L_\rho)$ as
$$\|\Phi\|_{\mathcal{C}_\rho} = \inf\big\{t > 0; \rho_{\mathcal{C}_\rho}\big(\frac{1}{t}\Phi\big) \leq 1\big\}. \tag{3.3}$$
Assume that ρ satisfies the Δ_2-type condition, then we can introduce the supremum-norm of an element $\Phi \in \mathcal{C}_\rho([0,1], L_\rho)$ by
$$\|\Phi\|_\infty = \sup_{\theta\in[0,1]} \|\Phi(\theta)\|_\rho. \tag{3.4}$$
Before we state the main theorem, we will need the following lemma.

Lemma 3.3 ([7]). *Let ρ be a convex regular modular function that satisfies the Δ_2-type condition. Let $\{\Phi_n\}$ be sequence in $\mathcal{C}_\rho([0,1], L_\rho)$ and $\Phi \in \mathcal{C}_\rho([0,1], L_\rho)$. Then the following statements are equivalent:*

(a) $\lim_{n\to\infty} \rho_{\mathcal{C}_\rho}(\Phi_n - \Phi) = 0$,
(b) $\lim_{n\to\infty} \|\Phi_n - \Phi\|_{\mathcal{C}_\rho} = 0$,
(c) $\lim_{n\to\infty} \|\Phi_n - \Phi\|_\infty = 0$.

Let $f : [0,1]\times[0,1]\times L_\rho \to \mathbb{R}$ and $\Phi, g \in \mathcal{C}_\rho([0,1], L_\rho)$ such that for each $\theta \in [0,1]$, $f(\theta, \cdot, \Phi(\cdot)) \in L_\rho$, and

(H1) $\sigma \to f(\theta, \sigma, \Phi(\sigma))$ is Lebesgue measurable over [0,1];
(H2) $\theta \to \int_0^1 f(\theta, \sigma, \Phi(\sigma))\, d\sigma \in \mathcal{C}_\rho([0,1], L_\rho)$.

We will say that f is ρ-strongly continuous with respect to the first variable if
$$\lim_{\theta\to\theta_0} \sup_{\sigma\in[0,1]} \rho(f(\theta, \sigma, \Phi(\sigma)) - f(\theta_0, \sigma, \Phi(\sigma))) = 0,$$
for any $\theta_0 \in [0,1]$, and $\Phi \in \mathcal{C}_\rho([0,1], B)$ where B is a nonempty ρ-bounded subset of L_ρ.

To the Fredholm integral equation (2.1) in the modular function spaces L_ρ, we associate the integral operator $\mathcal{A} : \mathcal{C}_\rho([0,1], L_\rho) \to \mathcal{C}_\rho([0,1], L_\rho)$ defined by
$$(\mathcal{A}\Phi)(\cdot) = g(\cdot) + \int_0^1 f(\cdot, \sigma, \Phi(\sigma))\, d\sigma. \tag{3.5}$$
It is clear that the solutions to (2.1) are exactly the fixed points of $\mathcal{A}$, i.e., $\mathcal{A}\Phi = \Phi$.

In the proof of the main result, we use Schaeffer's fixed point theorem. We state it here for the reader's convenience.

Theorem 3.4 ([5, 13, 14]). *Let X be a normed space, T a continuous mapping of X into X, such that the closure of $T(B)$ is compact for any bounded subset B of X. Then either*

(i) *the equation $x = \lambda Tx$ has a solution for $\lambda = 1$, or*
(ii) *the set of all such solutions x, for $0 < \lambda < 1$, is unbounded.*

Note that if a map is continuous and the closure of bounded sets are compact, it is known as completely continuous. More details on completely continuous mappings may be found in Deimling, [3, p. 55]. Similarly, we will say that the operator $\mathcal{A}$, defined by (3.5), is $\rho_{\mathcal{C}_\rho}$-completely continuous if it is $\rho_{\mathcal{C}_\rho}$-continuous and the image of any $\rho_{\mathcal{C}_\rho}$-bounded subset is $\rho_{\mathcal{C}_\rho}$-relatively compact. Now we are ready to state the main result of our work.

Theorem 3.5. *Let ρ be a convex regular modular function that satisfies the Δ_2-type condition. Assume that $g \in \mathcal{C}_\rho([0,1], L_\rho)$ and $f : [0,1] \times [0,1] \times L_\rho \to \mathbb{R}$ satisfies the following properties:*

- (SC) *f is ρ-strongly continuous with respect to the first variable and satisfies (H1) and (H2),*
- (LC) *there exists $L \geq 0$ such that*
$$\rho(f(\theta,\cdot,\Phi(\cdot)) - f(\theta,\cdot,\Psi(\cdot))) \leq L\ \rho(\Phi(\cdot) - \Psi(\cdot)),$$
- (BC) *there exist $M < 2/\omega_\rho(2)$ and $N \geq 0$ such that*
$$\rho(f(\theta,\cdot,\Phi(\cdot))) \leq M\rho_{\mathcal{C}_\rho}(\Phi) + N,$$

for any $\Phi, \Psi \in \mathcal{C}_\rho([0,1], L_\rho)$ and $\theta \in [0,1]$. Then (2.1) has a solution in $\mathcal{C}_\rho([0,1], L_\rho)$.

Proof. First, let us prove that the operator $\mathcal{A} : \mathcal{C}_\rho([0,1], L_\rho) \to \mathcal{C}_\rho([0,1], L_\rho)$ is $\rho_{\mathcal{C}_\rho}$-completely continuous. Indeed, over the interval $[0,1]$, we choose the mesh points
$$\sigma_i^n = \frac{i}{n}, \quad i = 0, \dots, n,$$
where $n > 1$ is any positive integer. It is clear that $\sigma_{i+1}^n - \sigma_i^n = \frac{1}{n}$ approaches 0 as n approaches ∞, and
$$\sum_{i=0}^{n-1} (\sigma_{i+1}^n - \sigma_i^n)(f(\theta,\sigma_i^n,\Phi(\sigma_i^n)) - f(\theta,\sigma_i^n,\Psi(\sigma_i^n)))$$
is $\|\cdot\|_\rho$-convergent to $\int_0^1 f(\theta,\sigma,\Phi(\sigma)) - f(\theta,\sigma,\Psi(\sigma))\,d\sigma$ in $(L_\rho, \|\cdot\|_\rho)$, and consequently, by using the Δ_2-type condition, is also ρ-convergent. Since
$$\sum_{i=0}^{n-1} \sigma_{i+1}^n - \sigma_i^n = \sum_{i=0}^{n-1} \frac{1}{n} = 1,$$
and using (LC) property of f, Fatou property and the convexity of ρ, we obtain
$$\begin{aligned}
\rho((\mathcal{A}\Phi)(\theta) - (\mathcal{A}\Psi)(\theta)) &\leq \liminf_{n\to\infty} \rho\Big(\sum_{i=0}^{n-1} \frac{1}{n} f(\theta,\sigma_i^n,\Phi(\sigma_i^n)) - f(\theta,\sigma_i^n,\Psi(\sigma_i^n))\Big) \\
&\leq L \liminf_{n\to\infty} \sum_{i=0}^{n-1} \frac{1}{n} \rho(\Phi(\sigma_i^n) - \Psi(\sigma_i^n)) \\
&\leq L\rho_{\mathcal{C}_\rho}(\Phi - \Psi) \liminf_{n\to\infty} \sum_{i=0}^{n-1} \frac{1}{n} \\
&= L\rho_{\mathcal{C}_\rho}(\Phi - \Psi),
\end{aligned}$$
which implies
$$\rho_{\mathcal{C}_\rho}(\mathcal{A}(\Phi) - \mathcal{A}(\Psi)) \leq L\ \rho_{\mathcal{C}_\rho}(\Phi - \Psi),$$

for any $\Phi, \Psi \in \mathcal{C}_\rho([0,1], L_\rho)$. This will imply that the operator $\mathcal{A}$ is $\rho_{\mathcal{C}_\rho}$-continuous and the image of any nonempty ρ-bounded subset $B \subset L_\rho$ is ρ-bounded. Fix $B \subset L_\rho$ a ρ-bounded nonempty subset. Let us prove that $\mathcal{A}(B)$ is $\rho_{\mathcal{C}_\rho}$-relatively compact. Using the property (BTI), we have

$$\begin{aligned}
&\rho((\mathcal{A}\Phi)(\theta) - (\mathcal{A}\Psi)(\tilde{\theta})) \\
&\leq \frac{\omega_\rho(2)}{2}\rho(g(\theta) - g(\tilde{\theta})) + \frac{\omega_\rho(2)}{2}\rho\Big(\int_0^1 f(\theta, \sigma, \Phi(\sigma)) - f(\tilde{\theta}, \sigma, \Psi(\sigma))\, d\sigma\Big) \\
&\leq \frac{\omega_\rho(2)}{2}\rho(g(\theta) - g(\tilde{\theta})) + \frac{\omega_\rho(2)}{2}\liminf_{n\to\infty}\rho\Big(\sum_{i=0}^{n-1}\frac{1}{n} f(\theta, \sigma_i^n, \Phi(\sigma_i^n)) - f(\tilde{\theta}, \sigma_i^n, \Psi(\sigma_i^n))\Big) \\
&\leq \frac{\omega_\rho(2)}{2}\rho(g(\theta) - g(\tilde{\theta})) + \frac{\omega_\rho(2)}{2}\sup_{\sigma\in[0,1]}\rho(f(\theta, \sigma, \Phi(\sigma)) - f(\tilde{\theta}, \sigma, \Psi(\sigma)))\sum_{i=0}^{n-1}\frac{1}{n} \\
&= \frac{\omega_\rho(2)}{2}\rho(g(\theta) - g(\tilde{\theta})) + \frac{\omega_\rho(2)}{2}\sup_{\sigma\in[0,1]}\rho(f(\theta, \sigma, \Phi(\sigma)) - f(\tilde{\theta}, \sigma, \Psi(\sigma))),
\end{aligned}$$

for any $\theta, \tilde{\theta} \in [0,1]$ and any ρ-bounded subset $B \subset L_\rho$. Since f is ρ-strongly continuous with respect to the first variable and g is ρ-continuous, then by using the Δ_2-type condition and Lemma 3.3, we conclude that the family $\mathcal{A}(B)$ will be equicontinuous with respect to the Luxemburg norm $\|\cdot\|_{\mathcal{C}_\rho}$. Arzelà-Ascoli theorem implies that $\mathcal{A}(B)$ is $\|\cdot\|_{\mathcal{C}_\rho}$-relatively compact and then $\rho_{\mathcal{C}_\rho}$-relatively compact. Therefore, $\mathcal{A}$ is $\rho_{\mathcal{C}_\rho}$-completely continuous. Next, we prove that $\mathcal{A}$ has a fixed point. Consider the set

$$S = \{\Phi \in \mathcal{C}^\rho([0,1], L_\rho) : \Phi = \nu\mathcal{A}\Phi, \text{ for some } \nu \text{ in } [0,1]\}.$$

Let us prove that S is $\rho_{\mathcal{C}_\rho}$-bounded. Note that the zero-function is in S. Hence S is not empty. Next, let $\Phi \in S$. We have $\Phi = \nu\ \mathcal{A}\Phi$, for some $\nu \in [0,1]$, which implies

$$\begin{aligned}
\rho(\Phi(\theta)) &\leq \rho(\nu\ g(\theta) + \nu\int_0^1 f(\theta, \sigma, \Phi(\sigma)\, d\sigma) \\
&\leq \nu\rho(\ g(\theta) + \int_0^1 f(\theta, \sigma, \Phi(\sigma))\, d\sigma) \\
&\leq \nu\frac{\omega_\rho(2)}{2}\Big(\rho(\ g(\theta)) + \rho\Big(\int_0^1 f(\theta, \sigma, \Phi(\sigma))\, d\sigma\Big)\Big),
\end{aligned}$$

by using $\nu \leq 1$, the Δ_2-type condition. Hence the Fatou property implies

$$\begin{aligned}
\rho(\Phi(\theta)) &\leq \frac{\omega_\rho(2)}{2}\rho(\ g(\theta)) + \frac{\omega_\rho(2)}{2}\liminf_{n\to\infty}\sum_{i=0}^{n-1}(\sigma_{i+1}^n - \sigma_i^n)\rho(f(\theta, \sigma_i^n, \Phi(\sigma_i^n)), \\
&\leq \frac{\omega_\rho(2)}{2}\rho(g(\theta)) + \frac{\omega_\rho(2)}{2}\ (M\rho_{\mathcal{C}_\rho}(\Phi) + N)\liminf_{n\to\infty}\sum_{i=0}^{n-1}(\sigma_{i+1}^n - \sigma_i^n), \\
&= \frac{\omega_\rho(2)}{2}\rho(g(\theta)) + \frac{\omega_\rho(2)}{2}(M\rho_{\mathcal{C}_\rho}(\Phi) + N),
\end{aligned}$$

which implies

$$\rho_{\mathcal{C}_\rho}(\Phi) \leq \frac{\omega_\rho(2)}{2}\rho_{\mathcal{C}_\rho}(g) + \frac{\omega_\rho(2)}{2}(M\rho_{\mathcal{C}_\rho}(\Phi) + N).$$

Since $M < \frac{2}{\omega_\rho(2)}$, we obtain

$$\rho_{\mathcal{C}_\rho}(\Phi) \le \frac{\omega_\rho(2)}{2} \frac{\rho_{\mathcal{C}_\rho}(g) + N}{1 - \frac{\omega_\rho(2)}{2} M},$$

which implies that S is $\rho_{\mathcal{C}_\rho}$-bounded. Using Schaefer's fixed point theorem, we conclude that $\mathcal{A}$ has a fixed point, i.e., the equation (FIE) has a solution in $\mathcal{C}_\rho([0,1], L_\rho)$. $\square$

4. Applications

Electrorheological (ER) fluids are suspensions of extremely fine non-conducting but electrically active particles (up to 50 micrometres diameter) in an electrically insulating fluid. The apparent viscosity of these fluids changes reversibly by an order of up to 100,000 in response to an electric field. For example, a typical ER fluid can go from the consistency of a liquid to that of a gel, and back, with response times on the order of milliseconds. The effect is sometimes called the Winslow effect[16].

The giant electrorheological (GER) fluid was discovered in 2003, and is able to sustain higher yield strengths than many other ER fluids. A mathematical model associated to the study of the eigenvalues of the p-Laplacian for these kind of fluids [10] is done via the variable exponent spaces introduced by Orlicz as early as 1931 [12].

Let $p : [0,1] \to [1,\infty]$ be a Lebesgue measurable function finite almost everywhere. We define

$$p^- := p^-_{[0,1]} := \operatorname{ess\,inf}_{x\in[0,1]} p(x). \quad p^+ := p^+_{[0,1]} := \operatorname{ess\,sup}_{x\in[0,1]} p(x).$$

We define the variable exponent Lebesgue space $L^{p(\cdot)}([0,1])$ by

$$L^{p(\cdot)} := L^{p(\cdot)}([0,1]) = \{\phi \in L^0([0,1]) : \varrho_{L^{p(\cdot)}}(\lambda\phi) < \infty \quad \text{for some } \lambda > 0\},$$

where $L^0([0,1])$ denote the space of all $\mathbb{R}$-valued, Lebesgue measurable functions on $[0,1]$ and

$$\varrho_{L^{p(\cdot)}}(\phi) = \int_0^1 \mid \phi(x) \mid^{p(x)} dx. \tag{4.1}$$

Note that $\varrho_{L^{p(\cdot)}}$ is a convex modular as p is finite almost everywhere [4]. If $p^+ < \infty$, then $\varrho_{L^{p(\cdot)}}$ satisfies the Δ_2-type condition. Indeed, we have

$$\begin{aligned}
\varrho_{L^{p(\cdot)}}(2\phi) &= \int_0^1 \mid 2\phi(x) \mid^{p(x)} dx \\
&= \int_0^1 2^{p(x)} \, |\phi(x)|^{p(x)} dx \\
&\le 2^{p^+} \int_0^1 |\phi(x)|^{p(x)} dx \\
&\le 2^{p^+} \varrho_{L^{p(\cdot)}}(\phi),
\end{aligned}$$

for any $\phi \in L^{p(\cdot)}$, which does imply $\omega_{\varrho_{L^{p(\cdot)}}}(2) \le 2^{p^+}$. Consider the Fredholm integral equation

$$\Phi(\theta) = g(\theta) + \int_0^1 |\sin(1 + \theta + \Phi(\sigma))| \, d\sigma. \tag{4.2}$$

Assume that Φ and g are in $\mathcal{C}_{\varrho_{L^{p(\cdot)}}}([0,1],L^{p(\cdot)})$. Then (4.2) has a solution in $\mathcal{C}_{\varrho_{L^{p(\cdot)}}}([0,1],L^{p(\cdot)})$. Indeed, let $\Psi,\Phi\in\mathcal{C}_{\varrho_{L^{p(\cdot)}}}([0,1],L^{p(\cdot)})$. Set

$$\phi := \Phi(\cdot)\in L^{p(\cdot)},\quad \psi := \Psi(\cdot)\in L^{p(\cdot)}.$$

Since

$$\big||\sin(1+\theta+\Phi(\cdot))| - |\sin(1+\theta+\Psi(\cdot))|\big| \le \big|\Phi(\cdot)-\Psi(\cdot)\big|,$$

we have

$$\varrho_{L^{p(\cdot)}}\Big(|\sin(1+\theta+\Phi(\cdot))| - |\sin(1+\theta+\Psi(\cdot))|\Big) \le \varrho_{L^{p(\cdot)}}(\Phi(\cdot)) - \Psi(\cdot)).$$

Moreover,

$$\begin{aligned}
\varrho_{L^{p(\cdot)}}(|\sin(1+\theta+\Phi(\cdot))|) &= \varrho_{L^{p(\cdot)}}(|\sin(1+\theta+\phi)|)\\
&= \int_0^1 |\sin(1+\theta+\phi(x))|^{p(x)}dx\\
&\le \int_0^1 1\ \, d\sigma\\
&= M\rho_{\mathcal{C}_{\varrho_{L^{p(\cdot)}}}}(\Phi)+N,
\end{aligned}$$

where

$$\varrho_{\mathcal{C}_{\varrho_{L^{p(\cdot)}}}}(\Phi) = \sup_{\theta\in[0,1]}\varrho_{L^{p(\cdot)}}(\Phi(\theta)),\quad M=0,\quad N=1.$$

Clearly, we have $M<\omega(2)/2 = 2^{p^+-1}$. Therefore Theorem 3.5 implies that (4.2) has a solution in $\mathcal{C}_{\varrho_{L^{p(\cdot)}}}([0,1],L^{p(\cdot)})$.

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