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BOUNDARY REGULARITY FOR NONDIVERGENCE ELLIPTIC EQUATION WITH UNBOUNDED DRIFT

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Abstract. We obtain the pointwise boundary differentiability of strong solutions for elliptic equations with the lower order coefficients, the boundary, and the right-hand side term satisfying a Dini type condition. Furthermore, we establish a pointwise estimate of strong solutions and show that the gradients of the strong solutions are continuous along the boundary if the drift term, the boundary, and the right-hand side term satisfy a uniform Dini type condition on the boundary.

1. Introduction

In this article, we will study the boundary regularity of strong solutions of elliptic equation with unbounded lower order coefficients. Suppose that $u \in W^{2,n}_{\rm loc}(\Omega)\cap C(\overline{\Omega})$ satisfies

$$\begin{gathered} Lu := -a_{ij}(x)D_{ij}u + b_i(x)D_iu = f(x) \quad \text{in } \Omega; \\ u(x) = 0 \quad \text{on } \partial\Omega; \end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n \ge 2)$. We use the summation convention over repeated indices and the notations $D_i := \frac{\partial}{\partial x_i}$; $D_{ij} := D_iD_j$. We assume that a_{ij}, b_i and f are measurable functions on Ω, the matrix $(a_{ij}(x))_{n\times n}$ is symmetric and satisfies the uniformly elliptic condition

$$\lambda|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \lambda^{-1}|\xi|^2, \quad \text{for all } \xi \in \mathbb{R}^n, \quad \text{a.e. } x \in \Omega, \tag{1.2}$$

with a constant $\lambda \in (0,1]$, and $b_i, f \in L^n(\Omega)$. Throughout this article, the operator L in (1.1) is applied to functions u in the class $W(\Omega) := W^{2,n}_{\rm loc}(\Omega) \cap C(\overline{\Omega})$.

In the following, we extend the results in [15] to elliptic equations with unbounded lower order term. The boundary differentiability is shown for strong solution of nondivergence elliptic equation on $C^{1,Dini}$ domain with unbounded drift satisfies Dini type condition. Furthermore, we prove that boundary first order derivative is continuous along the boundary.

As for the boundary regularity of nondivergence elliptic equations: If the drift term $|\mathbf{b}|$ is bounded, Krylov [8, 9] showed that the solution is $C^{1,\alpha}$ along the boundary if $\partial\Omega$ is $C^{1,1}$. Lieberman [13] gave a more general estimates. Wang [19] proved

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a similar pointwise result as in [8, 9] by an iteration method that will be adopted in this paper. Ma and Wang [15] proved a boundary $C^{1,\psi}$ estimate for fully nonlinear elliptic equations on $C^{1,Dini}$ domain. Li and Wang [11, 12] showed the boundary differentiability of solutions of elliptic equations on convex domains. If $|\mathbf{b}|$ is unbounded, Ladyzhenskaya and Ural'tseva in [10] proved boundary $C^{1,\alpha}$ estimate of elliptic and parabolic inequalities on $W^{2,q}$ domain with $\mathbf{b} \in L^q$, $\Phi \in L^q$, $q > n$ and nonlinear term $\mu_1|Du|^2$. Apushkinskaya and Nazarov [1] proved the boundary $C^{1,\alpha}$ estimate for nondivergence parabolic equation with composite righthand side and lower order coefficients, and in [2] they gave a counterexample of Hopf-Oleinik lemma in the elliptic case. Safonov [18] obtained the Hopf-Oleinik lemma on a flat domain for elliptic equations and gave the counterexample which indicated that the Dini condition on $|\mathbf{b}|$ can not be removed for our theorem. Nazarov [16] proved the Hopf-Oleinik Lemma and boundary gradient estimate under minimal restrictions on lower-order coefficients. Braga, Moeira and Wang [3] generalized the elliptic case in [10] to L^n viscosity solutions with $\mu_1 = 0$ and $C^{1,Dini}$ boundary value. Some related results concerning Dini continuity can be found in [4, 6, 7, 17, 20, 21].

The following Alexandroff-Bakelman-Pucci maximum principle and Harnack inequality are our main tools.

Theorem 1.1 ([5, 18]). *Let Ω be a bounded domain in $\mathbb{R}^n$, and let u be a function in $W(\Omega)$ such that $Lu \le f$ in Ω. Suppose that the matrix $(a_{ij}(x))_{n\times n}$ is symmetric and satisfies the uniformly elliptic condition* (1.2)*, and $b_i, f \in L^n(\Omega)$. Then*

$$\sup_{\Omega} u \le \sup_{\partial\Omega} u + N \operatorname{diam}\Omega \cdot e^{N\|\mathbf{b}\|^n_{L^n(\Omega)}} \|f^+\|_{L^n(\Omega)}, \tag{1.3}$$

where

$$\|\mathbf{b}\|_{L^n(\Omega)} = \Big(\int_\Omega |\mathbf{b}|^n\,dx\Big)^{1/n}, \quad \mathbf{b} = (b_1, b_2, \dots, b_n), \tag{1.4}$$

and N is a positive constant depending only on n and λ.

Theorem 1.2 (Harnack Inequality). *Let u be a nonnegative function in $W(B_8)$, $Lu = f$ in B_8 and $b_i, f \in L^n(B_8)$. There exists a positive constant ϵ_0 depending only on λ and n, such that if $\|\mathbf{b}\|_{L^n(B_8)} \le \epsilon_0$, then*

$$\sup_{B_1} u \le C(\inf_{B_1} u + \|f\|_{L^n(B_8)}), \tag{1.5}$$

where C is constant depending only on λ and n.

Theorem 1.2 follows from the the proof in [18] clearly. The most important thing is that the quantity $\|\mathbf{b}\|_{L^n}$ is scaling invariant(see [18, Remark 1.4]) and the Harnack constant is invariant in the iteration procedure. Before we state out our main theorem, for convenience, we give the following notation and definitions.

$\{e_i\}_{i=1}^n$ is the standard basis of $\mathbb{R}^n$.

$$|x| := \Big(\sum_{i=1}^n x_i^2\Big)^{1/2}$$

is the Euclidean norm of $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. $a^+ := \max\{0, a\}$. $B_r := \{x \in \mathbb{R}^n : |x| < r\}$. $B_r(x) := x + B_r$. $\Omega_r := \Omega \cap B_r$. $\Omega_r(x) := \Omega \cap B_r(x)$. $\operatorname{diam}(\Omega) := \sup_{x,y\in\Omega} |x-y|$.

$$Q_r := \{x \in \mathbb{R}^n : |x_i| < r,\ i = 1, 2, \dots, n\}.$$

$\|f\|_{L^n(\Omega)} := \big(\int_\Omega |f(x)|^n\,dx\big)^{1/n}$. $W(\Omega) := W^{2,n}_{\rm loc}(\Omega) \cap C(\overline{\Omega})$.

Definition 1.3. We say that $\partial\Omega$ is $C^{1,Dini}$ at $x \in \partial\Omega$, if there exist a unit vector $\vec{n}$ and a positive constant r_0 such that

$$\frac{1}{r} \sup_{y \in \partial\Omega, |y-x| \le r} |(y-x)\cdot \vec{n}| \le \omega(r), \quad \text{for } 0 < r \le r_0,$$

where $\omega(r)$ is a nonnegative nondecreasing function and satisfies $\int_0^{r_0} \frac{\omega(r)}{r} dr < \infty$. We say that $\partial\Omega$ is $C^{1,Dini}$ if for any $x \in \partial\Omega$, $\partial\Omega$ is $C^{1,Dini}$ at $x \in \partial\Omega$.

If $\partial\Omega$ satisfies the pointwise $C^{1,Dini}$ condition at any $x \in \partial\Omega$ with the same r_0, it follows that $\partial\Omega$ is $C^{1,Dini}$ in the classical sense, i.e., $\partial\Omega$ can be locally represented as a C^1 graph with the gradient being Dini continuous.

Definition 1.4. We say that the function $g \in L^n(\Omega)$ is $C_n^{-1,Dini}$ at $x \in \partial\Omega$, if there exists a positive constant r_0 such that

$$\Big(\frac{1}{|B_r(x)\cap\Omega|}\int_{B_r(x)\cap\Omega} |g(y)|^n dy\Big)^{1/n} \le r^{-1}\omega(r)$$

for each $0 < r \le r_0$, where $\omega(r)$ is a nonnegative nondecreasing function and satisfies $\int_0^{r_0} \frac{\omega(r)}{r} dr < \infty$. Obviously, we have $\|g\|_{L^n(\Omega\cap B_r(x))} \le |B_1(0)|^{1/n}\omega(r) \le 2\omega(r)$. We say that g is $C_n^{-1,Dini}$ on $\partial\Omega$ if for any $x \in \partial\Omega$, g is $C_n^{-1,Dini}$ at $x \in \partial\Omega$.

Generally, for any function in $L^p(\Omega)(1 \le p \le \infty)$, we can define the pointwise $C_p^{k,\mathrm{Dini}}$ $(k \in \mathbb{Z})$. We say that the function $g \in L^p(\Omega)$ is $C_p^{k,\mathrm{Dini}}$ at $x \in \partial\Omega$, if there exists a positive constant r_0 and a $k-th$ order polynomial $P_k^x(y)$ ($P_k^x(y) \equiv 0$ if $k < 0$) such that

$$\Big(\frac{1}{|B_r(x)\cap\Omega|}\int_{B_r(x)\cap\Omega} |g(y) - P_k^x(y)|^p dy\Big)^{1/p} \le r^k\omega(r)$$

for each $0 < r \le r_0$, where $\omega(r)$ is a nonnegative nondecreasing function and satisfies $\int_0^{r_0} \frac{\omega(r)}{r} dr < \infty$.

The main results of this paper are Theorems 1.5, 1.9, and Corollary 1.7 below.

Theorem 1.5. *Assume that*

1. $0 \in \partial\Omega$, $r_0 > 0$, $u \in W(\Omega_{r_0})$, $u|_{\partial\Omega\cap B_{r_0}} = 0$, $Lu = f$ *in* Ω_{r_0}, $|\mathbf{b}|$, $f \in L^n(\Omega_{r_0})$ *and* $\int_0^{r_0} \frac{\|f\|_{L^n(\Omega_r)}}{r} dr < \infty$*;*
2. $\partial\Omega$ *is* $C^{1,Dini}$ *at* 0 *and* $|\mathbf{b}|$ *is* $C_n^{-1,Dini}$ *at* 0 *with the modulus of continuity* $\omega(r)$ *satisfies*
$$\omega(r_0) \le \min\{\frac{\delta}{6}, \frac{1}{2}, \frac{\epsilon_0}{2}\} \quad \text{and} \quad \int_0^{r_0} \frac{\omega(r)}{r} dr \le \min\{1, \frac{\delta \ln\frac{1}{\delta}}{72M\sqrt{n}A_2}\}, \tag{1.6}$$
where δ, M *and* A_2 *are constants depending only on* λ *and* n *(see Lemma 2.2), and* ϵ_0 *is the constant in Theorem 1.2.*

Then u *is differentiable at* 0*, furthermore, there exist a linear function* $L(x)$ *and constants* $\hat{\alpha} > 0$, $\Lambda > 1$, $C > 0$ *such that*

$$\begin{aligned}|u(x) - L(x)| \le C\Big\{ & r^{\hat{\alpha}} + \omega(\Lambda r) + \|f\|_{L^n(\Omega_{\Lambda r})} + r^{\hat{\alpha}}\int_r^{r_0} \frac{\omega(s) + \|f\|_{L^n(\Omega_s)}}{s^{1+\hat{\alpha}}} ds \\ & + \int_0^{\Lambda r} \frac{\omega(s) + \|f\|_{L^n(\Omega_s)}}{s} ds\Big\} r,\end{aligned} \tag{1.7}$$

for any $x \in \Omega_r$ and $0 < r \leq \frac{r_0}{\Lambda}$, where C depends on $\|u\|_{L^\infty(\Omega_{r_0})}$, $\|f\|_{L^n(\Omega_{r_0})}$, $\int_0^{r_0} \frac{\|f\|_{L^n(\Omega_s)}}{s} ds$, r_0, λ and n.

Remark 1.6. (1) The condition (1.6) will always be satisfied for small r_0 if the modulus of continuity $\omega(r)$ satisfies the Dini condition, which will guarantee that the slopes of hyperplanes in the iteration procedure are uniformly bounded (see (2.22)).

(2) We can also deduce pointwise boundary differentiability with nonhomogeneous pointwise $C^{1,Dini}$ boundary value as in [15]. Here we only consider the homogeneous boundary value just for convenience.

(3) The modulus of continuity $\omega(r)$ is nondecreasing can be replaced by $\omega(r)$ satisfies the doubling condition(see [14, Definition 2.3]).

The following corollary is a direct consequence of Theorems 1.5 and 2.1.

Corollary 1.7. *Assume that*

(1) *$0 \in \partial\Omega$, $r_0 > 0$, $u \in W(\Omega_{r_0})$, $Lu = f$ in Ω_{r_0}, $u|_{\partial\Omega\cap B_{r_0}} = 0$ and $|\mathbf{b}|$, $f \in L^n(\Omega_{r_0})$;*
(2) *$\partial\Omega$ is $C^{1,Dini}$ at 0, $|\mathbf{b}|$ is $C_n^{-1,Dini}$ at 0 and f is $C_n^{-1,Dini}$ at 0 with the modulus of continuity $\omega(r)$ satisfies*
$$\omega(r_0) \leq \min\big\{\frac{\delta}{6}, \frac{1}{2}, \frac{\epsilon_0}{2}\big\}, \quad \int_0^{r_0} \frac{\omega(r)}{r} dr \leq \min\big\{1, \frac{\delta \ln\frac{1}{\delta}}{72M\sqrt{n}A_2}\big\},$$
where δ, M and A_2 are the constants in Lemma 2.2, and ϵ_0 is the constant in Theorem 1.2.

Then u is differentiable at 0, furthermore, there exist a linear function $L(x)$ and constants $\hat{\alpha} > 0$, $\Lambda > 1$, $C > 0$ such that for any $x \in \Omega_r$ and $0 < r \leq r_0/\Lambda$,
$$|u(x) - L(x)| \leq C\Big(r^{\hat{\alpha}} + \omega(\Lambda r) + r^{\hat{\alpha}} \int_r^{r_0} \frac{\omega(s)}{s^{1+\hat{\alpha}}} ds + \int_0^{\Lambda r} \frac{\omega(s)}{s} ds\Big) r, \tag{1.8}$$
where C depends on $\|u\|_{L^\infty(\Omega_{r_0})}$, r_0, λ and n.

Remark 1.8. If $\partial\Omega$ is $C^{1,\alpha}$ at 0, $|\mathbf{b}|$ is $C_n^{-1,\alpha}$ at 0 and f is $C_n^{-1,\alpha}$ at 0 with $\omega(r) = r^\alpha (0 < \alpha < 1)$, then u is $C^{1,\hat{\beta}}$ at 0 with $\hat{\beta} = \min\{\alpha, \hat{\alpha}\}$ if $\alpha \neq \hat{\alpha}$ and $0 < \hat{\beta} < \min\{\alpha, \hat{\alpha}\}$ if $\alpha = \hat{\alpha}$.

Theorem 1.9. *Assume that*

(1) *$0 \in \partial\Omega$, $r_0 > 0$, $u \in W(\Omega_{3r_0})$, $Lu = f$ in Ω_{3r_0}, $u|_{\partial\Omega\cap B_{3r_0}} = 0$ and $|\mathbf{b}|$, $f \in L^n(\Omega_{3r_0})$;*
(2) *$\partial\Omega$ is $C^{1,Dini}$, $|\mathbf{b}|$ is $C_n^{-1,Dini}$ and f is $C_n^{-1,Dini}$ on $\partial\Omega \cap B_{r_0}$ uniformly with the modulus of continuity $\omega(r)$ satisfies*
$$\omega(r_0) \leq \min\big\{\frac{\delta}{6}, \frac{1}{2}, \frac{\epsilon_0}{2}\big\}, \quad \int_0^{r_0} \frac{\omega(r)}{r} dr \leq \min\big\{1, \frac{\delta \ln\frac{1}{\delta}}{72M\sqrt{n}A_2}\big\},$$
where δ, M and A_2 are constants in Lemma 2.2, and ϵ_0 is the constant in Theorem 1.2.

Then there exist constants $\hat{\alpha} > 0$, $\Lambda > 1$, $C > 0$ such that for any $y, z \in \partial\Omega \cap B_{r_0}$ and $0 < |y - z| = r \leq \frac{r_0}{\Lambda}$,
$$|\nabla u(y) - \nabla u(z)| \leq C\Big(r^{\hat{\alpha}} + \omega(\Lambda r) + r^{\hat{\alpha}} \int_r^{r_0} \frac{\omega(s)}{s^{1+\hat{\alpha}}} ds + \int_0^{\Lambda r} \frac{\omega(s)}{s} ds\Big),$$

where $\hat{\alpha}$ and Λ are the constants in Corollary 1.7, and C is a constant depending on $\|u\|_{L^\infty(\Omega_{3r_0})}$, r_0, λ and n.

Remark 1.10. If $\partial\Omega$ is $C^{1,\alpha}$ on $\partial\Omega\cap B_{r_0}$, $|\mathbf{b}|$ is $C_n^{-1,\alpha}$ on $\partial\Omega\cap B_{r_0}$ and f is $C_n^{-1,\alpha}$ on $\partial\Omega\cap B_{r_0}$ with $\omega(r)=r^\alpha(0<\alpha<1)$, then ∇u is $C^{\hat{\beta}}$ along $\partial\Omega\cap B_{r_0}$ with $\hat{\beta}=\min\{\alpha,\hat{\alpha}\}$ if $\alpha\neq\hat{\alpha}$ and $0<\hat{\beta}<\min\{\alpha,\hat{\alpha}\}$ if $\alpha=\hat{\alpha}$.

We shall prove Theorems 1.5 and 1.9 in the next section.

2. Boundary estimates

By standard normalization, it is enough to prove Theorem 2.1, below, instead of proving Theorem 1.5. Since $\partial\Omega$ is $C^{1,Dini}$ at $0\in\partial\Omega$, without loss of generality, we assume $\vec{n}=e_n$ as the inward normal direction in the following Theorem 2.1. Consider the normalization of solution,

$$\tilde{u}_\epsilon(x)=\frac{u(r_0x)}{\|u\|_{L^\infty(\Omega_{r_0})}+\epsilon+r_0\|f\|_{L^n(\Omega_{r_0})}+r_0\int_0^{r_0}\frac{\|f\|_{L^n(\Omega_r)}}{r}dr},$$

for $\epsilon>0$ and $x\in\tilde{\Omega}\cap B_1$, with the normalized domain $\tilde{\Omega}:=\{x\in\mathbb{R}^n:\ r_0x\in\Omega\}$. Obviously, $\tilde{u}_\epsilon(x)$ satisfies

$$\|\tilde{u}_\epsilon\|_{L^\infty(\tilde{\Omega}_1)}\le 1\quad\text{and}\quad -\tilde{a}_{ij}(x)D_{ij}\tilde{u}_\epsilon(x)+\tilde{b}_i(x)D_i\tilde{u}_\epsilon(x)=\tilde{f}(x)$$

for $x\in\tilde{\Omega}\cap B_1$, where

$$\tilde{a}_{ij}(x)=a_{ij}(r_0x),\quad \tilde{b}_i(x)=r_0b_i(r_0x),$$
$$\tilde{f}(x)=\frac{r_0^2f(r_0x)}{\|u\|_{L^\infty(\Omega_{r_0})}+\epsilon+r_0\|f\|_{L^n(\Omega_{r_0})}+r_0\int_0^{r_0}\frac{\|f\|_{L^n(\Omega_r)}}{r}dr}.$$

Let $\tilde{\omega}(r)=\omega(r_0r)$. Obviously,

$$\frac{1}{r}\sup_{y\in\partial\tilde{\Omega},|y|\le r}|y\cdot e_n|\le\tilde{\omega}(r),\quad \|\tilde{\mathbf{b}}\|_{L^n(\tilde{\Omega}_r)}=\|\mathbf{b}\|_{L^n(\Omega_{r_0r})}\le 2\tilde{\omega}(r)$$

for $0<r\le 1$, and

$$\int_0^1\frac{\tilde{\omega}(r)}{r}dr=\int_0^{r_0}\frac{\omega(r)}{r}dr.$$

Theorem 2.1. *Assume that*

1. *$0\in\partial\Omega$, $u\in W(\Omega_1)$, $u|_{\partial\Omega\cap B_1}=0$, $Lu=f$ in Ω_1, and $\|u\|_{L^\infty(\Omega_1)}\le 1$;*
2. *$f\in L^n(\Omega_1)$ with $\|f\|_{L^n(\Omega_1)}\le 1$ and $\int_0^1\frac{\|f\|_{L^n(\Omega_r)}}{r}dr\le 1$;*
3. *$\partial\Omega$ is $C^{1,Dini}$ at 0 and $|\mathbf{b}|$ is $C_n^{-1,Dini}$ at 0 with the modulus of continuity $\omega(r)$ satisfies the normalized conditions*

$$\omega(1)\le\min\Big\{\frac{\delta}{6},\frac{1}{2},\frac{\epsilon_0}{2}\Big\},\quad \int_0^1\frac{\omega(r)}{r}dr\le\min\Big\{1,\frac{\delta\ln\frac{1}{\delta}}{72MA_2}\Big\},\tag{2.1}$$

where ϵ_0 is the constant in Theorem 1.2, and δ, M, A_2 are constants in Lemma 2.2.

Then there exist the three positive constants C, $\hat{\alpha}$ and $\Lambda(\geq 324n)$ depending only on λ and n, and there exists a constant θ such that

$$\begin{aligned}|u(x)-\theta x_n| \leq C\Big\{ & r^{\hat{\alpha}}+\omega(\Lambda r)+\|f\|_{L^n(\Omega_{\Lambda r})}+r^{\hat{\alpha}}\int_r^1 \frac{\omega(s)+\|f\|_{L^n(\Omega_s)}}{s^{1+\hat{\alpha}}}ds \\ & +\int_0^{\Lambda r}\frac{\omega(s)+\|f\|_{L^n(\Omega_s)}}{s}ds\Big\}r,\end{aligned} \tag{2.2}$$

for any $x\in\Omega_r$ and $r\leq \frac{1}{\Lambda}$.

We shall establish Theorem 2.1 by an iteration method which is based on Lemmas 2.2 and 2.3 below. For convenience, we define

$$\gamma(r)=\frac{1}{r}\sup_{y\in\partial\Omega,|y|\leq r}|y\cdot\ e_n| \quad \text{for } 0<r\leq 1.$$

Obviously,

$$\gamma(r)\leq\omega(r),\quad \|\mathbf{b}\|_{L^n(\Omega_r)}\leq 2\omega(r)\quad \text{for } 0<r\leq 1.$$

Lemma 2.2. *Suppose that $0\in\partial\Omega$, $u\in W(\Omega_1)$, $u|_{\partial\Omega\cap B_1}=0$, $Lu=f$ in Ω_1, $f\in L^n(\Omega_1)$, $\gamma(1)\leq\delta/6$ and $\|\mathbf{b}\|_{L^n(\Omega_1)}\leq\min\{\epsilon_0,1\}$, where ϵ_0 is the constant in Theorem 1.2 and $\delta(<1)$ will be chosen in (2.3). Then there exist positive constants $\mu<1$, M, A_1 and A_2 depending only on λ and n. If*

$$kx_n-l\leq u(x)\leq Kx_n+B\quad \text{in } \Omega_1, \tag{2.3}$$

for some constants $l\geq 0$, $B(\geq 0)$, k and K with $k\leq K$, then there exist constants $\tilde{k}$ and $\tilde{K}$ such that

$$\begin{aligned}&\tilde{k}x_n-A_1\|f\|_{L^n(\Omega_1)}-A_2(|K|+|k|+l)(\gamma(1)+\|\mathbf{b}\|_{L^n(\Omega_1)})\\ &\leq u(x)\leq \tilde{K}x_n+A_1\|f\|_{L^n(\Omega_1)}+A_2(|K|+|k|+B)(\gamma(1)+\|\mathbf{b}\|_{L^n(\Omega_1)})\end{aligned} \tag{2.4}$$

in Ω_δ, where either

$$\tilde{k}=k-3M\sqrt{n}l+\mu(K-k)\quad \text{and}\quad \tilde{K}=K+3M\sqrt{n}B, \tag{2.5}$$

or

$$\tilde{k}=k-3M\sqrt{n}l\quad \text{and}\quad \tilde{K}=K+3M\sqrt{n}B-\mu(K-k). \tag{2.6}$$

Obviously, we have $\tilde{k}\leq\tilde{K}$.

Proof of Lemma 2.2. First we proof the following.

Claim. There exist positive constants M, $\tilde{\delta}$ and C_1 depending only on λ and n, such that

$$\begin{aligned}&(k-3M\sqrt{n}l)x_n-C_1\|f\|_{L^n(\Omega_1)}-3M\sqrt{n}(|k|+l)\gamma(1)-C_1(|k|+l)\|\mathbf{b}\|_{L^n(\Omega_1)}\\ &\leq u(x)\\ &\leq (K+3M\sqrt{n}B)x_n+C_1\|f\|_{L^n(\Omega_1)}+3M\sqrt{n}(|K|+B)\gamma(1)\\ &\quad +C_1(|K|+B)\|\mathbf{b}\|_{L^n(\Omega_1)}\quad \text{in } \Omega\cap Q_{\tilde{\delta}}.\end{aligned}$$

Proof. Let $M=1+\frac{2\sqrt{n-1}}{\lambda}(\geq 3)$ and $\epsilon(>0)$ be small enough, such that

$$3-(1+\epsilon)(2+\epsilon)(M-1)^{\epsilon}\geq 0. \tag{2.7}$$

Let

$$\tilde{\delta}=\frac{1}{M\sqrt{n}}(\leq\frac{1}{3\sqrt{n}}),\quad \delta=\frac{\tilde{\delta}}{2M}=\frac{1}{2\sqrt{n}(1+\frac{2\sqrt{n-1}}{\lambda})^2} \tag{2.8}$$

and

$$\tilde{\psi}(x) = \frac{4}{3}\Big(\frac{2(x_n+\gamma(1))}{\tilde{\delta}} - \frac{(x_n+\gamma(1))^2}{\tilde{\delta}^2}\Big) + \frac{\lambda^2}{2(n-1)}\sum_{i=1}^{n-1}\Big(\Big(\frac{|x_i|}{\tilde{\delta}}-1\Big)^+\Big)^{2+\epsilon}.$$

The barrier function $\tilde{\psi}(x)$ is C^2 and satisfies the following conditions (observe that $1 \le \frac{\tilde{\delta}+\gamma(1)}{\tilde{\delta}} \le 3/2$):

$$\begin{gathered}
\tilde{\psi}(x) \ge 1 \quad \text{on } \overline{Q}_{1/\sqrt{n}} \cap \{x \in \mathbb{R}^n : x_n = \tilde{\delta}\};\\
\tilde{\psi}(x) \ge 0 \quad \text{on } \overline{Q}_{1/\sqrt{n}} \cap \{x \in \mathbb{R}^n : x_n = -\gamma(1)\};\\
\tilde{\psi}(x) \ge 1 \quad \text{on } \partial Q_{\frac{1}{\sqrt{n}}} \cap \{x \in \mathbb{R}^n :\ -\gamma(1) < x_n < \tilde{\delta}\};\\
-a_{ij}(x)D_{ij}\tilde{\psi}(x) \ge 0 \quad \text{a.e. in } Q_{1/\sqrt{n}} \cap \{x \in \mathbb{R}^n : -\gamma(1) < x_n < \tilde{\delta}\} \cap \Omega;\\
\tilde{\psi}(x) \le \frac{3(x_n+\gamma(1))}{\tilde{\delta}} \quad \text{in } Q_{\tilde{\delta}} \cap \{x : x_n \ge -\gamma(1)\}.
\end{gathered} \tag{2.9}$$

Combining (**??**) and (2.4), we have

$$\begin{gathered}
L\big(kx_n - l\tilde{\psi}(x) - u(x)\big) \le b_i(x)D_i(kx_n - l\tilde{\psi}(x)) - f(x) \quad \text{in } \tilde{Q} \cap \Omega;\\
kx_n - l\tilde{\psi}(x) - u(x) \le |k|\gamma(1) \quad \text{on } \partial(\tilde{Q} \cap \Omega);
\end{gathered} \tag{2.10}$$

where $\tilde{Q} = Q_{1/\sqrt{n}} \cap \{x \in \mathbb{R}^n : -\gamma(1) < x_n < \tilde{\delta}\}$.

By the Alexandroff-Bakelman-Pucci maximum principle,

$$kx_n - l\tilde{\psi}(x) - u(x) \le |k|\gamma(1) + C_1(|k|+l)\|\mathbf{b}\|_{L^n(\Omega_1)} + C_1\|f\|_{L^n(\Omega_1)} \tag{2.11}$$

in $\tilde{Q} \cap \Omega$, where C_1 is a constant depending only on λ and n.

By (2.4) (fifth inequality), we have

$$\begin{aligned}
u(x) \ge\ & (k - 3M\sqrt{n}l)x_n - C_1\|f\|_{L^n(\Omega_1)} - 3M\sqrt{n}(|k|+l)\gamma(1)\\
& - C_1(|k|+l)\|\mathbf{b}\|_{L^n(\Omega_1)}
\end{aligned} \tag{2.12}$$

in $\Omega \cap Q_{\tilde{\delta}}$. As in (2.5), we have

$$\begin{gathered}
L\big(u(x) - Kx_n - B\tilde{\psi}(x)\big) \le f(x) - b_i(x)D_i(Kx_n + B\tilde{\psi}(x)) \quad \text{in } \tilde{Q} \cap \Omega;\\
u(x) - Kx_n - B\tilde{\psi}(x) \le |K|\gamma(1) \quad \text{on } \partial(\tilde{Q} \cap \Omega).
\end{gathered}$$

According to the Alexandroff-Bakelman-Pucci maximum principle,

$$u(x) - Kx_n - B\tilde{\psi}(x) \le |K|\gamma(1) + C_1\|f\|_{L^n(\Omega_1)} + C_1(|K|+B)\|\mathbf{b}\|_{L^n(\Omega_1)}$$

in $\tilde{Q} \cap \Omega$, where C_1 is a constant depending only on λ and n. By (2.4) (fifth inequality), we have

$$\begin{aligned}
u(x) \le\ & (K + 3M\sqrt{n}B)x_n + C_1\|f\|_{L^n(\Omega_1)} + 3M\sqrt{n}(|K|+B)\gamma(1)\\
& + C_1(|K|+B)\|\mathbf{b}\|_{L^n(\Omega_1)}
\end{aligned} \tag{2.13}$$

in $\Omega \cap Q_{\tilde{\delta}}$. By (2.7) and (2.8), the claim follows. □

Let $\Gamma = \overline{Q}_{M\delta} \cap \{x \in \mathbb{R}^n :\ x_n = \delta\}$. By $\gamma(1) \le \delta/6$, we have

$$\Gamma \subset \Omega \quad \text{and} \quad \operatorname{dist}(\Gamma, \partial\Omega) \ge \frac{\delta}{2}. \tag{2.14}$$

Next, we show (**??**) for the two cases: $u(\delta e_n) \ge \frac{1}{2}(K+k)\delta$ and $u(\delta e_n) < \frac{1}{2}(K+k)\delta$, corresponding to (**??**) and (**??**).

Case 1: $u(\delta e_n) \geq \frac{1}{2}(K+k)\delta$. Let

$$\begin{aligned} v(x) &= u(x) - (k - 3M\sqrt{n}l)x_n + C_1\|f\|_{L^n(\Omega_1)} + 3M\sqrt{n}(|k|+l)\gamma(1) \\ &\quad + C_1(|k|+l)\|\mathbf{b}\|_{L^n(\Omega_1)}. \end{aligned}$$

Then

$$\begin{aligned} v(\delta e_n) &\geq \Big(\frac{K-k}{2} + 3M\sqrt{n}l\Big)\delta + C_1\|f\|_{L^n(\Omega_1)} + 3M\sqrt{n}(|k|+l)\gamma(1) \\ &\quad + C_1(|k|+l)\|\mathbf{b}\|_{L^n(\Omega_1)}. \end{aligned} \tag{2.15}$$

Since $v(x) \geq 0$ for $x \in \Omega \cap Q_{\tilde{\delta}}$, from (2.9) and the interior Harnack inequality, it follows that

$$\sup_{\Gamma} v(x) \leq C_2\big(\inf_{\Gamma} v(x) + \|f\|_{L^n(\Omega_1)} + (|k|+l)\|\mathbf{b}\|_{L^n(\Omega_1)}\big), \tag{2.16}$$

where $C_2(\geq 1)$ is a constant depending only on λ and n. Combining (2.10),(2.11) and $v(x) \geq 0$, we have

$$\begin{aligned} \inf_{\Gamma} v(x) \geq \Big\{ &\frac{1}{C_2}\Big((\frac{K-k}{2} + 3M\sqrt{n}l)\delta + 3M\sqrt{n}(|k|+l)\gamma(1)\Big) \\ &+ (\frac{C_1}{C_2} - 1)\Big((|k|+l)\|\mathbf{b}\|_{L^n(\Omega_1)} + \|f\|_{L^n(\Omega_1)}\Big)\Big\}^+ := a. \end{aligned}$$

Let

$$\psi(x) = \frac{3}{8}\Big(\Big(\frac{x_n+\gamma(1)}{\delta}\Big) + \Big(\frac{x_n+\gamma(1)}{\delta}\Big)^2\Big) - \frac{\lambda^2}{4(n-1)}\sum_{i=1}^{n-1}\Big((\frac{|x_i|}{\delta} - 1)^+\Big)^{2+\epsilon}, \tag{2.17}$$

where ϵ satisfies (2.2).

The barrier function $\psi(x)$ is C^2 and satisfies the following conditions (observe that $1 \leq \frac{\delta+\gamma(1)}{\delta} \leq 7/6$):

$$\begin{gathered} \psi(x) \leq 1 \quad \text{on } Q_{M\delta} \cap \{x \in \mathbb{R}^n : x_n = \delta\}; \\ \psi(x) \leq 0 \quad \text{on } Q_{M\delta} \cap \{x \in \mathbb{R}^n : x_n = -\gamma(1)\}; \\ \psi(x) \leq 0 \quad \text{on } \partial Q_{M\delta} \cap \{x \in \mathbb{R}^n : -\gamma(1) \leq x_n \leq \delta\}; \\ -a_{ij}(x)D_{ij}\psi(x) \leq 0 \quad \text{a.e. in } Q_{M\delta} \cap \{x \in \mathbb{R}^n : -\gamma(1) < x_n < \delta\} \cap \Omega; \\ \psi(x) \geq \frac{x_n+\gamma(1)}{3\delta} \quad \text{in } Q_\delta \cap \{x :\ x_n \geq -\gamma(1)\}; \\ \psi(x) \leq \frac{x_n+\gamma(1)}{\delta} \quad \text{in } Q_{M\delta} \cap \{x \in \mathbb{R}^n : -\gamma(1) \leq x_n \leq \delta\}. \end{gathered} \tag{2.18}$$

We claim that

$$\begin{gathered} L\big(a\psi(x) - v(x)\big) \leq b_i(x)D_i(a\psi(x) + (k - 3M\sqrt{n}l)x_n) - f \quad \text{in } \tilde{\tilde{Q}} \cap \Omega; \\ a\psi(x) - v(x) \leq \frac{2+9M\sqrt{n}}{C_2}(|K|+|k|+l)\gamma(1) \quad \text{on } \partial(\tilde{\tilde{Q}} \cap \Omega); \end{gathered} \tag{2.19}$$

where $\tilde{\tilde{Q}} = Q_{M\delta} \cap \{x \in \mathbb{R}^n : -\gamma(1) < x_n < \delta\}$.

In fact, the first inequality is clear. For the second inequality, we separate the boundary $\partial(\tilde{\tilde{Q}} \cap \Omega)$ into three parts:

$$\partial\tilde{\tilde{Q}} \cap \{x \in \mathbb{R}^n : x_n = \delta\}, \quad \partial\tilde{\tilde{Q}} \cap \{x \in \mathbb{R}^n : -\delta < x_n < \delta\} \cap \overline{\Omega}, \quad \partial\Omega \cap \tilde{\tilde{Q}}.$$

The first part is just Γ where $v(x) \ge a$ and $\psi(x) \le 1$, then $a\psi(x) - v(x) \le 0$ on it. On the second part, since $v(x) \ge 0$ and $\psi(x) \le 0$, we have $a\psi(x) - v(x) \le 0$ on them. On the last part, since $\psi(x) \le \frac{x_n + \gamma(1)}{\delta} \le 1$ on it by (2.13)(6), we have

$$\begin{aligned}
a\psi(x) - v(x) &\le \frac{1}{C_2}\Big(\big(\frac{K-k}{2} + 3M\sqrt{n}l\big)\delta + 3M\sqrt{n}(|k|+l)\gamma(1)\Big)\frac{x_n + \gamma(1)}{\delta} \\
&\le \frac{1}{C_2}\Big(\big(\frac{|K|+|k|}{2} + 3M\sqrt{n}l\big)(x_n + \gamma(1)) + 3M\sqrt{n}(|k|+l)\gamma(1)\Big) \\
&\le \frac{2 + 9M\sqrt{n}}{C_2}(|K| + |k| + l)\gamma(1),
\end{aligned}$$

where we have used $-\gamma(1) \le x_n \le \gamma(1)$ for $x \in \partial\Omega \cap \tilde{\tilde{Q}}$. By the Alexandroff-Bakelman-Pucci maximum principle,

$$a\psi(x) - v(x) \le C_3(|K| + |k| + l)(\gamma(1) + \|\mathbf{b}\|_{L^n(\Omega_1)}) + C_3\|f\|_{L^n(\Omega_1)} \quad \text{in } \tilde{\tilde{Q}} \cap \Omega,$$

where we have used $\|\mathbf{b}\|_{L^n(\Omega_1)} \le 1$ and C_3 is a constant depending only on λ and n.

From (2.13) (fifth inequality), it follows that for all $x \in \Omega \cap Q_\delta$,

$$\begin{aligned}
a\psi(x) &\ge \frac{a}{3\delta}(x_n + \gamma(1)) \\
&\ge \frac{\frac{(K-k)\delta}{2C_2} - \|f\|_{L^n(\Omega_1)} - (|k|+l)\|\mathbf{b}\|_{L^n(\Omega_1)}}{3\delta}(x_n + \gamma(1)) \\
&\ge \frac{K-k}{6C_2}x_n - \|f\|_{L^n(\Omega_1)} - (|k|+l)\|\mathbf{b}\|_{L^n(\Omega_1)},
\end{aligned}$$

where we have used $K - k \ge 0$.

Therefore, for all $x \in \Omega_\delta$,

$$\begin{aligned}
u(x) &\ge a\psi(x) + (k - 3M\sqrt{n}l)x_n - (C_1 + C_3)\|f\|_{L^n(\Omega_1)} \\
&\quad - (C_3 + 3M\sqrt{n} + C_1)(|K| + |k| + l)(\gamma(1) + \|\mathbf{b}\|_{L^n(\Omega_1)}) \\
&\ge \Big(k - 3M\sqrt{n}l + \frac{1}{6C_2}(K-k)\Big)x_n - (C_1 + C_3 + 1)\|f\|_{L^n(\Omega_1)} \\
&\quad - (C_3 + 3M\sqrt{n} + C_1 + 1)(|K| + |k| + l)(\gamma(1) + \|\mathbf{b}\|_{L^n(\Omega_1)}).
\end{aligned} \tag{2.20}$$

Let

$$\mu = \frac{1}{6C_2}, \quad A_1 = C_1 + C_3 + 1, \quad A_2 = C_1 + C_3 + 3M\sqrt{n} + 1. \tag{2.21}$$

Combining (2.8),(2.15) and (2.16), we have (**??**) and (**??**).

Case 2: $u(\delta e_n) < \frac{1}{2}(K+k)\delta$. The proof is similar to that of Case 1. Let

$$\begin{aligned}
v(x) &= (K + 3M\sqrt{n}B)x_n + C_1\|f\|_{L^n(\Omega_1)} + 3M\sqrt{n}(|K| + B)\gamma(1) \\
&\quad + C_1(|K| + B)\|\mathbf{b}\|_{L^n(\Omega_1)} - u(x)
\end{aligned}$$

for $x \in \Omega \cap Q_{\tilde{\delta}}$. Then

$$\begin{aligned}
v(\delta e_n) > \big(&\frac{K-k}{2} + 3M\sqrt{n}B\big)\delta + C_1\|f\|_{L^n(\Omega_1)} + 3M\sqrt{n}(|K| + B)\gamma(1) \\
&+ C_1(|K| + B)\|\mathbf{b}\|_{L^n(\Omega_1)}).
\end{aligned}$$

By the interior Harnack inequality, we have

$$\sup_{\Gamma} v \leq C_2\big(\inf_{\Gamma} v + \|f\|_{L^n(\Omega_1)} + (|K|+B)\|\mathbf{b}\|_{L^n(\Omega_1)}\big),$$

where $C_2(\geq 1)$ is a constant depending only on λ and n. Then

$$\begin{aligned} \inf_{\Gamma} v \geq \Big\{ &\frac{1}{C_2}\Big(\big(\frac{K-k}{2}+3M\sqrt{n}B\big)\delta + 3M\sqrt{n}(|K|+B)\gamma(1)\Big) \\ &+ \big(\frac{C_1}{C_2}-1\big)(\|f\|_{L^n(\Omega_1)} + (|K|+B)\|\mathbf{b}\|_{L^n(\Omega_1)}))\Big\}^+ := a. \end{aligned} \tag{2.22}$$

Let $\psi(x)$ be defined by (2.12). As in (2.14), we have

$$\begin{aligned} L\big(a\psi(x)-v(x)\big) &\leq b_iD_i(a\psi(x) - (|K|+3M\sqrt{n}B)x_n) + f(x) \quad \text{in } \tilde{\tilde{Q}}\cap\Omega; \\ a\psi(x)-v(x) &\leq \frac{(2+9M)}{C_2}(|K|+|k|+B)\gamma(1) \quad \text{on } \partial(\tilde{\tilde{Q}}\cap\Omega); \end{aligned} \tag{2.23}$$

where $\tilde{\tilde{Q}} = Q_{M\delta}\cap\{x\in\mathbb{R}^n : -\gamma(1) < x_n < \delta\}$.

Therefore, by the Alexandroff-Bakelman-Pucci maximum principle,

$$a\psi(x)-v(x) \leq C_3(|K|+|k|+B)(\gamma(1)+\|\mathbf{b}\|_{L^n(\Omega_1)}) + C_3\|f\|_{L^n(\Omega_1)}, \tag{2.24}$$

in $\tilde{\tilde{Q}}\cap\Omega$, where we have used $\|\mathbf{b}\|_{L^n(\Omega_1)}\leq 1$, and C_3 is a constant depending only on λ and n.

By (2.13) (fifth inequality), we have that for any $x\in\Omega\cap Q_\delta$,

$$\frac{a}{3\delta}(x_n+\gamma(1)) - v(x) \leq C_3(|K|+|k|+B)(\gamma(1)+\|\mathbf{b}\|_{L^n(\Omega_1)}) + C_3\|f\|_{L^n(\Omega_1)}.$$

Combining (2.17) with (2.19), we have that for all $x\in\Omega_\delta$,

$$\begin{aligned} u(x) &\leq (K+3M\sqrt{n}B)x_n - \frac{a}{3\delta}(x_n+\gamma(1)) + (C_1+C_3)\|f\|_{L^n(\Omega_1)} \\ &\quad + (C_1+C_3+3M\sqrt{n})(|K|+|k|+B)(\gamma(1)+\|\mathbf{b}\|_{L^n(\Omega_1)}) \\ &\leq \big(K+3M\sqrt{n}B - \frac{1}{6C_2}(K-k)\big)x_n + (C_1+C_3+1)\|f\|_{L^n(\Omega_1)} \\ &\quad + (C_1+C_3+3M\sqrt{n}+1)(|K|+|k|+B)(\gamma(1)+\|\mathbf{b}\|_{L^n(\Omega_1)}). \end{aligned} \tag{2.25}$$

Let $\mu = \frac{1}{6C_2}$, $A_1 = C_1+C_3+1$ and $A_2 = C_1+C_3+3M\sqrt{n}+1$. Combining (2.7) and (2.20), we have that (**??**) and (**??**) hold. □

Using induction, the following lemma is a direct consequence of Lemma 2.2.

Lemma 2.3. *Suppose that $0\in\partial\Omega$, $u\in W(\Omega_1)$, $u|_{\partial\Omega\cap B_1}=0$, $Lu=f$ in Ω_1, $\|u\|_{L^\infty(\Omega_1)}\leq 1$, $f\in L^n(\Omega_1)$ and $\omega(1)\leq\min\{\epsilon_0/2, 1/2, \delta/6\}$. Then there exist nonnegative sequences $\{l_m\}_{m=0}^\infty$, $\{B_m\}_{m=0}^\infty$, and sequences $\{k_m\}_{m=0}^\infty$, $\{K_m\}_{m=0}^\infty$ with $k_0=K_0=0$, $l_0=B_0=1$, and for $m=0,1,2,\dots$,*

$$l_{m+1} = A_1\delta^m\|f\|_{L^n(\Omega_{\delta^m})} + A_2\delta^m(|K_m|+|k_m|+\frac{l_m}{\delta^m})(\gamma(\delta^m)+\|\mathbf{b}\|_{L^n(\Omega_{\delta^m})}),$$

$$B_{m+1} = A_1\delta^m\|f\|_{L^n(\Omega_{\delta^m})} + A_2\delta^m(|K_m|+|k_m|+\frac{B_m}{\delta^m})(\gamma(\delta^m)+\|\mathbf{b}\|_{L^n(\Omega_{\delta^m})}),$$

and

$$k_{m+1} = k_m - 3M\sqrt{n}\frac{l_m}{\delta^m} + \mu(K_m-k_m) \quad \text{and} \quad K_{m+1} = K_m + 3M\sqrt{n}\frac{B_m}{\delta^m},$$

or

$$k_{m+1}=k_m-3M\sqrt{n}\frac{l_m}{\delta^m}\quad \textit{and}\quad K_{m+1}=K_m+3M\sqrt{n}\frac{B_m}{\delta^m}-\mu(K_m-k_m),$$

such that

$$k_mx_n-l_m\le u(x)\le K_mx_n+B_m\quad \textit{in } \Omega_{\delta^m}, \tag{2.26}$$

where δ, μ, M, A_1 and A_2 are positive constants given by Lemma 2.2.

Proof of Theorem 2.1. Let $\{l_m\}_{m=0}^\infty$, $\{B_m\}_{m=0}^\infty$, $\{k_m\}_{m=0}^\infty$ and $\{K_m\}_{m=0}^\infty$ be defined by Lemma 2.3. We prove the following claim first.

Claim. There exists a constant C_1 depending only λ and n such that for all $m=0,1,2,\dots$,

$$|K_m|,|k_m|,\frac{B_m}{\delta^m},\frac{l_m}{\delta^m}\le C_1. \tag{2.27}$$

Proof. Firstly, notice that we take $K_0=k_0=0$ and $l_0=B_0=1$, then by induction, we have $K_m\ge k_m$ for all $m\ge 0$. For $m\ge 0$, we define $S_m=\sum_{i=0}^m\Big(\frac{B_i}{\delta^i}+\frac{l_i}{\delta^i}\Big)$. For any $m\ge 0$, since

$$K_{m+1}\le K_m+3M\sqrt{n}\frac{B_m}{\delta^m}\quad \text{and } K_0=0,$$

we have

$$K_{m+1}\le 3M\sqrt{n}S_m\quad \text{for } m\ge 0.$$

Similarly, we have

$$k_{m+1}\ge -3M\sqrt{n}S_m\quad \text{for } m\ge 0.$$

It follows that

$$|k_{m+1}|+|K_{m+1}|\le 6M\sqrt{n}S_m\quad \text{for } m\ge 0. \tag{2.28}$$

Since

$$\begin{aligned}\frac{B_{m+1}+l_{m+1}}{\delta^{m+1}}&=\frac{A_2}{\delta}(\gamma(\delta^m)+\|\mathbf{b}\|_{L^n(\Omega_{\delta^m})})(2|K_m|+2|k_m|+\frac{B_m+l_m}{\delta^m})\\&\quad+\frac{2A_1}{\delta}\|f\|_{L^n(\Omega_{\delta^m})},\end{aligned}$$

for $m\ge 1$, combining the above identity with (2.23), we obtain

$$\begin{aligned}\frac{B_{m+1}+l_{m+1}}{\delta^{m+1}}&\le\frac{A_2}{\delta}\big(\gamma(\delta^m)+\|\mathbf{b}\|_{L^n(\Omega_{\delta^m})}\big)\Big(12M\sqrt{n}S_{m-1}+\frac{B_m+l_m}{\delta^m}\Big)\\&\quad+\frac{2A_1}{\delta}\|f\|_{L^n(\Omega_{\delta^m})}\\&\le\frac{12M\sqrt{n}A_2}{\delta}(\gamma(\delta^m)+\|\mathbf{b}\|_{L^n(\Omega_{\delta^m})})S_m+\frac{2A_1}{\delta}\|f\|_{L^n(\Omega_{\delta^m})}.\end{aligned} \tag{2.29}$$

By the normalized condition, we have

$$\begin{aligned}\sum_{i=1}^\infty\frac{12M\sqrt{n}A_2}{\delta}\big(\gamma(\delta^i)+\|\mathbf{b}\|_{L^n(\Omega_{\delta^i})}\big)&\le\sum_{i=1}^\infty\frac{36M\sqrt{n}A_2}{\delta}\omega(\delta^i)\\&\le\frac{36M\sqrt{n}A_2}{\delta\ln\frac{1}{\delta}}\int_0^1\frac{\omega(r)}{r}dr\le\frac12,\end{aligned} \tag{2.30}$$

and

$$\frac{2A_1}{\delta}\sum_{i=1}^\infty\|f\|_{L^n(\Omega_{\delta^i})}\le\frac{2A_1}{\delta\ln\frac1\delta}\int_0^1\frac{\|f\|_{L^n(\Omega_r)}}{r}dr\le\frac{2A_1}{\delta\ln\frac1\delta}. \tag{2.31}$$

From (2.24)-(2.26), it follows that for any $m \ge 1$,

$$\begin{aligned} S_{m+1} - S_1 &= \sum_{i=1}^{m} \frac{B_{i+1} + l_{i+1}}{\delta^{i+1}} \\ &\le S_{m+1} \sum_{i=1}^{m} \frac{12M\sqrt{n}A_2}{\delta} \big(\gamma(\delta^i) + \|\mathbf{b}\|_{L^n(\Omega_{\delta^i})}\big) + \frac{2A_1}{\delta} \sum_{i=1}^{m} \|f\|_{L^n(\Omega_{\delta^i})} \\ &\le \frac{1}{2} S_{m+1} + \frac{2A_1}{\delta \ln \frac{1}{\delta}}. \end{aligned}$$

Therefore, for all $m \ge 1$,

$$S_{m+1} \le \frac{4A_1}{\delta \ln(1/\delta)} + 2S_1.$$

Since $S_0 = 2$, $0 \le S_1 \le A_1 + A_2$, we have

$$0 \le S_m \le 2A_1 + 2A_2 + 2 + \frac{4A_1}{\delta \ln \frac{1}{\delta}} \quad \text{for all } m \ge 0.$$

Let $C_1 = 3M\sqrt{n}(2A_1 + 2A_2 + 2 + \frac{4A_1}{\delta \ln \frac{1}{\delta}})$. This completes the proof of the claim. $\square$

Next we show estimate (**??**). By Lemma 2.3, we have that for all $m \ge 1$,

$$0 \le K_{m+1} - k_{m+1} \le (1-\mu)(K_m - k_m) + 3M\sqrt{n}\frac{l_m + B_m}{\delta^m}$$

or

$$|K_{m+1} - k_{m+1}| \le (1-\mu)|K_m - k_m| + C_2(\|f\|_{L^n(\Omega_{\delta^{m-1}})} + \omega(\delta^{m-1})),$$

where $C_2 = \big(3M\sqrt{n}(A_1 + 6A_2C_1)\big)/\delta$.

Let $1 - \mu = \delta^{\hat\alpha} (\hat\alpha > 0)$. By iteration, we have that for all $m \ge 1$,

$$|K_{m+1} - k_{m+1}| \le C_3 \delta^{\hat\alpha m}\Big(1 + \int_{\delta^m}^{1} \frac{\omega(r) + \|f\|_{L^n(\Omega_r)}}{r^{1+\hat\alpha}} dr\Big),$$

where C_3 is a constant depending only on λ and n.

For any $m \ge 1$,

$$\begin{aligned} K_{m+1} + k_{m+1} &\le K_m + k_m + \mu(K_m - k_m) + 3M\sqrt{n}\frac{B_m}{\delta^m}, \\ K_{m+1} + k_{m+1} &\ge K_m + k_m - \mu(K_m - k_m) - 3M\sqrt{n}\frac{l_m}{\delta^m}. \end{aligned}$$

Hence,

$$\begin{aligned} &|(K_{m+1} + k_{m+1}) - (K_m + k_m)| \\ &\le \mu|K_m - k_m| + 3M\sqrt{n}\frac{l_m + B_m}{\delta^m} \\ &\le \mu|K_m - k_m| + C_4(\omega(\delta^{m-1}) + \|f\|_{L^n(\Omega_{\delta^{m-1}})}), \end{aligned} \tag{2.32}$$

where C_4 is a constant depending only on λ and n. It follows that

$$\begin{aligned}
&\sum_{j=m}^{\infty} |(K_{j+1}+k_{j+1})-(K_j+k_j)| \\
&\le C_3\mu \sum_{j=m}^{\infty} (\delta^{j-1})^{\hat\alpha}\Big(1+\int_{\delta^{j-1}}^{1} \frac{\omega(r)+\|f\|_{L^n(\Omega_r)}}{r^{1+\hat\alpha}}dr\Big) \\
&\quad + C_4\sum_{j=m}^{\infty}(\omega(\delta^{j-1})+\|f\|_{L^n(\Omega_{\delta^{j-1}})}).
\end{aligned} \tag{2.33}$$

Let

$$F_r := \int_r^1 \frac{\omega(s)+\|f\|_{L^n(\Omega_s)}}{s^{1+\hat\alpha}}ds.$$

By

$$\begin{aligned}
&\sum_{j=m}^{\infty}(\delta^{j-1})^{\hat\alpha}\int_{\delta^{j-1}}^{1} \frac{\omega(r)+\|f\|_{L^n(\Omega_r)}}{r^{1+\hat\alpha}}dr \\
&= \sum_{j=m-1}^{\infty}(\delta^{\hat\alpha})^{j}F_{\delta^j} \\
&= \frac{1}{\delta^{\hat\alpha}(1-\delta)}\sum_{j=m-1}^{\infty}(\delta^{j+1})^{\hat\alpha}F_{\delta^j}\cdot\frac{\delta^j-\delta^{j+1}}{\delta^j} \\
&\le \frac{1}{\delta^{\hat\alpha}(1-\delta)}\sum_{j=m-1}^{\infty}\int_{\delta^{j+1}}^{\delta^j} r^{\hat\alpha-1}F_r dr \\
&= \frac{1}{\delta^{\hat\alpha}(1-\delta)}\int_0^{\delta^{m-1}} r^{\hat\alpha-1}F_r dr \\
&= \frac{1}{\delta^{\hat\alpha}(1-\delta)\hat\alpha}\Big(\int_0^{\delta^{m-1}} \frac{\omega(r)+\|f\|_{L^n(\Omega_r)}}{r}dr \\
&\quad + (\delta^{m-1})^{\hat\alpha}\int_{\delta^{m-1}}^{1} \frac{\omega(r)+\|f\|_{L^n(\Omega_r)}}{r^{1+\hat\alpha}}dr\Big)
\end{aligned}$$

and

$$\sum_{j=m}^{\infty}(\omega(\delta^{j-1})+\|f\|_{L^n(\Omega_{\delta^{j-1}})}) \le \frac{1}{1-\delta}\int_0^{\delta^{m-2}} \frac{\omega(r)+\|f\|_{L^n(\Omega_r)}}{r}dr,$$

it follows that

$$\begin{aligned}
&\sum_{j=m}^{\infty} |(K_{j+1}+k_{j+1})-(K_j+k_j)| \\
&\le C_5\Big\{(\delta^{m-1})^{\hat\alpha}+(\delta^{m-1})^{\hat\alpha}\int_{\delta^{m-1}}^{1} \frac{\omega(r)+\|f\|_{L^n(\Omega_r)}}{r^{1+\hat\alpha}}dr \\
&\quad + \int_0^{\delta^{m-2}} \frac{\omega(r)+\|f\|_{L^n(\Omega_r)}}{r}dr\Big\}.
\end{aligned} \tag{2.34}$$

where C_5 is a constant depending only on λ and n.

While $m \to \infty$, by $\lim_{r\to 0^+}\omega(r)=0$ and L'Hospital rule, we have the righthand side of (2.29) tends to 0. Hence $\{K_m+k_m\}_{m=0}^{\infty}$ is convergent. Let $\lim_{m\to\infty}\frac{K_m+k_m}{2}=\theta$. Then for all $m\geq 2$,

$$\begin{aligned} \big|\theta-\frac{K_m+k_m}{2}\big| &\leq \sum_{j=m}^{\infty}\big|\frac{K_{j+1}+k_{j+1}}{2}-\frac{K_j+k_j}{2}\big| \\ &\leq \frac{C_5}{2}\Big\{(\delta^{m-1})^{\hat\alpha}+(\delta^{m-1})^{\hat\alpha}\int_{\delta^{m-1}}^{1}\frac{\omega(r)+\|f\|_{L^n(\Omega_r)}}{r^{1+\hat\alpha}}dr \\ &\quad+\int_0^{\delta^{m-2}}\frac{\omega(r)+\|f\|_{L^n(\Omega_r)}}{r}dr\Big\}. \end{aligned} \tag{2.35}$$

For any $m\geq 0$ and any $x\in\Omega_{\delta^m}$, we have

$$|u(x)-\theta x_n|\leq |u(x)-\frac{K_m+k_m}{2}x_n|+|(\frac{K_m+k_m}{2}-\theta)x_n|. \tag{2.36}$$

From (2.21), it follows that

$$-\frac{|K_m-k_m|}{2}|x_n|-l_m\leq u(x)-\frac{K_m+k_m}{2}x_n\leq\frac{|K_m-k_m|}{2}|x_n|+B_m.$$

Then for any $m\geq 0$ and any $x\in\Omega_{\delta^m}$,

$$|u(x)-\frac{K_m+k_m}{2}x_n|\leq(|K_m-k_m|+\frac{l_m+B_m}{\delta^m})\delta^m. \tag{2.37}$$

By (2.30) and the inequality above, for all $x\in\Omega_{\delta^m}$, $m=2,3,\dots$,

$$\begin{aligned} &|u(x)-\theta x_n| \\ &\leq |u(x)-\frac{K_m+k_m}{2}x_n|+|(\frac{K_m+k_m}{2}-\theta)x_n| \\ &\leq \Big(|K_m-k_m|+\frac{B_m+l_m}{\delta^m}+|\frac{K_m+k_m}{2}-\theta|\Big)\delta^m \\ &\leq C_6\Big\{(\delta^{m-1})^{\hat\alpha}+(\delta^{m-1})^{\hat\alpha}\int_{\delta^{m-1}}^{1}\frac{\omega(r)+\|f\|_{L^n(\Omega_r)}}{r^{1+\hat\alpha}}dr \\ &\quad+\omega(\delta^{m-1})+\|f\|_{L^n(\Omega_{\delta^{m-1}})}+\int_0^{\delta^{m-2}}\frac{\omega(r)+\|f\|_{L^n(\Omega_r)}}{r}dr\Big\}\delta^m, \end{aligned} \tag{2.38}$$

where C_6 is a constant depending only on λ and n.

Let $\Lambda=1/\delta^2$ ($\geq 324n$). By (2.33), we have that for all $x\in\Omega_r$ and $r\leq 1/\Lambda$,

$$\begin{aligned} |u(x)-\theta x_n|\leq C_7\Big\{&r^{\hat\alpha}+\omega(\Lambda r)+r^{\hat\alpha}\int_r^1\frac{\omega(s)+\|f\|_{L^n(\Omega_s)}}{s^{1+\hat\alpha}}ds \\ &+\|f\|_{L^n(\Omega_{\Lambda r})}+\int_0^{\Lambda r}\frac{\omega(s)+\|f\|_{L^n(\Omega_s)}}{s}ds\Big\}r. \end{aligned}$$

This completes the proof of Theorem 2.1. □

Proof of Theorem 1.9. Consider $|\nabla u(y)-\nabla u(z)|$, where $y,z\in\partial\Omega\cap B_{r_0}$ and $0<|y-z|=r\leq\frac{r_0}{\Lambda}$. By Corollary 1.7, we have

$$\|u(x)-L_y(x)\|_{L^\infty(\Omega_r(y))}\leq C\Big\{r^{\hat\alpha}+\omega(\Lambda r)+r^{\hat\alpha}\int_r^{r_0}\frac{\omega(s)}{s^{1+\hat\alpha}}ds+\int_0^{\Lambda r}\frac{\omega(s)}{s}ds\Big\}r,$$

$$\|u(x)-L_z(x)\|_{L^\infty(\Omega_r(z))}\leq C\Big\{r^{\hat\alpha}+\omega(\Lambda r)+r^{\hat\alpha}\int_r^{r_0}\frac{\omega(s)}{s^{1+\hat\alpha}}ds+\int_0^{\Lambda r}\frac{\omega(s)}{s}ds\Big\}r.$$

Noticing that $\partial\Omega$ is $C^{1,Dini}$ and the normalization makes ω small enough, then there exist a point $p \in \Omega$ and a small positive constant $\eta(< \frac{1}{\Lambda})$ such that $\overline{B_{\eta r}(p)} \subset \Omega_r(y) \cap \Omega_r(z)$. Then by the triangle inequality, we have

$$\|L_y(x) - L_z(x)\|_{L^\infty(\overline{B_{\eta r}(p)})} \le 2C\Big\{r^{\hat{\alpha}} + \omega(\Lambda r) + r^{\hat{\alpha}} \int_r^{r_0} \frac{\omega(s)}{s^{1+\hat{\alpha}}} ds + \int_0^{\Lambda r} \frac{\omega(s)}{s} ds\Big\} r.$$

Since $L_y(x) - L_z(x)$ is an affine function, we obtain

$$|\nabla L_y(x) - \nabla L_z(x)| \le \frac{1}{\eta r} \|L_y(x) - L_z(x)\|_{L^\infty(\overline{B_{\eta r}(p)})}.$$

It follows that

$$|\nabla L_y(x) - \nabla L_z(x)| \le \frac{2C}{\eta}\Big(r^{\hat{\alpha}} + \omega(\Lambda r) + r^{\hat{\alpha}} \int_r^{r_0} \frac{\omega(s)}{s^{1+\hat{\alpha}}} ds + \int_0^{\Lambda r} \frac{\omega(s)}{s} ds\Big).$$

Hence, for $y, z \in \partial\Omega \cap B_{r_0}$, $0 < |y - z| = r \le \frac{r_0}{\Lambda}$, we have

$$|\nabla u(y) - \nabla u(z)| \le \frac{2C}{\eta}\Big(r^{\hat{\alpha}} + \omega(\Lambda r) + r^{\hat{\alpha}} \int_r^{r_0} \frac{\omega(s)}{s^{1+\hat{\alpha}}} ds + \int_0^{\Lambda r} \frac{\omega(s)}{s} ds\Big).$$

This completes the proof. □

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