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APPROXIMATE CONTROLLABILITY OF QUASILINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this article, we study the approximate controllability for quasilinear differential equations with deviating arguments in a Hilbert space. We establish sufficient conditions for the existence of a mild solution. We also obtain sufficient conditions for the controllability of quasilinear equations. Further, we use it to establish the approximate controllability for quasilinear differential equations with deviating arguments. We discuss examples in which the analytical results are applied.

1. Introduction

Controllability refers a mathematical problem that deals with the possibility of driving the system from an initial state to an arbitrary final state with the help of suitable control input. The concept of controllability is of great importance in the mathematical theory of control of finite or infinite dimensional dynamical systems [2, 4]. In the controllability theory, we try to find a control function that drives the system from its initial state to an arbitrary final state in a finite time. Basically, in an infinite dimensional system, the two types of controllability can occur, one is exact and another is approximate controllability. Exact controllability enables to steer the system to the arbitrary final state from the initial state of the system. On the other hand, approximate controllability enables to steer the system to the arbitrary small neighbourhood of the final state from the initial state of the system. As far as the applications are concerned, the approximate controllability is more applicable to dynamical systems and the area got much attention in the recent years [5, 6, 7, 27, 10, 11, 13, 14, 15, 21, 23, 24, 25, 30, 33].

If we want to analyze a problem that occurs in our surroundings mathematically, the very first tool that comes to our mind is the differential equations. Through a suitable mathematical formulation we try to convert the actual problem into a differential equation. The formulation as well as finding a solution to the differential equation is not a hard task if we do not bother about any external factor that effects the problem. To be more accurate we have to take care of those factors which gives a better situation to the problem. That is why the deviating arguments are considered

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in differential equations. Differential equations with deviating arguments are a special class of differential equations where the unknown quantity and its derivative appear in different values of their arguments [5, 6, 8, 16, 18, 15, 22, 26]. Such type of differential equations has had a huge importance from the 1960 onwards and some remarkable research has been done in this area. Differential equations with deviating arguments occur in the theory of self-oscillating systems, the theory of automatic control, the problems of long-term planning in economics, the systems in biophysics, the study of problems related with combustion in rocket engines, and many other areas of science and technology. Considering the plentiful applications of the differential equation with deviating arguments, many author have studied differential equation with deviating arguments extensively e.g. [5, 6, 8, 12, 16, 17, 18, 22, 26]. Differential equations with deviating arguments are also known as functional differential equations.

The simpler and the most important version of differential equations with deviated arguments are differential equations with retarded arguments. It is also known as delay differential equations. This class represents a differential equation in which highest order derivative of the unknown function is evaluated at a certain values and the lower order derivatives are evaluated at a lesser or equal values. One most familiar delay problem is the hot shower problem where delay is occurred due to the time required for water to become hot and to flow from the tap to the shower head. In some of the models, the information is transferred from the input to the output after a finite time. Such systems are called the system with finite delay. In this case, the deviated argument is the finite delay in the differential equations.

The class of quasi-linear differential equations is another most important class that arises in the study of gas dynamics, continuum mechanics, traffic flow models, nonlinear acoustics and groundwater flows. Thus the theory of quasi-linear differential equations and their generalizations become as one of the most rapidly developing areas in applied mathematics [1, 19, 31]. In this article we plan to study the approximate controllability for the following system in a complex Hilbert space $(H, \|\cdot\|)$. We consider the following system consisting of quasilinear differential equation with deviating arguments in H,

$$\begin{gathered}[\frac{d}{dt}+A(t,x(t))]x(t)=f(t,x(t),x([h(x(t),t)]))+Bu(t), \quad t\in I=[0,T],\\ x(0)=x_0.\end{gathered} \tag{1.1}$$

Here, we assume that $-A(t,x)$, for each $t\geq 0$ and $x\in H$ generates an analytic semigroup of bounded linear operators on H, $u(\cdot)$ is the control function in $L^2(I,U)$ for a Hilbert space U, B is a bounded linear operator on U into H. The functions f and h satisfy suitable conditions in their arguments stated in Section 2.

The approximate controllability for the following nonlocal delay system with deviating arguments in a Hilbert space X is recently studied by Das et al. [5],

$$\begin{gathered}\frac{dx}{dt}-Ax(t)=f(t,x_t,x([h(x(t),t)]))+Bu(t), \quad t\in J=[0,b],\\ x(t)=\phi(t),\ t\in[-a,0].\end{gathered} \tag{1.2}$$

Here, we assume that $-A$ generates a strongly continuous semigroup of bounded linear operators on X, $u(\cdot)$ is the control function in $L^2(J,U)$ for a Hilbert space U, B is a bounded linear operator on U into X. The functions $f:J\times X\times X\to X$ and $h:X\times J\to J$ satisfy Lipschitz conditions in their arguments[5] and $x_t(\tau)=x(t+\tau)$

for $\tau \in [-a, 0]$. The results for existence of mild solution is established by the Schauder fixed point theorem. They showed that the mild solution is approximately controllable. Further, Haloi [15] discuss the sufficient conditions for the approximate controllability of the following problem with deviating arguments in a Hilbert space X,

$$\begin{gathered}\frac{dx}{dt} + A(t)x(t) = f(t, x(t), x([h(x(t), t)])) + Bu(t), \quad t \in J = [0, b], \\ x(t) = \phi(t) + g(x)(t),\ t \in [-a, 0].\end{gathered} \tag{1.3}$$

Here, we assume that $-A(t)$, for each $t \geq 0$ generates a compact analytic semigroup of bounded linear operators on X, $u(\cdot)$ is the control function in $L^2(J, U)$ for a Hilbert space U, $B : U \to X$ is a bounded linear operator. The functions $f : J \times X \times X \to X$, $h : X \times J \to J$ and $g : C([-a, b], X) \to C([-a, 0], X)$ are assumed to satisfy suitable conditions in their arguments.

We mentioned that the approximate controllability of the quasilinear systems with deviating arguments have not been considered so far. We devote this article for the study of the approximate controllability of quasilinear systems with deviating arguments (1.1) in an arbitrary infinite dimensional Hilbert space. The results in this articles are new and contribute to the theory of mathematical control.

We organize the article as follows. In Section 2, we provide preliminaries, assumptions and lemmas that will be needed for proving the main results. We prove the local existence of a solution in Section 3. The approximate controllability results are established in Section 4. Finally, we provide an example to illustrate the application of the results.

2. Preliminaries and assumptions

In this section, we collect notation, assumptions and lemmas that are used in the remaining sections. We refer to the book by Bensoussan et al. [2], Curtain and Zwart [4], Friedman [9], Pazy [28], Tanabe [29] and Yosida[32] for more details.

Let H and U be two complex Hilbert spaces. Let $T \in [0, \infty)$ and $\{A(t) : 0 \leq t \leq T\}$ be a family of closed linear operators on H. Let $\Delta = \{(t, s) \in I \times I : 0 \leq s \leq t \leq T\}$ and $\mathcal{L}(H)$ denotes the Banach space of all bounded linear operator on H.

Definition 2.1. An operator $\mathcal{U} : \Delta \to \mathcal{L}(H)$ is said to be a compact evolution family if the following holds:

(a) $\mathcal{U}(s, s) = I$ is the identity operator in H for $s \in I$,
(b) $\mathcal{U}(t, r)U(r, s) = U(t, s)$, $0 \leq s \leq r \leq t \leq T$,
(c) $\mathcal{U}$ is strongly continuous on Δ,
(d) $\mathcal{U}(t, s)$ satisfies

$$\frac{\partial \mathcal{U}(t, s)}{\partial t} + A(t)\mathcal{U}(t, s) = 0, \frac{\partial \mathcal{U}(t, s)}{\partial s} - \mathcal{U}(t, s)A(s) = 0, \quad (t, s) \in \Delta,$$

(e) $\mathcal{U}(t, s)$ are completely continuous for $(t, s) \in \Delta$.

Suppose the family $\{A(t) : 0 \leq t \leq T\}$ satisfies the following properties,

(1) The domain $D(A(t))$ of $A(t)$ is dense in H and is independent of t.
(2) For each $t \in [0, T]$, the resolvent $R(\lambda; A(t))$ exists for all $\operatorname{Re} \lambda \leq 0$ and there exists a constant $C_1 > 0$ (independent of t and λ) such that

$$\|R(\lambda; A(t))\| \leq \frac{C_1}{|\lambda| + 1}, \quad \operatorname{Re} \lambda \leq 0, \quad t \in [0, T].$$

(3) There are constants $C_2 > 0$ and $\alpha \in (0, 1]$ such that
$$\|[A(t_1) - A(t_2)]A^{-1}(t_3)\| \le C_2|t_1 - t_2|^\alpha,$$
for $t_1, t_2, t_3 \in [0, T]$. Here C_2 and α are independent of t_1,t_2 and t_3.

With the properties (1)–(3), there exists a fundamental solution $\{\mathcal{U}(t, s) : 0 \le s \le t \le T\}$ corresponding to $A(t)$.

Remark 2.2. The evolution semigroup $\mathcal{U}(t, s)$ is strongly continuous on the compact set Δ, there exists a constant $K > 0$ such that
$$\|\mathcal{U}(t, s)\| \le K \quad \text{for any } (t, s) \in \Delta. \tag{2.1}$$

We make the following assumptions:

(A1) Let the operator $A_0 = A(0, x_0)$ be closed with dense domain D_0 in H and there exists a constant $C_3 > 0$ independent of λ such that
$$\|(\lambda I - A_0)^{-1}\| \le \frac{C_3}{1 + |\lambda|}; \quad \text{for all } \lambda \text{ with } \operatorname{Re}\lambda \le 0. \tag{2.2}$$

From this inequality it follows that the negative fractional powers $A_0^{-\rho}$, for $\rho > 0$ of A_0 is well defined [9] and
$$A_0^{-\rho} = \frac{1}{\Gamma(\rho)} \int_0^\infty e^{-tA_0} t^{\rho-1} dt.$$
Then $A_0^{-\rho}$ is bijective on H. Thus we define positive fractional powers of A_0 by $A_0^\rho = [A_0^{-\rho}]^{-1}$. Then A_0^ρ is closed linear operator with dense domain $D(A_0^\rho)$ in H and $D(A_0^{\mu_1}) \subset D(A_0^{\mu_2})$ if $\mu_1 > \mu_2$. For $0 < \rho \le 1$, let $H_\rho = D(A_0^\rho)$ and equip this space with the graph norm
$$\|x\|_\rho = \|A_0^\rho x\|.$$
Thus H_ρ becomes a Hilbert space with respect to the above norm. We define, for each $\rho > 0, H_{-\rho} = (H_\rho)^*$, the dual space of H_ρ, endowed with the natural norm
$$\|x\|_{-\rho} = \|A_0^{-\rho} x\|.$$
Let $M, \tilde{M} > 0$ and $W_\rho = \{x \in H_\rho : \|x\|_\rho < M\}$,
$W_{\rho-1} = \{y \in H_{\rho-1} : \|y\|_{\rho-1} < \tilde{M}\}$. We shall also use the following assumptions:

(A2) For some $\rho \in [0, 1)$ and for any $x \in W_\rho$, the operator $A(t, x)$ is well defined on D_0 for all $t \in I$. Furthermore, for any $t, s \in I$ and $x, y \in W_\rho$ the following condition holds
$$\|[A(t, x) - A(s, y)]A^{-1}(s, y)\| \le R(M)\|x - y\|_\rho. \tag{2.3}$$

(A3) For every $t, s \in I$; $x, y \in W_\rho$ and $x', y' \in W_{\rho-1}$ there exists constants $d_f = d_f(t, M, \tilde{M}) > 0$ s.t. the nonlinear map $f : I \times W_\rho \times W_{\rho-1} \to H$ satisfies the condition
$$\begin{gathered} \|f(t, x, x') - f(s, y, y')\| \le d_f(\|x - y\|_\rho + \|x' - y'\|_{\rho-1}), \\ \|f(0, x(0), x([h(x(0), 0)]))\| \le d_{f_0}, \text{ for all } t, s \in I. \end{gathered} \tag{2.4}$$

(A4) There exist constants $d_h = d_h(t, M) > 0$ such that
$$|h(x, t) - h(y, s)| \le d_h\|x - y\|_\rho, \quad \text{for all } x, y \in W_\rho; t, s \in I, h(\cdot, 0) = 0. \tag{2.5}$$

(A5) For some $\mu > \rho$ we assume that $x_0 \in H_\mu$ and $\|x_0\|_\rho < M$.

(A6) A_0^{-1} is completely continuous on H.

Remark 2.3. Since, H is a Hilbert space, the assumption (A6) implies that A_0^{-1} is compact on H. Thus, $R(\lambda; A_0)$ is compact for $\operatorname{Re}\lambda \leq 0$.

For some $0 \leq \rho \leq 1$, we define

$$X_\rho = \{\phi \in C(I; H_\rho) : \|\phi(t) - \phi(s)\|_{\rho-1} \leq d_\rho |t-s|, t, s \in I\}.$$

where d_ρ is a positive constant. Then X_ρ is a Hilbert space with the sup-norm of $C(I; H_\rho)$. Let $y \in X_\rho$. If $T > 0$ is sufficiently small, then we have

$$\|y(t)\|_\rho < M, \quad \text{for } t \in I.$$

Hence,

$$A_y(t) = A(t, y(t))$$

is well defined for each $t \in I$. Also we have

$$\|(\lambda I - A_y(t))^{-1}\| \leq \frac{\tilde{C}}{1 + |\lambda|}, \quad \text{for all } \lambda \text{ with } \operatorname{Re}\lambda \leq 0, \tag{2.6}$$

$$\|[A_y(t) - A_y(s)]A_y^{-1}(\eta)\| \leq \tilde{C}. \tag{2.7}$$

Thus there exists a fundamental solution $\mathcal{U}_y(t,s)$ corresponding to $A_y(t)$. We also use the following assumption.

(A7) Choose $T > 0$ small enough such that $R(\lambda; A_y(t))$ is compact for $t \in I$, Re $\lambda \leq 0$. So $A(t, y(t))$ generates a compact evolution family.

Let $x(T, x_0, u)$ be the state value of the system (1.1) at terminal time T corresponding to the initial value x_0 and the control function u. We define the following set

$$R(T, x_0) = \{x(T, x_0, u) : u \in L^2(I, U)\}.$$

The set $R(T, x_0)$ is called the reachable set of the system (1.1) at time T.

Definition 2.4. (1) A controllability map for the system (1.1) on I is the bounded linear map $\mathcal{B}^T : L^2(I, U) \to H$ which is defined as

$$\mathcal{B}^T u := \int_0^T \mathcal{U}_x(T,s)Bu(s)ds, \quad \text{for } u \in L^2(I, U). \tag{2.8}$$

(2) The system (1.1) is exactly controllable on I if $R(T, x_0) = H$, that is for all $y_0, y_1 \in H$, there exists $u \in L^2(I, U)$ such that the mild solution to the system (1.1) satisfies $x(0, x_0, u) = y_0$ and $x(T, x_0, u) = y_1$.

(3) The system (1.1) is approximately controllable on I if $\overline{R(T, x_0)} = H$, that is for given $\epsilon > 0$ and $y_0, y_1 \in X$, there exists a control $u \in L^2(I, U)$ steers from the point $x(0, x_0, u) = y_0$ to all points at time T within a distance of ϵ from y_1. More precisely,

$$x(0, x_0, u) = y_0, \ \|x(T, x_0, u) - y_1\| < \epsilon.$$

(4) The controllability Gramian of the system (1.1) on I is defined by

$$\Gamma_0^T := \mathcal{B}^T(\mathcal{B}^T)^*.$$

Lemma 2.5. [14] *The following properties hold for the controllability map:*

(a) $(\mathcal{B}^T)^* z(s) = B^* \mathcal{U}_x^*(T,s)z$, *for* $s \in [0, T]$, $z \in H$.

(b) $\Gamma_0^T = \mathcal{B}^T(\mathcal{B}^T)^* \in \mathcal{L}(H)$ *has the following representation:*

$$\Gamma_0^T z = \int_0^T \mathcal{U}_x(T,s)BB^*\mathcal{U}_x^*(T,s)z ds, \quad \text{for } z \in H \tag{2.9}$$

and $\Gamma_0^T \geq 0$, *where* B^* *and* $\mathcal{U}_x^*$ *denote the adjoint of* B *and* $\mathcal{U}_x$ *respectively.*

We consider the following control system in H,

$$\begin{gathered}[\frac{d}{dt} + A(t,x(t))]x(t) = Bu(t), \quad t \in I,\\ x(0) = x_0.\end{gathered} \tag{2.10}$$

We define the resolvent operator associated with (2.10) as

$$R(\lambda, \Gamma_0^T) = (\lambda I + \Gamma_0^T)^{-1}, \quad \lambda > 0.$$

We use the following assumption.

(A8) $\lambda R(\lambda; \Gamma_0^T) \to 0$ as $\lambda \to 0^+$ in the strong operator topology.

We use the following characterization.

Theorem 2.6 ([23]). *Let* X *be a separable Banach space with dual* X^*. *The following are equivalent for a symmetric operator* $P : X^* \to X$:

(i) P *is positive,*
(ii) $x_\epsilon(h) = \epsilon(\epsilon I + PQ)^{-1}(h) \to 0$ *as* $\epsilon \to 0^+$*in the strong operator topology, where* $Q : X \to X^*$ *denotes the duality map.*

We prove the following theorem of approximately controllability for (2.10).

Theorem 2.7. *The system* (2.10) *is approximately controllable on* I *if and only if condition* (A8) *holds.*

Proof. The system (2.10) is the homogeneous part of (1.1). We note that there exists a unique mild solution to (1.1) [18] and the mild solution x_h for (2.10) in X_ρ is given as follows

$$x_h(t) = \mathcal{U}_x(t,0)x_0 + \int_0^t \mathcal{U}_x(t,s)Bu(s)ds, \quad t \in I = [0,T]. \tag{2.11}$$

For $z \in H$ and $\lambda > 0$ we define the control function for (2.10) as

$$u_\lambda(t,x) = B^*U_x^*R(\lambda; \Gamma_0^T)\{z - \mathcal{U}_x(T,0)x_0\}. \tag{2.12}$$

Put

$$r_h(x) = z - \mathcal{U}_x(T,0)x_0.$$

Then we get

$$x_h(T) = z - \lambda R(\lambda; \Gamma_0^T)r_h(x).$$

Thus, $x_h(T) \to z$ if and only if $\lambda R(\lambda; \Gamma_0^T) \to 0$ as $\lambda \to 0$. $\square$

For $z \in H$ and $\lambda > 0$, we define the control function $u_\lambda(t,x)$ for (1.1) as

$$\begin{aligned}u_\lambda(t,x) = B^*\mathcal{U}_x^*(T,s)R(\lambda; \Gamma_0^T)\Big\{&z - U_x(T,0)x_0\\ &- \int_0^T \mathcal{U}_x(T,s)f(s,x(s),x([h(x(s),s)]))ds\Big\}.\end{aligned} \tag{2.13}$$

We also recall the Krasnoselskii's fixed point theorem. We refer the reader for proof to Burton [3].

Theorem 2.8. *Let P be a map from a closed bounded convex subset S of a Banach space X into S. Suppose that $Px = P_1x + P_2x$ for $x \in S$ and $P_1u + P_2v \in S$ for every pair $u, v \in S$. If P_1 is contraction and P_2 is compact, then the equation $P_1u + P_2u = u$ has a solution in S*

3. Existence of solution

In this section, we establish the existence and uniqueness of a local solution to (1.1) corresponding to a given control function u_ϵ. The proof of the theorem is based on the technique in [14, 21].

Definition 3.1. A function $x \in X_\rho$ is said to be a mild solution to problem (1.1) if $x(t)$ satisfies

$$\begin{aligned} x(t) = &\mathcal{U}_x(t,0)x_0 + \int_0^t \mathcal{U}_x(t,s)f(s,x(s),x([h(x(s),s)]))ds \\ &+ \int_0^t \mathcal{U}_x(t,s)Bu(s)ds, \quad t \in I = [0,T]. \end{aligned} \tag{3.1}$$

Theorem 3.2. *System* (1.1) *has a unique mild solution in X_ρ for each control $u_\lambda \in L^2(I,U)$ if assumptions* (A1)–(A8) *hold and*

$$K[(d_f + d_f d_{\rho-1} d_h) + d_{f_0}]T < 1.$$

Proof. We consider the ball

$$V_l = \{z \in X_\rho : \|z\|_\rho \le l\},$$

where l is a positive constant such that $l < M$. For each $y \in V_l$, we define the map G_λ as

$$\begin{aligned} G_\lambda y(t) = &\mathcal{U}_y(t,0)x_0 + \int_0^t \mathcal{U}_y(t,s)f(s,y(s),y([h(y(s),s)]))ds \\ &+ \int_0^t \mathcal{U}_y(t,s)Bu_\lambda(s,y)ds, \quad t \in I. \end{aligned}$$

We put

$$L = \frac{1}{\lambda}\|B\| \sup_{t\in I} \|B^*\mathcal{U}_y^*(T,t)\|.$$

For $t \in I$, we have the estimate

$$\begin{aligned} &\|Bu_\lambda(t,y)\|_\rho \\ &\le \frac{1}{\lambda}\|BB^*\mathcal{U}_y^*(T,s)\|\Big\{\|z\| + K\|x_0\|_\rho + K\int_0^T \Big\|\Big(f(s,y(s),y([h(y(s),s)])) \\ &\quad - f(0,y(0),y([h(y(0),0)])) + f(0,y(0),y([h(y(0),0)]))\Big)\Big\|ds\Big\} \\ &\le L\Big\{\|z\| + K\|x_0\|_\rho + K\int_0^T [d_f(\|y(s)-y(0)\|_\rho \\ &\quad + \|y([h(y(s),s)]) - y([h(y(0),0)])\|_{\rho-1})]ds + KTd_{f_0}\Big\} \\ &\le L\Big\{\|z\| + KM + K\int_0^T d_f(l + M + d_{\rho-1}d_h(l+M))ds + KTd_{f_0}\Big\} \\ &= L\{\|z\| + KM + K[(l+M)(d_f + d_f d_h d_{\rho-1}) + d_{f_0}]T\} = m, \end{aligned} \tag{3.2}$$

where,
$$m = L\{\|z\| + KM + K[(l+M)(d_f + d_f d_h d_{\rho-1}) + d_{f_0}]T\}.$$
Let $t_1, t_2 \in I$ with $t_1 < t_2$ and $y \in X_\rho$. Using [9, Lemma II.14.1 and Lemma 14.4], we obtain
$$\begin{aligned}
&\|G_\lambda y(t_1) - G_\lambda y(t_2)\|_{\rho-1}\\
&\leq \|[\mathcal{U}_y(t_1,0) - \mathcal{U}_y(t_2,0)]x_0\|_{\rho-1}\\
&\quad + \Big\| \int_0^{t_1} \mathcal{U}_y(t_1,s) f_y(s) ds - \int_0^{t_2} \mathcal{U}_y(t_2,s) f_y(s) ds \Big\|_{\rho-1}\\
&\quad + \Big\| \int_0^{t_1} \mathcal{U}_y(t_1,s) Bu_\lambda(s,y) ds - \int_0^{t_2} \mathcal{U}_y(t_2,s) Bu_\lambda(s,y) ds \Big\|_{\rho-1}\\
&\leq MM_1(t_2 - t_1) + M_2 M_3 (1 + |\log(t_2 - t_1)|)(t_2 - t_1)\\
&\quad + mM_4(1 + |\log(t_2 - t_1)|)(t_2 - t_1)\\
&\leq d_\rho (t_2 - t_1),
\end{aligned}$$
where $d_\rho = MM_1 + (M_2 M_3 + mM_4)(1 + |\log(t_2 - t_1)|)$, M_1,M_2, M_3 and M_4 are positive constants. Thus $G_\lambda \in X_\rho$. Using estimate (3.2), we obtain
$$\begin{aligned}
\|G_\lambda y(t)\|_\rho &\leq K\|x_0\|_\rho + \int_0^t \|\mathcal{U}_y(t,s) f_y(s)\|_\rho ds + \int_0^t \|\mathcal{U}_y(t,s) Bu_\lambda(s,y)\|_\rho ds\\
&\leq KM + K[(l+M)(1 + d_{\rho-1} d_h) d_f + d_{f_0}]T + KTM \leq l,
\end{aligned}$$
provided
$$KM + K[(l+M)(1 + d_{\rho-1} d_h) d_f + d_{f_0}]T + KTM \leq l$$
or
$$KM + KTm + Kl[(1 + d_{\rho-1} d_h) d_f + d_{f_0}]T + KM[(1 + d_{\rho-1} d_h) d_f + d_{f_0}]T \leq l$$
or
$$KM + KTm + KM[(1 + d_{\rho-1} d_h) d_f + d_{f_0}]T \leq l[1 - K\{(1 + d_{\rho-1} d_h) d_f + d_{f_0}\}T].$$
This is possible only if $K\{(1 + d_{\rho-1} d_h) d_f + d_{f_0}\}T < 1$. Thus we choose T such that
$$T < \frac{1}{K\{(1 + d_{\rho-1} d_h) d_f + d_{f_0}\}}.$$
So, G_λ maps V_l into itself. We decompose G_λ as $G_\lambda = G_{\lambda,1} + G_{\lambda,2}$, where
$$G_{\lambda,1} y(t) = \mathcal{U}_y(t,0)x_0 + \int_0^t \mathcal{U}_y(t,s) f_y(s) ds, \quad t \in I,$$
$$G_{\lambda,2} y(t) = \int_0^t \mathcal{U}_y(t,s) Bu_\lambda(s,y) ds, \ t \in I.$$

We begin by showing that $G_{\lambda,1}$ is contraction on V_l. For $y_1, y_2 \in V_l$ and $t \in [0.T]$, put $g_1(t) = G_{\lambda,1} y_1(t)$ and $g_2(t) = G_{\lambda,1} y_2(t)$. Thus for $i = 1, 2$, we have
$$\begin{gathered}
[\frac{d}{dt} + A_{y_i}(t)] g_i(t) = f_{y_i}(t), \quad t \in (0,T],\\
g_i(0) = x_0.
\end{gathered} \tag{3.3}$$
Thus,
$$\frac{d}{dt}(g_1 - g_2) + A_{y_1}(t)(g_1 - g_2) = [A_{y_2}(t) - A_{y_1}(t)] g_2 + f_{y_1}(t) - f_{y_2}(t).$$

Note that $A_0(t)g_2(t)$ is uniformly Hölder continuous for $0 < t \leq T$ and that $A_0 \int_o^t \mathcal{U}_{y_2}(t,s) f_{y_2}(s) ds$ is a bounded function, and hence we have the bound

$$\|A_0 G_{\lambda,1} y_2(t)\| \leq \tilde{M} t^{\delta-1}$$

The operator $[A_{y_2}(t) - A_{y_1}(t)]A_0^{-1}$ is uniformly Hölder continuous for $0 < \beta \leq t \leq T$. Hence $[A_{y_2}(t) - A_{y_1}(t)]g_2(t)$ is uniformly Hölder continuous for $0 < \beta \leq t \leq T$.

$$\begin{aligned} g_1(t) - g_2(t) = & \mathcal{U}_{y_1}(t,\beta)[g_1(\beta) - g_2(\beta)] \\ & + \int_\beta^t \mathcal{U}_{y_1}(t,s) \big\{ [A_{y_2}(s) - A_{y_1}(s)] g_2(t) + [f_{y_1}(s) - f_{y_2}(s)] \big\} ds. \end{aligned} \tag{3.4}$$

Letting $\beta \to 0$, we obtain

$$g_1(t) - g_2(t) = \int_0^t \mathcal{U}_{y_1}(t,s) \{ [A_{y_2}(s) - A_{y_1}(s)] g_2(t) + [f_{y_1}(s) - f_{y_2}(s)] \} ds.$$

So

$$\begin{aligned} \|G_{\lambda,1} y_1(t) - G_{\lambda,1} y_2(t)\|_\rho & \leq N_1 \tilde{M}(M) \int_0^t (t-s)^{-\rho} \|y_1(s) - y_2(s)\|_\rho s^{\delta-1} ds \\ & \quad + N_2 d_f \int_0^t (t-s)^{-\rho} \{ \|y_1(s) - y_2(s)\|_\rho \\ & \quad + \|y_1([h(y_1(s),s)]) - y_2([h(y_2(s),s)\|_{\rho-1}]) \} ds \\ & \leq N_1 \tilde{M}(M) \int_0^t (t-s)^{-\rho} \|y_1(s) - y_2(s)\|_\rho s^{\delta-1} ds \\ & \quad + \frac{N_2 \tilde{M}(M)}{1-\rho} d_f (2 + d_\rho d_h) T^{1-\rho} \sup \|y_1(t) - y_2(t)\|_\rho \\ & \leq \tilde{N} T^{\delta-\rho} \sup \|y_1(t) - y_2(t)\|_\rho, \end{aligned}$$

where

$$\tilde{N} = \max\{ \frac{N_1 \tilde{M}(M)}{1-\rho}, \frac{N_2 \tilde{M}(M)}{1-\rho} d_f (2 + d_\rho d_h) \}.$$

Choose $T > 0$ such that

$$\tilde{N} T^{\delta-\rho} < \frac{1}{2}.$$

Therefore,

$$\|G_{\lambda,1} y_1 - G_{\lambda,1} y_2\|_\rho \leq \|y_1 - y_2\|_\rho.$$

Hence $G_{\lambda,1}$ is contraction on V_l. We next show that the map $G_{\lambda,2}$ is completely continuous.

Step 1: Let $\{v_n\}$ be a sequence in V_l such that $v_n \to v \in V_l$ as $n \to \infty$. It follows from (A3) and (A4) that

(a) $\|Bu_\lambda(s,v_n) - Bu_\lambda(s,v)\|_\rho \to 0$ as $n \to \infty$.
(b) $\|Bu_\lambda(s,v_n) - Bu_\lambda(s,v)\|_\rho \leq 2m$.

The dominated convergence theorem allows us to obtain

$$\begin{aligned} & \|G_{\lambda,2} v_n(t) - G_{\lambda,2} v(t)\|_\rho \\ & \leq \int_0^t \|U_{v_n}(t,s) Bu_\lambda(s,v_n) - \mathcal{U}_v(t,s) Bu_\lambda(s,v)\|_\rho ds \\ & = \int_0^t \|U_{v_n}(t,s) Bu_\lambda(s,v_n) - \mathcal{U}_{v_n}(t,s) Bu_\lambda(s,v) \end{aligned}$$

$$
\begin{aligned}
&+\mathcal{U}_{v_n}(t,s)Bu_\lambda(s,v)-\mathcal{U}_v(t,s)Bu_\lambda(s,v)\|_\rho ds\\
&\le K\int_0^t \|Bu_\lambda(s,v_n)-Bu_\lambda(s,v)\|_\rho ds+\int_0^t \|Bu_\lambda(s,v)[\mathcal{U}_{v_n}(t,s)-\mathcal{U}_v(t,s)]\|_\rho ds\\
&\to 0 \quad \text{as } n\to\infty.
\end{aligned}
$$

Step 2: Let $t_1, t_2 \in I$ such that $t_1 < t_2$ and $v \in V_l$. It follows from [9, Lemma II, 14.1, 14.4] that

$$
\|G_{\lambda,2}v(t_2)-G_{\lambda,2}v(t_1)\|_\rho \le N_1(t_2-t_1)^\nu,
$$

for some constants $0 \le \nu \le 1$ and $N_1 > 0$. Thus $\{G_{\lambda,2}(V_l)\}$ is equicontinous on I.

Step 3: We show that $\{G_{\lambda,2}v(t) : v \in V_l\}$ is relatively compact in H. If $0 < \eta < t$, then we have

$$
\begin{aligned}
G^\eta_{\lambda,2}v(t)&=\int_0^{t-\eta}\mathcal{U}_v(t,s)Bu_\lambda(s,v)ds\\
&=\mathcal{U}_v(t,t-\eta)\int_0^{t-\eta}\mathcal{U}_v(t-\eta,s)Bu_\lambda(s,v)ds\\
&=\mathcal{U}_v(t,t-\eta)F(t,\eta),
\end{aligned}
$$

where $F(t,\eta)=\int_0^{t-\eta}\mathcal{U}_v(t-\eta,s)Bu_\lambda(s,v)ds$. We note that $F(t,\eta)$ is bounded on V_l. As $\mathcal{U}_v(t,s)$ is compact in H, so for each $t \in (0,T]$, the set $\{G^\eta_{\lambda,2}v(t) : v \in V_l\}$ is relatively compact in H. Indeed, we have

$$
\|G_{\lambda,2}v(t)-G^\eta_{\lambda,2}v(t)\|_\rho \le \int_{t-\eta}^t \|\mathcal{U}_v(t,s)Bu_\lambda(s,v)\|_\rho ds \le Km\eta \to 0 \quad \text{as } \eta\to 0^+.
$$

Thus the set $\{G_{\lambda,2}v(t) : v \in V_l\}$ is arbitrarily close to the relatively compact set $\{G^\eta_{\lambda,2}v(t) : v \in V_l\}$ for each $t \in I$. Hence, for all $t \in I$ the set $\{G_{\lambda,2}v(t) : v \in V_l\}$ is relatively compact in H.

By Ascoli-Arzela theorem, the set $\{G_{\lambda,2}v : v \in V_l\}$ is relatively compact in X_ρ. Thus the map $G_{\lambda,2}$ is completely continuous from V_l into V_l.

Thus the map G_λ has fixed point on V_l by Krasnoselskii's fixed point theorem. Hence for each $\lambda > 0$, the system (1.1) has a mild solution in V_l corresponding to each control $u_\lambda(s,x)$. □

4. Approximate controllability

We prove the following theorem of approximate controllability for system (1.1).

Theorem 4.1. *Let assumptions* (A1)–(A8) *hold. Let the functions* $f : I \times W_\rho \times W_{\rho-1} \to H$ *and* $h : W_\rho \times I \to I$ *be uniformly bounded. Then system* (1.1) *is approximately controllable on* I.

Proof. It follows from Theorem 3.2 that G_λ has fixed point y_λ in V_l. That is, y_λ is a mild solution for the control

$$
u_\lambda(t,y_\lambda)=B^*\mathcal{U}^*_{y_\lambda}(T,t)R(\lambda,\Gamma_0^T)r(y_\lambda),
$$

where,

$$
r(y_\lambda)=z-\mathcal{U}_{y_\lambda}(T,0)x_0-\int_0^T\mathcal{U}_{y_\lambda}(T,s)f_{y_\lambda}(s)ds,\quad t\in I.
$$

Further, we have

$$\begin{aligned} y_\lambda(T) &= \mathcal{U}_{y_\lambda}(T,0)x_0 + \int_0^T \mathcal{U}_{y_\lambda}(T,s)f_{y_\lambda}(s)ds \\ &\quad + \int_0^T \mathcal{U}_{y_\lambda}(T,s)Bu_\lambda(s,y_\lambda)ds, \\ &= z - r(y_\lambda) + \Gamma_0^T R(\lambda;\Gamma_0^T)r(y_\lambda) \\ &= z - \lambda R(\lambda;\Gamma_0^T)r(y_\lambda). \end{aligned} \tag{4.1}$$

As $f : I \times W_\rho \times W_{\rho-1} \to H$ and $h : W_\rho \times I \to I$ are uniformly bounded, $f_{y_\lambda}(s)$ is bounded in $L^2(I,H)$. Thus there exists a subsequence denoted by $f_{y_\lambda}(s)$ that converges to $f(s)$ say. We define

$$q = z - \mathcal{U}_{y_\lambda}(T,0)x_0 - \int_0^T \mathcal{U}_{y_\lambda}(T,s)f(s)ds.$$

By the compactness of $\mathcal{U}_{y_\lambda}(t,s)$ and Arzela-Ascoli theorem, we have

$$\|r(y_\lambda) - q\| \le K \int_0^T \|f_{y_\lambda}(s) - f(s)\|ds \to 0 \quad \text{as } \lambda \to 0+\,. \tag{4.2}$$

Again from (4.1), we have

$$\begin{aligned} \|y_\lambda(T) - z\| &\le \|\lambda R(\lambda,\Gamma_0^T)(q)\| + \|\lambda R(\lambda,\Gamma_0^T)(q - r(y_\lambda))\| \\ &\le \|\lambda R(\lambda,\Gamma_0^T)(q)\| + \|r(y_\lambda) - q\|. \end{aligned}$$

By assumption (A8) and (4.2), we have $\|y_\lambda(T) - z\| \to 0$ as $\lambda \to 0^+$. This completes the proof. $\square$

5. Applications

Example 5.1. Let $X = L^2([0,T] \times [0,1];\mathbb{R})$. We consider the following system with deviating arguments in X,

$$\begin{gathered} \frac{\partial y(t,x)}{\partial t} + [\kappa(t,x) + |y(t,x)|\frac{\partial^2}{\partial x^2}]y(t,x) \\ = Bu(x,t) + v(x,y(t,x)) + w(t,x,y(t,x)), \quad T > t > 0,\ x \in [0,1], \\ y(t,0) = y(t,1), \quad 0 \le t \le T, \\ y(0,x) = y_0(x). \end{gathered} \tag{5.1}$$

We assume that

(1) $\kappa \in C^1([0,T] \times [0,1],\mathbb{R})$;
(2) $v : [0,1] \to \mathbb{R} \to \mathbb{R}$ defined as

$$v(x,y(t,x)) = \int_0^x F(x,x_1)y(\tilde{g}(t)|y(t,x_1)|,x_1)dx_1,$$

for $(t,x) \in (0,T) \times [0,1]$, where $\tilde{g} : \mathbb{R}^+ \to \mathbb{R}^+$ is locally Hölder continuous in t with $\tilde{g}(0) = 0$ and $F \in C^1([0,1] \times [0,1];\mathbb{R})$;
(3) $w : \mathbb{R}^+ \times [0,1] \times \mathbb{R} \to \mathbb{R}$ is measurable in x, locally Hölder continuous in t, locally Lipschitz continuous in t, locally Lipschitz continuous in y, uniformly in x.

We define
$$A(t,y)y(t) = [\kappa(t,x) + |y(t,x)|\frac{\partial^2}{\partial x^2}]y(t,x),$$
where $\frac{\partial^2}{\partial x^2}$ is the distributional derivative of y.

Then $A_0 y = [\kappa(0,x) + y_0(x)\frac{\partial^2}{\partial x^2}]y$. So $D(A_0) = H^2(0,1) \cap H_0^1(0,1)$.

It is known that that $-A(t,y)$ generates a compact analytic evolution semigroup of bounded operators $\mathcal{U}(t,s)$ on $L^2[0,1]$ [9]. Now $X_{\frac{1}{2}} = D(A_0^{\frac{1}{2}}) = H_0^1(0,1), X_{-\frac{1}{2}} = H^{-1}(0,1)$. For $x \in (0,1)$ we define $f : \mathbb{R}^+ \times H_0^1(0,1) \times H^{-1}(0,1) \to X$ by
$$f(t,\phi_1,\phi_2) = v(x,\phi_2) + w(t,x,\phi_1),$$
where
$$v(x,\phi_2(x,t)) = \int_0^x F(x,x_1)\phi_2(x_1,t)dx_1.$$
Then we have
$$\|w(t_1,x,\phi) - w(t_2,x,\phi_2)\| \le C\|\phi_1 - \phi_2\|.$$
Thus f defined as above satisfies required assumptions (A3) and (A4), [15].

We consider the infinite dimensional control space
$$U = \{w : w = \sum_0^\infty w_n e_n(x), \sum_0^\infty |w_n|^2 < \infty\}$$
equipped with the norm $\|w\| = (\sum_0^\infty |w_n|^2)^{\frac{1}{2}}$. We define B as $B : U \to X$ as
$$Bw = 4w_2 e_1(x) + \sum_{n=2}^\infty w_n e_n(x).$$
Here B is a bounded linear map and the adjoint of B is
$$B^* u = (4u_1 + u_2)e_2(x) + \sum_{n=3}^\infty u_n e_n(x).$$
If $B^* U^*(t,s)u = 0$, then $u = 0$. Thus (5.1) is approximately controllable on $[0,T]$.

Example 5.2. Let $X = L^2([0,T] \times [0,1]; \mathbb{R})$ be the same space as defined in the first example. Let us consider the following system in X,
$$\begin{gathered}\frac{\partial w(t,x)}{\partial t} + \frac{\partial}{\partial x}[\psi(t,x)\frac{\partial}{\partial x}]w(t,x) = f(t,x,w(t,x),w(t-t_0,x)) + Bu(x,t),\\ T > t \ge t_0 > 0,\ x \in [0,1],\\ w(t,0) = w(t,1), \quad 0 \le t \le T,\\ w(0,x) = w_0(x),\end{gathered} \tag{5.2}$$
where
$$\begin{aligned}&f(t,x,w(t,x),w(t-t_0,x))\\ &= \int_0^1 K(x,y)[w(g(t)|w(t,y)|,y) + w(\tilde{g}(t-t_0)|w(t-t_0,y)|,y)]dy.\end{aligned}$$
We assume that

(a) ψ is a positive bounded function, Lipschitz continuous in the first variable and has bounded and continuous partial derivatives for all $0 \le t < T$ and $x \in [0,1]$.

(b) B is a bounded linear map from a Hilbert space U to X and we define B as exactly similar to the previous example.
(c) $K \in C^1([0,1]\times[0,1];\mathbb{R})$.
(d) $g, \tilde{g} : \mathbb{R}^+ \to \mathbb{R}^+$ are two Hölder continuous functions with $g(0) = 0 = \tilde{g}(0)$.

We define

$$A(t,w)w(t) = -\frac{\partial}{\partial x}[\psi(t,x)\frac{\partial}{\partial x}w(t,x)].$$

Then domain of the operator A_0 is $D(A_0) = H^2(0,1) \cap H_0^1(0,1)$, and domain of the fractional power operator $A_0^{1/2}$ is $X_{1/2} = D(A_0^{1/2}) = H_0^1(0,1)$. Also the family $\{A(t,w) : 0 \le t \le T\}$ satisfies all the conditions to generate a compact analytic evolution semigroup of bounded operators $\mathcal{U}(t,s)$ on $L^2(0,1)$ for the system 5.2 [17]. We consider the space U as defined in the first example, then we can show that the system 5.2 is approximate controllable in the interval $[0,T]$.

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