FINDING THE “BEST” n-DIE WHEN FIXING THE SUM

by

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FAIR USE AND AUTHOR'S PERMISSION STATEMENT

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ABSTRACT

This paper discusses results found by systematically listing every possible 6-sided die with a fixed sum of 21 and running a round-robin tournament to find the "best" die.

1 INTRODUCTION

This paper discusses what it means to be the "best" die with a fixed sum. The idea behind this paper came after reading [4] where a game with dice was played. The game consisted of children building their dice and rolling them against each other in a tournament. Conditions had to be set on how the children were allowed to build the dice since the easiest way to win would be to pick one extremely large number and place it on all the die's faces. Hence, why we are using n-sided dice with a fixed sum, where the sum is based off the standard die. This allows the game to have comparable dice.

In this paper, we systematically find all the n-dice with a fixed sum and organize them using lexicographical ordering. Then, we will be put all the dice into a round-robin tournament. The "best" die will be whichever die obtains the most tournament points. Thus, we will be conducting a couple tournaments and change the point assignment to wins, ties, and losses to analyze the potentially differing results.

When two dice are being compared, we are thinking of each die being rolled once and comparing the displayed face value as we see in [1]. Thus, to find out a "better" die, we will see that all 6 faces of the first die will be compared to each of the 6 faces of the second die. This simulates rolling each die once and comparing face value. This is an important disclaimer as we will analyze a question posed by Dr. Eugene Curtin regarding rolling each die twice and compare results from their max face value.

2 DEFINITIONS AND NOTATIONS

Section 2 will introduce definitions and notations used throughout the paper.

Definition 2.0.1 An *n-die* is a non-increasing sequence of non-negative integers denoted $D_\alpha = (f_1, f_2, \ldots, f_n)$ and, for $i \in \{1, 2, \ldots, n\}$ f_i is the ith face of D_α where α denotes the dice name. The set of all n-dice will be denoted by $\mathcal{D}_n$.

For our purposes we will be working with positive integers since we will not be using a face value of zero and working in $n = 6$ i.e. 6-sided dice. Thus, we will not make mention of $\mathcal{D}_6$ and assume we are discussing 6-sided dice. When $\alpha = st$, this will denote the standard dice D_{st} where $D_{st} = (6, 5, 4, 3, 2, 1)$. If $\alpha \in \mathbb{Z}$, then this will refer to the dice using lexicographical ordering i.e. D_1 will be the largest dice where $D_1 = (6, 6, 6, 1, 1, 1)$.

As to not get the fixed sum dice confused with the fixed product dice, we will denote the fixed product with a different letter. i.e. P_α will denote n-die with a fixed produt of 720.

If $\sigma \in \mathbb{N}$, a (σ, n)-die is an n-die D_α such that

$$\sum_{i=1}^{6} f_i = \sigma.$$

Since we are working with 6-sided die with fixed sum, then for the rest of the paper $\sigma = 21$. We see this is resulted from the standard die $D_{st} = (6, 5, 4, 3, 2, 1)$ since $6 + 5 + 4 + 3 + 2 + 1 = 21$.

If $\pi \in \mathbb{N}$, a (π, n)-die is an n-die D_α such that

$$\prod_{i=1}^{6} f_i = \pi.$$

Since we are working with 6-sided die with fixed product, then for the rest of the paper $\pi = 6! = 720$. We see this is resulted from the standard die D_{st} since

$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 6! = 720$.

Definition 2.0.2 The *cardinality* of a set $\{x_n\}$ denoted $\text{card}(\{x_n\})$ is the number of elements in that set.

Definition 2.0.3 Let $D = (f_1, f_2, \ldots, f_n)$ and $D' = (f'_1, f'_2, \ldots, f'_n)$ be two dice. D is *"better"* than D' denoted $D > D'$ means the number of times D rolls higher than D' is greater than the number of times D' rolls higher than D i.e.

$$\sum_{i=1}^{6} \text{card}(\{j \mid f_i > f'_j\}) - \sum_{i=1}^{6} \text{card}(\{j \mid f'_i > f_j\}) > 0.$$

We have quotes around "better" since the word may not mean one dice is actually better than the other rather it has an overall higher roll compared to another dice. A better dice we would think would beat out all dice which is our goal to see if we find a dice that has that quality and we will give that dice a different title of champion.

Similarly D is *"worse"* than D' is denoted $D < D'$ i.e.

$$\sum_{i=1}^{6} \text{card}(\{j \mid f_i > f'_j\}) - \sum_{i=1}^{6} \text{card}(\{j \mid f'_i > f_j\}) < 0.$$

D *"ties"* with D' is denoted $D \sim D'$ i.e.

$$\sum_{i=1}^{6} \text{card}(\{j \mid f_i > f'_j\}) - \sum_{i=1}^{6} \text{card}(\{j \mid f'_i > f_j\}) = 0.$$

Definition 2.0.4 A *round-robin tournament*, also known as an all-play-all tournament, is a competition where each participant will compete against all the other participants one at a time. No participant is eliminated since each participant needs to compete against all the others. The results from the tournament are gained by calculating the number of wins, ties, and losses. Depending on the number of points assigned to wins, ties, and losses will affect who is considered the winner of the tournament.

Definition 2.0.5 A relation R on a set X is *transitive* if, for all elements a, b, c $\in X$, whenever aRb and $bRc \implies aRc$ where aRb denotes a relates to b.

3 RESULTS

Section 3 will contain all the results we have found regarding 6-sided dice with a fixed sum of 21 as well as 6-sided dice with a fixed product of $6! = 720$.

3.1 Counting dice with fixed sum

As to count all the 6-sided dice with a fixed sum of 21 and not miss any, we will use the lexicographical order, or dictionary order. This ordering will treat each dice as a 6-digit number hence why we chose to have the dice be a non-increasing sequence.

When using lexicographical ordering on the set of 6-digit numbers, the most efficient approach is to pick as many of the largest digit in the highest place value without surpassing the fixed sum of 21. We see 6 is the largest digit and can be placed three times to obtain $6+6+6=18$ otherwise we would have $6+6+6+6=24$. Then our next step is to try and put the next largest digit in the next place value i.e. place a 5 in the thousands place. We see this breaks our fixed sum of 21 since $6+6+6+5=23$. Thus, we traverse the digits until we are below the fixed sum. We can pick the digit 3 to obtain $6+6+6+3=21$, but recall that we are not using any zeroes to fill the rest of the place values. Hence, we can only use the digit 1 to fill the rest of the place values. Therefore, we obtain $6+6+6+1+1+1=21$. Now we call this our largest die $D_1 = (6,6,6,1,1,1)$. We continue the process until our highest place value cannot be reduced to a lower digit. We get $D_{32} = (4,4,4,3,3,3)$ to acquire a total of 32 unique 6-sided dice with a fixed sum of 21.

3.1.1 Listing all dice with fixed sum

- $D_1 = (6,6,6,1,1,1)$
- $D_2 = (6,6,5,2,1,1)$
- $D_3 = (6,6,4,3,1,1)$

- $D_4 = (6, 6, 4, 2, 2, 1)$
- $D_5 = (6, 6, 3, 3, 2, 1)$
- $D_6 = (6, 6, 3, 2, 2, 2)$
- $D_7 = (6, 5, 5, 3, 1, 1)$
- $D_8 = (6, 5, 5, 2, 2, 1)$
- $D_9 = (6, 5, 4, 4, 1, 1)$
- $D_{10} = (6, 5, 4, 3, 2, 1)$, where $D_{10} = D_{st}$
- $D_{11} = (6, 5, 4, 2, 2, 2)$
- $D_{12} = (6, 5, 3, 3, 3, 1)$
- $D_{13} = (6, 5, 3, 3, 2, 2)$
- $D_{14} = (6, 4, 4, 4, 2, 1)$
- $D_{15} = (6, 4, 4, 3, 3, 1)$
- $D_{16} = (6, 4, 4, 3, 2, 2)$
- $D_{17} = (6, 4, 3, 3, 3, 2)$
- $D_{18} = (6, 3, 3, 3, 3, 3)$
- $D_{19} = (5, 5, 5, 4, 1, 1)$
- $D_{20} = (5, 5, 5, 3, 2, 1)$
- $D_{21} = (5, 5, 5, 2, 2, 2)$
- $D_{22} = (5, 5, 4, 4, 2, 1)$
- $D_{23} = (5, 5, 4, 3, 3, 1)$

- $D_{24} = (5, 5, 4, 3, 2, 2)$
- $D_{25} = (5, 5, 3, 3, 3, 2)$
- $D_{26} = (5, 4, 4, 4, 3, 1)$
- $D_{27} = (5, 4, 4, 4, 2, 2)$
- $D_{28} = (5, 4, 4, 3, 3, 2)$
- $D_{29} = (5, 4, 3, 3, 3, 3)$
- $D_{30} = (4, 4, 4, 4, 4, 1)$
- $D_{31} = (4, 4, 4, 4, 3, 2)$
- $D_{32} = (4, 4, 4, 3, 3, 3)$

3.2 Counting dice with fixed product

As to count all the 6-sided dice with a fixed product of 720 and not miss any, we will use the prime factorization to group the combinations.

$$720 = 2^4 \cdot 3^2 \cdot 5$$

The prime factorization will be broken down into cases for the different combinations of 2^4 and 3^2, then at the end we will count how many ways the factor of 5 can be distributed amongst the faces without repetition.

1. 2^4 can be separated amongst the 6-sides in 5 different ways:

(i) $f_1 = 2^4$

(ii) $f_1 = 2^3,\ f_2 = 2^1$

(iii) $f_1 = 2^2,\ f_2 = 2^2$

(iv) $f_1 = 2^2,\ f_2 = 2^1,\ f_3 = 2^1$

(v) $f_1 = 2^1,\ f_2 = 2^1,\ f_3 = 2^1,\ f_4 = 2^1$

2. 3^2 can be separated amongst the 6-sides in 2 different ways:

(i) $f_1 = 3^2$

(ii) $f_1 = 3^1,\ f_2 = 3^1$

3. 5 can be separated amongst the 6-sides in 1 way:

(i) $f_1 = 5$

Case Ia: Using 1(i) and 2(i)

$f_1 = 2^4 \cdot 3^2$

$f_1 = 2^4,\ f_2 = 3^2$

So, we have 2 combinations. Then, taking into consideration the number of ways the factor 5 can be distributed amongst the faces without repetition, we get 2 ways for the first combination and 3 ways for the second combination totaling the combinations to $2 + 3 = 5$ different combinations using 1(i), 2(i), and 3(i) with a fixed product of 720.

Case Ib: Using 1(i) and 2(ii)

$f_1 = 2^4 \cdot 3,\ f_2 = 3$

$f_1 = 2^4,\ f_2 = 3,\ f_3 = 3$

Similarly counting the number of ways for Case Ia, we get a total of 6 combinations for using 1(i), 2(ii), and 3(i).

Case IIa: Using 1(ii) and 2(i). Notice we will no longer write 2^1 and just assume the power being used is 1.

$f_1 = 2^3 \cdot 3^2,\ f_2 = 2$

$f_1 = 2^3,\ f_2 = 2 \cdot 3^2$

$f_1 = 2^3,\ f_2 = 2^1,\ f_3 = 3^2$

Case IIa yields a total of 10 combinations.

Case IIb: Using 1(ii) and 2(ii)

$f_1 = 2^3 \cdot 3,\ f_2 = 2 \cdot 3$

$f_1 = 2^3,\ f_2 = 2 \cdot 3,\ f_3 = 3$

$f_1 = 2^3 \cdot 3,\ f_2 = 2,\ f_3 = 3$

$f_1 = 2^3,\ f_2 = 2,\ f_3 = 3,\ f_4 = 3$

Yields a total of 15 combinations.

Case IIIa: Using 1(iii) and 2(i)

6 combinations

Case IIIb: 1(iii) and 2(ii)

9 combinations

Case IVa: Using 1(iv) and 2(i)

11 combinations

Case IVb: 1(iv) and 2(ii)

16 combinations

Case Va: 1(v) and 2(i)

6 combinations

Case Vb: 1(v) and 2(ii)

10 combinations

Therefore, by Cases Ia through Vb, we get a total of 94 6-sided dice with a fixed product of 720.

3.2.1 Listing dice with fixed product

For the dice with fixed product, I will only be listing the dice used in the paper for reference.

- $P_5 = (16, 9, 5, 1, 1, 1)$
- $P_8 = (48, 5, 3, 1, 1, 1)$
- $P_{11} = (16, 5, 3, 3, 1, 1)$
- $P_{14} = (72, 5, 2, 1, 1, 1)$
- $P_{30} = (24, 10, 3, 1, 1, 1)$

3.3 Round-robin tournament with fixed sum

The total number of dice with a fixed sum of 21 came out to 32 total dice. These dice were then put into a round-robin tournament which we will call "Tournament 1." At this point, we assigned points as follows: win = +3 points, tie = +1 point, loss = 0 points. Changes to this point system results in different results as we will see in Section 3.3.4.

3.3.1 Tournament 1 results

Dice 19 Wins:	55
Dice 7 Wins:	54
Dice 23 Wins:	54
Dice 27 Wins:	53
Dice 22 Wins:	52
Dice 9 Wins:	51
Dice 20 Wins:	50
Dice 14 Wins:	48
Dice 26 Wins:	48
Dice 30 Wins:	47
Dice 31 Wins:	47
Dice 29 Wins:	44
Dice 2 Wins:	42
Dice 8 Wins:	42
Dice 15 Wins:	42
Dice 28 Wins:	42
Dice 32 Wins:	42
Dice 18 Wins:	41
Dice 1 Wins:	39
Dice 3 Wins:	39
Dice 12 Wins:	39
Dice 11 Wins:	38
Dice 6 Wins:	37
Dice 5 Wins:	36
Dice 13 Wins:	36
Dice 17 Wins:	36
Dice 24 Wins:	35
Dice 25 Wins:	35
Dice 4 Wins:	33
Dice 16 Wins:	33
Dice 21 Wins:	32
Dice 10 Wins:	31

Figure 1: Tournament 1 Results. The table sorts the results greatest to least by amount of tournament points.

3.3.2 "Best" die with fixed sum

The dice with the most points in the round-robin tournament will be denoted the **"best"** die.

- The "best" die is $D_{19} = (5,5,5,4,1,1)$ with a total of 55 points. Majority of the wins from D_{19} come from beating D_{22} through D_{32} which have lower face values spread throughout each die.

- $D_7 = (6,5,5,3,1,1)$ comes in second place with a total of 54 points. An interesting fact is $D_7 > D_{19}$. This makes us question who is actually the "champion" die. But, similarly enough, D_7 gets majority of its wins from D_{22} through D_{32}.

- In third place is an interesting die. $D_{23} = (5,5,4,3,3,1)$ which has a wider spread of wins from the larger dice in lexicographical ordering, the middle dice, and the smaller dice. Unfortunately $D_{23} < D_7$ and $D_{23} < D_{19}$ so D_{23} may not be the best candidate for the "champion" die but does show some interesting results.

3.3.3 "Worst" die with fixed sum

Similarly the dice with the least points in the round-robing tournament will be denoted the **"worst"** die.

- The "worst" die is the standard die $D_{st} = D_{10} = (6,5,4,3,2,1)$ with a total of 31 tournament points. The number of points is not surprising as we expected from previous results for D_{st} to tie with all other dice with fixed sum.

- Second to last place is $D_{16} = (6,4,4,3,2,2)$ with 33 points. Majority of its wins are from D_1 through D_6, then majority of its losses are from D_{22} through D_{32} which we see is the complete opposite of the "best" die.

- $D_4 = (6,6,4,2,2,1)$ ended up in a tie with D_{16} with 33 points, but had a wider spread of wins, losses, and ties. Not surprisingly enough, $D_4 < D_{16}$. An interesting result is $D_4 > D_7$ and $D_4 > D_{19}$ and $D_4 \sim D_{23}$. Does this mean D_4 could be a potential candidate for the "champion" die?

3.3.4 Tournament points affecting "best" die

We changed the point assignment to see how this affected the "best" die. Points were changed to the following: win = +2 points, tie = +1 point, loss = 0 points. We will call this new tournament, "Tournament 2." We did notice that the "best" and "worst" dice were different. The "best" die changed to D_7 and the "worst" die changed to D_{16}.

3.3.5 Tournament 2 results

Dice 7 Wins:	38
Dice 23 Wins:	38
Dice 19 Wins:	37
Dice 22 Wins:	37
Dice 27 Wins:	37
Dice 9 Wins:	36
Dice 20 Wins:	35
Dice 26 Wins:	35
Dice 14 Wins:	34
Dice 30 Wins:	33
Dice 31 Wins:	32
Dice 1 Wins:	31
Dice 2 Wins:	31
Dice 3 Wins:	31
Dice 8 Wins:	31
Dice 10 Wins:	31
Dice 15 Wins:	31
Dice 28 Wins:	31
Dice 32 Wins:	31
Dice 29 Wins:	30
Dice 18 Wins:	29
Dice 12 Wins:	28
Dice 24 Wins:	28
Dice 11 Wins:	27
Dice 17 Wins:	27
Dice 5 Wins:	26
Dice 13 Wins:	26
Dice 6 Wins:	25
Dice 21 Wins:	25
Dice 25 Wins:	25
Dice 4 Wins:	24
Dice 16 Wins:	24

Figure 2: Tournament 2 Results. The table sorts the results greatest to least by amount of tournament points.

3.4 Intransitive better, worse, and tie operations

Section 3.4 will show the how the different operations defined in Section 2 are all intransitive as exhibited under similar conditions using probabilities in [2] while maintaining the use of n-dice with a fixed sum as seen in [3].

3.4.1 Intransitivity with fixed sum

Section 3.4.1 will exhibit counterexamples for transitivity on the relations " $>$," " $<$," and " $\sim$ ".

Counterexample for "better" denoted " $>$ ":

$D_1 = (6,6,6,1,1,1),\ D_{16} = (6,4,4,3,2,2),\ D_{19} = (5,5,5,4,1,1).$

Comparing D_1 and D_{19}:

$$\sum_{n=1}^{6} \text{card}(\{j \mid f_i > f'_j\}) = 18 \text{ and } \sum_{n=1}^{6} \text{card}(\{j \mid f'_i > f_j\}) = 12$$

$$\sum_{n=1}^{6} \text{card}(\{j \mid f_i > f'_j\}) - \sum_{n=1}^{6} \text{card}(\{j \mid f'_i > f_j\})$$

$$18 - 12 = 6 > 0 \implies D_1 > D_{19}$$

Comparing D_{19} and D_{16}:

$$\sum_{n=1}^{6} \text{card}(\{j \mid f_i > f'_j\}) = 18 \text{ and } \sum_{n=1}^{6} \text{card}(\{j \mid f'_i > f_j\}) = 16$$

$$\sum_{n=1}^{6} \text{card}(\{j \mid f_i > f'_j\}) - \sum_{n=1}^{6} \text{card}(\{j \mid f'_i > f_j\})$$

$$18 - 16 = 2 > 0 \implies D_{19} > D_{16}$$

Comparing D_1 and D_{16}:

$$\sum_{n=1}^{6} \text{card}(\{j \mid f_i > f'_j\}) = 15 \text{ and } \sum_{n=1}^{6} \text{card}(\{j \mid f'_i > f_j\}) = 18$$

$$\sum_{n=1}^{6} \text{card}(\{j \mid f_i > f'_j\}) - \sum_{n=1}^{6} \text{card}(\{j \mid f'_i > f_j\})$$

$$15 - 18 = -3 < 0 \implies D_1 < D_{16}$$

$$\not\Rightarrow D_1 > D_{16}$$

Thus, $D_1 > D_{19}$ and $D_{19} > D_{16} \not\Rightarrow D_1 > D_{16}$ i.e. $D_1 < D_{16}$. Therefore " $>$ " is intransitive.

Counterexample for "worse" denoted " $<$ ":

Similarly, $D_1 < D_{16}$ and $D_{16} < D_{19} \not\Rightarrow D_1 < D_{19}$ because $D_{19} < D_1$. Therefore " $<$ " is intransitive.

Counterexample for ties denoted " $\sim$ ":

$D_1 = (6,6,6,1,1,1)$, $D_4 = (6,6,4,2,2,1)$, and $D_{10} = D_{st} = (6,5,4,3,2,1)$.

Comparing D_1 and D_{st}:

$$\sum_{n=1}^{6} \text{card}(\{j \mid f_i > f'_j\}) = \sum_{n=1}^{6} \text{card}(\{j \mid f'_i > f_j\}) = 15$$

$$\sum_{n=1}^{6} \text{card}(\{j \mid f_i > f'_j\}) - \sum_{n=1}^{6} \text{card}(\{j \mid f'_i > f_j\})$$

$$15 - 15 = 0 \implies D_1 \sim D_{st}$$

Comparing D_{st} and D_4:

$$\sum_{n=1}^{6} \text{card}(\{j \mid f_i > f'_j\}) = \sum_{n=1}^{6} \text{card}(\{j \mid f'_i > f_j\}) = 15$$

$$\sum_{n=1}^{6} \text{card}(\{j \mid f_i > f'_j\}) - \sum_{n=1}^{6} \text{card}(\{j \mid f'_i > f_j\})$$

$$15 - 15 = 0 \implies D_{st} \sim D_4$$

Comparing D_1 and D_4:

$$\sum_{n=1}^{6} \text{card}(\{j \mid f_i > f'_j\}) = 12 \text{ and } \sum_{n=1}^{6} \text{card}(\{j \mid f'_i > f_j\}) = 15$$

$$\sum_{n=1}^{6} \text{card}(\{j \mid f_i > f'_j\}) - \sum_{n=1}^{6} \text{card}(\{j \mid f'_i > f_j\})$$

$$12 - 15 = -3 < 0 \implies D_1 < D_4$$

$$\implies D_1 \nsim D_4$$

Thus, $D_1 \sim D_{st}$ and $D_{st} \sim D_4 \nRightarrow D_1 \sim D_4$ i.e. $D_1 \nsim D_4$. Therefore " $\sim$ " is intransitive.

3.4.2 Intransitivity with fixed product

Section 3.4.2 will exhibit counterexamples for transitivity on the relations " $>$," " $<$," and " $\sim$ ".

Counterexample for "better" denoted " $>$ ":

$P_5 = (16, 9, 5, 1, 1, 1)$, $P_{11} = (16, 5, 3, 3, 1, 1)$, $P_{30} = (24, 10, 3, 1, 1, 1)$

Comparing P_{11} and P_{30}:

$$\sum_{n=1}^{6} \text{card}(\{j \mid f_i > f'_j\}) = 15 \text{ and } \sum_{n=1}^{6} \text{card}(\{j \mid f'_i > f_j\}) = 13$$

$$\sum_{n=1}^{6} \text{card}(\{j \mid f_i > f'_j\}) - \sum_{n=1}^{6} \text{card}(\{j \mid f'_i > f_j\})$$

$$15 - 13 = 2 > 0 \implies P_{11} > P_{30}$$

Comparing P_{30} and P_5:

$$\sum_{n=1}^{6} \text{card}(\{j \mid f_i > f'_j\}) = 14 \text{ and } \sum_{n=1}^{6} \text{card}(\{j \mid f'_i > f_j\}) = 13$$

$$\sum_{n=1}^{6} \text{card}(\{j \mid f_i > f'_j\}) - \sum_{n=1}^{6} \text{card}(\{j \mid f'_i > f_j\})$$

$$14 - 13 = 1 > 0 \implies P_{30} > P_5$$

Comparing P_{11} and P_5:

$$\sum_{n=1}^{6} \text{card}(\{j \mid f_i > f'_j\}) = 14 \text{ and } \sum_{n=1}^{6} \text{card}(\{j \mid f'_i > f_j\}) = 14$$

$$\sum_{n=1}^{6} \text{card}(\{j \mid f_i > f'_j\}) - \sum_{n=1}^{6} \text{card}(\{j \mid f'_i > f_j\})$$

$$14 - 14 = 0 = 0 \implies P_{11} \sim P_5$$

$$\nRightarrow P_{11} > P_5$$

Thus, $P_{11} > P_{30}$ and $P_{30} > P_5 \not\Rightarrow P_{11} > P_5$ since $P_{11} \sim P_5$. Therefore " $>$ " is intransitive.

Counterexample for "worse" denoted " $<$ ":

Similarly, $P_5 < P_{30}$ and $P_{30} < D_{11} \not\Rightarrow P_5 < P_{11}$ because $P_5 < P_{11}$. Therefore " $<$ " is intransitive.

Counterexample for ties denoted " $\sim$ ":

$P_8 = (48, 5, 3, 1, 1, 1)$, $P_{14} = (72, 5, 2, 1, 1, 1)$, and $P_{30} = (24, 10, 3, 1, 1, 1)$.

Comparing P_{30} and P_8:

$$\sum_{n=1}^{6} \mathrm{card}(\{j \mid f_i > f'_j\}) = \sum_{n=1}^{6} \mathrm{card}(\{j \mid f'_i > f_j\}) = 15$$
$$\sum_{n=1}^{6} \mathrm{card}(\{j \mid f_i > f'_j\}) - \sum_{n=1}^{6} \mathrm{card}(\{j \mid f'_i > f_j\})$$
$$13 - 13 = 0 \implies P_{30} \sim P_8$$

Comparing P_8 and P_{14}:

$$\sum_{n=1}^{6} \mathrm{card}(\{j \mid f_i > f'_j\}) = \sum_{n=1}^{6} \mathrm{card}(\{j \mid f'_i > f_j\}) = 15$$
$$\sum_{n=1}^{6} \mathrm{card}(\{j \mid f_i > f'_j\}) - \sum_{n=1}^{6} \mathrm{card}(\{j \mid f'_i > f_j\})$$
$$13 - 13 = 0 \implies P_8 \sim P_{14}$$

Comparing P_{30} and P_{14}:

$$\sum_{n=1}^{6} \mathrm{card}(\{j \mid f_i > f'_j\}) = 14 \text{ and } \sum_{n=1}^{6} \mathrm{card}(\{j \mid f'_i > f_j\}) = 13$$
$$\sum_{n=1}^{6} \mathrm{card}(\{j \mid f_i > f'_j\}) - \sum_{n=1}^{6} \mathrm{card}(\{j \mid f'_i > f_j\})$$
$$14 - 13 = 1 > 0 \implies P_{30} > P_{14}$$
$$\implies P_{30} \nsim P_{14}$$

Thus, $P_{30} \sim P_8$ and $P_8 \sim P_{14} \not\Rightarrow P_{30} \sim P_{14}$ since $P_{30} \nsim P_{14}$ because $P_{30} > P_{14}$. Therefore " $\sim$ " is intransitive.

3.5 Rolling twice

Dr. Eugene Curtin posed the question: If we roll each die twice, how will this affect our "best" die?

Rolling a die twice will result in a face value of x and y where $x = f_i$ for some $i \in [1,6]$ and $y = f_j$ for some $j \in [1,6]$ denoted (x,y). The maximum value of the two rolls will be taken as the score to compare to other dice i.e. $max\{x,y\}$. By taking the largest die, D_1 from the lexicographic ordering of the dice with fixed sum of 21, we obtain the following results:

$D_1 = (6,6,6,1,1,1)$ rolled twice has 36 cases where the first roll has 6 cases and the second roll similarly has 6 cases resulting in $6 \cdot 6 = 36$ cases. We see rolling a die twice results in $(a,b) : a \in \{6,6,6,1,1,1\}$, $b \in \{6,6,6,1,1,1\}$.The possible scores from D_1 are $(1,1)$, $(1,6)$, $(6,1)$, and $(6,6)$. We see the $max\{x,y\}$ for D_1 can only result in 1 or 6. Thus, we obtain a score of 1 or 6. To obtain a score of 1 can be resulted from 9 cases since there are 3 possible values of 1 on the first roll and second roll. Then, we obtain a score of 6 from $36 - 9 = 27$ cases. We see that we have a 25% chance of obtaining a score of 1 and a 75% chance of obtaining a score of 6.

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