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A BREZIS-NIRENBERG PROBLEM ON HYPERBOLIC SPACES

PAULO CÉSAR CARRIÃO, RAQUEL LEHRER,
OLÍMPIO HIROSHI MIYAGAKI, ANDRÉ VICENTE

Abstract. We consider a Brezis-Nirenberg problem on the hyperbolic space $\mathbb{H}^n$. By using the stereographic projection, the problem becomes a singular problem on the boundary of the open ball $B_1(0) \subset \mathbb{R}^n$. Thanks to the Hardy inequality, in a version due to Brezis-Marcus, the difficulty involving singularities can be overcame. We use the mountain pass theorem due to Ambrosetti-Rabinowitz and Brezis-Nirenberg arguments to obtain a nontrivial solution.

1. Introduction

The main purpose of this article is to study the following Brezis-Nirenberg problem on the hyperbolic space $\mathbb{H}^n$, for $n \geq 3$,

$$-\Delta_{\mathbb{H}^n} u = \lambda u^q + u^{2^*-1} \quad \text{in } \mathbb{H}^n, \tag{1.1}$$

where $\lambda > 0$ is a real parameter, $\Delta_{\mathbb{H}^n}$ denotes the Laplace-Beltrami operator on $\mathbb{H}^n$, and $1 < q < 2^* - 1$, where $2^* := \frac{2n}{n-2}$. $\mathbb{H}^n$ is the hyperbolic space defined as

$$\mathbb{H}^n = \{x \in \mathbb{R}^{n+1} : x_1^2 + x_2^2 + \cdots + x_n^2 - x_{n+1}^2 = -1 \text{ and } x_{n+1} > 0\}.$$

The corresponding equation in the Euclidean space arises in geometry and physics problems, and the above equation is a natural generalization of the Brezis-Nirenberg equation, introduced in the beautiful paper [13]. In the past years, many authors have n treated this type of equations, in the Euclidean space, extending or complementing it in several directions. We would like to cite the papers [1, 21, 29], as well as the survey papers [11, 24, 28].

A result similar to the one in [13] for a Euclidean space, was obtained in [25] for the hyperbolic space. More exactly, the author discussed problem (1.1) in a bounded domain of $\mathbb{H}^n$ with $q = 1$ (homogeneous case). Also for the homogeneous case of the above problem, in [20] it were studied the existence and nonexistence of solutions and of an entire solution, i.e. a solution that belongs to the closure of $C_c^\infty(\mathbb{H}^n)$. We would like to mention [4, 7] for the existence of radial solutions, and [17, 18] for sign changing solutions and nondegeneracy properties of solutions.

Some eigenvalue problems in an unbounded domain on the hyperbolic space have been studied in [9], and some supercritical problems in [19]. We also mention the papers [3, 5, 6] which studied problem (1.1) in the sphere $\mathbb{S}^{n-1}$.

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We use the stereographic projection $E:\mathbb{H}^n\to\mathbb{R}^n$, where each point $P'\in\mathbb{H}^n$ is projected to $P\in\mathbb{R}^n$, where P is the intersection of the straight line connecting P' and the point $(0,\dots,0,-1)$. More exactly, we have the explicit projections $G:\mathbb{R}^n\to\mathbb{H}^n$ and $G^{-1}:\mathbb{H}^n\to\mathbb{R}^n$ given by

$$G(x)=(xp(x),(1+|x|^2)p/2),\quad G^{-1}(y)=\frac{1}{y_{n+1}}y,\quad x,y\in\mathbb{R}^n,$$

where $p(x)=\frac{2}{1-|x|^2}$. This projection takes $\mathbb{H}^n$ into the open ball $B_1(0)\subset\mathbb{R}^n$ (see [23, 26]). Considering $B_1(0)$ endowed with the Riemannian metric g given by $g_{ij}=p^2\delta_{ij}$ (see [17, 18, 25]) the gradient and the Laplace-Beltrami operator corresponding to this metric are given by

$$\nabla_{\mathbb{H}^n}u=\frac{\nabla u}{p},\quad \Delta_{\mathbb{H}^n}u=p^{-n}div(p^{n-2}\nabla u)=p^{-2}\Delta+\frac{(n-2)}{p}\langle x,\nabla\rangle.$$

Therefore, if u is a solution of (1.1), then v, defined by $v=p^{\frac{n-2}{2}}u$, satisfies the problem

$$\begin{gathered}-\Delta v+\frac{n(n-2)}{4}p^2v=\lambda p^\alpha v^q+v^{2^*-1},\quad \text{in } B_1(0)\\ v=0,\quad \text{on } \partial B_1(0),\end{gathered}\tag{1.2}$$

where $\alpha=n-(q+1)\frac{n-2}{2}$.

From now on, we will consider $\Omega:=B_1(0)$. We denote by $H^1_{0,r}(\Omega)$ the subspace of $H^1_0(\Omega)$ of the radial functions which is endowed with the norm given by $\|v\|=\|\nabla v\|_2$, where $\|\cdot\|_2$ is the usual norm of $L^2(\Omega)$. Since the Euclidean sphere with center at the origin $0\in\mathbb{R}^N$ is also a hyperbolic sphere with center at the origin $0\in\mathbb{H}^n$, $H^1_{0,r}(\Omega)$ also can be seen as the subspace of $H^1_0(\Omega)$ consisting of hyperbolic radial functions; see [7, Appendix].

We have the following functional $I:H^1_{0,r}(\Omega)\to\mathbb{R}$ associated with problem (1.2),

$$I(v)=\frac12\int_\Omega|\nabla v|^2+\frac{n(n-2)}{8}\int_\Omega p^2v^2-\frac{\lambda}{q+1}\int_\Omega p^\alpha v^{q+1}-\frac{1}{2^*}\int_\Omega v^{2^*},$$

whose Gateaux derivative is

$$I'(v)w=\int_\Omega\nabla v\cdot\nabla w+\frac{n(n-2)}{4}\int_\Omega p^2vw-\lambda\int_\Omega p^\alpha|v|^{q-1}vw-\int_\Omega|v|^{2^*-2}vw.$$

Our main result is the following theorem.

Theorem 1.1. *Problem* (1.1) *has a nontrivial solution* $u\in H^1(\mathbb{H}^n)$*, provided that the following conditions hold:*

- (i) $1<q<2^*-1$, $n\ge4$ *and for all* $\lambda>0$.
- (ii) $3<q<5$, $n=3$ *and for all* $\lambda>0$.
- (iii) $1<q\le3$, $n=3$ *and* λ *sufficiently large.*

In this work we consider a Brezis-Nirenberg problem on the hyperbolic space $\mathbb{H}^n$. To the best of our knowledge, the way to solve this class of problems is to work directly in the hyperbolic space $\mathbb{H}^n$ and/or to use the projection G and to work in a subset of $\mathbb{R}^n$ endowed with the Riemannian metric g. See the references [4, 7, 17, 19, 20]. The main purpose of this work is to show a new way to deal with the problem. We use the stereographic projection, G, to change the original problem in $\mathbb{H}^n$ into the singular problem (1.2) in $B_1(0)$. After this, we work in $B_1(0)$ with the Euclidean metric. Precisely, after applying the stereographic projection, the problem in $\mathbb{H}^n$ becomes a singular problem on the boundary of $B_1(0)$. Therefore, the function p, given by the projection, is considered as a non-constant coefficient. The main difficulty of the paper is control the terms involving p, close to the boundary of $B_1(0)$. Our main tool to overcome this problem is the Hardy inequality, in a version of Brezis-Marcus (see Lemma 2.1). Finally, the criticality of the Sobolev immersion is handled by adapting some arguments made in Brezis-Nirenberg [13], as well as in Miyagaki [21]. Thus, the mountain pass theorem due to Ambrosetti-Rabinowitz is used to obtain a nontrivial solution in $H^1_{0,r}(\Omega)$, the subspace of $H^1_0(\Omega)$ consisting of radial functions. The Principle of Symmetric Criticality of Palais (see [22]) is used to prove that the nontrivial solution is into $H^1_0(\Omega)$.

2. Variational framework

We start with one of main parts of the paper.

Lemma 2.1. *The following two inequalities hold*

$$\int_\Omega p^2h^2 \le C\int_\Omega |\nabla h|^2, \quad \forall h\in H^1_0(\Omega), \tag{2.1}$$

$$\int_\Omega p^\alpha h^{q+1} \le C\Big(\int_\Omega |\nabla h|^2\Big)^{\frac{q+1}{2}}, \quad \forall h\in H^1_0(\Omega) \tag{2.2}$$

Proof. In this proof we use the Hardy inequality (see [12])

$$\int_{\Omega_\beta} \big(\frac{u}{\delta}\big)^2 \le 4\int_{\Omega_\beta} |\nabla u|^2, \quad \forall u\in H^1_0(\Omega), \tag{2.3}$$

where $\Omega_\beta=\{x\in\Omega;\delta(x)<\beta\}$, for β sufficiently small and $\delta(x)=d(x,\partial\Omega)$. If Ω is convex, then the best constant is 4. In our case, we have $\delta(x)=1-|x|$. Thus, taking $h\in H^1_0(\Omega)$ we have

$$\int_{\Omega_\beta} p^2h^2 = \int_{\Omega_\beta}\frac{4h^2}{(1+|x|)^2(1-|x|)^2} \le 4\int_{\Omega_\beta}\frac{h^2}{\delta^2} \le 16\int_{\Omega_\beta}|\nabla h|^2. \tag{2.4}$$

On the other hand, we have

$$\int_{\Omega^c_\beta} p^2h^2 \le C_\beta\int_{\Omega^c_\beta} h^2 \le \frac{C_\beta}{\lambda_1}\int_{\Omega^c_\beta}|\nabla h|^2, \tag{2.5}$$

where Ω^c_β is the complementary set of Ω_β on $B_1(0)$ and λ_1 is the first eigenvalue of the Laplace operator. Therefore, from (2.4) and (2.5) we conclude that (2.1) holds.

Now, we prove the Hardy-Sobolev type inequality

$$\int_{\Omega_\beta}\frac{u^{q+1}}{\delta^\alpha} \le C\Big(\int_{\Omega_\beta}|\nabla u|^2\Big)^{\frac{q+1}{2}}, \quad \forall u\in H^1_0(\Omega). \tag{2.6}$$

Indeed, we have

$$\int_{\Omega_\beta}\frac{u^{q+1}}{\delta^\alpha} = \int_{\Omega_\beta} u^q u^{1-\alpha}\frac{u^\alpha}{\delta^\alpha} \le \Big(\int_{\Omega_\beta} u^{(q+1-\alpha)r}\Big)^{1/r}\Big(\int_{\Omega_\beta}\frac{u^2}{\delta^2}\Big)^{\alpha/2}, \tag{2.7}$$

where $r = 2/(2-\alpha)$. Also, since $(q+1-\alpha)r = 2^*$, we can use (2.3) and the Sobolev immersion to obtain

$$\begin{aligned}\Big(\int_{\Omega_\beta} u^{2^*}\Big)^{1/r}\Big(4\int_{\Omega_\beta}|\nabla u|^2\Big)^{\alpha/2} &\le 4^{\alpha/2}S^{-\frac{2^*}{2r}}\Big(\int_{\Omega_\beta}|\nabla u|^2\Big)^{\frac{2^*}{2r}}\Big(\int_{\Omega_\beta}|\nabla u|^2\Big)^{\alpha/2}\\ &= C\Big(\int_{\Omega_\beta}|\nabla u|^2\Big)^{\frac{q+1}{2}}.\end{aligned} \tag{2.8}$$

Therefore, combining (2.7) and (2.8) we conclude that (2.6) holds. Similarly as what was done for (2.1), we conclude that (2.2) holds. □

Lemma 2.2 (Mountain Pass Geometry). *(a) There exist $\beta > 0$ and $\rho > 0$ such that $I(v) \ge \beta$ when $\|v\| = \rho$.*
(b) $I(tv) \to -\infty$ as $t \to +\infty$, i.e., there exists $e \in H^1_{0,r}(\Omega)$ such that $I(e) < 0$.

Proof. For item (a), we observe that

$$I(v) \ge \frac{1}{2}\int_\Omega |\nabla v|^2 - \frac{\lambda}{q+1}\int_\Omega p^\alpha v^{q+1} - \frac{1}{2^*}\int_\Omega v^{2^*}.$$

Thus, using (2.2) and the Sobolev immersion result, we have

$$I(v) \ge \frac{1}{2}\int_\Omega |\nabla v|^2 - \frac{\lambda C}{q+1}\Big(\int_\Omega |\nabla v|^2\Big)^{\frac{q+1}{2}} - \frac{\tilde{C}}{2^*}\Big(\int_\Omega |\nabla v|^2\Big)^{2^*/2} \ge \beta > 0,$$

for $\|v\| = \rho$ sufficiently small. The proof of item (b) is trivial so we omit it. □

Lemma 2.2 and Ekeland's Variational Principle [2] allow us to use the general minimax principle[27, Theorem 2.9] which gives us a Palais-Smale sequence, $(u_k) \subset H^1_{0,r}(\Omega)$, at the level c, i.e.,

$$I(u_k) \to c \quad \text{and} \quad \|I'(u_k)\|_{H^1_{0,r}(\Omega)^*} \to 0, \tag{2.9}$$

where

$$c = \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} I(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], H^1_{0,r}(\Omega)); \gamma(0) = 0, I(\gamma(1)) < 0\}$.

Lemma 2.3. *The sequence $(u_k) \subset H^1_{0,r}(\Omega)$ defined above is bounded.*

Proof. Since (u_k) is a Palais-Smale sequence at level c, we can assume that

$$I(u_k) - \frac{1}{q+1}I'(u_k)u_k \le c + 1 + \|u_k\|.$$

Therefore,

$$\Big(\frac{1}{2} - \frac{1}{q+1}\Big)\|u_k\|^2 \le c + 1 + \|u_k\|,$$

and the sequence is bounded. □

In the next proof, we follow some arguments from [13, 21]. In [13] the authors considered a problem in a bounded domain of $\mathbb{R}^n$, but without the presence of singularities on the neighbourhood of the boundary. On the other hand, in the present work, unlike in [21], the domain is bounded and we cannot use directly the results of [21]. Therefore, some adaptations are necessary, specially in the proof of the $n = 3$ case.

Lemma 2.4. *We have* $c < \frac{S^{n/2}}{n}$, *where*

$$S := \inf_{u \in H^1_{0,r}(\Omega)} \frac{\int_\Omega |\nabla u|^2}{\big(\int_\Omega u^{2^*}\big)^{2/2^*}}.$$

Proof. First we observe that it is sufficient to show that there exists a $v_0 \in H^1_{0,r}(\Omega)$, $v_0 \neq 0$ such that

$$\sup_{t \geq 0} I(tv_0) < \frac{S^{n/2}}{n}. \tag{2.10}$$

Indeed, observing that $I(tv_0) \to -\infty$ as $t \to \infty$, there exists $R > 0$ such that $I(Rv_0) < 0$. Now, we write $u_1 := Rv_0$, and from Lemma 2.2, we have

$$0 < \beta \leq c = \inf_{\gamma \in \Gamma} \max_{\tau \in [0,1]} I(\gamma(\tau)) \leq \sup_{t \geq 0} I(tv_0) < \frac{S^{n/2}}{n}.$$

Therefore, we are going to prove the existence of a function v_0 such that (2.10) holds.

Let $0 < R < 1$ be fixed, chose in a way that $0 < 2R < 1$, and let $\varphi \in C_0^\infty(\Omega)$ be a cut-off function with support at B_{2R}, such that φ is identically 1 on B_R and $0 \leq \varphi \leq 1$ on B_{2R}, where B_r denotes the ball in $\mathbb{R}^n$ with center at the origin and radius r.

Given $\varepsilon > 0$ we set $\psi_\varepsilon(x) := \varphi(x)\omega_\varepsilon(x)$, where

$$\omega_\varepsilon(x) = (n(n-2)\varepsilon)^{\frac{n-2}{4}} \frac{1}{(\varepsilon + |x|^2)^{\frac{n-2}{2}}},$$

and ω_ε satisfies

$$\int_{\mathbb{R}^n} |\nabla \omega_\varepsilon|^2 = \int_{\mathbb{R}^n} |\omega_\varepsilon|^{2^*} = S^{n/2}. \tag{2.11}$$

From the definition of ω_ε, it can be shown that

$$\int_{B_R} |\nabla \omega_\varepsilon|^2 \leq \int_{B_R} |\omega_\varepsilon|^{2^*}, \tag{2.12}$$

$$\int_{B_1 - B_R} |\nabla \psi_\varepsilon|^2 = O(\varepsilon^{\frac{n-2}{2}}) \quad \text{as } \varepsilon \to 0. \tag{2.13}$$

Now, we define

$$v_\varepsilon := \frac{\psi_\varepsilon}{\big(\int_{B_{2R}} \psi_\varepsilon^{2^*}\big)^{1/2^*}}, \quad X_\varepsilon := \int_{B_1} |\nabla v_\varepsilon|^2.$$

Therefore, as [21], we have

$$X_\varepsilon \leq S + O(\varepsilon^\delta). \tag{2.14}$$

On the other hand, we have

$$\lim_{t \to +\infty} I(tv_\varepsilon) = -\infty, \forall \varepsilon > 0.$$

This implies that there exists $t_\varepsilon > 0$ such that $\sup_{t \geq 0} I(tv_\varepsilon) = I(t_\varepsilon v_\varepsilon)$. Now, we find an estimate for this t_ε. First, we consider the functional J, applied on tv_ε, where J is given by

$$J(tv_\varepsilon) = \frac{t_\varepsilon^2}{2}\Big(X_\varepsilon + \frac{n(n-2)}{4} \int_{B_{2R}} p^2 v_\varepsilon^2\Big) - \frac{t^{2^*}}{2^*} \int_{B_{2R}} v_\varepsilon^{2^*}.$$

Taking the derivative with respect to t and finding its critical points, we obtain

$$t\Big(X_\varepsilon + \frac{n(n-2)}{4}\int_{B_{2R}} p^2 v_\varepsilon^2\Big) - t^{2^*-1}\int_{B_{2R}} v_\varepsilon^{2^*} = 0.$$

Therefore, since $\int_{B_{2R}} v_\varepsilon^{2^*} = 1$, we obtain that

$$t_0 := \Big(X_\varepsilon + \frac{n(n-2)}{4}\int_{B_{2R}} p^2 v_\varepsilon^2\Big)^{\frac{1}{2^*-2}}$$

is the point for which the path $\mu(t) = J(tv_\varepsilon)$ attains its maximum value. Since the functional I differs from the functional J only by the negative term

$$-\frac{\lambda t^{q+1}}{q+1}\int_{B_{2R}} p^\alpha v_\varepsilon^{q+1},$$

we can conclude that the point t_ε for which the path $\gamma(t) = I(tv_\varepsilon)$ attains its maximum satisfies the inequality

$$t_\varepsilon \le t_0.$$

Since the function $t \mapsto \frac{1}{2}t^2 t_0^{2^*-2} - \frac{1}{2^*}t^{2^*}$ is increasing on $[0, t_0)$, and using (2.14) we obtain

$$\begin{aligned} I(t_\varepsilon v_\varepsilon) &= \frac{1}{n}\Big(X_\varepsilon + \frac{n(n-2)}{4}\int_{B_{2R}} p^2 v_\varepsilon^2\Big)^{\frac{2^*}{2^*-2}} - \frac{\lambda t_\varepsilon^{q+1}}{q+1}\int_{B_{2R}} p^\alpha v_\varepsilon^{q+1} \\ &\le \frac{1}{n}\Big(S + O(\varepsilon^\delta + \frac{n(n-2)}{4})\int_{B_{2R}} p^2 v_\varepsilon^2\Big)^{n/2} - \frac{\lambda t_\varepsilon^{q+1}}{q+1}\int_{B_{2R}} p^\alpha v_\varepsilon^{q+1}. \end{aligned}$$

Therefore,

$$I(t_\varepsilon v_\varepsilon) \le \frac{1}{n}\Big(S + O(\varepsilon^\delta) + \frac{n(n-2)}{4}\int_{B_{2R}} p^2 v_\varepsilon^2\Big)^{n/2} - \lambda C_\varepsilon \int_{B_{2R}} p^\alpha v_\varepsilon^{q+1},$$

where $C_\varepsilon = \frac{t_\varepsilon^{q+1}}{q+1}$.

At this point, we can assume that there exists a positive constant C_0 such that $C_\varepsilon \ge C_0 > 0, \forall \varepsilon > 0$. If that was not the case, we could find a sequence $\varepsilon_n \to 0$ as $n \to \infty$, such that $t_{\varepsilon_n} \to 0$ as $n \to \infty$, since $C_\varepsilon \ge 0$. Now, up to a subsequence, that we still denote by ε_n, we have $t_{\varepsilon_n} v_{\varepsilon_n} \to 0$ as $n \to \infty$. Therefore,

$$0 < c \le \sup_{t\ge 0} I(tv_{\varepsilon_n}) = I(t_{\varepsilon_n} v_{\varepsilon_n}) = I(0) = 0,$$

which is a contradiction.

Now, considering the inequality

$$(a+b)^\beta \le a^\beta + \beta(a+b)^{\beta-1}b,$$

for all $\beta \ge 1$ and $a, b > 0$, and observing that $\int_{B_{2R}} p^2 v_\varepsilon^2 < \infty$, we conclude

$$I(t_\varepsilon v_\varepsilon) \le \frac{S^{n/2}}{n} + O(\varepsilon^\delta) + \int_{B_{2R}} \Big(C\frac{n(n-2)}{4} p^2 v_\varepsilon^2 - C_\varepsilon \lambda p^\alpha v_\varepsilon^{q+1}\Big), \tag{2.15}$$

for some constant $C > 0$.

To complete the proof it is necessary to prove that

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^\delta}\int_{B_{2R}} \Big(C\frac{n(n-2)}{4} p^2 v_\varepsilon^2 - C_\varepsilon \lambda p^\alpha v_\varepsilon^{q+1}\Big) = -\infty. \tag{2.16}$$

In fact, assuming that (2.16) is proved, from (2.15) we have

$$I(t_\varepsilon v_\varepsilon) < \frac{S^{n/2}}{n},$$

for some $\varepsilon > 0$ sufficiently small, and the proof is complete.

Now, we prove (2.16). As in [13], we obtain

$$\int_{B_{2R}} |\psi_\varepsilon|^{2^*} = (n(n-2))^{n/2} \int_{\mathbb{R}^n} \frac{dx}{(1+|x|^2)^n} + O(\varepsilon^{n/2}). \tag{2.17}$$

So, it is sufficient to show that

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^\delta}\Big(\int_{B_R} \Big(C\frac{n(n-2)}{4} p^2\omega_\varepsilon^2 - C_\varepsilon \lambda p^\alpha \omega_\varepsilon^{q+1}\Big)\Big) = -\infty, \tag{2.18}$$

$$\int_{B_{2R}-B_R} \Big(C\frac{n(n-2)}{4} p^2 v_\varepsilon^2 - C_\varepsilon \lambda p^\alpha v_\varepsilon^{q+1}\Big) = O(\varepsilon^\delta). \tag{2.19}$$

First, we will consider (2.18) and recalling that $\delta = \frac{n-2}{2}$, we have

$$\begin{aligned} I_\varepsilon &= \frac{1}{\varepsilon^\delta}\int_{B_R} \Big(C\frac{n(n-2)}{4} p^2\omega_\varepsilon^2 - C_\varepsilon \lambda p^\alpha \omega_\varepsilon^{q+1}\Big) \\ &= C\int_{B_R} \Big(\frac{2}{1-|x|^2}\Big)^2 \frac{1}{(\varepsilon+|x|^2)^{n-2}} \\ &\quad - \lambda C \varepsilon^{\delta\frac{(q-1)}{2}} \int_{B_R} \Big(\frac{2}{1-|x|^2}\Big)^\alpha \frac{1}{(\varepsilon+|x|^2)^{\delta(q+1)}} \\ &= I_1 - I_2. \end{aligned} \tag{2.20}$$

We observe that on B_R,

$$2 < \frac{2}{1-|x|^2} \le \frac{2}{1-R^2}. \tag{2.21}$$

Therefore, making the change of variables $x = \varepsilon^{1/2}y$ and later using polar coordinates, we obtain

$$\begin{aligned} I_1 &\le C\frac{4}{(1-R^2)^2}\int_{B_R} \frac{1}{(\varepsilon+|x|^2)^{n-2}} \\ &= \frac{4C}{(1-R^2)^2}\int_{B_{R\varepsilon^{-1/2}}} \frac{\varepsilon^{n/2}}{(\varepsilon+\varepsilon|y|^2)^{n-2}} \\ &= \frac{4C}{(1-R^2)^2}\omega\varepsilon^{1-\delta}\int_0^{R\varepsilon^{-1/2}} \frac{r^{n-1}}{(1+r^2)^{n-2}}dr. \end{aligned} \tag{2.22}$$

Now, for I_2, considering again (2.21), the change of variables $x = \varepsilon^{1/2}y$ and later the change for polar coordinates, we have

$$\begin{aligned} I_2 &\ge \lambda C\varepsilon^{\delta\frac{(q-1)}{2}} \int_{B_R} \Big(\frac{2}{1-|x|^2}\Big)^\alpha \frac{1}{(\varepsilon+|x|^2)^{\delta(q+1)}} \\ &\ge \lambda C\varepsilon^{\delta\frac{(q-1)}{2}} 2^\alpha \int_{B_{R\varepsilon^{-1/2}}} \frac{\varepsilon^{n/2}}{(\varepsilon+\varepsilon|y|^2)^{\delta(q+1)}} \\ &= \lambda C\omega\varepsilon^{-\delta\frac{(q+1)}{2}+1}\int_0^{R\varepsilon^{-1/2}} \frac{r^{n-1}}{(1+r^2)^{\delta(q+1)}}dr. \end{aligned} \tag{2.23}$$

Thus, combining (2.20), (2.22) and (2.23) we obtain

$$
\begin{aligned}
I_\varepsilon &\le C\varepsilon^{1-\delta}\int_0^{R\varepsilon^{-1/2}} \frac{r^{n-1}}{(1+r^2)^{n-2}}dr \\
&\quad - \lambda C\varepsilon^{-\delta\frac{(q+1)}{2}+1}\int_0^{R\varepsilon^{-1/2}} \frac{r^{n-1}}{(1+r^2)^{\delta(q+1)}}dr \\
&= I_3 - I_4.
\end{aligned}
\tag{2.24}
$$

At this point we divide our proof into three cases: $n \ge 5$, $n = 4$ and $n = 3$.
Case $n \ge 5$. We observe that

$$
I_3 \le C\varepsilon^{1-\delta}\int_0^{\infty} \frac{r^{n-1}}{(1+r^2)^{n-2}}dr.
$$

Since the integral $\int_0^\infty \frac{r^{n-1}}{(1+r^2)^{n-2}}dr$ is convergent if $n \ge 5$, we conclude

$$
I_3 \le \frac{C}{(1-R^2)^2}\varepsilon^{1-\delta}. \tag{2.25}
$$

Again, since the integral $\int_0^\infty \frac{r^{n-1}}{(1+r^2)^{n-2}}dr$ converges when $n \ge 5$ and $q > 1$, it follows that we have the estimate

$$
\int_0^{R\varepsilon^{-1/2}} \frac{r^{n-1}}{(1+r^2)^{\delta(q+1)}}dr \ge \frac{C}{2}.
$$

Then there exists a constant $C > 0$ such that

$$
I_4 \ge C\varepsilon^{-\delta\frac{(q+1)}{2}+1}. \tag{2.26}
$$

Thus, with the estimates (2.24), (2.25) and (2.26) we obtain

$$
I_\varepsilon \le C\varepsilon^{1-\delta}\Big(1-\varepsilon^{-\delta\frac{(q-1)}{2}}
$$

Now, observing that $q > 1$ and taking the limit when $\varepsilon \to 0$, we obtain (2.18), since the exponent $-\delta\frac{(q-1)}{2}$ is negative.
Case $n = 4$. Since $\delta = 1$ and $q + 1 < 4 = 2^*$, from (2.24) we obtain

$$
I_\varepsilon \le C\int_0^{R\varepsilon^{-1/2}} \frac{r^3}{(1+r^2)^2}dr - C\varepsilon^{-\frac{q}{2}+\frac{1}{2}}\int_0^{R\varepsilon^{-1/2}} \frac{r^3}{(1+r^2)^4}dr.
$$

Observing that

$$
\begin{gathered}
\int_0^{R\varepsilon^{-1/2}} \frac{r^3}{(1+r^2)^2}dr = \ln\Big(1+\frac{R^2}{\varepsilon}\Big) + \frac{\varepsilon}{\varepsilon+R^2} - 1, \\
\int_0^{R\varepsilon^{-1/2}} \frac{r^3}{(1+r^2)^4}dr = \frac{-\varepsilon^2(\varepsilon+3R^2)}{12(\varepsilon+R^2)^3} + \frac{1}{12} := a(\varepsilon),
\end{gathered}
$$

we infer that

$$
I_\varepsilon \le C\ln\Big(1+\frac{R^2}{\varepsilon}\Big)\Big[1-a(\varepsilon)b(\varepsilon)\Big] + c(\varepsilon),
$$

where

$$
b(\varepsilon) = \frac{\varepsilon^{-\frac{q}{2}+\frac{1}{2}}}{\ln\big(1+\frac{R^2}{\varepsilon}\big)} \quad \text{and} \quad c(\varepsilon) = C\Big(\frac{\varepsilon}{\varepsilon+R^2} - 1\Big).
$$

As $\lim_{\varepsilon\to 0^+} a(\varepsilon) = \frac{1}{12}$, $\lim_{\varepsilon\to 0^+} b(\varepsilon) = \infty$ and $\lim_{\varepsilon\to 0^+} c(\varepsilon) = -C$ we conclude that (2.18) holds.

Case $n = 3$. Since $\delta = \frac{1}{2}$, from (2.24) we infer that

$$\begin{aligned} I_\varepsilon &\le C\varepsilon^{1/2} \int_0^{R\varepsilon^{-1/2}} \frac{r^2}{1+r^2} dr - \lambda C\varepsilon^{-\frac{q-1}{4}} \int_0^{R\varepsilon^{-1/2}} \frac{r^2}{(1+r^2)^{\frac{q+1}{2}}} dr \\ &\le C - C\varepsilon^{1/2} \tan^{-1}(R\varepsilon^{-1/2}) - \lambda C\varepsilon^{-\frac{q-1}{4}} \int_0^{R\varepsilon^{-1/2}} \frac{r^2}{(1+r^2)^{\frac{q+1}{2}}} dr. \end{aligned} \tag{2.27}$$

Now, if $q > 3$, then the integral in (2.27) converges and, as $\lim_{\varepsilon\to 0^+} \varepsilon^{-\frac{q-1}{4}} = \infty$, we conclude that (2.18) holds. If $1 < q \le 3$, then

$$\int_0^{R\varepsilon^{-1/2}} \frac{r^2}{(1+r^2)^{\frac{q+1}{2}}} dr \ge \int_0^{R\varepsilon^{-1/2}} \frac{1}{1+r^2} dr \ge C > 0,$$

for all $\varepsilon < \varepsilon_0$, with ε_0 small enough. Therefore, taking $\lambda = \varepsilon^{-\frac{1}{2}}$, we conclude that (2.18) also holds in this case. Therefore, we can conclude that (2.18) is true for $n \ge 3$.

Now, we prove (2.19), for $n \ge 3$. First, we observe that we can find fix a $\varepsilon > 0$ sufficiently small such that $O(\varepsilon^\delta) + \varepsilon^\delta I_\varepsilon < 0$. From (2.17) we obtain

$$\frac{1}{\varepsilon^\delta} \int_{B_{2R}-B_R} \Big(C\frac{n(n-2)}{4} p^2 v_\varepsilon^2 - \lambda C_\varepsilon p^\alpha v_\varepsilon^{q+1} \Big) \le \frac{C}{\varepsilon^\delta} \int_{B_{2R}-B_R} p^2 \varphi^2 \omega_\varepsilon^2.$$

We define $\Theta = B_{2R} - B_R$. Since $R \le |x| \le 2R$, we have

$$\frac{2}{1-R^2} \le p(x) \le \frac{2}{1-4R^2};$$

therefore

$$I_5 := \frac{C}{\varepsilon^\delta} \int_\Theta p^2 \varphi^2 \omega_\varepsilon^2 \le \frac{4C}{\varepsilon^\delta (1-4R^2)^2} \int_\Theta \varphi^2 \frac{\varepsilon^{\frac{n-2}{2}}}{(\varepsilon + |x|^2)^{n-2}}.$$

Making the change of variables $x = \varepsilon^{1/2} y$ and later changing to polar coordinates we obtain

$$\begin{aligned} I_5 &\le \frac{4C}{(1-4R^2)^2} \int_{\Theta'} \varphi^2(\varepsilon^{1/2} y) \frac{\varepsilon^{n/2}}{(\varepsilon + \varepsilon|y|^2)^{n-2}} \\ &\le \frac{4C\omega\varepsilon^{n/2}}{(1-4R^2)^2 \varepsilon^{n-2}} \int_{R\varepsilon^{-1/2}}^{2R\varepsilon^{-1/2}} \frac{r^{n-1}}{(1+r^2)^{n-2}} dr \\ &:= \frac{C\varepsilon^{n/2}}{(1-R^2)^2 \varepsilon^{n-2}} I_6, \end{aligned}$$

where $\Theta' = B_{2R\varepsilon^{-1/2}} - B_{R\varepsilon^{-1/2}}$.

By the Mean Value Theorem for integrals, there exists $r_0 \in [R\varepsilon^{-1/2}, 2R\varepsilon^{-1/2}]$ such that

$$I_6 = \frac{R r_0^{n-1} \varepsilon^{-1/2}}{(1+r_0^2)^{n-2}} \le \frac{2^{n-1} R^n \varepsilon^{-\frac{n-1}{2}} \varepsilon^{-1/2}}{(1+\frac{R^2}{\varepsilon})^{n-2}}.$$

Thus,

$$I_5 \le \frac{C\varepsilon^{n/2}}{(1-4R^2)^2 \varepsilon^{n-2}} \frac{2^{n-1} R^n \varepsilon^{-\frac{n-1}{2}} \varepsilon^{-1/2}}{(1+\frac{R^2}{\varepsilon})^{n-2}} = \frac{C(R)}{(\varepsilon + R^2)^{n-2}}.$$

Since $0 < \varepsilon \le 1$, we have

$$\frac{1}{(1+R^2)^{n-1}} \le \frac{1}{(\varepsilon+R^2)^{n-2}} \le \frac{1}{R^{2(n-2)}}.$$

Therefore,

$$I_5 \le \frac{C(R)}{R^{2(n-2)}},$$

and this allows us to complete the proof. □

3. Proof of the main result

To prove the main result we need the following lemma which is inspired by [7, Theorem 3.1]; see also [14].

Lemma 3.1. *Let (u_k) be a sequence in $H^1_{0,r}(\Omega)$ such that $\|u_k\| \le M$, for all $k \in \mathbb{N}$. Then*

(a) *there exists a constant K, independent of k, such that*

$$|u_k(|x|)| \le \frac{K}{|x|^{n/2}}\Big(\frac{1-|x|^2}{2}\Big)^{1/2}, \quad a.\ e.\ in\ \Omega.$$

(b) *If $u_k \rightharpoonup u$ in $H^1_{0,r}(\Omega)$, then $\int_\Omega p^\alpha u_k^{q+1}\,dx \to \int_\Omega p^\alpha u^{q+1}\,dx$, where $1 < q < 2^* - 1$.*

Proof. From polar coordinates, for each $k \in \mathbb{N}$, we have

$$\int_\Omega |\nabla u_k|^2\,dx = w_{n-1}\int_0^1 (u_k'(s))^2 s^{n-1}\,ds,$$

where w_{n-1} is the surface area of S^{n-1}. Thus, from Hölder inequality,

$$\begin{aligned} |u_k(|x|)| &= -\int_{|x|}^1 u_k'(s)\,ds \le \Big(\int_0^1 (u_k'(s))^2 s^{n-1}\,ds\Big)^{1/2}\Big(\int_{|x|}^1 s^{-(n-1)}\,ds\Big)^{1/2} \\ &\le \frac{w_{n-1}^{-\frac12}}{|x|^{n/2}}\|u_k\|\Big(\frac{1-|x|^2}{2}\Big)^{1/2}, \end{aligned}$$

which proves item (a). To prove item (b), we observe that

$$\int_\Omega p^\alpha u_k^{q+1}\,dx = \int_{\Omega_1} p^\alpha u_k^{q+1}\,dx + \int_{\Omega_2} p^\alpha u_k^{q+1}\,dx := I_k^1 + I_k^2, \tag{3.1}$$

where $\Omega_1 = \{x; |x| \le \frac23\}$ and $\Omega_2 = \Omega \cap \{x; |x| > \frac23\}$. As p is bounded in $\{x; |x| \le \frac23\}$ and $q < 2^* - 1$, Rellich's Theorem gives us the convergence of I_k^1.

To prove the convergence of I_k^2 we use the Dominated Convergence Theorem of Lebesgue. From the assumption, we have

$$p^\alpha u_k \to p^\alpha u \quad \text{a.e. in } \Omega_2. \tag{3.2}$$

On the other hand, by item (a), we observe that

$$p^\alpha |u(|x|)|^{q+1} \le C\Big(\frac{1-|x|^2}{2}\Big)^\beta, \tag{3.3}$$

a. e. in Ω_2, where $\beta = -n + \frac{q+1}{2}(n-1)$. We have

$$\int_{\Omega_2}\Big(\frac{1-|x|^2}{2}\Big)^\beta dx = w_{n-1}\int_{\frac23}^1 \Big(\frac{1-s^2}{2}\Big)^\beta s^{n-1}\,ds \le w_{n-1}\int_0^{\frac{5}{18}} z^\beta\,dz. \tag{3.4}$$

As $q > 1$, we obtain that $\beta+1 > 0$, thus the last integral of (3.4) converges. Therefore, (3.2)–(3.4) and Dominated Convergence Theorem give us the convergence of I_k^2, which concludes the proof. □

Now, we can prove the main result.

Proof of Theorem 1.1. By Lemma 2.3, we have that the sequence (u_k) is bounded, i.e., there exists a constant $C > 0$ such that

$$\|u_k\| \leq C, \forall k \in \mathbb{N}. \tag{3.5}$$

Then, there exists a subsequence, still denoted by (u_k), such that

$$u_k \rightharpoonup u \quad \text{weakly in } H^1_{0,r}(\Omega). \tag{3.6}$$

By the Sobolev immersion, we obtain that

$$u_k \to u \quad \text{strongly in } L^s(\Omega), 1 < s < 2^*$$

and we find $h \in L^s(\Omega)$ such that, going to a subsequence, if necessary

$$u_k \to u \quad \text{a.e. in } \Omega,$$
$$|u_k| \leq h \text{ a.e. in } \Omega$$

(see [10]). Since (2.9) holds, we have

$$I'(u_k)v = o(1), \quad \forall v \in H^1_{0,r}(\Omega). \tag{3.7}$$

Now, we prove that

$$|I'(u_k)v - I'(u)v| \to 0, \tag{3.8}$$

as $k \to \infty$, for all $v \in C_c^\infty(\Omega)$. In fact, for v fixed, we have

$$\begin{aligned}
&|I'(u_k)v - I'(u)v| \\
&\leq \Big|\int_\Omega (\nabla u_k - \nabla u)\cdot\nabla v\Big| + \frac{n(n-2)}{4}\max_{\text{supp}v} p^2\Big|\int_\Omega (u_k - u)v\Big| \\
&\quad + \lambda \max_{\text{supp } v} p^\alpha \Big|\int_\Omega (|u_k|^{q-1}u_k - |u|^{q-1}u)v\Big| + \Big|\int_\Omega (|u_k|^{2^*-2}u_k - |u|^{2^*-2}u)v\Big| \\
&:= I_7 + I_8 + I_9 + I_{10}.
\end{aligned}$$

From (3.6), $I_7 = o(1)$ and by the Dominated Convergence Theorem, $I_8 = o(1)$ and $I_9 = o(1)$. Now, from the boundedness of (u_k) in $L^{2^*}(\Omega)$, it follows that

$$|u_k|^{2^*-2}u_k \rightharpoonup |u|^{2^*-2}u \quad \text{weakly in } L^{\frac{2^*}{2^*-1}}(\Omega), \tag{3.9}$$

thus $I_{10} = o(1)$. Therefore (3.8) holds. From (3.7) and (3.8) it follows that $I'(u)v = 0$, for all $v \in C^\infty_{c,rad}(\Omega)$. By density we conclude that

$$I'(u)v = 0, \quad \forall v \in H^1_{0,r}(\Omega), \tag{3.10}$$

and u is a critical point of the functional I.

Now, we suppose that $u \equiv 0$. Considering $v = u_k$ in (3.7) we obtain

$$I'(u_k)u_k = \int_\Omega |\nabla u_k|^2 + \frac{n(n-2)}{4}\int_\Omega p^2 u_k^2 - \lambda\int_\Omega p^\alpha u_k^{q+1} - \int_\Omega u_k^{2^*} = o(1). \tag{3.11}$$

As $u \equiv 0$, from (3.6) we have

$$u_k \rightharpoonup 0 \quad \text{weakly in } H^1_{0,r}(\Omega). \tag{3.12}$$

Therefore, (3.12) and Lemma 3.1 give us

$$\int_\Omega p^\alpha u_k^{q+1} \to 0. \tag{3.13}$$

Now, we define

$$L = \lim \int_\Omega u_k^{2^*}. \tag{3.14}$$

From (3.11), (3.13) and (3.14), we have

$$L = \lim \Big(\int_\Omega |\nabla u_k|^2 + \frac{n(n-2)}{4} \int_\Omega p^2 u_k^2 \Big). \tag{3.15}$$

From the definition of S, given by Lemma 2.4, we have

$$\Big(\int_\Omega u_k^{2^*} \Big)^{2/2^*} S \le \int_\Omega |\nabla u_k|^2 \le \int_\Omega |\nabla u_k|^2 + \frac{n(n-2)}{4} \int_\Omega p^2 u_k^2,$$

thus $L^{2/2^*} S \le L$, and this gives us that

$$L \ge S^{n/2}. \tag{3.16}$$

On the other hand, from (2.9), (3.13), (3.14) and (3.15), we infer

$$(\frac{1}{2} - \frac{1}{2^*}) L = \frac{L}{n} = c. \tag{3.17}$$

From (3.16) and (3.17) we obtain $c \ge S^{n/2}/n$, which is a contradiction with Lemma 2.4. Therefore, we conclude that $u \ne 0$.

Now, we follow the ideas in [8, 15, 16] (see also [22]). Since $H^1_{0,r}(\Omega)$ is a closed subspace of $H^1_0(\Omega)$, we can write

$$H^1_0(\Omega) = H^1_{0,r}(\Omega) \oplus H^1_{0,r}(\Omega)^\perp,$$

where $\cdot^\perp$ denotes the orthogonal complement of the space. Therefore, for each $w \in H^1_0(\Omega)$, there exist $\vartheta \in H^1_{0,r}(\Omega)$ and $\vartheta^\perp \in H^1_{0,r}(\Omega)^\perp$ such that

$$w = \vartheta + \vartheta^\perp. \tag{3.18}$$

As $H^1_{0,r}(\Omega)$ is a Hilbert space and $I'(u) \in H^1_{0,r}(\Omega)^*$, from the Riesz Representation Theorem there exists $z \in H^1_{0,r}(\Omega)$ such that

$$I'(u)v = \int_\Omega \nabla z \cdot \nabla v, \quad \text{for all } v \in H^1_{0,r}(\Omega).$$

Thus, as $z \in H^1_{0,r}(\Omega)$ and $\vartheta^\perp \in H^1_{0,r}(\Omega)^\perp$, we have

$$I'(u)\vartheta^\perp = 0. \tag{3.19}$$

From (3.10), (3.18) and (3.19), for each $w \in H^1_0(\Omega)$, we obtain $I'(u)w = I'(u)\vartheta + I'(u)\vartheta^\perp = 0$. This allows us to conclude that u is a critical point of the functional I in $H^1_0(\Omega)$ and consequently a nontrivial weak solution for (1.2). This completes the proof. □

4. Final remarks

Arguing as in [13], we can consider a more general problem involving a lower-order perturbation, namely

$$-\Delta_{\mathbb{H}^n} u = f(u) + u^{2^*-1} \text{ in } \mathbb{H}^n, \tag{4.1}$$

where $f : [0,\infty] \to \mathbb{R}$ is a Caratheodory function satisfying the following conditions:

(1) $f(0) = 0$ and $\lim_{u\to\infty} \frac{f(u)}{u^q} = 0$, and $1 < q < 2^* - 1$.
(2) $\sup_{0\le u\le M} |f(u)| < \infty$ for all $M > 0$.
(3) $F(s) \le \theta s f(s)$, for some $\theta > 2$ for all $s > 0$,
(4) $f(u) \ge 0$ for all $u \ge 0$.

Theorem 4.1. *In addition to assumptions* (1)–(4)*, suppose*

$$\lim_{\epsilon\to 0} \epsilon \int_0^{\epsilon^{-1/2}} F\big[\big(\frac{\epsilon^{-1/2}}{1+s^2}\big)^{\frac{n-2}{2}}\big] s^{n-1} ds = \infty. \tag{4.2}$$

Then, problem (4.1) *has a nontrivial solution* $u \in H^1(\mathbb{H}^n)$.

The proof is made by variational method. First of all, by stereographic projection the problem (4.1) is equivalent to a problem in $B_1(0)$, namely,

$$\begin{gathered} -\Delta v + \frac{n(n-2)}{4} p^2 v = p^{\frac{n+2}{2}} f(p^{\frac{2-n}{2}} v) + v^{2^*-1}, \quad \text{in } B_1(0) \\ v = 0, \quad \text{on } \partial B_1(0), \end{gathered} \tag{4.3}$$

where $v := p^{\frac{n-2}{2}} u$.

The functional $I : H^1_{0,r}(\Omega) \to \mathbb{R}$ associated with problem (4.3) is

$$I(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 + \frac{n(n-2)}{8}\int_\Omega p^2 v^2 - \int_\Omega p^{\frac{n+2}{2}} F(p^{\frac{2-n}{2}} v) - \frac{1}{2^*}\int_\Omega v^{2^*},$$

whose Gateaux derivative is

$$I'(v)w = \int_\Omega \nabla v.\nabla w + \frac{n(n-2)}{4}\int_\Omega p^2 v.w - \int_\Omega p^{\frac{n+2}{2}} f(p^{\frac{2-n}{2}} v)\cdot w - \int_\Omega |v|^{2^*-2} v \cdot w,$$

where $\Omega := B_1(0)$.

As I satisfies the Mountain Pass Geometry, similarly to lemma 2.2, by Ekeland's Variational Principle [2] there exists a sequence $(u_k) \subset H^1_{0,r}(\Omega)$ which is a Palais-Smale sequence at the level c, i.e.,

$$I(u_k) \to c \quad \text{and} \quad \|I'(u_k)\|_{H^1_{0,r}(\Omega)^*} \to 0,$$

where

$$c = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} I(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], H^1_{0,r}(\Omega)); \gamma(0) = 0, I(\gamma(1)) < 0\}$.

The sketch of proof is the following:

(i) (u_k) is bounded in $H^1_{0,r}(\Omega)$ and $u_n \rightharpoonup u$ weakly in $H^1_{0,r}(\Omega)$.
(ii) $0 < c < \frac{S^{n/2}}{n}$.
(iii) u is a nontrivial solution for (4.1), that is, $I'(u)v = 0$ for all $v \in H^1_0(\Omega)$.

The proof of item (i) follows by (f_3). Item (ii) is obtained using assumptions (1)–(4), (4.2) together with the arguments made in Lemma 2.4. Finally, the item (iii) follows by applying the principle of symmetric criticality due to Palais [22].

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Paulo César Carrião

Universidade Federal de Minas Gerais - ICEX - DM, CEP: 31270-901, Belo Horizonte, MG, Brazil

Email address: `pauloceca@gmail.com`

Raquel Lehrer

Universidade Estadual do Oeste do Paraná - CCET, Rua Universitária, 2069, Jd. Universitário, CEP: 85819-110, Cascavel, PR, Brazil

Email address: `rlehrer@gmail.com`

Olímpio Hiroshi Miyagaki

Universidade Federal de Juiz de Fora - DM, CEP: 36036-330, Juiz de Fora, MG, Brazil

Email address: `ohmiyagaki@gmail.com`

André Vicente

Universidade Estadual do Oeste do Paraná - CCET, Rua Universitária, 2069, Jd. Universitário, CEP: 85819-110, Cascavel, PR, Brazil

Email address: `andre.vicente@unioeste.br`