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COUPLED SYSTEMS OF FRACTIONAL DIFFERENTIAL INCLUSIONS WITH COUPLED BOUNDARY CONDITIONS

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Abstract. We investigate the existence of solutions for a boundary-value problem of coupled fractional differential inclusions supplemented with coupled boundary conditions. By applying standard fixed point theorems for multi-valued maps, we derive some existence results for the given problem when the multi-valued maps involved have convex and non-convex values. Several results follow from the ones obtained in this article by specializing the parameters involved in the problem at hand.

1. Introduction

Fractional differential equations arise naturally in the mathematical modeling of several real-world phenomena and have recently gained great importance in view of their varied applications in scientific and engineering problems. Fractional derivatives help to take care of the hereditary properties of processes under investigation and give rise to more realistic models than the ones based on integer-order derivatives. For further details and explanations see, for instance, [2, 24, 31].

Fractional-order boundary value problems (BVPs) have been extensively studied by many researchers. For the recent development of the topic, we refer the reader to a series of articles [4, 17, 21, 28, 37, 39, 40, 42] and the references cited therein. In particular, coupled systems of fractional-order differential equations have attracted special attention in view of their occurrence in the mathematical modeling of physical phenomena like chaos synchronization [18, 20, 41], anomalous diffusion [35], ecological effects [23], disease models [11, 16, 30], etc. For some recent theoretical results on coupled systems of fractional-order differential equations, for example, see [3, 5, 6, 7, 8, 34, 36, 38].

Differential inclusions are found to be of great utility in studying dynamical systems and stochastic processes. Examples include sweeping processes [1, 29, 32], granular systems [33], nonlinear dynamics of wheeled vehicles [9], control problems [26], etc. The details of pressing issues in stochastic processes, control, differential games, optimization and their application in finance, manufacturing, queueing networks, and climate control can be found in the text [25]. For application of fractional differential inclusions in synchronization processes, see [14].

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In this article, motivated by [8], we consider a new boundary value problem of coupled Caputo (Liouville-Caputo) type fractional differential inclusions:

$$\begin{aligned} &{}^cD^{\alpha}x(t) \in F(t, x(t), y(t)), \quad t \in [0, T],\ 1 < \alpha \le 2, \\ &{}^cD^{\beta}y(t) \in G(t, x(t), y(t)), \quad t \in [0, T],\ 1 < \beta \le 2, \end{aligned} \tag{1.1}$$

subject to the coupled boundary conditions:

$$\begin{aligned} x(0) = \nu_1 y(T), \quad x'(0) = \nu_2 y'(T), \\ y(0) = \mu_1 x(T), \quad y'(0) = \mu_2 x'(T), \end{aligned} \tag{1.2}$$

where ${}^cD^{\alpha}, {}^cD^{\beta}$ denote the Caputo fractional derivatives of order α and β respectively, $F, G : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ are given multivalued maps, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, and $\nu_i, \mu_i, i = 1, 2$ are real constants with $\nu_i\mu_i \neq 1$, $i = 1, 2$.

The objective of the present work is to establish existence criteria for solutions of the problem (1.1)-(1.2) for convex and nonconvex valued multivalued maps F and G by applying the standard fixed-point theorems for multivalued maps. The rest of the paper is organized as follows. We present background material about multivalued analysis and fractional calculus in Section 2, while the main results are derived in Section 3. We emphasize that the tools of the fixed point theory employed in our analysis are standard, however their application to systems of fractional differential inclusions is new.

2. Preliminaries

Let us begin this section with some basic concepts of multivalued maps [15, 22].

Let $(\mathcal{X}, \|\cdot\|)$ be a normed space and define $\mathcal{P}_{cl}(\mathcal{X}) = \{\mathcal{Y} \in \mathcal{P}(\mathcal{X}) : \mathcal{Y} \text{ is closed}\}$, $\mathcal{P}_{cp,c}(\mathcal{X}) = \{\mathcal{Y} \in \mathcal{P}(\mathcal{X}) : \mathcal{Y} \text{ is compact and convex}\}$.

A multi-valued map $\mathcal{G} : \mathcal{X} \to \mathcal{P}(\mathcal{X})$ is

(a) convex (closed) valued if $\mathcal{G}(x)$ is convex (closed) for all $x \in \mathcal{X}$;
(b) upper semi-continuous (u.s.c.) on $\mathcal{X}$ if the set $\mathcal{G}(x_0)$ is a nonempty closed subset of $\mathcal{X}$ for each $x_0 \in \mathcal{X}$, and there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$ for each open set N of $\mathcal{X}$ containing $\mathcal{G}(x_0)$;
(c) lower semi-continuous (l.s.c.) if the set $\{y \in X : \mathcal{G}(y) \cap B \neq \emptyset\}$ is open for any open set B in E.
(d) completely continuous if $\mathcal{G}(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in \mathcal{P}_b(\mathcal{X}) = \{\mathcal{Y} \in \mathcal{P}(\mathcal{X}) : \mathcal{Y} \text{ is bounded}\}$.

Remark 2.1. If the multi-valued map $\mathcal{G}$ is completely continuous with nonempty compact values, then $\mathcal{G}$ is u.s.c. if and only if $\mathcal{G}$ has a closed graph, i.e., $x_n \to x_*$, $y_n \to y_*$, $y_n \in \mathcal{G}(x_n)$ imply $y_* \in \mathcal{G}(x_*)$; the set $Gr(\mathcal{G}) = \{(x, y) \in X \times Y, y \in G(x)\}$ defines the graph of $\mathcal{G}$.

A multivalued map $\mathcal{G} : [a, b] \to \mathcal{P}_{cl}(\mathbb{R})$ is said to be measurable if the function $t \mapsto d(y, \mathcal{G}(t)) = \inf\{|y - z| : z \in \mathcal{G}(t)\}$ is measurable for every $y \in \mathbb{R}$.

A multivalued map $\mathcal{G} : [a, b] \times \mathbb{R}^2 \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if (i) $t \mapsto G(t, x, y)$ is measurable for all $x, y \in \mathbb{R}$ and (ii) $(x, y) \mapsto F(t, x, y)$ is upper semicontinuous for almost all $t \in [a, b]$.

Further a Carathéodory function $\mathcal{G}$ is called L^1-Carathéodory if (i) for each $\rho > 0$, there exists $\varphi_\rho \in L^1([a, b], \mathbb{R}^+)$ such that $\|\mathcal{G}(t, x, y)\| = \sup\{|v| : v \in \mathcal{G}(t, x, y)\} \le \varphi_\rho(t)$ for all $x, y \in \mathbb{R}$ with $\|x\|, \|y\| \le \rho$ and for a.e. $t \in [a, b]$.

Next, we recall some basic definitions of fractional calculus.

Definition 2.2. The fractional integral of order r with the lower limit zero for a function f is defined as

$$I^r f(t) = \frac{1}{\Gamma(r)} \int_0^t \frac{f(s)}{(t-s)^{1-r}} ds, \quad t > 0, \quad r > 0,$$

provided the right hand-side is point-wise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function, which is defined by $\Gamma(r) = \int_0^\infty t^{r-1} e^{-t} dt$.

Definition 2.3. The Riemann-Liouville fractional derivative of order $r > 0$, $n-1 < r < n$, $n \in \mathbb{N}$, is defined as

$$D_{0+}^r f(t) = \frac{1}{\Gamma(n-r)} \Big(\frac{d}{dt}\Big)^n \int_0^t (t-s)^{n-r-1} f(s) ds,$$

where the function $f(t)$ has absolutely continuous derivative up to order $(n-1)$.

Definition 2.4. The Caputo derivative of order r for a function $f : [0, \infty) \to \mathbb{R}$ can be written as

$${}^c D_{0+}^r f(t) = D_{0+}^r \Big(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \Big), \quad t > 0, \quad n-1 < r < n.$$

In the rest of this article, we will use ${}^c D^r$ instead of ${}^c D_{0+}^r$ for the sake of convenience.

Remark 2.5. If $f(t) \in C^n[0, \infty)$, then

$${}^c D^r f(t) = \frac{1}{\Gamma(n-r)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{r+1-n}} ds = I^{n-r} f^{(n)}(t), \quad t > 0, \ n-1 < q < n.$$

Now we present an auxiliary lemma which plays a key role in the forthcoming analysis; see [8] for its proof.

Lemma 2.6. *Let* $\phi, h \in C([0,T], \mathbb{R})$ *and* $\nu_i \mu_i \neq 1$, $i = 1, 2$. *Then the solution of the linear fractional differential system*

$$\begin{gathered}
{}^c D^\alpha x(t) = \phi(t), \quad t \in [0,T], \ 1 < \alpha \leq 2, \\
{}^c D^\beta y(t) = h(t), \quad t \in [0,T], \ 1 < \beta \leq 2, \\
x(0) = \nu_1 y(T), \quad x'(0) = \nu_2 y'(T), \\
y(0) = \mu_1 x(T), \quad y'(0) = \mu_2 x'(T),
\end{gathered} \tag{2.1}$$

is equivalent to the system of integral equations

$$\begin{aligned}
& x(t) \\
& = \frac{\mu_2}{1-\nu_2\mu_2} \Big(\frac{\nu_1 T(\mu_1\nu_2 + 1)}{1-\nu_1\mu_1} + \nu_2 t \Big) B_2 + \frac{\nu_2}{1-\nu_2\mu_2} \Big(\frac{T(\mu_1+\mu_2)\nu_1}{1-\nu_1\mu_1} + t \Big) A_2 \\
& \quad + \frac{\nu_1}{1-\nu_1\mu_1} (A_1 + \mu_1 B_1) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi(s) ds
\end{aligned} \tag{2.2}$$

and

$$\begin{aligned} y(t) &= \frac{\mu_2}{1-\nu_2\mu_2}\Big(\frac{T\mu_1(\nu_1+\nu_2)}{1-\nu_1\mu_1}+t\Big)B_2+\frac{\nu_2}{1-\nu_2\mu_2}\Big(\frac{T\mu_1(\nu_1\mu_2+1)}{1-\nu_1\mu_1}+\mu_2 t\Big)A_2 \\ &\quad+\frac{\mu_1}{1-\nu_1\mu_1}(\nu_1A_1+B_1)+\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}h(s)ds, \end{aligned} \tag{2.3}$$

where

$$A_1=\int_0^T\frac{(T-s)^{\beta-1}}{\Gamma(\beta)}h(s)ds,\quad B_1=\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\phi(s)ds, \tag{2.4}$$

$$A_2=\int_0^T\frac{(T-s)^{\beta-2}}{\Gamma(\beta-1)}h(s)ds,\quad B_2=\int_0^T\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}\phi(s)ds. \tag{2.5}$$

Definition 2.7. A function $(x,y)\in C^2([0,T],\mathbb{R})\times C^2([0,T],\mathbb{R})$ is a solution of the coupled system (1.1)-(1.2) if it satisfies the coupled boundary conditions (1.2) and there exist functions $f,g\in L^1([0,T],\mathbb{R})$ such that $f(t)\in F(t,x(t),y(t))$, $g(t)\in G(t,x(t),y(t))$ a.e. on $[0,T]$ and

$$\begin{aligned} x(t) &= \frac{\mu_2}{1-\nu_2\mu_2}\Big(\frac{\nu_1T(\mu_1\nu_2+1)}{1-\nu_1\mu_1}+\nu_2 t\Big)\int_0^T\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s)ds \\ &\quad+\frac{\nu_2}{1-\nu_2\mu_2}\Big(\frac{T(\mu_1+\mu_2)\nu_1}{1-\nu_1\mu_1}+t\Big)\int_0^T\frac{(T-s)^{\beta-2}}{\Gamma(\beta-1)}g(s)ds \\ &\quad+\frac{\nu_1}{1-\nu_1\mu_1}\int_0^T\frac{(T-s)^{\beta-1}}{\Gamma(\beta)}g(s)ds+\frac{\nu_1\mu_1}{1-\nu_1\mu_1}\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)ds \\ &\quad+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)ds \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} y(t) &= \frac{\mu_2}{1-\nu_2\mu_2}\Big(\frac{T\mu_1(\nu_1+\nu_2)}{1-\nu_1\mu_1}+t\Big)\int_0^T\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s)ds \\ &\quad+\frac{\nu_2}{1-\nu_2\mu_2}\Big(\frac{T\mu_1(\nu_1\mu_2+1)}{1-\nu_1\mu_1}+\mu_2 t\Big)\int_0^T\frac{(T-s)^{\beta-2}}{\Gamma(\beta-1)}g(s)ds \\ &\quad+\frac{\nu_1\mu_1}{1-\nu_1\mu_1}\int_0^T\frac{(T-s)^{\beta-1}}{\Gamma(\beta)}g(s)ds \\ &\quad+\frac{\mu_1}{1-\nu_1\mu_1}\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)ds+\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}g(s)ds. \end{aligned} \tag{2.7}$$

3. Main results

Let us introduce the space $X=\{x(t)|x(t)\in C([0,T],\mathbb{R})\}$ endowed with the norm $\|x\|=\sup\{|x(t)|,t\in[0,T]\}$. Obviously $(X,\|\cdot\|)$ is a Banach space. Also the product space $(X\times X,\|(x,y)\|)$ is a Banach space equipped with norm $\|(x,y)\|=\|x\|+\|y\|$.

For each $(x,y)\in X\times X$, define the sets of selections of F,G by

$$S_{F,(x,y)}=\{f\in L^1([0,T],\mathbb{R}):f(t)\in F(t,x(t),y(t))\text{ for a.e. }t\in[0,T]\}$$

and

$$S_{G,(x,y)} = \{g \in L^1([0,T],\mathbb{R}) : g(t) \in G(t,x(t),y(t)) \text{ for a.e. } t \in [0,T]\}.$$

In view of Lemma 2.6, we define the operators $\mathcal{K}_1, \mathcal{K}_2 : X \times X \to \mathcal{P}(X \times X)$ by

$$\begin{aligned}\mathcal{K}_1(x,y) = \big\{&h_1 \in X \times X : \text{ there exist } f \in S_{F,(x,y)}, g \in S_{G,(x,y)} \text{ such that} \\ &h_1(x,y)(t) = Q_1(t,x,y), \forall t \in [0,T]\big\}\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}\mathcal{K}_2(x,y) = \big\{&h_2 \in X \times X : \text{ there exists } f \in S_{F,(x,y)}, g \in S_{G,(x,y)} \text{ such that} \\ &h_2(x,y)(t) = Q_2(t,x,y), \forall t \in [0,T]\big\},\end{aligned} \tag{3.2}$$

where

$$\begin{aligned}Q_1(x,y)(t) &= \frac{\mu_2}{1-\nu_2\mu_2}\Big(\frac{\nu_1 T(\mu_1\nu_2+1)}{1-\nu_1\mu_1} + \nu_2 t\Big)\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s)ds \\ &+ \frac{\nu_2}{1-\nu_2\mu_2}\Big(\frac{T(\mu_1+\mu_2)\nu_1}{1-\nu_1\mu_1} + t\Big)\int_0^T \frac{(T-s)^{\beta-2}}{\Gamma(\beta-1)} g(s)ds \\ &+ \frac{\nu_1}{1-\nu_1\mu_1}\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} g(s)ds \\ &+ \frac{\nu_1\mu_1}{1-\nu_1\mu_1}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds\end{aligned}$$

and

$$\begin{aligned}Q_2(x,y)(t) &= \frac{\mu_2}{1-\nu_2\mu_2}\Big(\frac{T\mu_1(\nu_1+\nu_2)}{1-\nu_1\mu_1} + t\Big)\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s)ds \\ &+ \frac{\nu_2}{1-\nu_2\mu_2}\Big(\frac{T\mu_1(\nu_1\mu_2+1)}{1-\nu_1\mu_1} + \mu_2 t\Big)\int_0^T \frac{(T-s)^{\beta-2}}{\Gamma(\beta-1)} g(s)ds \\ &+ \frac{\nu_1\mu_1}{1-\nu_1\mu_1}\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} g(s)ds \\ &+ \frac{\mu_1}{1-\nu_1\mu_1}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g(s)ds.\end{aligned}$$

Then we define an operator $\mathcal{K} : X \times X \to \mathcal{P}(X \times X)$ by

$$\mathcal{K}(x,y)(t) = \begin{pmatrix}\mathcal{K}_1(x,y)(t) \\ \mathcal{K}_2(x,y)(t)\end{pmatrix},$$

where $\mathcal{K}_1$ and $\mathcal{K}_2$ are respectively defined by (3.1) and (3.2).

For the sake of computational convenience, we set

$$\begin{aligned}M_1 = &\frac{|\mu_2|}{|1-\nu_2\mu_2|}\Big(\frac{|\nu_1|(|\mu_1||\nu_2|+1)}{|1-\nu_1\mu_1|} + |\nu_2|\Big)\frac{T^\alpha}{\Gamma(\alpha)} \\ &+ \Big(\frac{|\nu_1||\mu_1|}{|1-\nu_1\mu_1|} + 1\Big)\frac{T^\alpha}{\Gamma(\alpha+1)},\end{aligned} \tag{3.3}$$

$$M_2 = \frac{|\nu_2|}{|1-\nu_2\mu_2|}\Big(\frac{(|\mu_1|+|\mu_2|)|\nu_1|}{|1-\nu_1\mu_1|} + 1\Big)\frac{T^\beta}{\Gamma(\beta)} + \frac{|\nu_1|}{|1-\nu_1\mu_1|}\frac{T^\beta}{\Gamma(\beta+1)}, \tag{3.4}$$

$$M_3 = \frac{|\mu_2|}{|1-\nu_2\mu_2|}\Big(\frac{|\mu_1|(|\nu_1|+|\nu_2|)}{|1-\nu_1\mu_1|} + 1\Big)\frac{T^\alpha}{\Gamma(\alpha)} + \frac{|\mu_1|}{|1-\nu_1\mu_1|}\frac{T^\alpha}{\Gamma(\alpha+1)}, \tag{3.5}$$

$$
\begin{aligned}
M_4 = & \frac{|\nu_2|}{|1-\nu_2\mu_2|}\Big(\frac{|\mu_1|(|\nu_1||\mu_2|+1)}{|1-\nu_1\mu_1|}+|\mu_2|\Big)\frac{T^\beta}{\Gamma(\beta)} \\
& + \Big(\frac{|\nu_1||\mu_1|}{|1-\nu_1\mu_1|}+1\Big)\frac{T^\beta}{\Gamma(\beta+1)}.
\end{aligned} \tag{3.6}
$$

3.1. **The Carathéodory case.** To prove our first result dealing with the convex values of F and G, we need the following known results.

Lemma 3.1 ([15, Proposition 1.2])**.** *If $\mathcal{G}: X \to \mathcal{P}_{cl}(Y)$ is u.s.c., then $Gr(\mathcal{G})$ is a closed subset of $X \times Y$; i.e., for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{y_n\}_{n\in\mathbb{N}} \subset Y$, if $x_n \to x_*$, $y_n \to y_*$ when $n \to \infty$ and $y_n \in \mathcal{G}(x_n)$, then $y_* \in \mathcal{G}(x_*)$. Conversely, if $\mathcal{G}$ is completely continuous and has a closed graph, then it is upper semi-continuous.*

Lemma 3.2 ([27])**.** *Let X be a separable Banach space. Let $\mathcal{G}: [0,T]\times\mathbb{R}^2 \to \mathcal{P}_{cp,c}(\mathbb{R})$ be an L^1- Carathéodory multivalued map and let χ be a linear continuous mapping from $L^1([0,T],\mathbb{R})$ to $C([0,T],\mathbb{R})$. Then the operator*

$$
\chi \circ S_{\mathcal{G},x}: C([0,T],\mathbb{R}) \to \mathcal{P}_{cp,c}(C([0,T],\mathbb{R})), \quad x \mapsto (\chi\circ S_{\mathcal{G},x})(x) = \chi(S_{\mathcal{G},x})
$$

is a closed graph operator in $C([0,T],\mathbb{R}) \times C([0,T],\mathbb{R})$.

Lemma 3.3 (Nonlinear alternative for Kakutani maps [19])**.** *Let $\mathcal{E}_1$ be a closed convex subset of a Banach space $\mathcal{E}$, and $\mathcal{U}$ be an open subset of $\mathcal{E}_1$ with $0 \in \mathcal{U}$. Suppose that $F: \overline{\mathcal{U}} \to \mathcal{P}_{cp,c}(\mathcal{E}_1)$ is an upper semicontinuous compact map. Then either*

(i) *F has a fixed point in $\overline{\mathcal{U}}$, or*
(ii) *there is a $u \in \partial\mathcal{U}$ and $\lambda \in (0,1)$ with $u \in \lambda F(u)$.*

For the next Theorem we use the following assumptions:

(H1) the maps $F, G: [0,T]\times\mathbb{R}^2 \to \mathcal{P}(\mathbb{R})$ are L^1-Carathéodory and have convex values;
(H2) there exist continuous nondecreasing functions $\psi_1, \psi_2, \phi_1, \phi_2 : [0,\infty) \to (0,\infty)$ and functions $p_1, p_2 \in C([0,T],\mathbb{R}^+)$ such that
$$
\|F(t,x,y)\|_{\mathcal{P}} := \sup\{|f| : f \in F(t,x,y)\} \le p_1(t)[\psi_1(\|x\|) + \phi_1(\|y\|)],
$$
and
$$
\|G(t,x,y)\|_{\mathcal{P}} := \sup\{|g| : g \in G(t,x,y)\} \le p_2(t)[\psi_2(\|x\|) + \phi_2(\|y\|)],
$$
for each $(t,x,y) \in [0,T]\times\mathbb{R}^2$;
(H3) there exists a positive number N such that
$$
\frac{N}{(M_1+M_3)\|p_1\|(\psi_1(N)+\phi_1(N)) + (M_2+M_4)\|p_2\|(\psi_2(N)+\phi_2(N))} > 1,
$$
where M_i $(i=1,2,3,4)$ are given by (3.3)-(3.6).

Theorem 3.4. *Under assumptions* (H1)–(H3)*, the coupled system* (1.1)-(1.2) *has at least one solution on* $[0,T]$.

Proof. Consider the operators $\mathcal{K}_1, \mathcal{K}_2 : X\times X \to \mathcal{P}(X\times X)$ defined by (3.1) and (3.2). From (H_1), it follows that the sets $S_{F,(x,y)}$ and $S_{G,(x,y)}$ are nonempty for each $(x,y) \in X\times X$. Then, for $f \in S_{F,(x,y)}$, $g \in S_{G,(x,y)}$ for $(x,y) \in X\times X$, we have

$$
h_1(x,y)(t) = \frac{\mu_2}{1-\nu_2\mu_2}\Big(\frac{\nu_1 T(\mu_1\nu_2+1)}{1-\nu_1\mu_1}+\nu_2 t\Big)\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s)ds
$$

$$
\begin{aligned}
&+ \frac{\nu_2}{1-\nu_2\mu_2}\Big(\frac{T(\mu_1+\mu_2)\nu_1}{1-\nu_1\mu_1}+t\Big)\int_0^T \frac{(T-s)^{\beta-2}}{\Gamma(\beta-1)}g(s)ds \\
&+ \frac{\nu_1}{1-\nu_1\mu_1}\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)}g(s)ds \\
&+ \frac{\nu_1\mu_1}{1-\nu_1\mu_1}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)ds
\end{aligned}
$$

and

$$
\begin{aligned}
h_2(x,y)(t) = {}& \frac{\mu_2}{1-\nu_2\mu_2}\Big(\frac{T\mu_1(\nu_1+\nu_2)}{1-\nu_1\mu_1}+t\Big)\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s)ds \\
&+ \frac{\nu_2}{1-\nu_2\mu_2}\Big(\frac{T\mu_1(\nu_1\mu_2+1)}{1-\nu_1\mu_1}+\mu_2 t\Big)\int_0^T \frac{(T-s)^{\beta-2}}{\Gamma(\beta-1)}g(s)ds \\
&+ \frac{\nu_1\mu_1}{1-\nu_1\mu_1}\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)}g(s)ds \\
&+ \frac{\mu_1}{1-\nu_1\mu_1}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)ds + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}g(s)ds,
\end{aligned}
$$

where $h_1 \in \mathcal{K}_1(x,y), h_2 \in \mathcal{K}_2(x,y)$ and so $(h_1,h_2) \in \mathcal{K}(x,y)$.

Now we verify that the operator $\mathcal{K}$ satisfies the assumptions of the nonlinear alternative of Leray-Schauder type. It will be done in several steps. In the first step, we show that $\mathcal{K}(x,y)$ is convex valued. Let $(h_i,\bar{h}_i) \in (\mathcal{K}_1,\mathcal{K}_2), i=1,2$. Then there exist $f_i \in S_{F,(x,y)}$, $g_i \in S_{G,(x,y)}$, $i=1,2$ such that, for each $t \in [0,T]$, we have

$$
\begin{aligned}
h_i(t) = {}& \frac{\mu_2}{1-\nu_2\mu_2}\Big(\frac{\nu_1 T(\mu_1\nu_2+1)}{1-\nu_1\mu_1}+\nu_2 t\Big)\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}f_i(s)ds \\
&+ \frac{\nu_2}{1-\nu_2\mu_2}\Big(\frac{T(\mu_1+\mu_2)\nu_1}{1-\nu_1\mu_1}+t\Big)\int_0^T \frac{(T-s)^{\beta-2}}{\Gamma(\beta-1)}g_i(s)ds \\
&+ \frac{\nu_1}{1-\nu_1\mu_1}\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)}g_i(s)ds + \frac{\nu_1\mu_1}{1-\nu_1\mu_1}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}f_i(s)ds \\
&+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f_i(s)ds
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{h}_i(t) = {}& \frac{\mu_2}{1-\nu_2\mu_2}\Big(\frac{T\mu_1(\nu_1+\nu_2)}{1-\nu_1\mu_1}+t\Big)\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}f_i(s)ds \\
&+ \frac{\nu_2}{1-\nu_2\mu_2}\Big(\frac{T\mu_1(\nu_1\mu_2+1)}{1-\nu_1\mu_1}+\mu_2 t\Big)\int_0^T \frac{(T-s)^{\beta-2}}{\Gamma(\beta-1)}g_i(s)ds \\
&+ \frac{\nu_1\mu_1}{1-\nu_1\mu_1}\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)}g_i(s)ds + \frac{\mu_1}{1-\nu_1\mu_1}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}f_i(s)ds \\
&+ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}g_i(s)ds.
\end{aligned}
$$

Let $0 \le \omega \le 1$. Then, for each $t \in [0,T]$, we have

$$[\omega h_1 + (1-\omega)h_2](t)$$

$$
\begin{aligned}
&= \frac{\mu_2}{1-\nu_2\mu_2}\Big(\frac{\nu_1 T(\mu_1\nu_2+1)}{1-\nu_1\mu_1}+\nu_2 t\Big)\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}[\omega f_1(s)+(1-\omega)f_2(s)]ds\\
&\quad + \frac{\nu_2}{1-\nu_2\mu_2}\Big(\frac{T(\mu_1+\mu_2)\nu_1}{1-\nu_1\mu_1}+t\Big)\int_0^T \frac{(T-s)^{\beta-2}}{\Gamma(\beta-1)}[\omega g_1(s)+(1-\omega)g_2(s)]ds\\
&\quad + \frac{\nu_1}{1-\nu_1\mu_1}\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)}[\omega g_1(s)+(1-\omega)g_2(s)]ds\\
&\quad + \frac{\nu_1\mu_1}{1-\nu_1\mu_1}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}[\omega f_1(s)+(1-\omega)f_2(s)]ds\\
&\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[\omega f_1(s)+(1-\omega)f_2(s)]ds
\end{aligned}
$$

and

$$
\begin{aligned}
&[\omega\bar{h}_1+(1-\omega)\bar{h}_2](t)\\
&= \frac{\mu_2}{1-\nu_2\mu_2}\Big(\frac{T\mu_1(\nu_1+\nu_2)}{1-\nu_1\mu_1}+t\Big)\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}[\omega f_1(s)+(1-\omega)f_2(s)]ds\\
&\quad + \frac{\nu_2}{1-\nu_2\mu_2}\Big(\frac{T\mu_1(\nu_1\mu_2+1)}{1-\nu_1\mu_1}+\mu_2 t\Big)\int_0^T \frac{(T-s)^{\beta-2}}{\Gamma(\beta-1)}[\omega g_1(s)+(1-\omega)g_2(s)]ds\\
&\quad + \frac{\nu_1\mu_1}{1-\nu_1\mu_1}\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)}[\omega g_1(s)+(1-\omega)g_2(s)]ds\\
&\quad + \frac{\mu_1}{1-\nu_1\mu_1}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}[\omega f_1(s)+(1-\omega)f_2(s)]ds\\
&\quad + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}[\omega g_1(s)+(1-\omega)g_2(s)]ds.
\end{aligned}
$$

Since F,G are convex valued, we deduce that $S_{F,(x,y)}$, $S_{G,(x,y)}$ are convex valued. Obviously $\omega h_1+(1-\omega)h_2\in\mathcal{K}_1$, $\omega\bar{h}_1+(1-\omega)\bar{h}_2\in\mathcal{K}_2$ and hence $\omega(h_1,\bar{h}_1)+(1-\omega)(h_2,\bar{h}_2)\in\mathcal{K}$.

Now we show that $\mathcal{K}$ maps bounded sets into bounded sets in $X\times X$. For a positive number r, let $B_r=\{(x,y)\in X\times X:\|(x,y)\|\le r\}$ be a bounded set in $X\times X$. Then, there exist $f\in S_{F,(x,y)}, g\in S_{G,(x,y)}$ such that

$$
\begin{aligned}
h_1(x,y)(t) &= \frac{\mu_2}{1-\nu_2\mu_2}\Big(\frac{\nu_1 T(\mu_1\nu_2+1)}{1-\nu_1\mu_1}+\nu_2 t\Big)\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s)ds\\
&\quad + \frac{\nu_2}{1-\nu_2\mu_2}\Big(\frac{T(\mu_1+\mu_2)\nu_1}{1-\nu_1\mu_1}+t\Big)\int_0^T \frac{(T-s)^{\beta-2}}{\Gamma(\beta-1)}g(s)ds\\
&\quad + \frac{\nu_1}{1-\nu_1\mu_1}\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)}g(s)ds + \frac{\nu_1\mu_1}{1-\nu_1\mu_1}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)ds\\
&\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)ds
\end{aligned}
$$

and

$$
\begin{aligned}
h_2(x,y)(t) &= \frac{\mu_2}{1-\nu_2\mu_2}\Big(\frac{T\mu_1(\nu_1+\nu_2)}{1-\nu_1\mu_1}+t\Big)\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s)ds\\
&\quad + \frac{\nu_2}{1-\nu_2\mu_2}\Big(\frac{T\mu_1(\nu_1\mu_2+1)}{1-\nu_1\mu_1}+\mu_2 t\Big)\int_0^T \frac{(T-s)^{\beta-2}}{\Gamma(\beta-1)}g(s)ds
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{\nu_1\mu_1}{1-\nu_1\mu_1}\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} g(s)ds + \frac{\mu_1}{1-\nu_1\mu_1}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds \\
&+ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g(s)ds.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
|h_1(x,y)(t)| &\le \frac{|\mu_2|T}{|1-\nu_2\mu_2|}\Big(\frac{|\nu_1|(|\mu_1||\nu_2|+1)}{|1-\nu_1\mu_1|} + |\nu_2|\Big)\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s)|ds \\
&\quad + \frac{|\nu_2|T}{|1-\nu_2\mu_2|}\Big(\frac{(|\mu_1|+|\mu_2|)|\nu_1|}{|1-\nu_1\mu_1|} + 1\Big)\int_0^T \frac{(T-s)^{\beta-2}}{\Gamma(\beta-1)}|g(s)|ds \\
&\quad + \frac{|\nu_1|}{|1-\nu_1\mu_1|}\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)}|g(s)|ds \\
&\quad + \frac{|\nu_1||\mu_1|}{|1-\nu_1\mu_1|}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s)|ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s)|ds \\
&\le \frac{|\mu_2|}{|1-\nu_2\mu_2|}\Big(\frac{|\nu_1|(|\mu_1||\nu_2|+1)}{|1-\nu_1\mu_1|} + |\nu_2|\Big)\frac{T^\alpha}{\Gamma(\alpha)}\|p_1\|(\psi_1(r)+\phi_1(r)) \\
&\quad + \frac{|\nu_2|}{|1-\nu_2\mu_2|}\Big(\frac{(|\mu_1|+|\mu_2|)|\nu_1|}{|1-\nu_1\mu_1|} + 1\Big)\frac{T^\beta}{\Gamma(\beta)}\|p_2\|(\psi_2(r)+\phi_2(r)) \\
&\quad + \frac{|\nu_1|}{|1-\nu_1\mu_1|}\frac{T^\beta}{\Gamma(\beta+1)}\|p_2\|(\psi_2(r)+\phi_2(r)) \\
&\quad + \frac{|\nu_1||\mu_1|}{|1-\nu_1\mu_1|}\frac{T^\alpha}{\Gamma(\alpha+1)}\|p_1\|(\psi_1(r)+\phi_1(r)) \\
&\quad + \frac{T^\alpha}{\Gamma(\alpha+1)}\|p_1\|(\psi_1(r)+\phi_1(r)) \\
&= M_1\|p_1\|(\psi_1(r)+\phi_1(r)) + M_2\|p_2\|(\psi_2(r)+\phi_2(r))
\end{aligned}
$$

and

$$
\begin{aligned}
|h_2(x,y)(t)| &\le \frac{|\mu_2|T}{|1-\nu_2\mu_2|}\Big(\frac{|\mu_1|(|\nu_1|+|\nu_2|)}{|1-\nu_1\mu_1|} + 1\Big)\frac{T^\alpha}{\Gamma(\alpha)}\|p_1\|(\psi_1(r)+\phi_1(r)) \\
&\quad + \frac{|\nu_2|T}{|1-\nu_2\mu_2|}\Big(\frac{|\mu_1|(|\nu_1||\mu_2|+1)}{|1-\nu_1\mu_1|} + |\mu_2|\Big)\frac{T^\beta}{\Gamma(\beta)}\|p_2\|(\psi_2(r)+\phi_2(r)) \\
&\quad + \frac{|\nu_1||\mu_1|}{|1-\nu_1\mu_1|}\frac{T^\beta}{\Gamma(\beta+1)}\|p_2\|(\psi_2(r)+\phi_2(r)) \\
&\quad + \frac{|\mu_1|}{|1-\nu_1\mu_1|}\frac{T^\alpha}{\Gamma(\alpha+1)}\|p_1\|(\psi_1(r)+\phi_1(r)) \\
&\quad + \frac{T^\beta}{\Gamma(\beta+1)}\|p_2\|(\psi_2(r)+\phi_2(r)) \\
&= M_3\|p_1\|(\psi_1(r)+\phi_1(r)) + M_4\|p_2\|(\psi_2(r)+\phi_2(r)).
\end{aligned}
$$

Thus,

$$
\begin{gathered}
\|h_1(x,y)\| \le M_1\|p_1\|(\psi_1(r)+\phi_1(r)) + M_2\|p_2\|(\psi_2(r)+\phi_2(r)), \\
\|h_2(x,y)\| \le M_3\|p_1\|(\psi_1(r)+\phi_1(r)) + M_4\|p_2\|(\psi_2(r)+\phi_2(r)).
\end{gathered}
$$

Hence we obtain

$$\begin{aligned}
\|(h_1,h_2)\| &= \|h_1(x,y)\| + \|h_2(x,y)\| \\
&\le (M_1+M_3)\|p_1\|(\psi_1(r)+\phi_1(r)) + (M_2+M_4)\|p_2\|(\psi_2(r)+\phi_2(r)) \\
&= \ell \quad \text{(a constant)}.
\end{aligned}$$

Next, we show that $\mathcal{K}$ is equicontinuous. Let $t_1, t_2 \in [0,T]$ with $t_1 < t_2$. Then, there exist $f \in S_{F,(x,y)}, g \in S_{G,(x,y)}$ such that

$$\begin{aligned}
h_1(x,y)(t) &= \frac{\mu_2}{1-\nu_2\mu_2}\Big(\frac{\nu_1 T(\mu_1\nu_2+1)}{1-\nu_1\mu_1} + \nu_2 t\Big)\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s)ds \\
&\quad + \frac{\nu_2}{1-\nu_2\mu_2}\Big(\frac{T(\mu_1+\mu_2)\nu_1}{1-\nu_1\mu_1} + t\Big)\int_0^T \frac{(T-s)^{\beta-2}}{\Gamma(\beta-1)} g(s)ds \\
&\quad + \frac{\nu_1}{1-\nu_1\mu_1}\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} g(s)ds + \frac{\nu_1\mu_1}{1-\nu_1\mu_1}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds \\
&\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds
\end{aligned}$$

and

$$\begin{aligned}
h_2(x,y)(t) &= \frac{\mu_2}{1-\nu_2\mu_2}\Big(\frac{T\mu_1(\nu_1+\nu_2)}{1-\nu_1\mu_1} + t\Big)\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s)ds \\
&\quad + \frac{\nu_2}{1-\nu_2\mu_2}\Big(\frac{T\mu_1(\nu_1\mu_2+1)}{1-\nu_1\mu_1} + \mu_2 t\Big)\int_0^T \frac{(T-s)^{\beta-2}}{\Gamma(\beta-1)} g(s)ds \\
&\quad + \frac{\nu_1\mu_1}{1-\nu_1\mu_1}\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} g(s)ds + \frac{\mu_1}{1-\nu_1\mu_1}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds \\
&\quad + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g(s)ds.
\end{aligned}$$

Then we have

$$\begin{aligned}
&|h_1(x,y)(t_2) - h_1(x,y)(t_1)| \\
&\le \frac{T^\alpha\|p_1\|(\psi_1(r)+\phi_1(r))|\nu_2||\mu_2|}{|1-\nu_2\mu_2|\Gamma(\alpha)}(t_2-t_1) + \frac{T^\beta\|p_2\|(\psi_2(r)+\phi_2(r))|\nu_2|}{|1-\nu_2\mu_2|\Gamma(\beta)}(t_2-t_1) \\
&\quad + \|p_1\|(\psi_1(r)+\phi_1(r))\Big|\frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}ds - \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}ds\Big| \\
&\le \Big[\frac{T^\alpha\|p_1\|(\psi_1(r)+\phi_1(r))|\nu_2||\mu_2|}{|1-\nu_2\mu_2|\Gamma(\alpha)} + \frac{T^\beta\|p_2\|(\psi_2(r)+\phi_2(r))|\nu_2|}{|1-\nu_2\mu_2|\Gamma(\beta)}\Big](t_2-t_1) \\
&\quad + \frac{\|p_1\|(\psi_1(r)+\phi_1(r))}{\Gamma(\alpha+1)}[2(t_2-t_1)^\alpha + |t_2^\alpha - t_1^\alpha|].
\end{aligned}$$

Analogously, we can obtain

$$\begin{aligned}
&|h_2(x,y)(t_2) - h_2(x,y)(t_1)| \\
&\le \Big[\frac{T^\alpha\|p_1\|(\psi_1(r)+\phi_1(r))|\mu_2|}{|1-\nu_2\mu_2|\Gamma(\alpha)} + \frac{T^\beta\|p_2\|(\psi_2(r)+\phi_2(r))|\nu_2||\mu_2|}{|1-\nu_2\mu_2|\Gamma(\beta)}\Big](t_2-t_1) \\
&\quad + \frac{\|p_2\|(\psi_2(r)+\phi_2(r))}{\Gamma(\beta+1)}[2(t_2-t_1)^\beta + |t_2^\beta - t_1^\beta|].
\end{aligned}$$

From the foregoing arguments, it follows that the operator $\mathcal{K}(x,y)$ is equicontinuous, and so it is completely continuous by the Ascoli-Arzelá theorem.

In our next step, we show that the operator $\mathcal{K}(x,y)$ has a closed graph. Let $(x_n,y_n)\to(x_*,y_*)$, $(h_n,\bar{h}_n)\in\mathcal{K}(x_n,y_n)$ and $(h_n,\bar{h}_n)\to(h_*,\bar{h}_*)$, then we need to show $(h_*,\bar{h}_*)\in\mathcal{K}(x_*,y_*)$. Observe that $(h_n,\bar{h}_n)\in\mathcal{K}(x_n,y_n)$ implies that there exist $f_n\in S_{F,(x_n,y_n)}$ and $g_n\in S_{G,(x_n,y_n)}$ such that

$$\begin{aligned}
h_n(x_n,y_n)(t) &= \frac{\mu_2}{1-\nu_2\mu_2}\Big(\frac{\nu_1T(\mu_1\nu_2+1)}{1-\nu_1\mu_1}+\nu_2t\Big)\int_0^T\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}f_n(s)ds\\
&\quad+\frac{\nu_2}{1-\nu_2\mu_2}\Big(\frac{T(\mu_1+\mu_2)\nu_1}{1-\nu_1\mu_1}+t\Big)\int_0^T\frac{(T-s)^{\beta-2}}{\Gamma(\beta-1)}g_n(s)ds\\
&\quad+\frac{\nu_1}{1-\nu_1\mu_1}\int_0^T\frac{(T-s)^{\beta-1}}{\Gamma(\beta)}g_n(s)ds\\
&\quad+\frac{\nu_1\mu_1}{1-\nu_1\mu_1}\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}f_n(s)ds+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f_n(s)ds
\end{aligned}$$

and

$$\begin{aligned}
\bar{h}_n(x_n,y_n)(t) &= \frac{\mu_2}{1-\nu_2\mu_2}\Big(\frac{T\mu_1(\nu_1+\nu_2)}{1-\nu_1\mu_1}+t\Big)\int_0^T\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}f_n(s)ds\\
&\quad+\frac{\nu_2}{1-\nu_2\mu_2}\Big(\frac{T\mu_1(\nu_1\mu_2+1)}{1-\nu_1\mu_1}+\mu_2t\Big)\int_0^T\frac{(T-s)^{\beta-2}}{\Gamma(\beta-1)}g_n(s)ds\\
&\quad+\frac{\nu_1\mu_1}{1-\nu_1\mu_1}\int_0^T\frac{(T-s)^{\beta-1}}{\Gamma(\beta)}g_n(s)ds\\
&\quad+\frac{\mu_1}{1-\nu_1\mu_1}\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}f_n(s)ds+\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}g_n(s)ds.
\end{aligned}$$

Let us consider the continuous linear operators $\Phi_1,\Phi_2: L^1([0,T],X\times X)\to C([0,T],X\times X)$ given by

$$\begin{aligned}
\Phi_1(x,y)(t) &= \frac{\mu_2}{1-\nu_2\mu_2}\Big(\frac{\nu_1T(\mu_1\nu_2+1)}{1-\nu_1\mu_1}+\nu_2t\Big)\int_0^T\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s)ds\\
&\quad+\frac{\nu_2}{1-\nu_2\mu_2}\Big(\frac{T(\mu_1+\mu_2)\nu_1}{1-\nu_1\mu_1}+t\Big)\int_0^T\frac{(T-s)^{\beta-2}}{\Gamma(\beta-1)}g(s)ds\\
&\quad+\frac{\nu_1}{1-\nu_1\mu_1}\int_0^T\frac{(T-s)^{\beta-1}}{\Gamma(\beta)}g(s)ds+\frac{\nu_1\mu_1}{1-\nu_1\mu_1}\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)ds\\
&\quad+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)ds
\end{aligned}$$

and

$$\begin{aligned}
\Phi_2(x,y)(t) &= \frac{\mu_2}{1-\nu_2\mu_2}\Big(\frac{T\mu_1(\nu_1+\nu_2)}{1-\nu_1\mu_1}+t\Big)\int_0^T\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s)ds\\
&\quad+\frac{\nu_2}{1-\nu_2\mu_2}\Big(\frac{T\mu_1(\nu_1\mu_2+1)}{1-\nu_1\mu_1}+\mu_2t\Big)\int_0^T\frac{(T-s)^{\beta-2}}{\Gamma(\beta-1)}g(s)ds\\
&\quad+\frac{\nu_1\mu_1}{1-\nu_1\mu_1}\int_0^T\frac{(T-s)^{\beta-1}}{\Gamma(\beta)}g(s)ds+\frac{\mu_1}{1-\nu_1\mu_1}\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)ds
\end{aligned}$$

$$+\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}g(s)ds.$$

From Lemma 3.2, we know that $(\Phi_1,\Phi_2)\circ(S_F,S_G)$ is a closed graph operator. Further, we have $(h_n,\bar{h}_n)\in(\Phi_1,\Phi_2)\circ(S_{F,(x_n,y_n)},S_{G,(x_n,y_n)})$ for all n. Since $(x_n,y_n)\to(x_*,y_*)$, $(h_n,\bar{h}_n)\to(h_*,\bar{h}_*)$, it follows that $f_*\in S_{F,(x,y)}$ and $g_*\in S_{G,(x,y)}$ such that

$$\begin{aligned}
h_*(x_*,y_*)(t) &= \frac{\mu_2}{1-\nu_2\mu_2}\Big(\frac{\nu_1T(\mu_1\nu_2+1)}{1-\nu_1\mu_1}+\nu_2t\Big)\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}f_*(s)ds\\
&\quad+\frac{\nu_2}{1-\nu_2\mu_2}\Big(\frac{T(\mu_1+\mu_2)\nu_1}{1-\nu_1\mu_1}+t\Big)\int_0^T \frac{(T-s)^{\beta-2}}{\Gamma(\beta-1)}g_*(s)ds\\
&\quad+\frac{\nu_1}{1-\nu_1\mu_1}\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)}g_*(s)ds\\
&\quad+\frac{\nu_1\mu_1}{1-\nu_1\mu_1}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}f_*(s)ds+\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f_*(s)ds
\end{aligned}$$

and

$$\begin{aligned}
\bar{h}_*(x_*,y_*)(t) &= \frac{\mu_2}{1-\nu_2\mu_2}\Big(\frac{T\mu_1(\nu_1+\nu_2)}{1-\nu_1\mu_1}+t\Big)\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}f_*(s)ds\\
&\quad+\frac{\nu_2}{1-\nu_2\mu_2}\Big(\frac{T\mu_1(\nu_1\mu_2+1)}{1-\nu_1\mu_1}+\mu_2t\Big)\int_0^T \frac{(T-s)^{\beta-2}}{\Gamma(\beta-1)}g_*(s)ds\\
&\quad+\frac{\nu_1\mu_1}{1-\nu_1\mu_1}\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)}g_*(s)ds\\
&\quad+\frac{\mu_1}{1-\nu_1\mu_1}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}f_*(s)ds+\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}g_*(s)ds,
\end{aligned}$$

that is, $(h_n,\bar{h}_n)\in\mathcal{K}(x_*,y_*)$.

Finally, we discuss a priori bounds on solutions. Let $(x,y)\in\nu\mathcal{K}(x,y)$ for $\nu\in(0,1)$. Then there exist $f\in S_{F,(x,y)}$ and $g\in S_{G,(x,y)}$ such that

$$\begin{aligned}
x(t) &= \nu\frac{\mu_2}{1-\nu_2\mu_2}\Big(\frac{\nu_1T(\mu_1\nu_2+1)}{1-\nu_1\mu_1}+\nu_2t\Big)\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s)ds\\
&\quad+\nu\frac{\nu_2}{1-\nu_2\mu_2}\Big(\frac{T(\mu_1+\mu_2)\nu_1}{1-\nu_1\mu_1}+t\Big)\int_0^T \frac{(T-s)^{\beta-2}}{\Gamma(\beta-1)}g(s)ds\\
&\quad+\nu\frac{\nu_1}{1-\nu_1\mu_1}\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)}g(s)ds+\frac{\nu_1\mu_1}{1-\nu_1\mu_1}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)ds\\
&\quad+\nu\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)ds
\end{aligned}$$

and

$$\begin{aligned}
y(t) &= \nu\frac{\mu_2}{1-\nu_2\mu_2}\Big(\frac{T\mu_1(\nu_1+\nu_2)}{1-\nu_1\mu_1}+t\Big)\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s)ds\\
&\quad+\nu\frac{\nu_2}{1-\nu_2\mu_2}\Big(\frac{T\mu_1(\nu_1\mu_2+1)}{1-\nu_1\mu_1}+\mu_2t\Big)\int_0^T \frac{(T-s)^{\beta-2}}{\Gamma(\beta-1)}g(s)ds\\
&\quad+\nu\frac{\nu_1\mu_1}{1-\nu_1\mu_1}\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)}g(s)ds+\frac{\mu_1}{1-\nu_1\mu_1}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)ds
\end{aligned}$$

$$+\nu\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g(s)ds.$$

Using the arguments employed in Step 2, for each $t\in[0,T]$, we obtain

$$\|x\| \le M_1\|p_1\|(\psi_1(\|x\|)+\phi_1(\|y\|)) + M_2\|p_2\|(\psi_2(\|x\|)+\phi_2(\|y\|)),$$

and

$$\|y\| \le M_3\|p_1\|(\psi_1(\|x\|)+\phi_1(\|y\|)) + M_4\|p_2\|(\psi_2(\|x\|)+\phi_2(\|y\|)).$$

In consequence, we have

$$\begin{aligned}\|(x,y)\| &= \|x\|+\|y\| \\ &\le (M_1+M_3)\|p_1\|(\psi_1(\|x\|)+\phi_1(\|y\|)) \\ &\quad + (M_2+M_4)\|p_2\|(\psi_2(\|x\|)+\phi_2(\|y\|)),\end{aligned}$$

which implies that

$$\frac{\|(x,y)\|}{(M_1+M_3)\|p_1\|(\psi_1(\|x\|)+\phi_1(\|y\|)) + (M_2+M_4)\|p_2\|(\psi_2(\|x\|)+\phi_2(\|y\|))} \le 1.$$

In view of (H3), there exists N such that $\|(x,y)\|\ne N$. Let us set

$$U=\{(x,y)\in X\times X : \|(x,y)\|<N\}.$$

Note that the operator $\mathcal{K}:\overline{U}\to\mathcal{P}_{cp,cv}(X)\times\mathcal{P}_{cp,cv}(X)$ is upper semicontinuous and completely continuous. From the choice of U, there is no $(x,y)\in\partial U$ such that $(x,y)\in\nu\mathcal{K}(x,y)$ for some $\nu\in(0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type [19], we deduce that $\mathcal{K}$ has a fixed point $(x,y)\in\overline{U}$ which is a solution of the problem (1.1)-(1.2). This completes the proof. □

3.2. **The lower semi-continuous case.** Here we discuss the case when F and G are not necessarily convex valued by applying the nonlinear alternative of Leray Schauder type together with a selection theorem due to Bressan and Colombo [10] for lower semi-continuous maps with decomposable values. Before presenting our main result in this section, we recall some definitions.

(i) $A_1\subset[0,T]\times\mathbb{R}$ is $\mathcal{L}\otimes\mathcal{D}$ measurable if A_1 belongs to the $\sigma-$algebra generated by all sets of the form $\mathcal{J}\times\mathcal{D}$, where $\mathcal{J}$ is Lebesgue measurable in $[0,T]$ and $\mathcal{D}$ is Borel measurable in $\mathbb{R}$.

(ii) $A_2\subset L^1([0,T],\mathbb{R})$ is decomposable if, for all $u,v\in A_2$ and measurable $\mathcal{J}\subset[0,T]=J$, the function $u\chi_{\mathcal{J}}+v\chi_{J-\mathcal{J}}\in A_2$, where $\chi_{\mathcal{J}}$ stands for the characteristic function of $\mathcal{J}$.

Theorem 3.5. *Assume that* (H2), (H3) *and the following condition hold:*

(H4) $F,G:[0,T]\times\mathbb{R}^2\to\mathcal{P}(\mathbb{R})$ *are nonempty compact-valued multivalued maps such that*
  (a) $(t,x,y)\mapsto F(t,x,y)$ *and* $(t,x,y)\mapsto G(t,x,y)$ *are* $\mathcal{L}\otimes\mathcal{D}\otimes\mathcal{D}$ *measurable,*
  (b) $(x,y)\mapsto F(t,x,y)$ *and* $(x,y)\mapsto G(t,x,y)$ *are lower semicontinuous for a.e.* $t\in[0,T]$.

Then the system (1.1)-(1.2) *has at least one solution on* $[0,T]$.

Proof. It follows from (H2) and (H4) that the maps

$$\begin{aligned}&\mathcal{F}_1: X\to\mathcal{P}(L^1([0,T],\mathbb{R})),\quad x\to\mathcal{F}_1(x,y)=S_{F,(x,y)},\\ &\mathcal{F}_2: X\to\mathcal{P}(L^1([0,T],\mathbb{R})),\quad y\to\mathcal{F}_2(x,y)=S_{G,(x,y)},\end{aligned}$$

are lower semicontinuous and have nonempty closed and decomposable values. Then, by selection theorem due to Bressan and Colombo, there exist continuous functions $f : X \to L^1([0,T],\mathbb{R})$ and $g : X \to L^1([0,T],\mathbb{R})$ such that $f \in \mathcal{F}_1(x,y)$ and $g \in \mathcal{F}_2(x,y)$ for all $x, y \in X$. Thus we have $f(t,x(t),y(t)) \in F(t,x(t),y(t))$ and $g(t,x(t),y(t)) \in G(t,x(t),y(t))$ for a.e. $t \in [0,T]$.

Consider the problem

$$\begin{gathered} {}^cD^\alpha x(t) = f(t,x(t),y(t)), \quad t \in [0,T],\ 1 < \alpha \le 2, \\ {}^cD^\beta y(t) = g(t,x(t),y(t)), \quad t \in [0,T],\ 1 < \beta \le 2, \end{gathered} \tag{3.7}$$

subject to the coupled boundary conditions (1.2).

Obviously, if $(x,y) \in X \times X$ is a solution of the system (3.7) with the boundary conditions (1.2), then (x,y) is a solution to the problem (1.1)-(1.2). In order to transform the problem (3.7)-(1.2) into a fixed point problem, we define the operator $\bar{\mathcal{K}} : X \times X \to X \times X$ by

$$\bar{\mathcal{K}}(x,y)(t) = \begin{pmatrix} \bar{\mathcal{K}}_1(x,y)(t) \\ \bar{\mathcal{K}}_2(x,y)(t) \end{pmatrix},$$

where

$$\begin{aligned} \bar{\mathcal{K}}_1(x,y)(t) &= \frac{\mu_2}{1-\nu_2\mu_2}\Big(\frac{\nu_1 T(\mu_1\nu_2+1)}{1-\nu_1\mu_1} + \nu_2 t\Big)\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(t,x(t),y(t))ds \\ &\quad + \frac{\nu_2}{1-\nu_2\mu_2}\Big(\frac{T(\mu_1+\mu_2)\nu_1}{1-\nu_1\mu_1} + t\Big)\int_0^T \frac{(T-s)^{\beta-2}}{\Gamma(\beta-1)} g(t,x(t),y(t))ds \\ &\quad + \frac{\nu_1}{1-\nu_1\mu_1}\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} g(t,x(t),y(t))ds \\ &\quad + \frac{\nu_1\mu_1}{1-\nu_1\mu_1}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(t,x(t),y(t))ds \\ &\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(t,x(t),y(t))ds \end{aligned}$$

and

$$\begin{aligned} \bar{\mathcal{K}}_2(x,y)(t) &= \frac{\mu_2}{1-\nu_2\mu_2}\Big(\frac{T\mu_1(\nu_1+\nu_2)}{1-\nu_1\mu_1} + t\Big)\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(t,x(t),y(t))ds \\ &\quad + \frac{\nu_2}{1-\nu_2\mu_2}\Big(\frac{T\mu_1(\nu_1\mu_2+1)}{1-\nu_1\mu_1} + \mu_2 t\Big)\int_0^T \frac{(T-s)^{\beta-2}}{\Gamma(\beta-1)} g(t,x(t),y(t))ds \\ &\quad + \frac{\nu_1\mu_1}{1-\nu_1\mu_1}\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} g(t,x(t),y(t))ds \\ &\quad + \frac{\mu_1}{1-\nu_1\mu_1}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(t,x(t),y(t))ds \\ &\quad + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g(t,x(t),y(t))ds. \end{aligned}$$

It can easily be shown that $\bar{\mathcal{K}}$ is continuous and completely continuous. The remaining part of the proof is similar to that of Theorem 3.4. So we omit it. This completes the proof. □

3.3. **The Lipschitz case.** This subsection is concerned with the case when the multivalued maps in the system (1.1) have nonconvex values. Let us first recall some auxiliary material.

Let (X, d) be a metric space induced from the normed space $(X; \|\cdot\|)$ and let $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ be defined by

$$H_d(U, V) = \max\{\sup_{u\in U} d(u, V), \sup_{v\in V} d(U, v)\},$$

where $d(U, v) = \inf_{u\in U} d(u, v)$ and $d(u, V) = \inf_{v\in V} d(u, v)$. Then $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space (see [25]).

Definition 3.6. A multivalued operator $\mathcal{G} : X \to \mathcal{P}_{cl}(X)$ is called (i) $\gamma-$Lipschitz if and only if there exists $\gamma > 0$ such that $H_d(\mathcal{G}(a), \mathcal{G}(b)) \le \gamma d(a, b)$ for each $a, b \in X$; and (ii) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

Lemma 3.7 (Covitz and Nadler [13])**.** *Let (X, d) be a complete metric space. If $\mathcal{G} : X \to \mathcal{P}_{cl}(X)$ is a contraction, then the fixed point set of $\mathcal{G}$ is not empty.*

Theorem 3.8. *Assume that the following conditions hold:*

- (H5) *$F, G : [0, T] \times \mathbb{R}^2 \to \mathcal{P}_{cp}(\mathbb{R})$ are such that $F(\cdot, x, y) : [0, T] \to \mathcal{P}_{cp}(\mathbb{R})$ and $G(\cdot, x, y) : [0, T] \to \mathcal{P}_{cp}(\mathbb{R})$ are measurable for all $x, y \in \mathbb{R}$;*
- (H6)
$$H_d(F(t, x, y), F(t, \bar{x}, \bar{y}) \le m_1(t)(|x - \bar{x}| + |y - \bar{y}|)$$
and
$$H_d(G(t, x, y), G(t, \bar{x}, \bar{y}) \le m_2(t)(|x - \bar{x}| + |y - \bar{y}|)$$
for almost all $t \in [0, T]$ and $x, y, \bar{x}, \bar{y} \in \mathbb{R}$ with $m_1, m_2 \in C([0, T], \mathbb{R}^+)$ and $d(0, F(t, 0, 0)) \le m_1(t)$, $d(0, G(t, 0, 0)) \le m_2(t)$ for almost all $t \in [0, T]$.

Then the problem (1.1)-(1.2) *has at least one solution on $[0, T]$ if*

$$(M_1 + M_3)\|m_1\| + (M_2 + M_4)\|m_2\| < 1, \tag{3.8}$$

where M_i $(i = 1, 2, 3, 4)$ are given by (3.3)-(3.6).

Proof. Observe that the sets $S_{F,(x,y)}$ and $S_{G,(x,y)}$ are nonempty for each $(x, y) \in X \times Y$ by the assumption (H_5), so F and G have measurable selections (see [12, Theorem III.6]). Now we show that the operator $\mathcal{K}$ satisfies the hypothesis of Lemma 3.7.

First we show that $\mathcal{K}(x, y) \in \mathcal{P}_{cl}(X) \times \mathcal{P}_{cl}(X)$ for each $(x, y) \in X \times X$. Let $(h_n, \bar{h}_n) \in \mathcal{K}(x_n, y_n)$ such that $(h_n, \bar{h}_n) \to (h, \bar{h})$ in $X \times X$. Then $(h, \bar{h}) \in X \times X$ and there exist $f_n \in S_{F,(x_n,y_n)}$ and $g_n \in S_{G,(x_n,y_n)}$ such that

$$\begin{aligned} h_n(x_n, y_n)(t) &= \frac{\mu_2}{1 - \nu_2\mu_2}\Big(\frac{\nu_1 T(\mu_1\nu_2 + 1)}{1 - \nu_1\mu_1} + \nu_2 t\Big) \int_0^T \frac{(T - s)^{\alpha-2}}{\Gamma(\alpha - 1)} f_n(s)ds \\ &\quad + \frac{\nu_2}{1 - \nu_2\mu_2}\Big(\frac{T(\mu_1 + \mu_2)\nu_1}{1 - \nu_1\mu_1} + t\Big) \int_0^T \frac{(T - s)^{\beta-2}}{\Gamma(\beta - 1)} g_n(s)ds \\ &\quad + \frac{\nu_1}{1 - \nu_1\mu_1} \int_0^T \frac{(T - s)^{\beta-1}}{\Gamma(\beta)} g_n(s)ds \\ &\quad + \frac{\nu_1\mu_1}{1 - \nu_1\mu_1} \int_0^T \frac{(T - s)^{\alpha-1}}{\Gamma(\alpha)} f_n(s)ds + \int_0^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} f_n(s)ds \end{aligned}$$

and

$$
\begin{aligned}
\bar{h}_n(x_n,y_n)(t) = &\frac{\mu_2}{1-\nu_2\mu_2}\Big(\frac{T\mu_1(\nu_1+\nu_2)}{1-\nu_1\mu_1}+t\Big)\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}f_n(s)ds \\
&+\frac{\nu_2}{1-\nu_2\mu_2}\Big(\frac{T\mu_1(\nu_1\mu_2+1)}{1-\nu_1\mu_1}+\mu_2 t\Big)\int_0^T \frac{(T-s)^{\beta-2}}{\Gamma(\beta-1)}g_n(s)ds \\
&+\frac{\nu_1\mu_1}{1-\nu_1\mu_1}\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)}g_n(s)ds \\
&+\frac{\mu_1}{1-\nu_1\mu_1}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}f_n(s)ds+\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}g_n(s)ds.
\end{aligned}
$$

Since F and G have compact values, we pass onto a subsequences (denoted as sequences) to get that f_n and g_n converge to f and g in $L^1([0,T],\mathbb{R})$ respectively. Thus $f\in S_{F,(x,y)}$ and $g\in S_{G,(x,y)}$ for each $t\in[0,T]$ and

$$
\begin{aligned}
&h_n(x_n,y_n)(t)\to h(x,y)(t) \\
&=\frac{\mu_2}{1-\nu_2\mu_2}\Big(\frac{\nu_1 T(\mu_1\nu_2+1)}{1-\nu_1\mu_1}+\nu_2 t\Big)\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s)ds \\
&\quad+\frac{\nu_2}{1-\nu_2\mu_2}\Big(\frac{T(\mu_1+\mu_2)\nu_1}{1-\nu_1\mu_1}+t\Big)\int_0^T \frac{(T-s)^{\beta-2}}{\Gamma(\beta-1)}g(s)ds \\
&\quad+\frac{\nu_1}{1-\nu_1\mu_1}\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)}g_n(s)ds+\frac{\nu_1\mu_1}{1-\nu_1\mu_1}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)ds \\
&\quad+\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)ds
\end{aligned}
$$

and

$$
\begin{aligned}
&\bar{h}_n(x_n,y_n)(t)\to \bar{h}(x,y)(t) \\
&=\frac{\mu_2}{1-\nu_2\mu_2}\Big(\frac{T\mu_1(\nu_1+\nu_2)}{1-\nu_1\mu_1}+t\Big)\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s)ds \\
&\quad+\frac{\nu_2}{1-\nu_2\mu_2}\Big(\frac{T\mu_1(\nu_1\mu_2+1)}{1-\nu_1\mu_1}+\mu_2 t\Big)\int_0^T \frac{(T-s)^{\beta-2}}{\Gamma(\beta-1)}g(s)ds \\
&\quad+\frac{\nu_1\mu_1}{1-\nu_1\mu_1}\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)}g_n(s)ds+\frac{\mu_1}{1-\nu_1\mu_1}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)ds \\
&\quad+\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}g(s)ds.
\end{aligned}
$$

Hence $(h,\bar{h})\in\mathcal{K}$, which implies that $\mathcal{K}$ is closed.

Next we show that there exists $\gamma<1$ such that

$$
H_d(\mathcal{K}(x,y),\mathcal{K}(\bar{x},\bar{y}))\le\gamma(\|x-\bar{x}\|+\|y-\bar{y}\|)\quad\text{for all } x,\bar{x},y,\bar{y}\in X.
$$

Let $(x,\bar{x}),(y,\bar{y})\in X\times X$ and $(h_1,\bar{h_1})\in\mathcal{K}(x,y)$. Then there exists $f_1\in S_{F,(x,y)}$ and $g_1\in S_{G,(x,y)}$ such that, for each $t\in[0,T]$, we have

$$
\begin{aligned}
h_1(x,y)(t)=&\frac{\mu_2}{1-\nu_2\mu_2}\Big(\frac{\nu_1 T(\mu_1\nu_2+1)}{1-\nu_1\mu_1}+\nu_2 t\Big)\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}f_1(s)ds \\
&+\frac{\nu_2}{1-\nu_2\mu_2}\Big(\frac{T(\mu_1+\mu_2)\nu_1}{1-\nu_1\mu_1}+t\Big)\int_0^T \frac{(T-s)^{\beta-2}}{\Gamma(\beta-1)}g_1(s)ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\nu_1}{1-\nu_1\mu_1}\int_0^T\frac{(T-s)^{\beta-1}}{\Gamma(\beta)}g_1(s)ds+\frac{\nu_1\mu_1}{1-\nu_1\mu_1}\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}f_1(s)ds\\
&+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f_1(s)ds
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{h}_1(x,y)(t)=&\frac{\mu_2}{1-\nu_2\mu_2}\Big(\frac{T\mu_1(\nu_1+\nu_2)}{1-\nu_1\mu_1}+t\Big)\int_0^T\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}f_1(s)ds\\
&+\frac{\nu_2}{1-\nu_2\mu_2}\Big(\frac{T\mu_1(\nu_1\mu_2+1)}{1-\nu_1\mu_1}+\mu_2t\Big)\int_0^T\frac{(T-s)^{\beta-2}}{\Gamma(\beta-1)}g_1(s)ds\\
&+\frac{\nu_1\mu_1}{1-\nu_1\mu_1}\int_0^T\frac{(T-s)^{\beta-1}}{\Gamma(\beta)}g_1(s)ds+\frac{\mu_1}{1-\nu_1\mu_1}\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}f_1(s)ds\\
&+\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}g_1(s)ds.
\end{aligned}
$$

By (H6), we have

$$
H_d(F(t,x,y),F(t,\bar{x},\bar{y})\le m_1(t)(|x(t)-\bar{x}(t)|+|y(t)-\bar{y}(t)|),
$$

and

$$
H_d(G(t,x,y),G(t,\bar{x},\bar{y})\le m_2(t)(|x(t)-\bar{x}(t)|+|y(t)-\bar{y}(t)|).
$$

So, there exists $f\in F(t,x(t),y(t))$ and $g\in G(t,x(t),y(t))$ such that

$$
\begin{aligned}
|f_1(t)-f|\le m_1(t)(|x(t)-\bar{x}(t)|+|y(t)-\bar{y}(t)|),\\
|g_1(t)-g|\le m_2(t)(|x(t)-\bar{x}(t)|+|y(t)-\bar{y}(t)|).
\end{aligned}
$$

Define $V_1,V_2:[0,T]\to\mathcal{P}(\mathbb{R})$ by

$$
\begin{aligned}
V_1(t)=\{f\in L^1([0,T],\mathbb{R}):|f_1(t)-f|\le m_1(t)(|x(t)-\bar{x}(t)|+|y(t)-\bar{y}(t)|)\},\\
V_2(t)=\{g\in L^1([0,T],\mathbb{R}):|g_1(t)-g|\le m_2(t)(|x(t)-\bar{x}(t)|+|y(t)-\bar{y}(t)|)\}.
\end{aligned}
$$

Since the multivalued operators $V_1(t)\cap F(t,x(t),y(t))$ and $V_2(t)\cap G(t,x(t),y(t))$ are measurable [12, Proposition III.4], there exists functions $f_2(t),g_2(t)$ which are measurable selections for V_1,V_2 and $f_2(t)\in F(t,x(t),y(t))$, $g_2(t)\in G(t,x(t),y(t))$ such that, for a.e. $t\in[0,T]$, we have

$$
\begin{aligned}
|f_1(t)-f_2(t)|\le m_1(t)(|x(t)-\bar{x}(t)|+|y(t)-\bar{y}(t)|),\\
|g_1(t)-g_2(t)|\le m_g(t)(|x(t)-\bar{x}(t)|+|y(t)-\bar{y}(t)|).
\end{aligned}
$$

Let

$$
\begin{aligned}
h_2(x,y)(t)=&\frac{\mu_2}{1-\nu_2\mu_2}\Big(\frac{\nu_1T(\mu_1\nu_2+1)}{1-\nu_1\mu_1}+\nu_2t\Big)\int_0^T\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}f_2(s)ds\\
&+\frac{\nu_2}{1-\nu_2\mu_2}\Big(\frac{T(\mu_1+\mu_2)\nu_1}{1-\nu_1\mu_1}+t\Big)\int_0^T\frac{(T-s)^{\beta-2}}{\Gamma(\beta-1)}g_2(s)ds\\
&+\frac{\nu_1}{1-\nu_1\mu_1}\int_0^T\frac{(T-s)^{\beta-1}}{\Gamma(\beta)}g_2(s)ds\\
&+\frac{\nu_1\mu_1}{1-\nu_1\mu_1}\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}f_2(s)ds+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f_2(s)ds
\end{aligned}
$$

and

$$\begin{aligned}\bar{h}_2(x,y)(t) &= \frac{\mu_2}{1-\nu_2\mu_2}\Big(\frac{T\mu_1(\nu_1+\nu_2)}{1-\nu_1\mu_1}+t\Big)\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}f_2(s)ds \\ &\quad + \frac{\nu_2}{1-\nu_2\mu_2}\Big(\frac{T\mu_1(\nu_1\mu_2+1)}{1-\nu_1\mu_1}+\mu_2 t\Big)\int_0^T \frac{(T-s)^{\beta-2}}{\Gamma(\beta-1)}g_2(s)ds \\ &\quad + \frac{\nu_1\mu_1}{1-\nu_1\mu_1}\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)}g_2(s)ds \\ &\quad + \frac{\mu_1}{1-\nu_1\mu_1}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}f_2(s)ds + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}g_2(s)ds.\end{aligned}$$

Then

$$\begin{aligned}&|h_1(x,y)(t)-h_2(x,y)(t)| \\ &\le \frac{|\mu_2|T}{|1-\nu_2\mu_2|}\Big(\frac{|\nu_1|(|\mu_1||\nu_2|+1)}{|1-\nu_1\mu_1|}+|\nu_2|\Big)\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f_1(s)-f_2(s)|ds \\ &\quad + \frac{|\nu_2|T}{|1-\nu_2\mu_2|}\Big(\frac{(|\mu_1|+|\mu_2|)|\nu_1|}{|1-\nu_1\mu_1|}+1\Big)\int_0^T \frac{(T-s)^{\beta-2}}{\Gamma(\beta-1)}|g_1(s)-g_2(s)|ds \\ &\quad + \frac{|\nu_1|}{|1-\nu_1\mu_1|}\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)}|g_1(s)-g_2(s)|ds \\ &\quad + \frac{|\nu_1||\mu_1|}{|1-\nu_1\mu_1|}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}|f_1(s)-f_2(s)|ds \\ &\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f_1(s)-f_2(s)|ds \\ &\le \frac{|\mu_2|T}{|1-\nu_2\mu_2|}\Big(\frac{|\nu_1|(|\mu_1||\nu_2|+1)}{|1-\nu_1\mu_1|}+|\nu_2|\Big) \\ &\quad \times \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}m_1(s)(|x(s)-\bar{x}(s)|+|y(s)-\bar{y}(s)|)ds \\ &\quad + \frac{|\nu_2|T}{|1-\nu_2\mu_2|}\Big(\frac{T(\mu_1+\mu_2)\nu_1}{1-\nu_1\mu_1}+t\Big) \\ &\quad \times \int_0^T \frac{(T-s)^{\beta-2}}{\Gamma(\beta-1)}m_2(s)(|x(s)-\bar{x}(s)|+|y(s)-\bar{y}(s)|)ds \\ &\quad + \frac{|\nu_1|}{|1-\nu_1\mu_1|}\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)}m_2(s)(|x(s)-\bar{x}(s)|+|y(s)-\bar{y}(s)|)ds \\ &\quad + \frac{|\nu_1||\mu_1|}{|1-\nu_1\mu_1|}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}m_1(s)(|x(s)-\bar{x}(s)|+|y(s)-\bar{y}(s)|)ds \\ &\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}m_1(s)(|x(s)-\bar{x}(s)|+|y(s)-\bar{y}(s)|)ds \\ &\le (M_1\|m_1\|+M_2\|m_2\|)(\|x-\bar{x}\|+\|y-\bar{y}\|),\end{aligned}$$

which implies

$$\|h_1(x,y)-h_2(x,y)\| \le (M_1\|m_1\|+M_2\|m_2\|)(\|x-\bar{x}\|+\|y-\bar{y}\|).$$

In a similar manner, one can find that

$$\|\bar{h}_1(x,y) - \bar{h}_2(x,y)\| \le (M_3\|m_1\| + M_4\|m_2\|)(\|x-\bar{x}\| + \|y-\bar{y}\|).$$

Thus

$$\|(h_1,\bar{h}_1),(h_2,\bar{h}_2)\| \le [(M_1+M_3)\|m_1\| + (M_2+M_4)\|m_2\|](\|x-\bar{x}\| + \|y-\bar{y}\|).$$

Analogously, interchanging the roles of (x,y) and $(\bar{x},\bar{y})$, we can obtain

$$H_d(\mathcal{K}(x,y),\mathcal{K}(\bar{x},\bar{y})) \le [(M_1+M_3)\|m_1\| + (M_2+M_4)\|m_2\|](\|x-\bar{x}\| + \|y-\bar{y}\|).$$

Therefore $\mathcal{K}$ is a contraction in view of the assumption (3.8). Hence it follows by Lemma 3.7 that $\mathcal{K}$ has a fixed point (x,y), which is a solution of problem (1.1)-(1.2). This completes the proof. □

Special Cases. Several new existence results follow as special cases from the work presented in this paper by fixing the parameters involved in the problem. For instance, if we choose $\nu_1 = 1 = \nu_2$ and $\mu_1 = -1 = \mu_2$ or $\nu_1 = -1 = \nu_2$ and $\mu_1 = 1 = \mu_2$, we obtain the results for coupled fractional differential inclusions equipped with a combination of coupled periodic and anti-periodic boundary conditions of the form: $x(0) = y(T)$, $x'(0) = y'(T)$, $y(0) = -x(T)$, $y'(0) = -x'(T)$ or $x(0) = -y(T)$, $x'(0) = -y'(T)$, $y(0) = x(T)$, $y'(0) = x'(T)$. For $\nu_1 = 1 = -\nu_2$ and $-\mu_1 = 1 = \mu_2$, our results correspond to a coupled system of fractional differential inclusions with the boundary conditions of the form: $x(0) = y(T)$, $x'(0) = -y'(T)$, $y(0) = -x(T)$, $y'(0) = x'(T)$, while the results for a coupled system of fractional differential inclusions supplemented with the boundary conditions: $x(0) = -y(T)$, $x'(0) = y'(T)$, $y(0) = x(T)$, $y'(0) = -x'(T)$ follow by setting $-\nu_1 = 1 = \nu_2$ and $\mu_1 = 1 = -\mu_2$ in the results of this paper. Letting $\nu_1 = 0 = \mu_1$ in the results of this paper, we obtain the ones for a coupled system of fractional differential inclusions equipped with coupled flux type conditions.

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