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CRANK-NICOLSON LEGENDRE SPECTRAL APPROXIMATION FOR SPACE-FRACTIONAL ALLEN-CAHN EQUATION

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Abstract. In this article, we consider spectral methods for solving the initial-boundary value problem of the space fractional-order Allen-Cahn equation. A fully discrete scheme based on the modified Crank-Nicolson scheme in time and the Legendre spectral method in space is established. The existence and uniqueness of the fully discrete scheme are derived, and the stability and convergence analysis of the fully discrete scheme are proved rigorously. By constructing a fractional duality argument, the corresponding optimal error estimates in L^2 and H^α norm are derived, respectively. Also, numerical experiments are performed to support the theoretical results.

1. Introduction

Research of fractional differential equations has been a lively topic in mathematical theory and real applications in the last few decades. For most fractional differential equations, however, we cannot obtain the exact solutions, it is natural to resort to numerical solutions. Up to now, there are several numerical techniques to solve fractional differential equations, such as finite difference methods [8, 17], finite element methods [7, 15, 21], spectral methods [9, 10, 12, 19].

Allen-Cahn equation was introduced in 1979 [2] to model phase transitions in iron alloys, it has become a basic model equation for the diffuse interface approach developed to study phase transitions and interfacial dynamics in materials science. There has been an increasing interests in Allen-Cahn equation from local to memory (time fractional) or nonlocal (space fractional) case [6, 20]. The fractional Allen-Cahn equation replaced the standard temporal or/and spatial integer order differential operator by a corresponding fractional order one, such as Riemann-Liouville, Caputo, Riesz, fractional Laplacian operators, etc.

In this article, we study the spectral approximation to the following space-fractional Allen-Cahn equation (SFACE)

$$\begin{gathered} u_t - \epsilon^2 \mathcal{L}^{(\alpha)} u + f(u) = 0, \quad x \in \Lambda,\ t \in (0,T], \\ u(x,0) = u_0(x), \quad x \in \Lambda, \\ u(\pm 1, t) = 0, \quad t \in [0,T], \end{gathered} \tag{1.1}$$

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where $\Lambda = (-1,1)$, $\alpha \in (1/2,1)$, $u = u(x,t)$ represents the concentration of one of the species of the alloy, the parameter ϵ represents the diffuse interface width, $f(u) = u^3 - u$, the nonlinear term, is the derivative of a free energy double-well potential $F(u) = \frac{1}{4}(u^2-1)^2$. Operator $\mathcal{L}^{(\alpha)}$ in the Riesz case is defined by

$$\mathcal{L}^{(\alpha)}u = \frac{\partial^{2\alpha}u}{\partial|x|^{2\alpha}} = -\frac{1}{2\cos\pi\alpha}\left({}_{-1}D_x^{2\alpha}u + {}_xD_1^{2\alpha}u\right),$$

where ${}_{-1}D_x^{2\alpha}$, ${}_xD_1^{2\alpha}$ represent the left and right Riemann-Liouville (R-L) fractional derivatives operators, respectively. For $n-1<\beta<n$, $n\in\mathbb{N}$, the operators ${}_{-1}D_x^{\beta}$ and ${}_xD_1^{\beta}$ are defined as

$$\begin{aligned}
{}_{-1}D_x^{\beta}u &= \frac{1}{\Gamma(n-\beta)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_{-1}^{x}(x-s)^{n-\beta-1}u(s)\,\mathrm{d}s,\\
{}_xD_1^{\beta}u &= \frac{(-1)^n}{\Gamma(n-\beta)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_{x}^{1}(s-x)^{n-\beta-1}u(s)\,\mathrm{d}s,
\end{aligned} \tag{1.2}$$

where $\Gamma(\cdot)$ is the standard Gamma function.

The fractional Allen-Cahn equation (1.1) can be viewed as an L^2-gradient flow of the following fractional version of Ginzburg-Landau-Wilson free energy functional

$$E(u) = \int_\Lambda \Big(F(u) - \frac{\epsilon^2}{2}u\mathcal{L}^{(\alpha)}u\Big)\mathrm{d}x = \frac{\epsilon^2}{2}|u|_\alpha^2 + \int_\Lambda F(u)\,\mathrm{d}x.$$

Recently, there have been several studies on the SFACE. Akagi, et al. [1] proved the existence and uniqueness of weak solutions to the related initial-boundary value problem of the SFACE after setting a proper functional framework. Hou et al. [8] considered Crank-Nicolson scheme in time and second order central difference approach in space for solving the SFACE with small perturbation and strong nonlinearity, a nonlinear iteration algorithm is proposed and the unique solvability, energy stability and convergence are proved. Burrage et al. [5] solved the SFACE by implicit finite element method on both structured and unstructured grids. Bueno-Orovio et al. [4] provided a numerical algorithm based on Fourier spectral method in space and backward Euler discretization in time to solve the SFACE. However, there is no theoretical analysis has been provided in [4, 5].

In this article, we construct a numerical approach by applying the modified Crank-Nicolson scheme in temporal and the Legendre Galerkin spectral method in spatial discretizations to (1.1). The existence and uniqueness of the fully discrete scheme are proved. The stability and convergence are derived strictly by introducing a fractional duality argument. It will be shown that the convergence rate of the numerical scheme is $\mathrm{O}(\tau^2 + N^{-m})$ in L^2-norm.

The organization of this article is as follows. We commence by reviewing some preliminaries of fractional order functional spaces endowed with inner products and norms, and give some useful lemmas in the next section. In section 3, The fully discrete spectral scheme is constructed by applying Crank-Nicolson difference scheme to temporal discretization and Legendre spectral method to the spatial component, the existence and uniqueness of the fully discrete scheme are derived. In section 4, the stability and convergence analysis of the fully discrete scheme are strictly proved, respectively. We present some numerical experiments in section 5, which support the theoretical results. We conclude by summary and discussion in the last section.

2. Preliminaries

In this section, we introduce some definitions and notation of fractional derivative spaces endowed with inner products and norms, then give some basic properties of fractional derivative and some lemmas, which will be used in the context.

The $L^2(\Lambda)$ inner product is denoted by $(\cdot,\cdot)$ and the $L^p(\Lambda)$ norm by $\|\cdot\|_{L^p}$ with the special case of $L^2(\Lambda)$ and $L^\infty(\Lambda)$ norms being written as $\|\cdot\|$ and $\|\cdot\|_\infty$, respectively. For $k\in\mathbb{N}$, we denote the seminorm and norm associated with the Sobolev space $H^k(\Lambda)$ by $|\cdot|_k$ and $\|\cdot\|_k$, respectively. For nonnegative real number $r\in\mathbb{R}^+\backslash\mathbb{Z}^+$, we use $H^r(\Lambda)$ to denote the fractional Sobolev spaces, the semi-norm $|\cdot|_r$ and norm $\|\cdot\|_r$ will defined below. Let E be a Sobolev space, we define space-time functional space $L^2(0,T;E)$ as

$$L^2(0,T;E) := \Big\{u:(0,T)\mapsto E: \int_0^T \|u\|_E^2\,\mathrm{d}t<\infty,\ u \text{ is measurable}\Big\},$$

and similarly we can define some other spaces for space-time functions. Throughout this article we use C to denote a generic nonnegative constant whose actual value may change from line to line.

Definition 2.1 (see [7, 15])**.** *Let $r>0$. Define the semi-norm*

$$|u|_r = \big\|\,|\omega|^r\hat{u}\big\|_{L^2(\mathbb{R})} = \Big(\int_{\mathbb{R}}|\omega|^{2r}|\hat{u}|^2\,\mathrm{d}\omega\Big)^{1/2},$$

and the norm

$$\|u\|_r = \big(\|u\|^2+|u|_r^2\big)^{1/2},$$

where $\hat{u}$ denote the Fourier transform of u. Define $H_0^r(\Lambda)$ as the closure of $C_0^\infty(\Lambda)$ in $H^r(\Lambda)$ with respect to norm $\|\cdot\|_r$, and use $H^{-r}(\Lambda)$ to denote the dual space of $H_0^r(\Lambda)$, with norm denoted by $\|\cdot\|_{-r}$.

Remark 2.2 (see [15])**.** Let $\tilde{u}$ be the expansion of u by zero outside of Λ, then $|u|_r = |\tilde{u}|_{H^r(\mathbb{R})}$.

Next, we introduce some useful fractional derivative spaces and related properties, which are used in the formulation of the numerical analysis, one can refer to [7, 15] for more details.

Definition 2.3 (see [7, 15])**.** Let $\mu>0$. Define the semi-norm

$$|u|_{J_L^\mu(\Lambda)} = \|{}_{-1}D_x^\mu u\|,$$

and the norm

$$\|u\|_{J_L^\mu(\Lambda)} = \Big(\|u\|^2+|u|^2_{J_L^\mu(\Lambda)}\Big)^{1/2}.$$

Denote $J_{L,0}^\mu(\Lambda)$ as the closure of $C_0^\infty(\Lambda)$ with respect to norm $\|\cdot\|_{J_L^\mu(\Lambda)}$.

Definition 2.4 (see [7, 15])**.** Let $\mu>0$. Define the semi-norm

$$|u|_{J_R^\mu(\Lambda)} = \|{}_xD_1^\mu u\|,$$

and the norm

$$\|u\|_{J_R^\mu(\Lambda)} = \big(\|u\|^2+|u|^2_{J_R^\mu(\Lambda)}\big)^{1/2}.$$

Denote $J_{R,0}^\mu(\Lambda)$ as the closure of $C_0^\infty(\Lambda)$ with respect to norm $\|\cdot\|_{J_R^\mu(\Lambda)}$.

Definition 2.5 (see [7, 15]). Let $\mu > 0$, $\mu \neq n - \frac{1}{2}, n \in \mathbb{N}$. Define the semi-norm

$$|u|_{J_S^\mu(\Lambda)} = \left|({}_{-1}D_x^\mu u,\ {}_xD_1^\mu u)\right|^{1/2},$$

and the norm

$$\|u\|_{J_S^\mu(\Lambda)} = \left(\|u\|^2 + |u|^2_{J_S^\mu(\Lambda)}\right)^{1/2}.$$

Define $J_{S,0}^\mu(\Lambda)$ as the closure of $C_0^\infty(\Lambda)$ with respect to norm $\|\cdot\|_{J_S^\mu(\Lambda)}$.

Remark 2.6. If the domain Λ in definitions 2.3–2.5 replaced by the entire line $\mathbb{R}$, the corresponding semi-norms should be denoted, respectively, by

$$|u|_{J_L^\mu(\mathbb{R})} = \|{}_{-\infty}D_x^\mu u\|_{L^2(\mathbb{R})},$$
$$|u|_{J_R^\mu(\mathbb{R})} = \|{}_xD_\infty^\mu u\|_{L^2(\mathbb{R})},$$
$$|u|_{J_S^\mu(\mathbb{R})} = \left(|({}_{-\infty}D_x^\mu u, {}_xD_\infty^\mu u)|\right)^{1/2}.$$

Let $J_L^\mu(\mathbb{R})$, $J_R^\mu(\mathbb{R})$, $J_S^\mu(\mathbb{R})$, and $H^\mu(\mathbb{R})$ denote the closure of $C_0^\infty(\mathbb{R})$ with respect to $\|u\|_{J_L^\mu(\mathbb{R})}$, $\|u\|_{J_R^\mu(\mathbb{R})}$, $\|u\|_{J_S^\mu(\mathbb{R})}$ and $\|u\|_{H^\mu(\mathbb{R})}$, respectively.

Lemma 2.7 (see [7, 15]). *Let $\mu > 0$, $\mu \neq n - 1/2$, $n \in \mathbb{N}$. Then*

(1) *$J_{L,0}^\mu(\Lambda)$, $J_{R,0}^\mu(\Lambda)$, $J_{S,0}^\mu(\Lambda)$, and $H_0^\mu(\Lambda)$ are equal, with equivalent semi-norms and norms;*
(2) *$J_L^\mu(\mathbb{R})$, $J_R^\mu(\mathbb{R})$, $J_S^\mu(\mathbb{R})$, and $H^\mu(\mathbb{R})$ are equal, with equivalent semi-norms and norms;*
(3) *A function $u \in L^2(\mathbb{R})$ belongs to $J_L^\mu(\mathbb{R})$ if and only if $|\omega|^\mu \hat{u} \in L^2(\mathbb{R})$, specifically $|u|_{J_L^\mu(\mathbb{R})} = \||\omega|^\mu \hat{u}\|_{L^2(\mathbb{R})} = |u|_{H^\mu(\mathbb{R})}$. Similarly, $|u|_{J_R^\mu(\mathbb{R})} = |u|_{H^\mu(\mathbb{R})}$.*

In what follows, we will use $H_0^\alpha(\Lambda)$ uniformly by the equivalent property of $J_{L,0}^\alpha(\Lambda)$, $J_{R,0}^\alpha(\Lambda)$ and $H_0^\alpha(\Lambda)$, and make no distinction between the three of them unless otherwise stated.

Lemma 2.8 (see [7, 15]). *Let $\mu > 0$ be given. Then*

$$\begin{aligned}({}_{-1}D_x^\mu u,\ {}_xD_1^\mu u) &= ({}_{-\infty}D_x^\mu \tilde{u},\ {}_xD_\infty^\mu \tilde{u})\\ &= \cos(\pi\mu)\|{}_{-\infty}D_x^\mu \tilde{u}\|^2_{L^2(\mathbb{R})}\\ &= \cos(\pi\mu)\|{}_xD_\infty^\mu \tilde{u}\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Hence we have the following relations.

Lemma 2.9 (see [7, 15]). *Let $\mu > 0$, $\Lambda = (-1, 1)$, $u \in H_0^\mu(\Lambda)$. Then*

$$({}_{-1}D_x^\mu u,\ {}_xD_1^\mu u) = \cos(\pi\mu)|u|^2_\mu.$$

Proof. We can obtain the result by Remark 2.2 and Lemmas 2.7, 2.8, immediately. □

Via integration by parts, one can verify readily the following result.

Lemma 2.10 (see [16]). *Let $0 < s < 1$, $u \in H_0^{2s}(\Lambda)$, $v \in H_0^s(\Lambda)$. Then we have*

$$({}_{-1}D_x^{2s} u, v) = ({}_{-1}D_x^s u, {}_xD_1^s v), \quad ({}_xD_1^{2s} u, v) = ({}_xD_1^s u, {}_{-1}D_x^s v).$$

Lemma 2.11 (Fractional Poincaré-Friedrichs inequality [7, 15]). *For $u \in H_0^\mu(\Lambda)$,*

$$\|u\| \leqslant C|u|_\mu,$$

and for $0 < s < \mu$, $s \neq n - 1/2, n \in \mathbb{N}$, $|u|_s \leqslant C|u|_\mu$.

Lemma 2.12 (Gagliardo-Nirenberg inequality [13]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain having the cone property and let $u \in L^q(\Omega)$ and its derivatives of order m, $D^m u$, belong to $L^r(\Omega)$, $1 \leqslant q, r \leqslant \infty$. For the derivatives $D^j u$, $0 \leqslant j < m$, we have*

$$\|D^j u\|_{L^p} \leqslant c\big(\|D^m u\|_{L^r} + \|u\|_{L^q}\big)^s \|u\|_{L^q}^{1-s},$$

where

$$\frac{1}{p} = \frac{j}{n} + s\Big(\frac{1}{r} - \frac{m}{n}\Big) + (1-s)\frac{1}{q}$$

for all s in the interval $\frac{j}{m} \leqslant s \leqslant 1$, (the constant c depending only on n, m, j, q, r, s), with the following exceptional case:
If $1 < r < \infty$, and $m - j - n/r$ is a nonnegative integer then (2.12) holds only for s satisfying $j/m \leqslant s < 1$.

The following discrete Gronwall's inequality is also used in the theoretical analysis.

Lemma 2.13 (Discrete Gronwall Lemma [14]). *Assume that k_n is a non-negative sequence, and that the sequence ϕ_n satisfies*

$$\begin{gathered} \phi_0 \leqslant g_0, \\ \phi_n \leqslant g_0 + \sum_{s=0}^{n-1} p_s + \sum_{s=0}^{n-1} k_s \phi_s, \quad n \geqslant 1. \end{gathered}$$

Then if $g_0 \geqslant 0$ and $p_n \geqslant 0$ for $n \geqslant 0$, it follows that

$$\phi_n \leqslant \Big(g_0 + \sum_{s=0}^{n-1} p_s\Big) \exp\Big(\sum_{s=0}^{n-1} k_s\Big), \quad n \geqslant 1.$$

The following lemma will be used in the proof of the existence of numerical solutions.

Lemma 2.14 ([18]). *Let X be a finite dimensional Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$, and Let P be a continuous mapping from X into itself such that*

$$(P(\xi), \xi) > 0 \quad \text{for } \|\xi\| = K > 0.$$

Then there exists $\xi \in X$, $\|\xi\| \leqslant K$, such that $P(\xi) = 0$.

We define

$$a(u, v) \triangleq \frac{1}{2\cos \pi\alpha}\big(({}_{-1}D_x^\alpha u, {}_x D_1^\alpha v) + ({}_x D_1^\alpha u, {}_{-1}D_x^\alpha v)\big), \quad \forall u, v \in H_0^\alpha(\Lambda), \tag{2.1}$$

for convenience. By the linearity of the left and right R-L derivatives, we can verify readily that $a(u, v)$ is a symmetric bilinear form, which has the following property.

Lemma 2.15. *The bilinear form $a(\cdot,\cdot)$ is continuous and coercive.*

Proof. By Hölder inequality, Lemmas 2.7 and 2.11, yield

$$\begin{aligned} |a(u, v)| &\leqslant \frac{1}{2|\cos \pi\alpha|}(\|{}_{-1}D_x^\alpha u\| \, \|{}_x D_1^\alpha v\| + \|{}_x D_1^\alpha u\| \, \|{}_{-1}D_x^\alpha v)\|) \\ &\leqslant C_1 |u|_\alpha |v|_\alpha, \quad \forall u, v \in H_0^\alpha(\Lambda), \end{aligned}$$

i.e., $a(\cdot,\cdot)$ is continuous on $H_0^\alpha(\Lambda) \times H_0^\alpha(\Lambda)$.

On the other hand, by Lemmas 2.9 and 2.11, we have

$$a(u,u)=\frac{1}{2\cos\pi\alpha}\big(({}_{-1}D_x^{\alpha}u,{}_xD_1^{\alpha}u)+({}_xD_1^{\alpha}u,{}_{-1}D_x^{\alpha}u)\big)=|u|_{\alpha}^2,\quad \forall u\in H_0^{\alpha}(\Lambda),$$

viz., $a(\cdot,\cdot)$ is coercive on $H_0^{\alpha}(\Lambda)$. The proof is complete. □

Let $P_N(\Lambda)$ be the set of all algebraic polynomials defined on domain Λ with the degree less than or equal to $N\in\mathbb{Z}^+$. $V_N^0=P_N(\Lambda)\cap H_0^1(\Lambda)$. The following projector $\Pi_N^{1,0}$, which is used below, can be found in [3].

Let $\Pi_N^{1,0}:H_0^1(\Lambda)\mapsto V_N^0$ be the orthogonal projection operator such that

$$\big(\partial_x(u-\Pi_N^{1,0}u),\partial_x\varphi\big)=0,\quad \forall\varphi\in V_N^0.$$

Lemma 2.16 (see [3])**.** *Let s be a real number. For any nonnegative real number r, $0\leqslant s\leqslant 1\leqslant r$, there exists a positive constant C depending only on r such that for any function u in $H_0^s(\Lambda)\cap H^r(\Lambda)$, the following estimate holds*

$$\|u-\Pi_N^{1,0}u\|_s\leqslant CN^{s-r}\|u\|_r.$$

We define projection $\Pi_N^{\alpha,0}:H_0^{\alpha}(\Lambda)\mapsto V_N^0$, such that

$$a(u-\Pi_N^{\alpha,0}u,v)=0,\quad \forall v\in V_N^0. \tag{2.2}$$

For the operation $\Pi_N^{\alpha,0}$, we have the following result.

Lemma 2.17. *Let $\alpha\in(1/2,1)$, $r\geqslant 1$ be a real number. There exists a positive constant C depending only on r, such that for any $u\in H_0^{\alpha}(\Lambda)\cap H^r(\Lambda)$, the following estimates hold*

$$\begin{gathered}|u-\Pi_N^{\alpha,0}u|_{\alpha}\leqslant CN^{\alpha-r}\|u\|_r,\\ \|u-\Pi_N^{\alpha,0}u\|\leqslant CN^{-r}\|u\|_r.\end{gathered}$$

Proof. By (2.2) and the continuous and coercivity of the bilinear form $a(\cdot,\cdot)$, we have

$$\begin{aligned}|u-\Pi_N^{\alpha,0}u|_{\alpha}^2&=a(u-\Pi_N^{\alpha,0}u,u-\Pi_N^{\alpha,0}u)\\&=a(u-\Pi_N^{\alpha,0}u,u-\Pi_N^{1,0}u)\\&\leqslant C|u-\Pi_N^{\alpha,0}u|_{\alpha}\|u-\Pi_N^{1,0}u\|_{\alpha},\quad \forall u\in V_N^0.\end{aligned}$$

Therefore, by Lemma 2.16, we obtain

$$|u-\Pi_N^{\alpha,0}u|_{\alpha}\leqslant C\|u-\Pi_N^{1,0}u\|_{\alpha}\leqslant CN^{\alpha-r}\|u\|_r,\quad \frac12<\alpha\leqslant r.$$

Next we estimate the error $\|u-\Pi_N^{\alpha,0}u\|$ by using a duality argument. For any $g\in L^2(\Lambda)$, we consider the auxiliary problem

$$\begin{gathered}-\mathcal{L}^{(\alpha)}w=g,\quad \text{in }\Lambda,\\ w=0,\quad \text{on }\partial\Lambda.\end{gathered}\tag{2.3}$$

By the definition of $\mathcal{L}^{(\alpha)}$ and Lemma 2.7, we obtain

$$\|w\|_{2\alpha}\leqslant C\|g\|. \tag{2.4}$$

The weak form of (2.3) is as follows:

$$a(\varphi,w)=(g,\varphi),\quad \forall\varphi\in H_0^{\alpha}(\Lambda).$$

Taking $\varphi = u - \Pi_N^{\alpha,0} u$, we obtain

$$\begin{aligned}(g, u - \Pi_N^{\alpha,0} u) &= a(u - \Pi_N^{\alpha,0} u, w)\\ &\leqslant C_1 \|u - \Pi_N^{\alpha,0} u\|_\alpha \|w - \Pi_N^{1,0} w\|_\alpha \\ &\leqslant CN^{-r} \|u\|_r \|w\|_{2\alpha}.\end{aligned} \tag{2.5}$$

Using (2.4) and (2.5), we have

$$\|u - \Pi_N^{\alpha,0} u\| = \sup_{g\in L^2(\Lambda),\, g\neq 0} \frac{|(g, u - \Pi_N^{\alpha,0} u)|}{\|g\|} \leqslant CN^{-r} \|u\|_r. \tag{2.6}$$

The proof is complete. □

3. Fully discrete scheme

In this section, we study the existence and uniqueness of the fully discrete scheme based on the modified Crank-Nicolson scheme in time and the Legendre spectral method in space .

Firstly, we introduce some notation. Let τ be the step size for time t, $t_k = k\tau$, $k = 0, 1, \dots, n_T$ and $T = n_T \tau$, $t_{k-\frac12} = (t_k + t_{k-1})/2$. For convenience, we introduce the following notation for the function $u(x, t)$,

$$u^k = u^k(\cdot) = u(\cdot, t_k), \quad u^{k-\frac12} = u(t_{k-\frac12}), \quad \bar\partial_t u^k = \frac{u^k - u^{k-1}}{\tau}, \quad u^{\bar k} = \frac{u^k + u^{k-1}}{2}.$$

The fully discrete spectral method for the problem (1.1) is: find $u_N^k \in V_N^0$, $k = 1, \dots n_T$, such that for all $\varphi \in V_N^0$,

$$\big(\bar\partial_t u_N^k, \varphi\big) + \epsilon^2 a(u_N^{\bar k}, \varphi) + \frac12 \Big(\big((u_N^k)^2 + (u_N^{k-1})^2\big) u_N^{\bar k}, \varphi\Big) = (u_N^{\bar k}, \varphi), \tag{3.1}$$

$$u_N^0 = \Pi_N^{\alpha,0} u_0. \tag{3.2}$$

For simplicity in what follows, we denote $g(u) = u^3$, $\tilde f(u, v) = \frac14 (u + v)(u^2 + v^2)$. A priori estimates are needed in the following analyses.

Lemma 3.1. *Suppose that u_N^k $(k = 0, 1, \dots, n_T)$ be the solution of (3.1)-(3.2), then we have*

$$\epsilon^2 |u_N^k|_\alpha^2 + \frac12 \|u_N^k\|_{L^4}^4 - \|u_N^k\|^2 \leqslant \epsilon^2 |u_N^0|_\alpha^2 + \frac12 \|u_N^0\|_{L^4}^4 - \|u_N^0\|^2.$$

Moreover, if $u_0 \in H^\alpha(\Lambda)$, we have

$$\|u_N^k\|_\infty \leqslant C(|u_N^k|_\alpha + \|u_N^k\|)^{\frac{1}{2\alpha}} \|u_N^k\|^{1-\frac{1}{2\alpha}} \leqslant C|u_0|_\alpha.$$

Proof. Taking $\varphi = u_N^k - u_N^{k-1}$ in (3.1), we obtain

$$\begin{aligned}&\tau \|\bar\partial_t u_N^k\|^2 + \epsilon^2 a(u_N^{\bar k}, u_N^k - u_N^{k-1}) + \Big(\tilde f(u_N^k, u_N^{k-1}), u_N^k - u_N^{k-1}\Big)\\ &- \frac12 (\|u_N^k\|^2 - \|u_N^{k-1}\|^2) = 0.\end{aligned} \tag{3.3}$$

For the middle two terms on the left hand side of (3.3), a simple computation yields

$$\begin{aligned}a(u_N^{\bar k}, u_N^k - u_N^{k-1}) &= \frac12 a(u_N^k, u_N^k) - \frac12 a(u_N^{k-1}, u_N^{k-1})\\ &= \frac12 (|u_N^k|_\alpha^2 - |u_N^{k-1}|_\alpha^2),\end{aligned} \tag{3.4}$$

$$\big(\tilde f(u_N^k,u_N^{k-1}),u_N^k-u_N^{k-1}\big)=\frac14\big(\|u_N^k\|_{L^4}^4-\|u_N^{k-1}\|_{L^4}^4\big). \tag{3.5}$$

Substituting (3.4) and (3.5) into (3.3), we obtain

$$\begin{aligned}&\tau\|\bar\partial_t u_N^k\|^2+\frac{\epsilon^2}{2}|u_N^k|_\alpha^2+\frac14\|u_N^k\|_{L^4}^4-\frac12\|u_N^k\|^2\\&=\frac{\epsilon^2}{2}|u_N^{k-1}|_\alpha^2+\frac14\|u_N^{k-1}\|_{L^4}^4-\frac12\|u_N^{k-1}\|^2.\end{aligned}$$

Then we can deduce that

$$\epsilon^2|u_N^k|_\alpha^2+\frac12\|u_N^k\|_{L^4}^4-\|u_N^k\|^2\leqslant \epsilon^2|u_N^0|_\alpha^2+\frac12\|u_N^0\|_{L^4}^4-\|u_N^0\|^2.$$

From above inequality, we have

$$|u_N^k|_\alpha^2\leqslant |u_N^0|_\alpha^2+\frac{1}{2\epsilon^2}(\|u_N^0\|_{L^4}^4+2).$$

By the definition of $\Pi_N^{\alpha,0}$, we obtain $|u_N^0|_\alpha\leqslant C|u_0|_\alpha$. Using Lemma 2.12, we have

$$\|u_N^0\|_{L^4}\leqslant C|u_0|_\alpha.$$

Therefore, we deduce that $|u_N^k|_\alpha^2\leqslant C(|u_0|_\alpha^2+1)$. By Lemma 2.12, we have

$$\|u_N^k\|_\infty\leqslant C|u_0|_\alpha\triangleq c_1.$$

The proof is complete. □

Remark 3.2. *Similarly, for the solution $u(x,t)$ of problem (1.1) we have $\|u\|_\infty\leqslant C|u_0|_\alpha$.*

Theorem 3.3 (Existence)**.** *For given $\{u_N^j\}_{j=0}^{k-1}$, there exists u_N^k satisfying (3.1).*

Proof. We defining the mapping $P:V_N^0\to V_N^0$, such that

$$\begin{aligned}(P(w),\varphi)=&(\frac{w}{\tau},\varphi)+\frac{\epsilon^2}{2}a(w+2u_N^{k-1},\varphi)+(\tilde f(w+u_N^{k-1},u_N^{k-1}),\varphi)\\&-\frac12(w+2u_N^{k-1},\varphi),\quad\forall\varphi\in V_N^0.\end{aligned}$$

Obviously P is continuous. Letting $\varphi=w$, we have

$$\begin{aligned}(P(w),w)=&\frac{\|w\|^2}{\tau}+\frac{\epsilon^2}{2}a(w+2u_N^{k-1},w)+(\tilde f(w+u_N^{k-1},u_N^{k-1}),w)\\&-\frac12(w+2u_N^{k-1},w).\end{aligned}\tag{3.6}$$

By a simple calculation, we obtain

$$\begin{aligned}(w+2u_N^{k-1},w)&=\big((w+u_N^{k-1})+u_N^{k-1},(w+u_N^{k-1})-u_N^{k-1}\big)\\&=\|w+u_N^{k-1}\|^2-\|u_N^{k-1}\|^2.\end{aligned}\tag{3.7}$$

Similarly, we have

$$a(w+2u_N^{k-1},w)=|w+u_N^{k-1}|_\alpha^2-|u_N^{k-1}|_\alpha^2,\tag{3.8}$$

$$(\tilde f(w+u_N^{k-1},u_N^{k-1}),w)=\frac14\big(\|w+u_N^{k-1}\|_{L^4}^4-\|u_N^{k-1}\|_{L^4}^4\big).\tag{3.9}$$

Substituting (3.7)-(3.9) into (3.6) yields

$$(P(w),w)=\frac{\|w\|^2}{\tau}+\frac12\Big(\epsilon^2|w+u_N^{k-1}|_\alpha^2+\frac12\|w+u_N^{k-1}\|_{L^4}^4-\|w+u_N^{k-1}\|^2\Big)$$

$$
\begin{aligned}
&- \frac{1}{2}\Big(\epsilon^2|u_N^{k-1}|_\alpha^2 + \frac{1}{2}\|u_N^{k-1}\|_{L^4}^4 - \|u_N^{k-1}\|^2\Big)\\
\geqslant& \frac{\|w\|^2}{\tau} - \frac{1}{2}\Big(\epsilon^2|u_N^{k-1}|_\alpha^2 + \frac{1}{2}\|u_N^{k-1}\|_{L^4}^4 - \|u_N^{k-1}\|^2 + 1\Big).
\end{aligned}
$$

By Lemma 3.1, we have

$$
(P(w), w) \geqslant \frac{\|w\|^2}{\tau} - \frac{1}{2}\Big(\epsilon^2|u_N^0|_\alpha^2 + \frac{1}{2}\|u_N^0\|_{L^4}^4 - \|u_N^0\|^2 + 1\Big).
$$

Thus we have $(P(w), w) > 0$, for $\|w\| = K > [\tau/2(\epsilon^2|u_N^0|_\alpha^2 + \frac{1}{2}\|u_N^0\|_{L^4}^4 - \|u_N^0\|^2 + 1)]^{1/2}$. By Lemma 2.14, there exists $w^{k-1} \in V_N^0$, $\|w^{k-1}\| \leqslant K$, such that $P(w^{k-1}) = 0$. Let $u_N^k = u_N^{k-1} + w^{k-1}$, therefore the existence of u_N^k is proved. $\square$

Theorem 3.4 (Uniqueness). *Suppose $\tau < 1$, then the solution of* (3.1)-(3.2) *is unique.*

Proof. Let $\{u_N^k\}$, $\{v_N^k\}$ be the two solutions of the discrete scheme (3.1)-(3.2) with the same initial condition. Let $w_N^k = u_N^k - v_N^k$, thus we have

$$
(\bar\partial_t w_N^k, \varphi) + \epsilon^2 a(w_N^{\bar k}, \varphi) + \big(\tilde f(u_N^k, u_N^{k-1}) - \tilde f(v_N^k, v_N^{k-1}), \varphi\big) = (w_N^{\bar k}, \varphi), \quad \forall \varphi \in V_N^0.
$$

Setting $\varphi = \bar\partial_t w_N^k$, we obtain

$$
\|\bar\partial_t w_N^k\|^2 + \frac{\epsilon^2}{2\tau}\big(|w_N^k|_\alpha^2 - |w_N^{k-1}|_\alpha^2\big) + \big(\tilde f(u_N^k, u_N^{k-1}) - \tilde f(v_N^k, v_N^{k-1}), \bar\partial_t w_N^k\big) \tag{3.10}
$$

$$
= (w_N^{\bar k}, \bar\partial_t w_N^k). \tag{3.11}
$$

Now, we estimate the last two terms of the above equation. For the last term, by Young inequality, we have

$$
\begin{aligned}
(w_N^{\bar k}, \bar\partial_t w_N^k) =& \frac{1}{2}\big(w_N^k - w_N^{k-1}, \bar\partial_t w_N^k\big) + (w_N^{k-1}, \bar\partial_t w_N^k)\\
\leqslant& \frac{\tau}{2}\|\bar\partial_t w_N^k\|^2 + \frac{1}{4}\|\bar\partial_t w_N^k\|^2 + \|w_N^{k-1}\|^2.
\end{aligned} \tag{3.12}
$$

For the penultimate term, noting that

$$
\tilde f(u, v) = \frac{1}{4}(u+v)(u^2+v^2) = \int_0^1 g(v + s(u-v))\,\mathrm{d}s, \quad g'(s) = 3s^2 \geqslant 0;
$$

therefore, utilizing the mean-value theorem of differentials, we obtain

$$
\begin{aligned}
&\big(\tilde f(u_N^k, u_N^{k-1}) - \tilde f(v_N^k, v_N^{k-1}), \bar\partial_t w_N^k\big)\\
&= \tau\Big(\int_0^1 g'(\xi) s\,\mathrm{d}s, (\bar\partial_t w_N^k)^2\Big) + \Big(\int_0^1 g'(\xi) w_N^{k-1}\,\mathrm{d}s, \bar\partial_t w_N^k\Big)\\
&\geqslant \Big(\int_0^1 g'(\xi) w_N^{k-1}\,\mathrm{d}s, \bar\partial_t w_N^k\Big),
\end{aligned} \tag{3.13}
$$

where ξ lies in the interval with endpoints $u_N^{k-1} + s(u_N^k - u_N^{k-1})$ and $v_N^{k-1} + s(v_N^k - v_N^{k-1})$. By Lemma 3.1, Hölder inequality and Young inequality yield

$$
\Big|\Big(\int_0^1 g'(\xi)\,\mathrm{d}s w_N^{k-1}, \bar\partial_t w_N^k\Big)\Big| \leqslant \frac{1}{4}\|\bar\partial_t w_N^k\|^2 + 9c_1^4\|w_N^{k-1}\|^2. \tag{3.14}
$$

Substituting (3.12), (3.14) into (3.10), noticing that $\tau < 1$, we deduce that

$$
\frac{\epsilon^2}{2\tau}\big(|w_N^k|_\alpha^2 - |w_N^{k-1}|_\alpha^2\big) \leqslant (9c_1^4 + 1)\|w_N^{k-1}\|^2.
$$

Summing for k from 1 to n, and by using Lemma 2.11, we deduce that

$$|w_N^n|_\alpha^2 \leqslant |w_N^0|_\alpha^2 + C(9c_1^4+1)\tau \sum_{i=0}^{n-1} |w_N^i|_\alpha^2.$$

Thus by Lemma 2.13, we obtain

$$|w_N^k|_\alpha^2 \leqslant e^{TC(9c_1^4+1)} |w_N^0|_\alpha^2 = 0.$$

Finally, using Lemma 2.11 once again, we have $\|w_N^k\| = 0$, i.e., $u_N^k = v_N^k$, $k = 0, 1, \dots, n_T$. The proof of the uniqueness is complete. $\square$

4. Stability and convergence of the fully discrete scheme

In this section, we give the stability and convergence analysis for the fully discrete scheme (3.1)–(3.2).

Theorem 4.1 (Stability)**.** *Assume $\tau < 1$, u_N^k, v_N^k $(k = 1, 2, \dots, n_T)$ be the solutions of the fully discrete scheme* (3.1) *with the initial value u_N^0, v_N^0, respectively. Then we have*

$$\|u_N^k - v_N^k\|_\alpha \leqslant e^{TC(9c_1^4+1)} \|u_N^0 - v_N^0\|_\alpha, \quad k = 1, 2, \dots, n_T.$$

Proof. Let $w_N^k = u_N^k - v_N^k$ in (3.1), then w_N^k satisfies

$$(\bar\partial_t w_N^k, \varphi) + \epsilon^2 a(w_N^{\bar k}, \varphi) + \big(\tilde f(u_N^k, u_N^{k-1}) - \tilde f(v_N^k, v_N^{k-1}), \varphi\big) = (w_N^{\bar k}, \varphi), \quad \forall \varphi \in V_N^0.$$

With the same line of the proof as for Theorem 3.4, we obtain the desired result. The proof is complete. $\square$

Now, we give the convergence result of the fully discrete scheme (3.1)-(3.2).

Theorem 4.2 (Convergence)**.** *Let u and u_N^n $(1 \leqslant n \leqslant n_T)$ be the solutions of* (1.1) *and* (3.1)-(3.2)*, respectively. Assume that $u \in L^\infty(0, T; H^m(\Lambda))$, $m > 2\alpha$, $u_t \in L^4(0, T; L^4(\Lambda)) \cap L^2(0, T; H^m(\Lambda))$, $u_{tt} \in L^2(0, T; H^{2\alpha}(\Lambda))$, $u_{ttt} \in L^2(0, T; L^2(\Lambda))$. Then for $\tau < 1$, there exists a positive constant c independent of τ and N, such that*

$$\|u^n - u_N^n\| \leqslant c(\tau^2 + N^{-m}) \quad \text{and} \quad |u^n - u_N^n|_\alpha \leqslant c(\tau^2 + N^{\alpha-m}).$$

Proof. Setting $u^k - u_N^k = (u^k - \Pi_N^{\alpha,0} u^k) + (\Pi_N^{\alpha,0} u^k - u_N^k) = \theta^k + \eta^k$. By (1.1), (3.1)-(3.2) and the definition of $\Pi_N^{\alpha,0}$, we have the error equation

$$\begin{gathered}
(\bar\partial_t \eta^k, v) + \epsilon^2 a(\eta^{\bar k}, v) = (\bar\partial_t u^k - u_t^{k-\frac12}, v) - (\bar\partial_t \theta^k, v) + (u^{k-\frac12} - u_N^{\bar k}, v) \\
+ \epsilon^2 a(u^{\bar k} - u^{k-\frac12}, v) + \big(\tilde f(u_N^k, u_N^{k-1}) - g(u^{k-\frac12}), v\big), \\
\eta^0 = 0.
\end{gathered} \tag{4.1}$$

Taking $v = \bar\partial_t \eta^k$ in (4.1), we have

$$\begin{aligned}
&\|\bar\partial_t \eta^k\|^2 + \frac{\epsilon^2}{2\tau}(|\eta^k|_\alpha^2 - |\eta^{k-1}|_\alpha^2) \\
&= (\bar\partial_t u^k - u_t^{k-\frac12}, \bar\partial_t \eta^k) - (\bar\partial_t \theta^k, \bar\partial_t \eta^k) + (u^{k-\frac12} - u_N^{\bar k}, \bar\partial_t \eta^k) \\
&\quad + \epsilon^2 a(u^{\bar k} - u^{k-\frac12}, \bar\partial_t \eta^k) + \big(\tilde f(u_N^k, u_N^{k-1}) - g(u^{k-\frac12}), \bar\partial_t \eta^k\big) \\
&\triangleq \sum_{i=1}^5 G_i.
\end{aligned} \tag{4.2}$$

Now we estimate terms on the right-hand side of (4.2). Via Taylor's theorem with integral remainder, Hölder inequality and Young inequality, we deduce that

$$
\begin{aligned}
&|G_1| \\
=&|(\bar{\partial}_t u^k - u_t^{k-\frac{1}{2}}, \bar{\partial}_t \eta^k)| \\
\leqslant&\frac{1}{2\tau}\|\bar{\partial}_t\eta^k\|\Big(\|\int_{t_{k-1}}^{t_{k-\frac{1}{2}}}(t_{k-1}-t)^2 u_{ttt}\,\mathrm{d}t\| + \|\int_{t_{k-\frac{1}{2}}}^{t_k}(t_k-t)^2 u_{ttt}\,\mathrm{d}t\|\Big) \\
\leqslant&\frac{1}{16}\|\bar{\partial}_t\eta^k\|^2 + \frac{2}{\tau^2}\Big(\|\int_{t_{k-1}}^{t_{k-\frac{1}{2}}}(t_{k-1}-t)^2 u_{ttt}\,\mathrm{d}t\|^2 + \|\int_{t_{k-\frac{1}{2}}}^{t_k}(t_k-t)^2 u_{ttt}\,\mathrm{d}t\|^2\Big).
\end{aligned}
\tag{4.3}
$$

By Hölder's inequality, we obtain

$$
\begin{aligned}
\|\int_{t_{k-1}}^{t_{k-\frac{1}{2}}}(t_{k-1}-t)^2 u_{ttt}\,\mathrm{d}t\|^2 &\leqslant \int_{-1}^{1}\Big(\int_{t_{k-1}}^{t_{k-\frac{1}{2}}}(t_{k-1}-t)^4\,\mathrm{d}t\Big)\Big(\int_{t_{k-1}}^{t_{k-\frac{1}{2}}}|u_{ttt}|^2\,\mathrm{d}t\Big)\,\mathrm{d}x \\
&= \frac{\tau^5}{5\cdot 2^5}\int_{t_{k-1}}^{t_{k-\frac{1}{2}}}\|u_{ttt}\|^2\,\mathrm{d}t,
\end{aligned}
$$

and

$$
\|\int_{t_{k-\frac{1}{2}}}^{t_k}(t_k-t)^2 u_{ttt}\,\mathrm{d}t\|^2 \leqslant \frac{\tau^5}{5\cdot 2^5}\int_{t_{k-\frac{1}{2}}}^{t_k}\|u_{ttt}\|^2\,\mathrm{d}t.
$$

Substituting two inequalities above into (4.3) yields

$$
|G_1| \leqslant \frac{1}{16}\|\bar{\partial}_t\eta^k\|^2 + \frac{\tau^3}{80}\int_{t_{k-1}}^{t_k}\|u_{ttt}\|^2\,\mathrm{d}t. \tag{4.4}
$$

By Hölder inequality and Young inequality, we have

$$
|G_2| = |(\bar{\partial}_t\theta^k, \bar{\partial}_t\eta^k)| \leqslant \frac{1}{16}\|\bar{\partial}_t\eta^k\|^2 + 4\|\bar{\partial}_t\theta^k\|^2.
$$

From Hölder inequality and Lemma 2.17, we obtain

$$
\|\bar{\partial}_t\theta^k\|^2 = \frac{1}{\tau^2}\Big\|\int_{t_{k-1}}^{t_k}\theta_t\mathrm{d}t\Big\|^2 \leqslant \frac{1}{\tau}\int_{t_{k-1}}^{t_k}\|\theta_t\|^2\,\mathrm{d}t \leqslant \frac{C}{\tau}N^{-2m}\int_{t_{k-1}}^{t_k}\|u_t\|_m^2\,\mathrm{d}t.
$$

Thus we obtain

$$
|G_2| \leqslant \frac{1}{16}\|\bar{\partial}_t\eta^k\|^2 + \frac{C}{\tau}N^{-2m}\int_{t_{k-1}}^{t_k}\|u_t\|_m^2\,\mathrm{d}t. \tag{4.5}
$$

Next, via a simple derivation, we have

$$
|G_3| = |(u^{k-\frac{1}{2}} - u_N^{\bar{k}}, \bar{\partial}_t\eta^k)| \leqslant |(u^{k-\frac{1}{2}} - u^{\bar{k}}, \bar{\partial}_t\eta^k)| + |(u^{\bar{k}} - u_N^{\bar{k}}, \bar{\partial}_t\eta^k)|. \tag{4.6}
$$

Similar to (4.3), by Taylor's theorem with integral remainder, Hölder and Young inequalities, we find that

$$
\begin{aligned}
&|(u^{k-\frac{1}{2}} - u^{\bar{k}}, \bar{\partial}_t\eta^k)| \\
&\leqslant \frac{1}{2}\|\bar{\partial}_t\eta^k\|\Big(\|\int_{t_{k-1}}^{t_{k-\frac{1}{2}}}(t-t_{k-1})u_{tt}\,\mathrm{d}t\| + \|\int_{t_{k-\frac{1}{2}}}^{t_k}(t_k-t)u_{tt}\,\mathrm{d}t\|\Big) \\
&\leqslant \frac{1}{32}\|\bar{\partial}_t\eta^k\|^2 + \frac{\tau^3}{6}\int_{t_{k-1}}^{t_k}\|u_{tt}\|^2\,\mathrm{d}t
\end{aligned}
$$

and

$$
\begin{aligned}
|(u^{\bar{k}} - u_N^{\bar{k}}, \bar{\partial}_t \eta^k)| \leqslant& |(\theta^{\bar{k}} + \eta^{k-1}, \bar{\partial}_t \eta^k)| + \big|\big(\frac{\tau}{2}\bar{\partial}_t \eta^k, \bar{\partial}_t \eta^k\big)\big| \\
\leqslant& \big(\frac{\tau}{2} + \frac{1}{32}\big)\|\bar{\partial}_t \eta^k\|^2 + 16(\|\theta^{\bar{k}}\|^2 + \|\eta^{k-1}\|^2).
\end{aligned}
$$

Thus, substituting above two inequalities into (4.6), we obtain

$$
|G_3| \leqslant \big(\frac{\tau}{2} + \frac{1}{16}\big)\|\bar{\partial}_t \eta^k\|^2 + 16(\|\theta^{\bar{k}}\|^2 + \|\eta^{k-1}\|^2) + \frac{\tau^3}{6}\int_{t_{k-1}}^{t_k} \|u_{tt}\|^2 \,\mathrm{d}t. \tag{4.7}
$$

Next, by using Lemma2.7 and analogous to the estimation of G_3, we deduce that

$$
\begin{aligned}
|G_4| =& \epsilon^2 |a(u^{k-\frac{1}{2}} - u^{\bar{k}}, \bar{\partial}_t \eta^k)| \\
=& \epsilon^2 \big|\big(\mathcal{L}^{(\alpha)}(u^{k-\frac{1}{2}} - u^{\bar{k}}), \bar{\partial}_t \eta^k\big)\big| \\
\leqslant& \frac{\epsilon^2}{|\cos \pi\alpha|}\|\bar{\partial}_t \eta^k\| |u^{k-\frac{1}{2}} - u^{\bar{k}}|_{2\alpha} \\
\leqslant& \frac{1}{16}\|\bar{\partial}_t \eta^k\|^2 + C\tau^3 \int_{t_{k-1}}^{t_k} |u_{tt}|_{2\alpha}^2 \,\mathrm{d}t.
\end{aligned} \tag{4.8}
$$

Now we estimate the last term on the right-hand side of (4.2). It is easy to obtain

$$
\begin{aligned}
G_5 =& \big(\tilde{f}(u_N^k, u_N^{k-1}) - g(u^{k-\frac{1}{2}}), \bar{\partial}_t \eta^k\big) \\
=& \big(\tilde{f}(u_N^k, u_N^{k-1}) - \tilde{f}(u^k, u^{k-1}), \bar{\partial}_t \eta^k\big) + \big(\tilde{f}(u^k, u^{k-1}) - g(u^{k-\frac{1}{2}}), \bar{\partial}_t \eta^k\big).
\end{aligned} \tag{4.9}
$$

For the first term of the right hand side of (4.9), analogous to (3.13) and (3.14), we find that

$$
\begin{aligned}
&\big(\tilde{f}(u_N^k, u_N^{k-1}) - \tilde{f}(u^k, u^{k-1}), \bar{\partial}_t \eta^k\big) \\
&= -\Big(\int_0^1 3\xi_1^2 \,\mathrm{d}s\theta^{k-1}, \bar{\partial}_t \eta^k\Big) - \Big(\int_0^1 3\xi_1^2 s \,\mathrm{d}s(\theta^k - \theta^{k-1}), \bar{\partial}_t \eta^k\Big) \\
&\quad - \Big(\int_0^1 3\xi_1^2 \,\mathrm{d}s\eta^{k-1}, \bar{\partial}_t \eta^k\Big) - \Big(\int_0^1 3\xi_1^2 s \,\mathrm{d}s(\eta^k - \eta^{k-1}), \bar{\partial}_t \eta^k\Big) \\
&\leqslant \frac{3}{16}\|\bar{\partial}_t \eta^k\|^2 + 36c_1^4(\|\theta^{k-1}\|^2 + \|\theta^k\|^2 + \|\eta^{k-1}\|^2),
\end{aligned}
$$

where c_1 depends on u.

For the last term of the right-hand side of (4.9), applying Taylor's formula for multivariate functions, we find that

$$
\begin{aligned}
&\big(\tilde{f}(u^k, u^{k-1}) - g(u^{k-\frac{1}{2}}), \bar{\partial}_t \eta^k\big) \\
&= 3\big((u^{k-\frac{1}{2}})^2(u^{\bar{k}} - u^{k-\frac{1}{2}}), \bar{\partial}_t \eta^k\big) + \frac{1}{4}\big((3\xi_2 + \xi_3)(u^k - u^{k-\frac{1}{2}})^2, \bar{\partial}_t \eta^k\big) \\
&\quad + \frac{1}{4}\big((\xi_2 + 3\xi_3)(u^{k-1} - u^{k-\frac{1}{2}})^2, \bar{\partial}_t \eta^k\big) \\
&\quad + \frac{1}{2}\big((\xi_2 + \xi_3)(u^{k-1} - u^{k-\frac{1}{2}})(u^k - u^{k-\frac{1}{2}}), \bar{\partial}_t \eta^k\big) \\
&\leqslant \frac{1}{16}\|\bar{\partial}_t \eta^k\|^2 + C\tau^3 \int_{t_{k-1}}^{t_k} \big(c_1^2\|u_t\|_{L^4}^4 + c_1^4\|u_{tt}\|^2\big) \,\mathrm{d}t.
\end{aligned}
$$

Substituting the two inequalities above into (4.9), we have

$$\begin{aligned} |G_5| \leqslant & \frac{1}{4}\|\bar{\partial}_t\eta^k\|^2 + 36c_1^4(\|\theta^{k-1}\|^2 + \|\theta^k\|^2 + \|\eta^{k-1}\|^2) && (4.10)\\ & + C\tau^3\int_{t_{k-1}}^{t_k}\big(c_1^2\|u_t\|_{L^4}^4 + c_1^4\|u_{tt}\|^2\big)\,\mathrm{d}t. && (4.11)\end{aligned}$$

Substituting (4.4), (4.5), (4.7), (4.8) and (4.10) into (4.2), in view of $\tau < 1$, we obtain

$$\begin{aligned}&\frac{\epsilon^2}{2\tau}(|\eta^k|_\alpha^2 - |\eta^{k-1}|_\alpha^2)\\ &\leqslant C(1+c_1^4)\|\eta^{k-1}\|^2 + CN^{-2m}\Big((1+c_1^4)\|u\|^2_{L^\infty(0,T;H^m(\Lambda))} + \frac{1}{\tau}\int_{t_{k-1}}^{t_k}\|u_t\|_m^2\,\mathrm{d}t\Big)\\ &\quad + C\tau^3\int_{t_{k-1}}^{t_k}\big(c_1^2\|u_t\|_{L^4}^4 + c_1^4\|u_{tt}\|^2 + \|u_{tt}\|_{2\alpha}^2 + \|u_{ttt}\|^2\big)\,\mathrm{d}t.\end{aligned}$$

Summing for k from 1 to n ($n \leqslant n_T$) we have

$$|\eta^n|_\alpha^2 \leqslant C(1+c_1^4)\tau\sum_{j=0}^{n-1}\|\eta^j\|^2 + c_2(\tau^4 + N^{-2m}),$$

where $c_2 = \max(c_3, c_4)$, and

$$\begin{aligned} c_3 =& C\big(c_1^2\|u_t\|^4_{L^4(0,T;L^4(\Lambda))} + c_1^4\|u_{tt}\|^2_{L^2(0,T;L^2(\Lambda))} + \|u_{tt}\|^2_{L^2(0,T;H^{2\alpha}(\Lambda))}\\ &+ \|u_{ttt}\|^2_{L^2(0,T;L^2(\Lambda))}\big),\\ c_4 =& C\big((1+c_1^4)\|u\|^2_{L^\infty(0,T;H^m(\Lambda))} + \|u_t\|^2_{L^2(0,T;H^m(\Lambda))}\big).\end{aligned}$$

Using Lemma 2.13, we have

$$|\eta^n|_\alpha^2 \leqslant c_2\mathrm{e}^{CT(1+c_1^4)}(\tau^4 + N^{-2m}).$$

Utilizing Lemma 2.11 and the triangle inequality, we have

$$\|u^n - u_N^n\| \leqslant c(\tau^2 + N^{-m}), \quad |u^n - u_N^n|_\alpha \leqslant c(\tau^2 + N^{\alpha-m}),$$

where $c = C(1 + c_2\mathrm{e}^{CT(1+c_1^4)})$. The proof is complete. $\square$

5. Numerical experiments

Example 5.1. To guarantee the exact solution have enough regularity, we add a forcing term on the equation.

$$\begin{gathered} u_t - \epsilon^2\mathcal{L}^{(\alpha)}u + u^3 - u = h(x,t), \quad x \in (-1,1), t \in (0,T],\\ u(x,0) = (1-x^2)^2, \quad x \in (-1,1),\\ u(\pm 1, t) = 0, \quad t \in [0,T]. \end{gathered} \tag{5.1}$$

where

$$\begin{aligned} h(x,t) = & \frac{4\epsilon^2 e^t}{\Gamma(5-2\alpha)\cos\pi\alpha}(1+x)^{2-2\alpha}\big[3(1+x)^2 - 3(4-2\alpha)(1+x)\\ & + (4-2\alpha)(3-2\alpha)\big] + \frac{4\epsilon^2 e^t}{\Gamma(5-2\alpha)\cos\pi\alpha}(1-x)^{2-2\alpha}\big[3(1-x)^2\\ & - 3(4-2\alpha)(1-x) + (4-2\alpha)(3-2\alpha)\big] + e^{3t}(1-x^2)^6. \end{aligned}$$

The exact solution is $u(x,t) = e^t(1-x^2)^2$. Here we select $\epsilon = 0.01$.

To confirm the temporal accuracy, we choose $N = 50$, which is large enough such that the spatial error is negligible compared with the temporal error. Table 1 lists the errors $\|u - u_N\|$ and temporal convergence orders at time $T = 2$ with different order α. From the table, we can check that temporal convergence order, almost second-order, are in accordance with the theoretical result in Theorem 4.2.

TABLE 1. L^2 errors and temporal convergence order for Example 5.1.

τ	$\alpha = 0.55$		$\alpha = 0.75$		$\alpha = 0.95$	
	Error	Order	Error	Order	Error	Order
1/10	1.1110e-02	–	1.1110e-02	–	1.1110e-02	–
1/20	2.7896e-03	1.9937	2.7896e-03	1.9937	2.7897e-03	1.9937
1/40	6.9817e-04	1.9984	6.9817e-04	1.9984	6.9817e-04	1.9984
1/80	1.7459e-04	1.9996	1.7459e-04	1.9996	1.7459e-04	1.9996
1/160	4.3650e-05	1.9999	4.3652e-05	1.9998	4.3651e-05	1.9999

Next, we investigate the spatial accuracy. We take $\alpha = 0.9$, $T = 2$, and $\tau = 0.001$ in order that the temporal discretization error is negligible compared with the spatial discretization error. As shown in Figure 1, the L^2 errors of the numerical solution decay exponentially as the polynomial degree N increased. Solution u is sufficient smooth with respect to spatial variable x, thus the numerical result is coincide with Theorem 4.2.

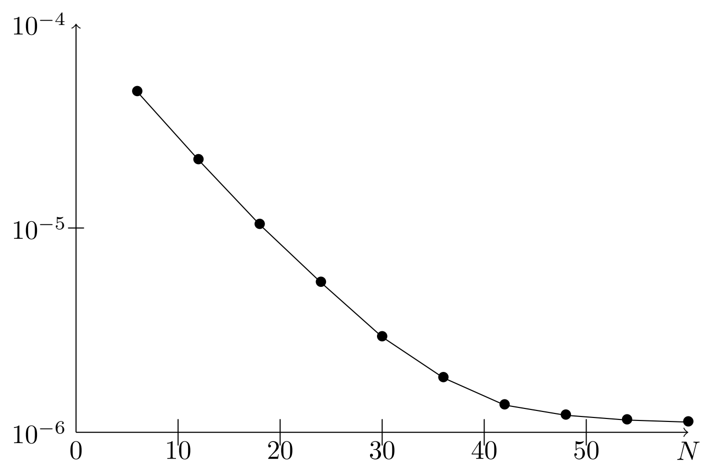

FIGURE 1. L^2-errors versus polynomial degree N, in Example 5.1 with $\alpha = 0.9$.

,

Example 5.2. Consider the problem

$$\begin{gathered} u_t - \mathcal{L}^{(\alpha)}u + f(u) = 0, \quad x \in (-1,1),\ t \in (0,2], \\ u(x,0) = (1-x^2)^2 x^{\frac{11}{3}}, \quad x \in (-1,1), \\ u(\pm 1, t) = 0, \quad t \in [0,2]. \end{gathered} \tag{5.2}$$

We can easily to verify that u_0 is finite regular. The reference solution is with $N = 70$ and $\tau = 0.001$. The temporal convergence orders are shown in Table 2. We observe that the convergence orders are almost 2.

TABLE 2. L^2 errors and temporal convergence orders for Example 5.2.

τ	$\alpha = 0.55$		$\alpha = 0.75$		$\alpha = 0.95$	
	Error	Order	Error	Order	Error	Order
1/10	4.2154e-04	–	4.2113e-04	–	4.2031e-04	–
1/20	1.0526e-04	2.0017	1.0516e-04	2.0017	1.0495e-04	2.0017
1/40	2.6277e-05	2.0020	2.6252e-05	2.0021	2.6201e-05	2.0021
1/80	6.5375e-06	2.0070	6.5312e-06	2.0070	6.5184e-06	2.0070
1/160	1.6032e-06	2.0278	1.6014e-06	2.0280	1.5981e-06	2.0282

Figure 2 shows the errors with respect to the polynomial degree N in a log-log scale with various α for Example 5.2. We can see from Figure 2 that the convergence rate is between 2 and 3.

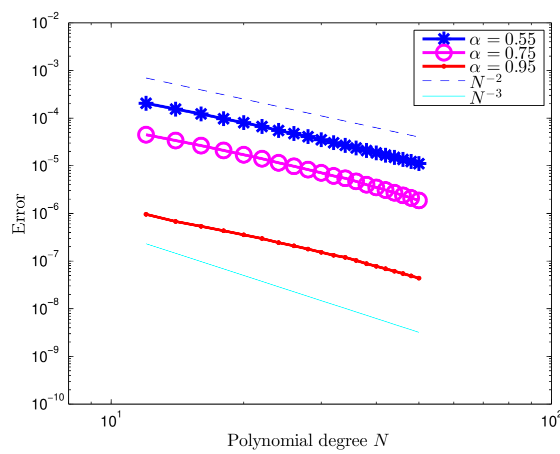

FIGURE 2. L^2-errors versus polynomial degree N in Example 5.2.

From above two numerical examples, we can see that when solution is sufficient smooth, the error is decay exponentially in space, when solution is finite regular, the error is Algebraic decay in space.

Conclusions. In this article, we studied the spectral approximation for an initial boundary value problem of the space fractional Allen-Cahn equation. A modified Crank-Nicolson Legendre spectral fully discrete scheme is established. We discussed the existence and uniqueness of the fully discrete scheme. The stability and convergence of the fully discrete scheme are proved strictly. By constructing a dual auxiliary problem, the convergence order of the scheme proved to be optimal in L^2 and H^α norm. The numerical experiments confirm the convergence analysis.

There are several avenues for further research, it is of interest to extend the results to other types of boundary conditions, time-/space-time fractional model and higher dimensional problems.

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